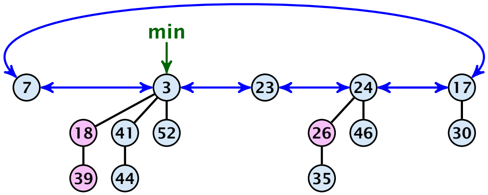
# 8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



# 8.3 Fibonacci Heaps

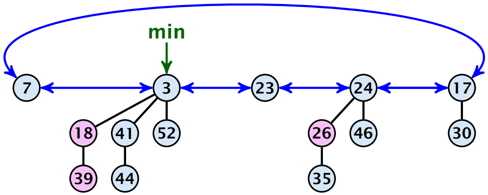
**Additional implementation details:**

- Every node $x$ stores its degree in a field $x.\mathrm{degree}$. Note that this can be updated in constant time when adding a child to $x$.
- Every node stores a boolean value $x.\mathrm{marked}$ that specifies whether $x$ is marked or not.

# 8.3 Fibonacci Heaps

**The potential function:**

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.




The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

# 8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use $c$ to denote the amount of work that a unit of potential can pay for.

# 8.3 Fibonacci Heaps

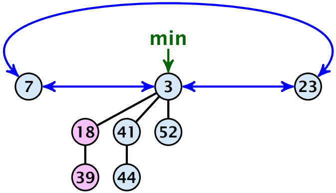
**$S$. minimum()**

- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

# 8.3 Fibonacci Heaps

## $S.\textbf{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer



# 8.3 Fibonacci Heaps

$S.\mathbf{merge}(S')$

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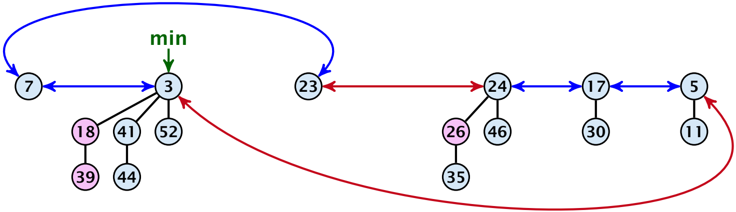
- Adjust the min-pointer




**Running time:**

- Actual cost $\mathcal{O}(1)$.

# 8.3 Fibonacci Heaps

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- Merge the root lists.
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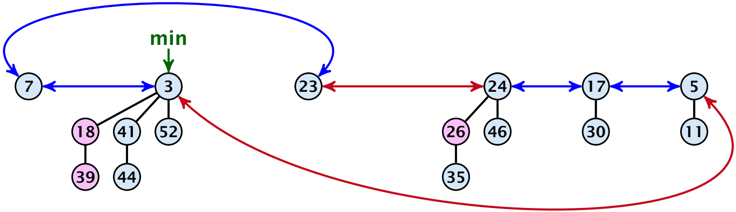


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# 8.3 Fibonacci Heaps

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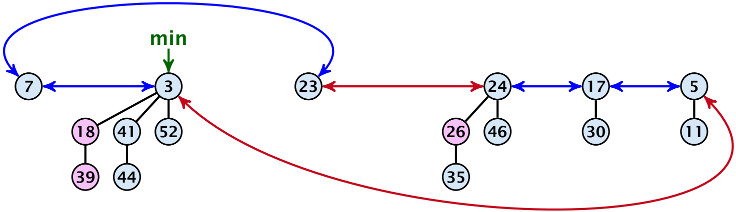


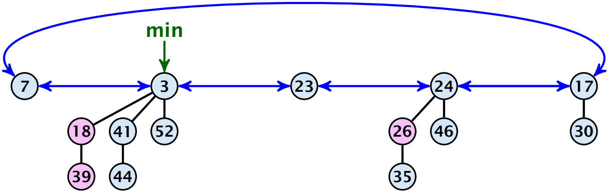

**Running time:**

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$.

