6.3 The Characteristic Polynomial

Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

This is the general form of a linear recurrence relation of order \( k \) with constant coefficients \((c_0, c_k \neq 0)\).

- \( T(n) \) only depends on the \( k \) preceding values. This means the recurrence relation is of order \( k \).
- The recurrence is linear as there are no products of \( T[n] \)’s.
- If \( f(n) = 0 \) then the recurrence relation becomes a linear, homogenous recurrence relation of order \( k \).

Note that we ignore boundary conditions for the moment.
Observations:

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

- In fact, any $k$ consecutive values completely determine the solution.

- $k$ non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.

- Then pick the right one by analyzing boundary conditions.

- First consider the homogenous case.
The Homogenous Case

The solution space

\[ S = \left\{ T = T[1], T[2], T[3], \ldots \mid T \text{ fulfills recurrence relation} \right\} \]

is a vector space. This means that if \( T_1, T_2 \in S \), then also \( \alpha T_1 + \beta T_2 \in S \), for arbitrary constants \( \alpha, \beta \).

How do we find a non-trivial solution?

We guess that the solution is of the form \( \lambda^n, \lambda \neq 0 \), and see what happens. In order for this guess to fulfill the recurrence we need

\[ c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0 \]

for all \( n \geq k \).
The Homogenous Case

Dividing by $\lambda^{n-k}$ gives that all these constraints are identical to

$$
\frac{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k}{\text{characteristic polynomial } P[\lambda]} = 0
$$

This means that if $\lambda_i$ is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the $k$ (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$
\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n
$$

is a solution for arbitrary values $\alpha_i$. 

The Homogenous Case

Lemma 1
Assume that the characteristic polynomial has $k$ distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n.$$ 

Proof.
There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.
The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha'_i$s such that these conditions are met:

\[
\begin{align*}
\alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2 + \cdots + \alpha_k \cdot \lambda_k &= T[1] \\
\alpha_1 \cdot \lambda_1^2 + \alpha_2 \cdot \lambda_2^2 + \cdots + \alpha_k \cdot \lambda_k^2 &= T[2] \\
\vdots \\
\alpha_1 \cdot \lambda_1^k + \alpha_2 \cdot \lambda_2^k + \cdots + \alpha_k \cdot \lambda_k^k &= T[k]
\end{align*}
\]
The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions \( T[i] \) and I want to see whether I can choose the \( \alpha'_i \)s such that these conditions are met:

\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\
\vdots & & & \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k 
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k 
\end{pmatrix}
= 
\begin{pmatrix}
T[1] \\
T[2] \\
\vdots \\
T[k] 
\end{pmatrix}
\]

We show that the column vectors are linearly independent. Then the above equation has a solution.
Computing the Determinant

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \ldots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \\
\begin{vmatrix}
1 & 1 & \ldots & 1 & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_{k-1} & \lambda_k \\
\vdots & \vdots & & \vdots & \vdots \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \\
\end{vmatrix}
\]

\[
= \prod_{i=1}^{k} \lambda_i \cdot \\
\begin{vmatrix}
1 & \lambda_1 & \ldots & \lambda_1^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \ldots & \lambda_2^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & \lambda_k & \ldots & \lambda_k^{k-2} & \lambda_k^{k-1} \\
\end{vmatrix}
\]

6.3 The Characteristic Polynomial
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 & \ldots & \lambda_1^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \ldots & \lambda_2^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \ldots & \lambda_k^{k-2} & \lambda_k^{k-1}
\end{vmatrix}
= \\
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \ldots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \ldots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \ldots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix}
\]
Computing the Determinant

\[
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix}
= 
\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix}
\]
Computing the Determinant

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix}
= \prod_{i=2}^{k} (\lambda_i - \lambda_1) \cdot 
\begin{vmatrix}
1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2}
\end{vmatrix}
\]
Computing the Determinant

Repeating the above steps gives:

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \ldots & \lambda_{k-1}^k & \lambda_k^k \\
\end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)
\]

Hence, if all $\lambda_i$’s are different, then the determinant is non-zero.
The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root $\lambda_i$ with multiplicity (Vielfachheit) at least 2. Then not only is $\lambda_i^n$ a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since $\lambda_i$ is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$.

