Algorithm 6 highest-label\((G, s, t)\)

1: initialize preflow
2: \textbf{foreach} \(u \in V \setminus \{s, t\}\) \textbf{do}
3: \hspace{1em} \textit{u.current-neighbour} \leftarrow \textit{u.neighbour-list-head}
4: \textbf{while} \exists\text{ active node } u \textbf{ do}
5: \hspace{1em} select active node \(u\) with highest label
6: \hspace{1em} discharge\( (u)\)
13.3 Highest Label

Lemma 1

*When using highest label the number of non-saturating pushes is only $O(n^3)$.*

A push from a node on level $\ell$ can only “activate” nodes on levels strictly less than $\ell$.

This means, after a non-saturating push from $u$ a relabel is required to make $u$ active again.

Hence, after $n$ non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(#\text{relabels} + 1) = O(n^3)$. 
Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $O(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:
How do we find the next node for a discharge operation?
Maintain lists $L_i, i \in \{0, \ldots, 2n\}$, where list $L_i$ contains active nodes with label $i$ (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node $u$ with label $k$, traverse the lists $L_k, L_{k-1}, \ldots, L_0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to $s$ or $t$ the list $k - 1$ must be non-empty (i.e., the search takes constant time).
Hence, the total time required for searching for active nodes is at most

\[ \Theta(n^3) + n(\#\text{non-saturating-pushes-to-s-or-t}) \]

**Lemma 2**

*The number of non-saturating pushes to s or t is at most \( \Theta(n^2) \).*

With this lemma we get

**Theorem 3**

*The push-relabel algorithm with the rule highest-label takes time \( \Theta(n^3) \).*
13.3 Highest Label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $O(n^2)$. A similar argument holds for the target.

- After a node $v$ (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before $v$ can become active again.

- This happens for every push that $v$ makes to the source. Since, every node can pass the threshold $n + 2$ at most once, $v$ can make at most $n$ pushes to the source.

- As this holds for every node the total number of pushes to the source is at most $O(n^2)$. 