Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- after access, an element is moved to the root; splay(x)
- repeated accesses are faster
- only amortized guarantee
- read-operations change the tree
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**find**($x$)

- search for $x$ according to a search tree
- let $\tilde{x}$ be last element on search-path
- splay($\tilde{x}$)
Splay Trees

**insert**(*x*)

- search for *x*; \(\tilde{x}\) is last visited element during search (successor or predecessor of *x*)
- splay(\(\tilde{x}\)) moves \(\tilde{x}\) to the root
- insert *x* as new root
delete($x$)

- search for $x$; splay($x$); remove $x$
- search largest element $\bar{x}$ in $A$
- splay($\bar{x}$) (on subtree $A$)
- connect root of $B$ as right child of $\bar{x}$
How to bring element to root?

- one (bad) option: \texttt{moveToRoot}(x)
- iteratively do rotation around parent of x until x is root
- if x is left child do right rotation otw. left rotation
Splay: Zig Case

better option splay(x):

- zig case: if \( x \) is child of root do left rotation or right rotation around parent
Splay: Zigzag Case

better option splay($x$):

- zigzag case: if $x$ is right child and parent of $x$ is left child (or $x$ left child parent of $x$ right child)
- do double right rotation around grand-parent (resp. double left left rotation)
Double Rotations

- $\text{LeftRotate}(y)$
- $\text{RightRotate}(x)$
- $\text{DoubleRightRotate}(x)$
better option splay(\textit{x}): 

- zigzig case: if \textit{x} is left child and parent of \textit{x} is left child (or \textit{x} right child, parent of \textit{x} right child)
- do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)
Splay vs. Move to Root

Splay Trees
Splay vs. Move to Root

7.3 Splay Trees
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Static Optimality

Suppose we have a sequence of $m$ find-operations. $\text{find}(x)$ appears $h_x$ times in this sequence.

The cost of a static search tree $T$ is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\Theta(\text{cost}(T_{\text{min}}))$, where $T_{\text{min}}$ is an optimal static search tree.
Dynamic Optimality

Let $S$ be a sequence with $m$ find-operations.

Let $A$ be a data-structure based on a search tree:
- the cost for accessing element $x$ is $1 + \text{depth}(x)$;
- after accessing $x$ the tree may be re-arranged through rotations;

**Conjecture:**
A splay tree that only contains elements from $S$ has cost $\Theta(\text{cost}(A, S))$, for processing $S$. 
Lemma 1

Splay Trees have an amortized running time of $O(\log n)$ for all operations.
Amortized Analysis

Definition 2
A data structure with operations \( \text{op}_1(), \ldots, \text{op}_k() \) has amortized running times \( t_1, \ldots, t_k \) for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most \( n \) elements, and let \( k_i \) denote the number of occurrences of \( \text{op}_i() \) within this sequence. Then the actual running time must be at most \( \sum_i k_i \cdot t_i(n) \).
Potential Method

Introduce a potential for the data structure.
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▶ Φ(D_i) is the potential after the i-th operation.
Potential Method

Introduce a potential for the data structure.

- $\Phi(D_i)$ is the potential after the $i$-th operation.
- Amortized cost of the $i$-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$
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- Show that $\Phi(D_i) \geq \Phi(D_0)$.
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Then

$$\sum_{i=1}^{k} c_i \leq k \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.
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Then

\[
\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0)
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\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}).
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- Show that \( \Phi(D_i) \geq \Phi(D_0) \).

Then

\[
\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i
\]

This means the amortized costs can be used to derive a bound on the total cost.
Example: Stack

Stack

- **S. push()**
- **S. pop()**
- **S. multipop(k):** removes $k$ items from the stack. If the stack currently contains less than $k$ items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- **S. push():** cost 1.
- **S. pop():** cost 1.
- **S. multipop(k):** cost $\min\{\text{size, } k\} = k$. 

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Use potential function $\Phi(S) = \text{number of elements on the stack.}$

Amortized cost:

- **S. push()** cost:
  
  $\hat{C}_{\text{push}} = C_{\text{push}} + \Delta \Phi = 1 + 1 \leq 2$.

- **S. pop()** cost:
  
  $\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta \Phi = 1 - 1 \leq 0$.

- **S. multipop(k)** cost:
  
  $\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta \Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0$.
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Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an \( n \)-bit binary counter may require to examine \( n \)-bits, and maybe change them.

**Actual cost:**

- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is \( k + 1 \), where \( k \) is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has \( k = 1 \)).
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Choose potential function $\Phi(x) = k$, where $k$ denotes the number of ones in the binary representation of $x$.

Amortized cost:

- Changing bit from $0$ to $1$:
  $\hat{C}_{0 \to 1} = C_{0 \to 1} + \Delta \Phi = 1 + 1 \leq 2$.