Calculating the derivative gives a polynomial that still has root $\lambda_i$. 
This means

\[ c_0 n \lambda_i^{n-1} + c_1 (n - 1) \lambda_i^{n-2} + \cdots + c_k (n - k) \lambda_i^{n-k-1} = 0 \]

Hence,

\[ \underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n - 1) \lambda_i^{n-1}}_{T[n-1]} + \cdots + \underbrace{c_k (n - k) \lambda_i^{n-k}}_{T[n-k]} = 0 \]
The Homogeneous Case

Suppose $\lambda_i$ has multiplicity $j$. We know that

$$c_0 n \lambda_i^n + c_1 (n - 1) \lambda_i^{n-1} + \cdots + c_k (n - k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with $\lambda$; plugging in $\lambda_i$)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n - 1)^2 \lambda_i^{n-1} + \cdots + c_k (n - k)^2 \lambda_i^{n-k} = 0$$

We can continue $j - 1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j - 1$. 

The Homogeneous Case

Lemma 2

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n - 1] + \cdots + c_k T[n - k] = 0$$

Let $\lambda_i, i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities $\ell_i$. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of $\alpha_{ij}$’s is a solution to the recurrence.
Example: Fibonacci Sequence

\[ T[0] = 0 \]
\[ T[1] = 1 \]
\[ T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2 \]

The characteristic polynomial is

\[ \lambda^2 - \lambda - 1 \]

Finding the roots, gives

\[ \lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} \left( 1 \pm \sqrt{5} \right) \]
Example: Fibonacci Sequence

Hence, the solution is of the form

\[ \alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

\[ T[0] = 0 \text{ gives } \alpha + \beta = 0. \]

\[ T[1] = 1 \text{ gives } \]

\[ \alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}} \]
Example: Fibonacci Sequence

Hence, the solution is

\[ \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \]
Consider the recurrence relation:

\[ c_0 T(n) + c_1 T(n - 1) + c_2 T(n - 2) + \cdots + c_k T(n - k) = f(n) \]

with \( f(n) \neq 0 \).

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.
The Inhomogeneous Case

The general solution of the recurrence relation is

\[ T(n) = T_h(n) + T_p(n) , \]

where \( T_h \) is any solution to the homogeneous equation, and \( T_p \) is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.
The Inhomogeneous Case

Example:

\[ T[n] = T[n - 1] + 1 \quad T[0] = 1 \]

Then,

\[ T[n - 1] = T[n - 2] + 1 \quad (n \geq 2) \]

Subtracting the first from the second equation gives,

\[ T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2) \]

or

\[ T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2) \]

I get a completely determined recurrence if I add \( T[0] = 1 \) and \( T[1] = 2 \).
The Inhomogeneous Case

Example: Characteristic polynomial:

\[
\lambda^2 - 2\lambda + 1 = 0
\]

Then the solution is of the form

\[
T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n
\]

\[
T[0] = 1 \text{ gives } \alpha = 1.
\]

\[
T[1] = 2 \text{ gives } 1 + \beta = 2 \implies \beta = 1.
\]
The Inhomogeneous Case

If \( f(n) \) is a polynomial of degree \( r \) this method can be applied \( r + 1 \) times to obtain a homogeneous equation:

\[
T[n] = T[n - 1] + n^2
\]

Shift:

\[
T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1
\]

Difference:

\[
\]

\[
T[n] = 2T[n - 1] - T[n - 2] + 2n - 1
\]
\[ T[n] = 2T[n - 1] - T[n - 2] + 2n - 1 \]

Shift:

\[ T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \]
\[ = 2T[n - 2] - T[n - 3] + 2n - 3 \]

Difference:

\[ T[n] - T[n - 1] = 2T[n - 1] - T[n - 2] + 2n - 1 \]
\[ - 2T[n - 2] + T[n - 3] - 2n + 3 \]
\[ T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2 \]

and so on...