# 8.3 Fibonacci Heaps

$S.\mathbf{insert}(x)$

- Create a new tree containing $x$.
- Insert $x$ into the root-list.
- Update min-pointer, if necessary.



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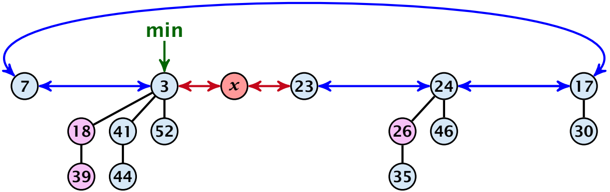

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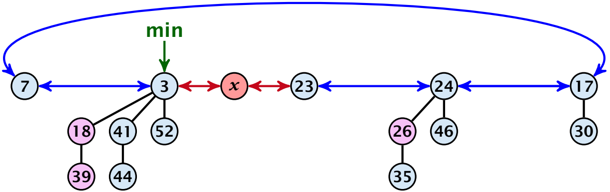
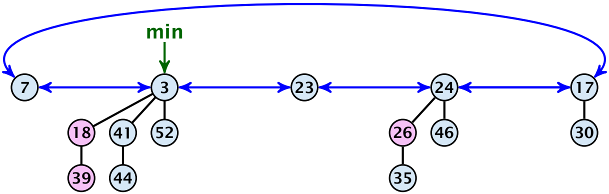



**Running time:**

- Actual cost $\mathcal{O}(1)$.
- Change in potential is $+1$.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

# 8.3 Fibonacci Heaps

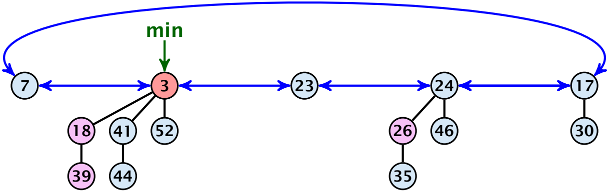
$S.$ **delete-min**$(x)$



## 8.3 Fibonacci Heaps

**$S$. delete-min$(x)$**

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.



# 8.3 Fibonacci Heaps

**$S$. delete-min($x$)**

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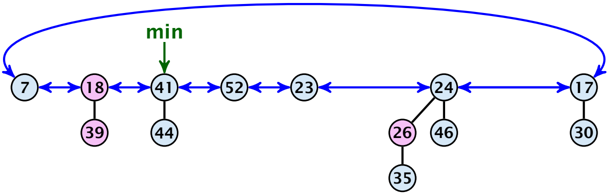
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



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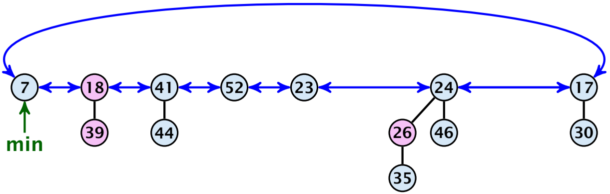

# 8.3 Fibonacci Heaps

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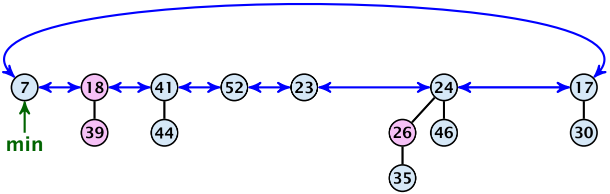
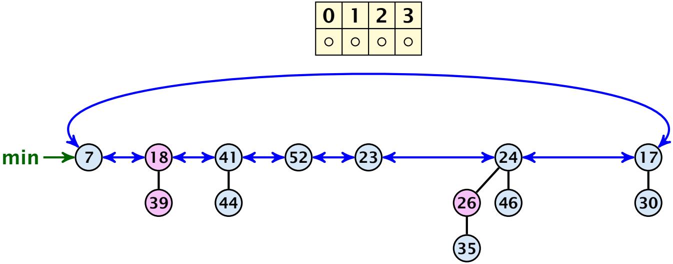



- Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

# 8.3 Fibonacci Heaps

**Consolidate:**



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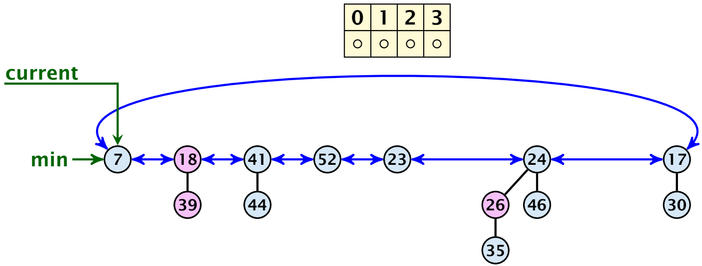

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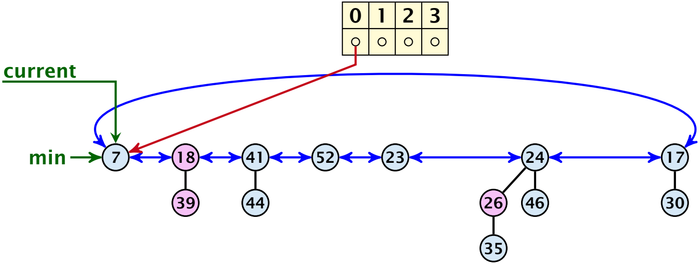

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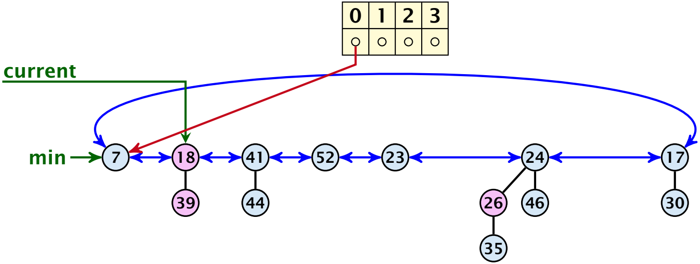

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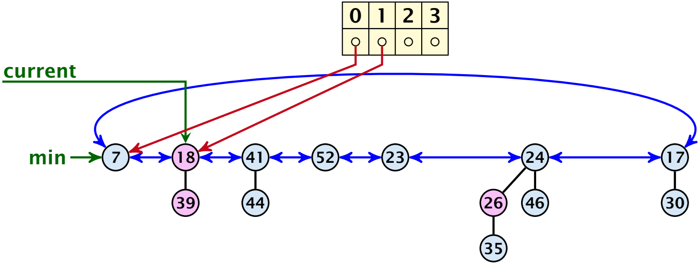

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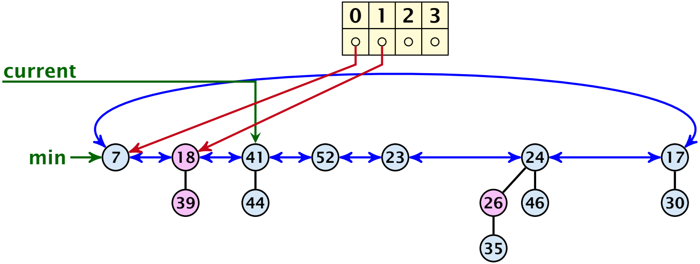

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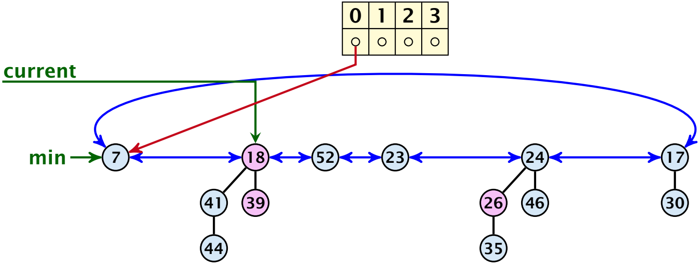

