7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- time for search $\Theta(n)$
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How can we improve the search-operation?

Let $|L_1|$ denote the number of elements in the "express lane", and $|L_0| = n$ the number of all elements (ignoring dummy elements).

Worst case search time: $|L_1| + |L_0| |L_1|$ (ignoring additive constants).

Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$. 
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Choose \( |L_1| = \sqrt{n} \). Then search time \( \Theta(\sqrt{n}) \).
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Add more express lanes. Lane $L_i$ contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list $L_{i-1}$.
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Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)
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- Find the largest item in list $L_k$ that is smaller than $x$. At most $|L_k| + 2$ steps.
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- At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.
Choose ratios between list-lengths evenly, i.e., \( \frac{|L_{i-1}|}{|L_i|} = r \), and, hence, \( L_k \approx r^{-k}n \).
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Worst case running time is: \( O(r^{-k}n + kr) \).
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Choosing \( k = \Theta(\log n) \) gives a logarithmic running time.
7.6 Skip Lists

How to do insert and delete?

If we want that in \( L_i \) we always skip over roughly the same number of elements in \( L_{i-1} \), an insert or delete may require a lot of re-organisation.

Use randomization instead!
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Use randomization instead!
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Insert:

- A search operation gives you the insert position for element $x$ in every list.
- Flip a coin until it shows head, and record the number $t \in \{1, 2, \ldots \}$ of trials needed.
- Insert $x$ into lists $L_0, \ldots, L_{t-1}$.

Delete:

- You get all predecessors via backward pointers.
- Delete $x$ in all lists it actually appears in.

The time for both operations is dominated by the search time.
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High Probability

Definition 1 (High Probability)
We say a randomized algorithm has running time $O(\log n)$ with high probability if for any constant $\alpha$ the running time is at most $O(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Here the $O$-notation hides a constant that may depend on $\alpha$. 
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We say a randomized algorithm has running time $\Theta(\log n)$ with high probability if for any constant $\alpha$ the running time is at most $\Theta(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Here the $\Theta$-notation hides a constant that may depend on $\alpha$. 
High Probability

Suppose there are *polynomially* many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. $E_i$ may be the event that the $i$-th search in a skip list takes time at most $\Theta(\log n)$).
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Then the probability that all $E_i$ hold is at least

$$\Pr[E_1 \land \cdots \land E_\ell]$$
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$$= 1 - n^{c-\alpha}.$$

This means $\Pr[E_1 \land \cdots \land E_\ell]$ holds with high probability.
Lemma 2

A search (and, hence, also insert and delete) in a skip list with $n$ elements takes time $O(\log n)$ with high probability (w. h. p.).
7.6 Skip Lists

Backward analysis:

At each point the path goes up with probability $\frac{1}{2}$ and left with probability $\frac{1}{2}$.

We show that w.h.p:

▶ A "long" search path must also go very high.
▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.
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7.6 Skip Lists

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-∞ - 5 - 8 - 10 - 12 - 14 - 18 - 23 - 26 - 28 - 35 - 43 - ∞

At each point the path goes up with probability $\frac{1}{2}$ and left with probability $\frac{1}{2}$.

We show that w.h.p:

- A "long" search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.
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We show that w.h.p:

- A “long” search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.
\[ \left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k \]
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\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}
\]
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\[
\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1}
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\]

\[
\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!}
\]
\[
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\[
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\[
= \left( \frac{n}{k} \right)^k \cdot \frac{k^k}{k!} \leq \left( \frac{en}{k} \right)^k
\]
Let $E_{z,k}$ denote the event that a search path is of length $z$ (number of edges) but does not visit a list above $L_k$. In particular, this means that during the construction in the backward analysis we see at most $k$ heads (i.e., coin flips that tell you to go up) in $z$ trials.
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7.6 Skip Lists

\[ \Pr[E_{z,k}] \]
Pr[$E_{z,k}$] ≤ Pr[at most $k$ heads in $z$ trials]
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≤ \binom{z}{k} 2^{-(z-k)}
7.6 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k}2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)}$$
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\[ \Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}] \]

\[ \leq \binom{z}{k} 2^{-(z-k)} \leq \left( \frac{ez}{k} \right)^k 2^{-(z-k)} \leq \left( \frac{2ez}{k} \right)^k 2^{-z} \]
Pr\[E_{z,k} \leq Pr[\text{at most } k \text{ heads in } z \text{ trials}] \]

\[
\leq \left( \frac{z}{k} \right)^{2^{-z-k}} \leq \left( \frac{ez}{k} \right)^k 2^{-z-k} \leq \left( \frac{2ez}{k} \right)^k 2^{-z}
\]

choosing \( k = \gamma \log n \) with \( \gamma \geq 1 \) and \( z = (\beta + \alpha)\gamma \log n \)
7.6 Skip Lists

