## 6.2 Master Theorem

Note that the cases do not cover all possibilities.

**Lemma 1**

*Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

*Case 1.*

*If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.*

*Case 2.*

*If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.*

*Case 3.*

*If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large $n$ $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.*



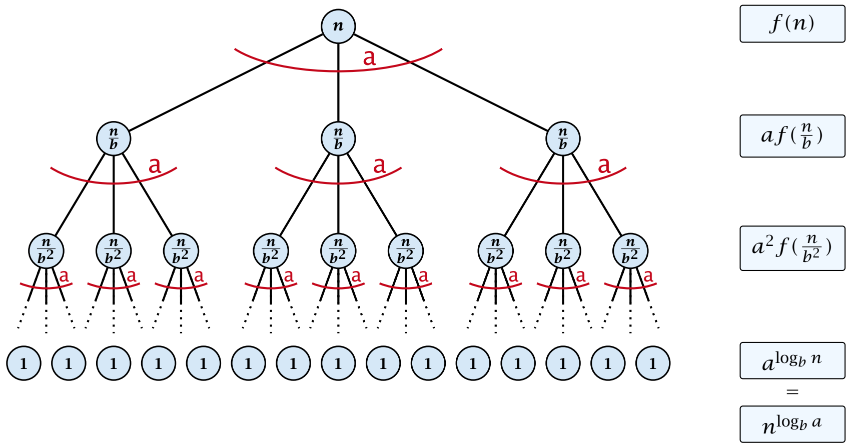
## 6.2 Master Theorem

We prove the Master Theorem for the case that $n$ is of the form $b^\ell$, and we assume that the non-recursive case occurs for problem size $1$ and incurs cost $1$.



## The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:






## 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1}-1}{q-1}} \quad = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \le \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$



Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$



Case 2. Now suppose that $f(n) \ge cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\ge c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$



Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad = cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k$$

$$= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \frac{1}{k}\ell^{k+1}$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large $n$: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$

Hence,

$$T(n) \leq \mathcal{O}(f(n)) \qquad \boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

Where did we use $f(n) \geq \Omega(n^{\log_b a+\epsilon})$?



## Example: Multiplying Two Integers

Suppose we want to multiply two $n$-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers $A$ and $B$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | $A$ |
| | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | $B$ |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | | |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | |

This gives that two $n$-bit integers can be added in time $\mathcal{O}(n)$.



## Example: Multiplying Two Integers

Suppose that we want to multiply an $n$-bit integer $A$ and an $m$-bit integer $B$ ($m \leq n$).

```
1 0 0 0 1 × 1 0 1 1
-------------------
          1 0 0 0 1
        1 0 0 0 1 0
      0 0 0 0 0 0 0
    1 0 0 0 1 0 0 0
-------------------
    1 0 1 1 1 0 1 1
```

- This is also nown as the "school method" for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

**Time requirement:**

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding $m$ numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.



## Example: Multiplying Two Integers

**A recursive approach:**
Suppose that integers $A$ and $B$ are of length $n = 2^k$, for some $k$.

| $B_1$ | $B_0$ | $\times$ | $A_1$ | $A_0$ |
|---|---|---|---|---|

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

## Example: Multiplying Two Integers

**Algorithm 3** mult$(A, B)$

| | |
|---|---|
| **if** $\lvert A\rvert = \lvert B\rvert = 1$ **then** | $\mathcal{O}(1)$ |
| **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
| split $B$ into $B_0$ and $B_1$ | $\mathcal{O}(n)$ |
| $Z_2 \leftarrow \text{mult}(A_1, B_1)$ | $T(\frac{n}{2})$ |
| $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$ | $2T(\frac{n}{2}) + \mathcal{O}(n)$ |
| $Z_0 \leftarrow \text{mult}(A_0, B_0)$ | $T(\frac{n}{2})$ |
| **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ | $\mathcal{O}(n)$ |

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$



## Example: Multiplying Two Integers

**Master Theorem:** Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$  $T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$  $T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

$\Longrightarrow$ Not better then the "school method".



## Example: Multiplying Two Integers

We can use the following identity to compute $Z_1$:

$$\begin{aligned} Z_1 &= A_1 B_0 + A_0 B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0} \end{aligned}$$

Hence,

**Algorithm 4** mult$(A, B)$

| | |
|---|---|
| **if** $\lvert A\rvert = \lvert B\rvert = 1$ **then** | $\mathcal{O}(1)$ |
| **return** $a_0 \cdot b_0$ | $\mathcal{O}(1)$ |
| split $A$ into $A_0$ and $A_1$ | $\mathcal{O}(n)$ |
| split $B$ into $B_0$ and $B_1$ | $\mathcal{O}(n)$ |
| $Z_2 \leftarrow \text{mult}(A_1, B_1)$ | $T(\frac{n}{2})$ |
| $Z_0 \leftarrow \text{mult}(A_0, B_0)$ | $T(\frac{n}{2})$ |
| $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$ | $T(\frac{n}{2}) + \mathcal{O}(n)$ |
| **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ | $\mathcal{O}(n)$ |

A more precise (correct) analysis would say that computing $Z_1$ needs time $T(\frac{n}{2} + 1) + \mathcal{O}(n)$.



## Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

**Master Theorem:** Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$  $T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$  $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".