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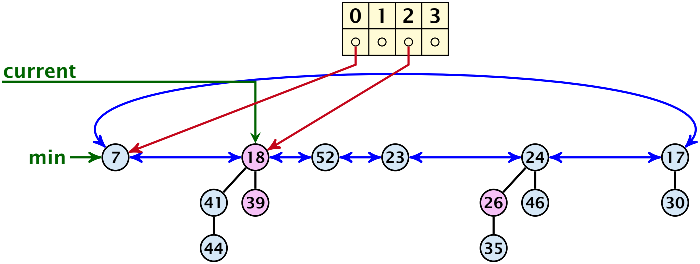

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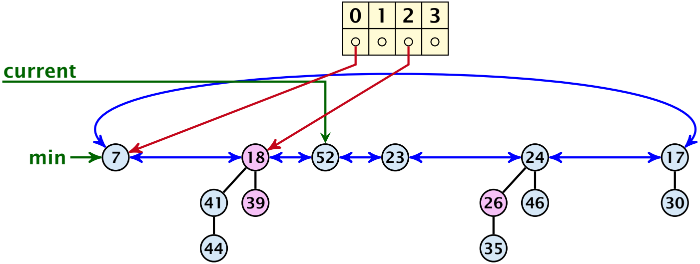

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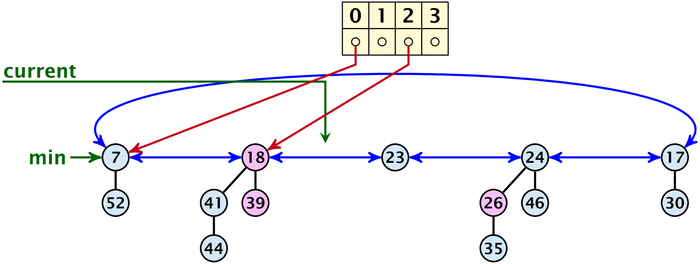

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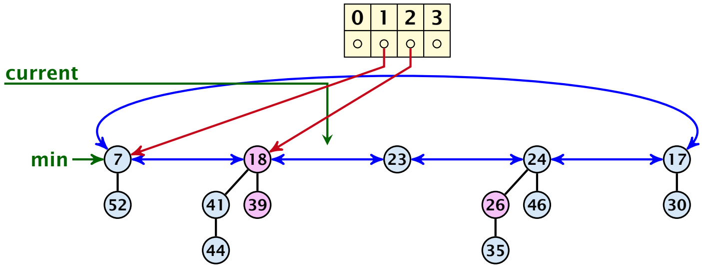

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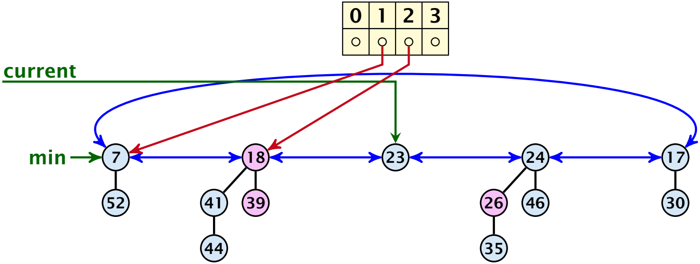

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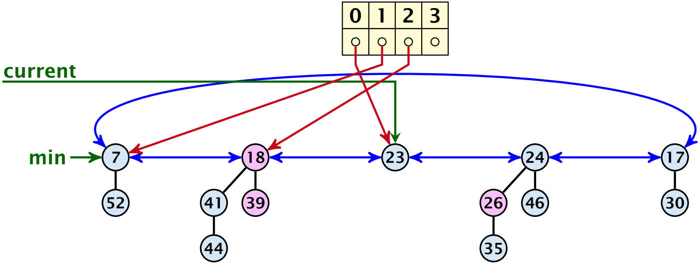

