Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- after access, an element is moved to the root; splay(x)
- repeated accesses are faster
- only amortized guarantee
- read-operations change the tree
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Splay Trees

**find($x$)**

- search for $x$ according to a search tree
- let $\tilde{x}$ be last element on search-path
- splay($\tilde{x}$)
Splay Trees

\textbf{insert}(x)

- search for \(x\); \(\hat{x}\) is last visited element during search (successor or predecessor of \(x\))
- \textbf{splay}(\(\hat{x}\)) moves \(\hat{x}\) to the root
- insert \(x\) as new root
Splay Trees

\textbf{delete}(x)

- search for \(x\); splay(\(x\)); remove \(x\)
- search largest element \(\tilde{x}\) in \(A\)
- splay(\(\tilde{x}\)) (on subtree \(A\))
- connect root of \(B\) as right child of \(\tilde{x}\)
How to bring element to root?

- one (bad) option: moveToRoot($x$)
- iteratively do rotation around parent of $x$ until $x$ is root
- if $x$ is left child do right rotation otw. left rotation
better option \texttt{splay}(x):

- zig case: if \( x \) is child of root do left rotation or right rotation around parent
better option splay(x):

- zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- do double right rotation around grand-parent (resp. double left left rotation)
Double Rotations
Splay: Zigzig Case

better option splay(x):

- zigzig case: if $x$ is left child and parent of $x$ is left child (or $x$ right child, parent of $x$ right child)
- do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)
Splay vs. Move to Root

7.3 Splay Trees
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7.3 Splay Trees
Static Optimality

Suppose we have a sequence of $m$ find-operations. $\text{find}(x)$ appears $h_x$ times in this sequence.

The cost of a *static* search tree $T$ is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\Theta(\text{cost}(T_{\text{min}}))$, where $T_{\text{min}}$ is an optimal static search tree.
Dynamic Optimality

Let $S$ be a sequence with $m$ find-operations.

Let $A$ be a data-structure based on a search tree:
- the cost for accessing element $x$ is $1 + \text{depth}(x)$;
- after accessing $x$ the tree may be re-arranged through rotations;

**Conjecture:**

A splay tree that only contains elements from $S$ has cost $O(\text{cost}(A, S))$, for processing $S$. 
Lemma 1

Splay Trees have an amortized running time of $O(\log n)$ for all operations.
Amortized Analysis

Definition 2
A data structure with operations \( \text{op}_1(), \ldots, \text{op}_k() \) has amortized running times \( t_1, \ldots, t_k \) for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most \( n \) elements, and let \( k_i \) denote the number of occurrences of \( \text{op}_i() \) within this sequence. Then the actual running time must be at most \( \sum_i k_i \cdot t_i(n) \).
Potential Method

Introduce a potential for the data structure.
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- $\Phi(D_i)$ is the potential after the $i$-th operation.
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$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}).$$
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- Show that $\Phi(D_i) \geq \Phi(D_0)$.
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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq k \sum_{i=1}^{k} \hat{c}_i = k \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0).$$

This means the amortized costs can be used to derive a bound on the total cost.
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Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0)$$
Potential Method

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$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) .$$

- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.
Example: Stack

Stack

- **S. push()**
- **S. pop()**
- **S. multipop(k):** removes $k$ items from the stack. If the stack currently contains less than $k$ items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- **S. push():** cost 1.
- **S. pop():** cost 1.
- **S. multipop(k):** cost $\min\{\text{size}, k\} = k$. 

7.3 Splay Trees
Example: Stack

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Actual cost:

- **S. push()**: cost \( 1 \).
- **S. pop()**: cost \( 1 \).
- **S. multipop**(k): cost \( \min\{\text{size}, k\} = k \).
Example: Stack

Use potential function $\Phi(S) = \text{number of elements on the stack}$. 

Amortized cost:

$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta \Phi = 1 + 1 \leq 2$.

$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta \Phi = 1 - 1 \leq 0$.

$\hat{C}_{\text{multipop}}(k) = C_{\text{multipop}} + \Delta \Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0$.  

7.3 Splay Trees

Ernst Mayr, Harald Räcke
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- **$S$. push():** cost
  
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  $$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta \Phi = 1 - 1 \leq 0$$

- **$S$. multipop($k$):** cost
  
  $$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta \Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0$$
Example: Stack

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  \hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta \Phi = \min\{\text{size, } k\} - \min\{\text{size, } k\} \leq 0 .
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Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an $n$-bit binary counter may require to examine $n$-bits, and maybe change them.

