Overview: Shortest Augmenting Paths

Lemma 1
The length of the shortest augmenting path never decreases.

Lemma 2
After at most $O(m)$ augmentations, the length of the shortest augmenting path strictly increases.
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These two lemmas give the following theorem:

**Theorem 3**
The shortest augmenting path algorithm performs at most $O(mn)$ augmentations. This gives a running time of $O(m^2n)$.

**Proof.**
- We can find the shortest augmenting paths in time $O(mn)$ via BFS.
- There are at most $O(mn)$ augmentations for paths of exactly $k < n$ edges.
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Ernst Mayr, Harald Räcke
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In the following we assume that the residual graph \( G_f \) does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.
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\begin{align*}
G_f & \quad L_G \\
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An $s$-$t$ path in $G_f$ that uses edges not in $E_L$ has length larger than $k$, even when considering edges added to $G_f$ during the round.
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Theorem 4
The shortest augmenting path algorithm performs at most $\Theta(mn)$ augmentations. Each augmentation can be performed in time $\Theta(m)$.

Theorem 5 (without proof)
There exist networks with $m = \Theta(n^2)$ that require $\Theta(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:
There always exists a set of $m$ augmentations that gives a maximum flow (why?).
Theorem 4

The shortest augmenting path algorithm performs at most $O(mn)$ augmentations. Each augmentation can be performed in time $O(m)$.

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When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $O(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $O(m)$ per augmentation for this).
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We maintain a subset $E_L$ of the edges of $G_f$ with the guarantee that a shortest $s$-$t$ path using only edges from $E_L$ is a shortest augmenting path.

With each augmentation some edges are deleted from $E_L$.

When $E_L$ does not contain an $s$-$t$ path anymore the distance between $s$ and $t$ strictly increases.

Note that $E_L$ is not the set of edges of the level graph but a subset of level-graph edges.
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Suppose that the initial distance between $s$ and $t$ in $G_f$ is $k$.

$E_L$ is initialized as the level graph $L_G$.

Perform a DFS search to find a path from $s$ to $t$ using edges from $E_L$.

Either you find $t$ after at most $n$ steps, or you end at a node $v$ that does not have any outgoing edges.

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between \( s \) and \( t \) strictly increases.

Initializing \( E_L \) for the phase takes time \( \Theta(m) \).

The total cost for searching for augmenting paths during a phase is at most \( \Theta(mn) \), since every search (successful (i.e., reaching \( t \)) or unsuccessful) decreases the number of edges in \( E_L \) and takes time \( \Theta(n) \).

The total cost for performing an augmentation during a phase is only \( \Theta(n) \). For every edge in the augmenting path one has to update the residual graph \( G_f \) and has to check whether the edge is still in \( E_L \) for the next search.

There are at most \( n \) phases. Hence, total cost is \( \Theta(mn^2) \).
Let a phase of the algorithm be defined by the time between two augmentations during which the distance between $s$ and $t$ strictly increases.

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