Pr[$E_{z,k}$] ≤ Pr[at most $k$ heads in $z$ trials]

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\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2\beta k}\right)^k \cdot n^{-\alpha}$
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\]

\[
\leq \left( \frac{2e(\beta + \alpha)}{2\beta} \right)^k n^{-\alpha}
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7.6 Skip Lists

Pr\[E_{z,k}\] ≤ Pr[at most $k$ heads in $z$ trials]

\[≤ \left(\frac{z}{k}\right) 2^{-(z-k)} ≤ \left(\frac{ez}{k}\right)^k 2^{-(z-k)} ≤ \left(\frac{2ez}{k}\right)^k 2^{-z}\]

choosing $k = \gamma \log n$ with $\gamma ≥ 1$ and $z = (\beta + \alpha) \gamma \log n$

\[≤ \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} ≤ \left(\frac{2ez}{2\beta k}\right)^k \cdot n^{-\alpha}\]

\[≤ \left(\frac{2e(\beta + \alpha)}{2\beta}\right)^k n^{-\alpha}\]

now choosing $\beta = 6\alpha$ gives
Pr[$E_{z,k}] \leq Pr[\text{at most } k \text{ heads in } z \text{ trials}]

\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}

choosing \( k = \gamma \log n \) with \( \gamma \geq 1 \) and \( z = (\beta + \alpha)\gamma \log n \)

\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2\beta k}\right)^k \cdot n^{-\alpha}

\leq \left(\frac{2e(\beta + \alpha)}{2\beta}\right)^k n^{-\alpha}

now choosing \( \beta = 6\alpha \) gives

\leq \left(\frac{42\alpha}{64\alpha}\right)^k n^{-\alpha}
7.6 Skip Lists

\[
\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]
\]

\[
\leq \left( \frac{z}{k} \right) 2^{-(z-k)} \leq \left( \frac{ez}{k} \right)^k 2^{-(z-k)} \leq \left( \frac{2ez}{k} \right)^k 2^{-z}
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\leq \left( \frac{2ez}{k} \right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left( \frac{2ez}{2^{\beta} k} \right)^k \cdot n^{-\alpha}
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7.6 Skip Lists

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choosing \( k = \gamma \log n \) with \( \gamma \geq 1 \) and \( z = (\beta + \alpha)\gamma \log n \)

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now choosing \( \beta = 6\alpha \) gives

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for \( \alpha \geq 1 \).
So far we fixed \( k = \gamma \log n \), \( \gamma \geq 1 \), and \( z = 7 \alpha \gamma \log n \), \( \alpha \geq 1 \).

This means that a search path of length \( \Omega(\log n) \) visits a list on a level \( \Omega(\log n) \), w.h.p.

Let \( A_{k+1} \) denote the event that the list \( L_{k+1} \) is non-empty. Then

\[
\Pr[A_{k+1}] \leq n^{2-(k+1)} \leq n - \alpha \left( \gamma - 1 \right).
\]

For the search to take at least \( z = 7 \alpha \gamma \log n \) steps either the event \( E_{z,k} \) or the event \( A_{k+1} \) must hold. Hence,

\[
\Pr[^{\text{search requires}} z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \leq n - \alpha + n - \left( \gamma - 1 \right).
\]

This means, the search requires at most \( z \) steps, w. h. p.
7.6 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha \gamma \log n$, $\alpha \geq 1$. 
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Let $A_{k+1}$ denote the event that the list $L_{k+1}$ is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)}.$$
So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha \gamma \log n$, $\alpha \geq 1$.

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$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)}.$$ 

For the search to take at least $z = 7\alpha \gamma \log n$ steps either the event $E_{z,k}$ or the event $A_{k+1}$ must hold.
7.6 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha \gamma \log n$, $\alpha \geq 1$.

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7.6 Skip Lists

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\[
\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]
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7.6 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha \gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

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$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)}.$$ 

For the search to take at least $z = 7\alpha \gamma \log n$ steps either the event $E_{z,k}$ or the event $A_{k+1}$ must hold.

Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$

$$\leq n^{-\alpha} + n^{-(\gamma-1)}$$
7.6 Skip Lists

So far we fixed \( k = \gamma \log n, \gamma \geq 1 \), and \( z = 7\alpha\gamma \log n, \alpha \geq 1 \).

This means that a search path of length \( \Omega(\log n) \) visits a list on a level \( \Omega(\log n) \), w.h.p.

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\]

For the search to take at least \( z = 7\alpha\gamma \log n \) steps either the event \( E_{z,k} \) or the event \( A_{k+1} \) must hold.

Hence,

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\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]
\leq n^{-\alpha} + n^{-(\gamma-1)}
\]

This means, the search requires at most \( z \) steps, w. h. p.