13.3 Highest Label

Algorithm 6 highest-label\((G, s, t)\)

1: initialize preflow
2: \textbf{foreach} \(u \in V \setminus \{s, t\} \textbf{ do}
3: \quad u.current-neighbour \leftarrow u.neighbour-list-head
4: \textbf{while} \exists \text{ active node } u \textbf{ do}
5: \quad \text{select active node } u \text{ with highest label}
6: \quad \text{discharge}(u)
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Lemma 1

When using highest label the number of non-saturating pushes is only $O(n^3)$.

A push from a node on level $\ell$ can only “activate” nodes on levels strictly less than $\ell$.

This means, after a non-saturating push from $u$ a relabel is required to make $u$ active again.

Hence, after $n$ non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#\text{relabels} + 1) = O(n^3)$. 
Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $O(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

**Question:**
How do we find the next node for a discharge operation?
Maintain lists $L_i$, $i \in \{0, \ldots, 2n\}$, where list $L_i$ contains active nodes with label $i$ (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node $u$ with label $k$, traverse the lists $L_k, L_{k-1}, \ldots, L_0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to $s$ or $t$ the list $k - 1$ must be non-empty (i.e., the search takes constant time).
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Hence, the total time required for searching for active nodes is at most

$$\Theta(n^3) + n(\#\text{non-saturating-pushes-to-s-or-t})$$

Lemma 2
The number of non-saturating pushes to s or t is at most $\Theta(n^2)$.

With this lemma we get

Theorem 3
The push-relabel algorithm with the rule highest-label takes time $\Theta(n^3)$. 
We only show that the number of pushes to the source is at most $O(n^2)$. A similar argument holds for the target.

After a node $v$ (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before $v$ can become active again.

This happens for every push that $v$ makes to the source. Since, every node can pass the threshold $n + 2$ at most once, $v$ can make at most $n$ pushes to the source.

As this holds for every node the total number of pushes to the source is at most $O(n^2)$. 