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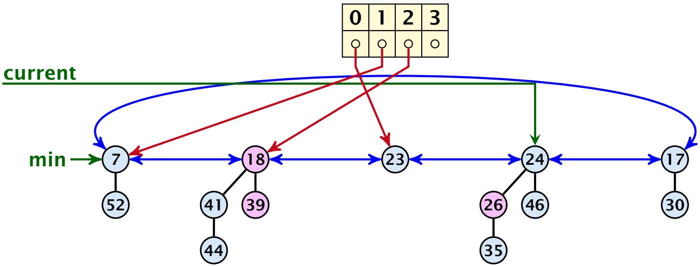

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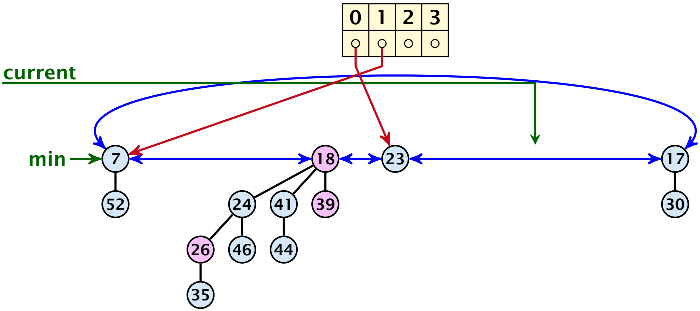

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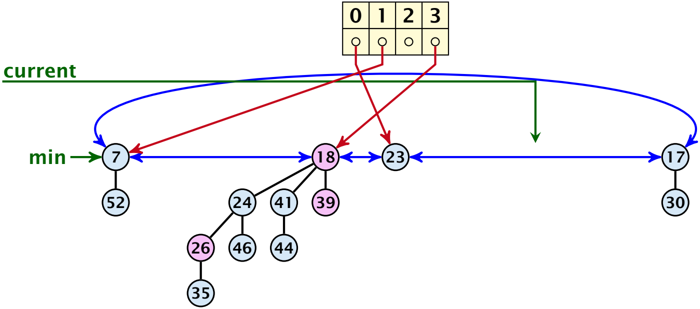

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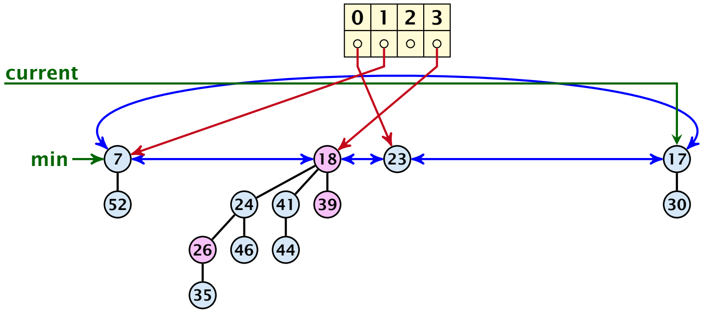

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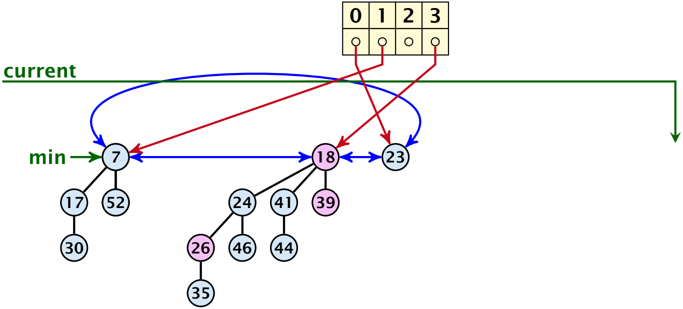

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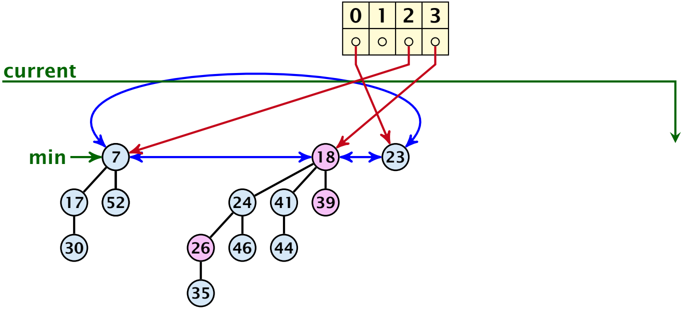