Actual cost:
- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is $k + 1$, where $k$ is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).
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Example: Binary Counter

Choose potential function $\Phi(x) = k$, where $k$ denotes the number of ones in the binary representation of $x$.

Amortized cost:

\[
\begin{align*}
\text{Changing bit from 0 to 1:} & \quad \hat{C}_{0 \rightarrow 1} = C_{0 \rightarrow 1} + \Delta \Phi = 1 + 1 \leq 2 \\
\text{Changing bit from 1 to 0:} & \quad \hat{C}_{1 \rightarrow 0} = C_{1 \rightarrow 0} + \Delta \Phi = 1 - 1 \leq 0 \\
\text{Increment:} & \quad \text{Let } k \text{ denotes the number of consecutive ones in the least significant bit-positions. An increment involves } k (1 \rightarrow 0)\text{-operations, and one } 0 \rightarrow 1\text{-operation.} \\
\text{Hence, the amortized cost is } \hat{C}_{1 \rightarrow 0} + k \leq 2.
\end{align*}
\]
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  \hat{C}_{1 \rightarrow 0} = C_{1 \rightarrow 0} + \Delta \Phi = 1 - 1 \leq 0 .
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- Increment: Let $k$ denotes the number of consecutive ones in the least significant bit-positions. An increment involves $k$ $(1 \rightarrow 0)$-operations, and one $(0 \rightarrow 1)$-operation.

Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$. 
Example: Binary Counter

Choose potential function $\Phi(x) = k$, where $k$ denotes the number of ones in the binary representation of $x$.

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- Increment: Let $k$ denotes the number of consecutive ones in the least significant bit-positions. An increment involves $k (1 \to 0)$-operations, and one (0 $\to$ 1)-operation.

Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \leq 2$. 
potential function for splay trees:

- size \( s(x) = |T_x| \)
- rank \( r(x) = \log_2(s(x)) \)
- \( \Phi(T) = \sum_{v \in T} r(v) \)

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.
\[ \Delta \Phi = r'(x) + r'(p) - r(x) - r(p) \]

\[ = r'(p) - r(x) \]

\[ \leq r'(x) - r(x) \]

\[ \text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x)) \]
Splay: Zig Case

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\[ = r'(p) + r'(g) - r(x) - r(p) \]

\[ \leq r'(x) + r'(g) - r(x) - r(x) \]

\[ = r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \]

\[ = -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \]

\[ \leq -2 + 3(r'(x) - r(x)) \quad \Rightarrow \quad \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
ΔΦ = r′(x) + r′(p) + r′(g) − r(x) − r(p) − r(g)

= r′(p) + r′(g) − r(x) − r(p)

≤ r′(x) + r′(g) − r(x) − r(x)

= r′(x) + r′(g) + r(x) − 3r′(x) + 3r′(x) − r(x) − 2r(x)

= −2r′(x) + r′(g) + r(x) + 3(r′(x) − r(x))

≤ −2 + 3(r′(x) − r(x)) ⇒ cost_{zigzag} ≤ 3(r′(x) − r(x))
\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]

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Splay: Zigzig Case

\[
\frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right)
= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right)
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right)
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
\[ \frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \]

\[ = \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \]

\[ = \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \]

\[ \leq \log \left( \frac{1}{2} s(x) + \frac{1}{2} s'(g) \right) \leq \log \left( \frac{1}{2} \right) = -1 \]
Splay: Zigzag Case

\[
\frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \\
= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
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\]

\[
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right)
\]

\[
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
\[
\frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \\
= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
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= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right)
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= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right)
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\]
Splay: Zigzag Case

\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]

\[ = r'(p) + r'(g) - r(x) - r(p) \]

\[ \leq r'(p) + r'(g) - r(x) - r(x) \]

\[ = r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \]

\[ \leq -2 + 2(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]
\[ = r'(p) + r'(g) - r(x) - r(p) \]
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Splay: Zigzag Case

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\Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)
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\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))
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Splay: Zigzag Case

\[
\frac{1}{2} \left( r'(p) + r'(g) - 2r'(x) \right)
\]

\[
= \frac{1}{2} \left( \log(s'(p)) + \log(s'(g)) - 2 \log(s'(x)) \right)
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\[
\leq \log \left( \frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
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Amortized cost of the whole splay operation:

\[
\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x))
\]

\[
= 2 + r(\text{root}) - r_0(x)
\]

\[
\leq \Theta(\log n)
\]