- Changing bit from $1$ to $0$:
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- Increment: Let $k$ denotes the number of consecutive ones in the least significant bit-positions. An increment involves $k (1 \to 0)$-operations, and one $(0 \to 1)$-operation.

Hence, the amortized cost is $k \cdot \hat{C}_{1 \to 0} + \hat{C}_{0 \to 1} \leq 2$.
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- Increment: Let $k$ denotes the number of consecutive ones in the least significant bit-positions. An increment involves $k$ ($1 \rightarrow 0$)-operations, and one ($0 \rightarrow 1$)-operation.

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  Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$. 
Splay Trees

potential function for splay trees:

- size \( s(x) = |T_x| \)
- rank \( r(x) = \log_2(s(x)) \)
- \( \Phi(T) = \sum_{v \in T} r(v) \)

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.
Splay: Zig Case

\[ \Delta \Phi = r'(x) + r'(p) - r(x) - r(p) \]
\[ = r'(p) - r(x) \]
\[ \leq r'(x) - r(x) \]

\[ \text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x)) \]
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\[ = r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \]
\[ = -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \]
\[ \leq -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
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\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]

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Splay: Zigzag Case

\[ \Delta \Phi = \frac{r'(x)}{x} + \frac{r'(p)}{p} + \frac{r'(g)}{g} - \frac{r(x)}{x} - \frac{r(p)}{p} - \frac{r(g)}{g} \]

\[ = \frac{r'(p)}{p} + \frac{r'(g)}{g} - \frac{r(x)}{x} - \frac{r(p)}{p} \]

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\[ \leq -2 + 3(\frac{r'(x)}{x} - \frac{r(x)}{x}) \]

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Splay: Zigzag Case

$\Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$

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Splay: Zigzag Case

\[ \frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \]

\[ = \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \]

\[ = \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \]

\[ \leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1 \]
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= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
Splay: Zigzig Case

\[
\frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right)
\]

\[
= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right)
\]

\[
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right)
\]

\[
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
\[
\frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \\
\quad = \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\
\quad = \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\
\quad \leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
Splay: Zigzag Case

\[
\frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \\
= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\
= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\
\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]
\[ = r'(p) + r'(g) - r(x) - r(p) \]
\[ \leq r'(p) + r'(g) - r(x) - r(x) \]
\[ = r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \]
\[ \leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
Splay: Zigzag Case

\[
\Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)
\]
\[
= r'(p) + r'(g) - r(x) - r(p)
\]
\[
\leq r'(p) + r'(g) - r(x) - r(x)
\]
\[
= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)
\]
\[
\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))
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Splay: Zigzag Case

\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]
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\[ \leq -2 + 2(r'(x) - r(x)) \]
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\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]

\[ = r'(p) + r'(g) - r(x) - r(p) \]

\[ \leq r'(p) + r'(g) - r(x) - r(x) \]

\[ = r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \]

\[ \leq -2 + 2(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]
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\[ \leq r'(p) + r'(g) - r(x) - r(x) \]
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\[ \leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{Cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
Splay: Zigzag Case

\[ \Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \]
\[ = r'(p) + r'(g) - r(x) - r(p) \]
\[ \leq r'(p) + r'(g) - r(x) - r(x) \]
\[ = r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \]
\[ \leq -2 + 2(r'(x) - r(x)) \quad \Rightarrow \quad \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x)) \]
Splay: Zigzag Case

\[
\frac{1}{2}
\left( r'(p) + r'(g) - 2r'(x) \right)
\]

\[
= \frac{1}{2}
\left( \log(s'(p)) + \log(s'(g)) - 2 \log(s'(x)) \right)
\]

\[
\leq \log \left( \frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
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Splay: Zigzag Case

\[ \frac{1}{2} \left( r'(p) + r'(g) - 2r'(x) \right) \]

\[ = \frac{1}{2} \left( \log(s'(p)) + \log(s'(g)) - 2 \log(s'(x)) \right) \]

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\[
\frac{1}{2} \left( r'(p) + r'(g) - 2r'(x) \right)
= \frac{1}{2} \left( \log(s'(p)) + \log(s'(g)) - 2 \log(s'(x)) \right)
\leq \log \left( \frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
Splay: Zigzag Case

\[
\frac{1}{2} \left( r'(p) + r'(g) - 2r'(x) \right) = \frac{1}{2} \left( \log(s'(p)) + \log(s'(g)) - 2 \log(s'(x)) \right) \leq \log \left( \frac{1}{2} \right) = -1
\]
Splay: Zigzag Case

\[ \frac{1}{2} \left( r'(p) + r'(g) - 2r'(x) \right) \]

\[ = \frac{1}{2} \left( \log(s'(p)) + \log(s'(g)) - 2 \log(s'(x)) \right) \]

\[ \leq \log \left( \frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1 \]
Amortized cost of the whole splay operation:

\[
\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
= 2 + r(\text{root}) - r_0(x) \\
\leq O(\log n)
\]