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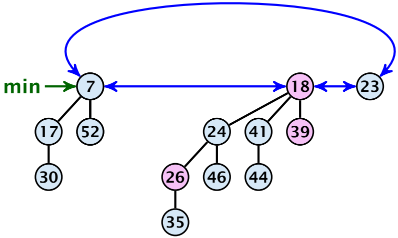

# 8.3 Fibonacci Heaps

**Actual cost for delete-min()**

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$$\begin{aligned} & c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ & \qquad \leq (c_1 + c)D_n + (c_1 - c)t + c \end{aligned}$$

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$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$

$$\le (c_1 + c)D_n + (c_1 - c)t + c \le 2c(D_n + 1) \le \mathcal{O}(D_n)$$

for $c \ge c_1$ .

# 8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$.

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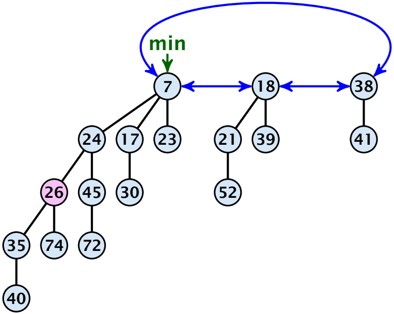
# Fibonacci Heaps: decrease-key(handle $h$, $v$)




**Case 1: decrease-key does not violate heap-property**

- Just decrease the key-value of element referenced by $h$. Nothing else to do.

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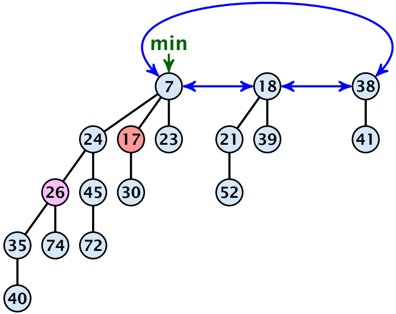


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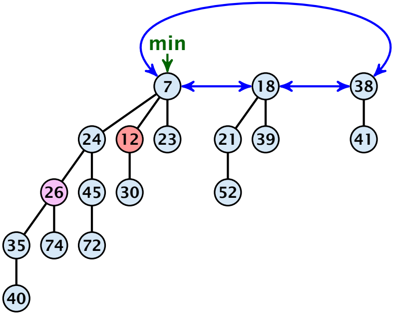


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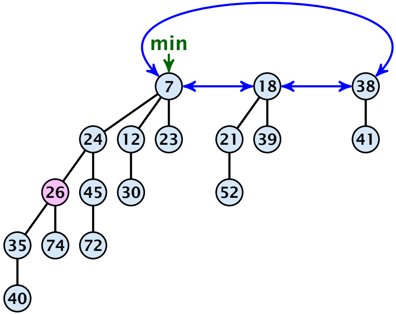


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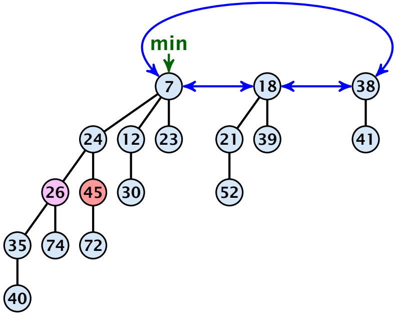


**Case 2: heap-property is violated, but parent is not marked**

- Decrease key-value of element $x$ reference by $h$.
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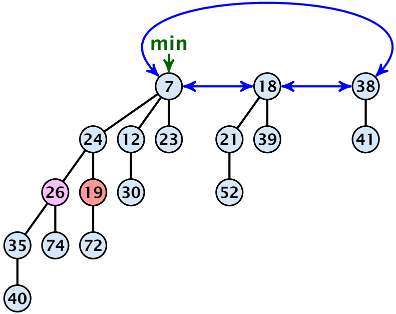


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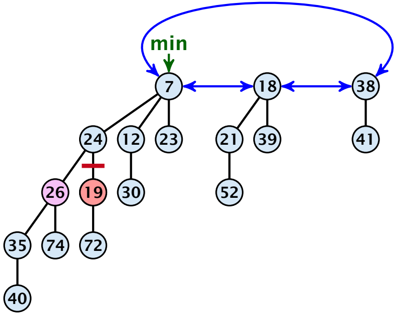


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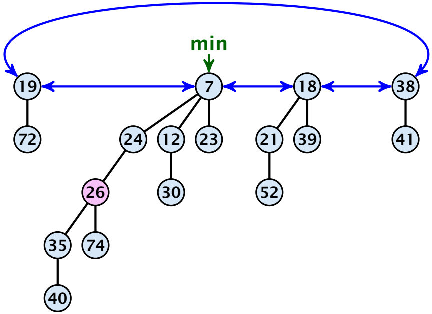


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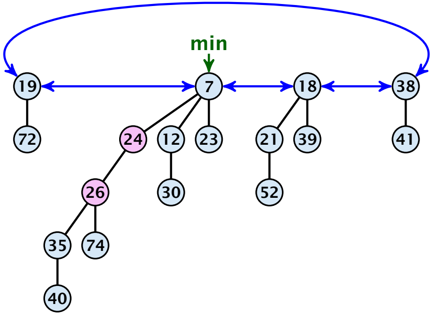


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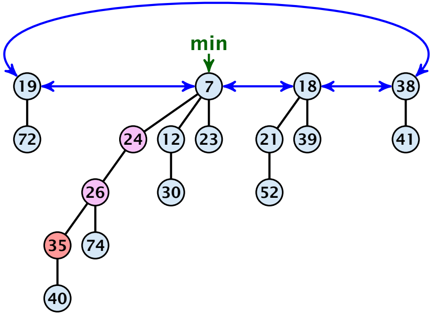


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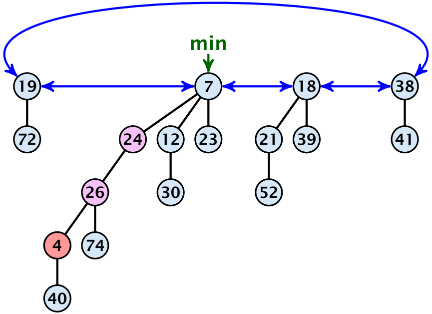


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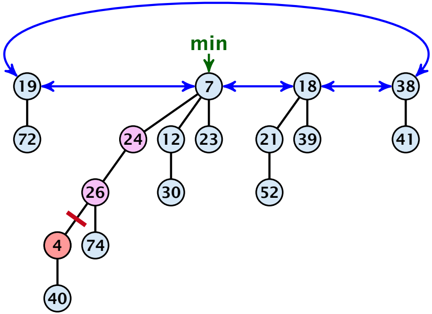


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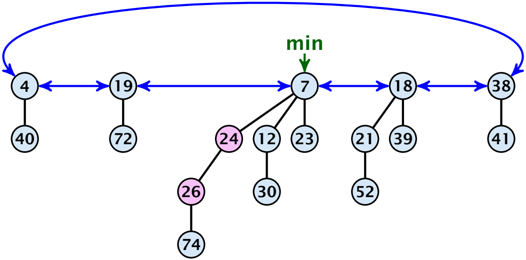


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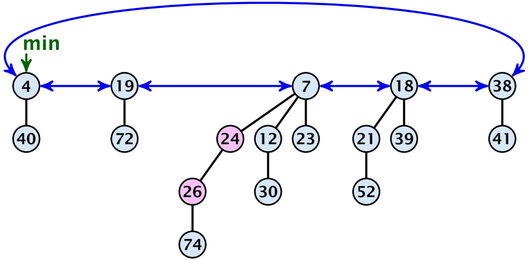


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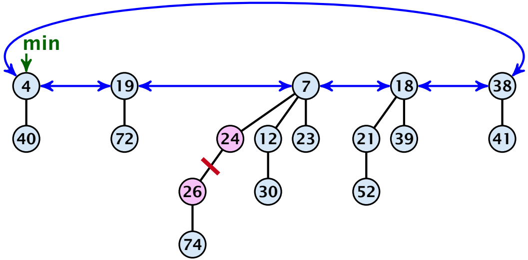


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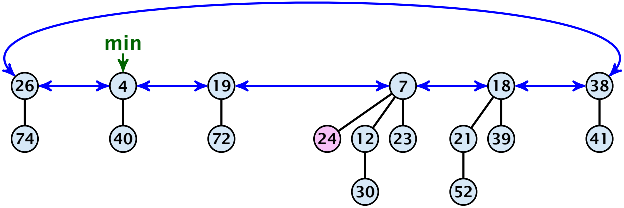


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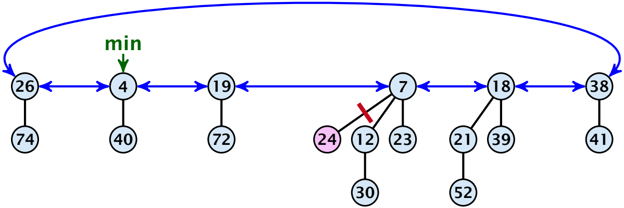


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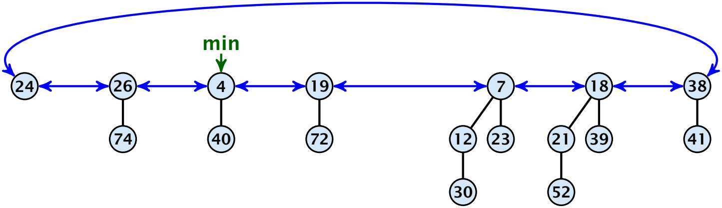


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- Cut the parent edge of $x$, and make $x$ into a root.
- Adjust min-pointers, if necessary.
- Execute the following:

  $p \leftarrow \mathrm{parent}[x]$;
  while ($p$ is marked)
      $pp \leftarrow \mathrm{parent}[p]$;
      cut of $p$; make it into a root; unmark it;
      $p \leftarrow pp$;
  if $p$ is unmarked and not a root mark it;

# Fibonacci Heaps: decrease-key(handle $h$, $v$)

**Actual cost:**

- Constant cost for decreasing the value.
- Constant cost for each of $\ell$ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant $c_2$.

**Amortized cost:**

- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
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# Delete node

$H.\textbf{delete}(x)$:

- decrease value of $x$ to $-\infty$.
- delete-min.

**Amortized cost:** $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(D_n)$ for delete-min.

# 8.3 Fibonacci Heaps

**Lemma 1**

*Let $x$ be a node with degree $k$ and let $y_1, \ldots, y_k$ denote the children of $x$ in the order that they were linked to $x$. Then*

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

# 8.3 Fibonacci Heaps

**Proof**

- When $y_i$ was linked to $x$, at least $y_1, \ldots, y_{i-1}$ were already linked to $x$.
- Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- Since, then $y_i$ has lost at most one child.
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# 8.3 Fibonacci Heaps

$\phi = \frac{1}{2}(1+\sqrt{5})$ denotes the *golden ratio*. Note that $\phi^2 = 1 + \phi$.

## Definition 2

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

**Facts:**

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0: $1 = F_0 \geq \Phi^0 = 1$

k=1: $2 = F_1 \geq \Phi^1 \approx 1.61$

k-2,k-1→ k: $F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi + 1)}^{\Phi^2} = \Phi^k$

k=2: $3 = F_2 = 2 + 1 = 2 + F_0$

k-1→ k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$