\section*{7.5 \((a, b)\)-trees}

\textbf{Definition 1}

For \(b \geq 2a - 1\) an \((a, b)\)-tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex \(v\) has at least \(a\) and at most \(b\) children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
5. there is a special dummy leaf node with key-value \(\infty\)
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5. there is a special dummy leaf node with key-value \(\infty\)
Each internal node \( v \) with \( d(v) \) children stores \( d - 1 \) keys \( k_1, \ldots, k_{d-1} \). The \( i \)-th subtree of \( v \) fulfills

\[
k_{i-1} < \text{key in } i\text{-th sub-tree} \leq k_i,
\]

where we use \( k_0 = -\infty \) and \( k_d = \infty \).
7.5 \((a, b)\)-trees

Example 2
7.5 \((a, b)\)-trees

Variants

- The dummy leaf element may not exist; it only makes implementation more convenient.

- Variants in which \(b = 2a\) are commonly referred to as \(B\)-trees.

- A \(B\)-tree usually refers to the variant in which keys and data are stored at internal nodes.

- A \(B^+\) tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.

- A \(B^*\) tree requires that a node is at least \(2/3\)-full as opposed to \(1/2\)-full (the requirement of a \(B\)-tree).
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Lemma 3

Let $T$ be an $(a, b)$-tree for $n > 0$ elements (i.e., $n + 1$ leaf nodes) and height $h$ (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \leq n + 1 \leq b^h$

2. $\log_b (n + 1) \leq h \leq 1 + \log_a (\frac{n+1}{2})$

Proof.

If $n > 0$, the root has degree at least 2 and all other nodes have degree at least $a$. This gives that the number of leaf nodes is at least $2a^{h-1}$.

Analogously, the degree of any node is at most $b$ and, hence, the number of leaf nodes at most $b^h$. 

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Time: $O(b \cdot h) = O(b \cdot \log n)$, if the individual nodes are organized as linear lists.
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Insert

Insert element $x$:

- Follow the path as if searching for $\text{key}[x]$.
- If this search ends in leaf $\ell$, insert $x$ before this leaf.
- For this add $\text{key}[x]$ to the key-list of the last internal node $v$ on the path.
- If after the insert $v$ contains $b$ nodes, do $\text{Rebalance}(v)$. 
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7.5 $(a, b)$-trees

Ernst Mayr, Harald Räcke
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**Insert**

**Rebalance(ν):**

- Let $k_i, i = 1, \ldots, b$ denote the keys stored in $ν$.  
- Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.  
- Create two nodes $ν_1$ and $ν_2$. $ν_1$ gets all keys $k_1, \ldots, k_{j-1}$ and $ν_2$ gets keys $k_{j+1}, \ldots, k_b$.  
- Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.  
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- The key $k_j$ is promoted to the parent of $ν$. The current pointer to $ν$ is altered to point to $ν_1$, and a new pointer (to the right of $k_j$) in the parent is added to point to $ν_2$.  
- Then, re-balance the parent.
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7.5 \((a, b)\)-trees
Insert

Insert(8)

7.5 \((a, b)\)-trees
Insert

Insert(8)
Insert

Insert(8)

10 19

1 3 5 8

1 3 5 8 10

14 28

10 19

14 19 28 ∞

8

7.5 \((a, b)\)-trees
Insert

Insert(8)
Insert

Insert(8)
7.5 \((a, b)\)-trees
Insert

Insert(6)
Insert

Insert(6)
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Insert

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7.5 \((a, b)\)-trees
Insert

Insert(7)

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Insert

Insert(7)
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\[
\begin{array}{c}
1 \\
3 \\
6 \\
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8 \\
10 \\
19 \\
3 \\
10 \\
19 \\
1 \\
5 \\
6 \\
7 \\
8 \\
14 \\
19 \\
28 \\
\infty
\end{array}
\]

7.5 \((a, b)\)-trees
Insert

Insert(7)
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Insert(7)

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Insert

Insert(7)

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Insert(7)

$7.5 \ (a,b)$-trees

Ernst Mayr, Harald Räcke

11. Apr. 2018

199/203
Delete element $x$ (pointer to leaf vertex):

- Let $v$ denote the parent of $x$. If $\text{key}[x]$ is contained in $v$, remove the key from $v$, and delete the leaf vertex.

- Otherwise delete the key of the predecessor of $x$ from $v$; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).

- If now the number of keys in $v$ is below $a - 1$ perform Rebalance$'$($v$).
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Delete element \( x \) (pointer to leaf vertex):

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- If now the number of keys in \( v \) is below \( a - 1 \) perform \( \text{Rebalance}'(v) \).
Delete

Rebalance’$(v)$:

- If there is a neighbour of $v$ that has at least $a$ keys take over the largest (if right neighbour) or smallest (if left neighbour) and the corresponding sub-tree.
- If not: merge $v$ with one of its neighbours.
- The merged node contains at most $(a - 2) + (a - 1) + 1$ keys, and has therefore at most $2a - 1 \leq b$ successors.
- Then rebalance the parent.
- During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.
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Rebalance′(v):

- If there is a neighbour of v that has at least a keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
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7.5 \((a, b)\)-trees
Delete

Delete(10)

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Delete

Delete(10)

7.5 \((a,b)\)-trees
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Delete(10)

7.5 \((a, b)\)-trees
7.5 \((a, b)\)-trees
Delete

Delete(14)
Delete

Delete(14)

7.5 \((a,b)\)-trees
Delete

Delete(14)
Delete(14)
Delete

Delete(14)
$7.5 \ (a, b)$-trees
Delete $(3)$
Delete

Delete(3)

7.5 \((a, b)\)-trees

Ernst Mayr, Harald Räcke
Delete

Delete(3)

7.5 \((a, b)\)-trees
Delete

Delete(3)

7.5 \((a, b)\)-trees
Delete

Delete(3)

\[ \begin{array}{c}
\text{19} \\
\downarrow \\
\text{1 5} \\
\downarrow \\
\text{1 5 19} \\
\downarrow \\
\text{1 5 19} \\
\downarrow \\
\text{1 5 19 28} \\
\downarrow \\
\text{1 5 19 28 }\infty
\end{array} \]
Delete

7.5 \((a, b)\)-trees
Delete

Delete$(1)$
Delete \(1\)
Delete(1)
7.5 \((a, b)\)-trees
Delete

Delete(19)
Delete(19)
Delete

Delete(19)

7.5 \((a, b)\)-trees

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Delete

Delete(19)

\[ \text{Diagram:}\]

7.5 \((a, b)\)-trees
Delete (19)

7.5 \((a, b)\)-trees
Delete (19)
(2, 4)-trees and red black trees

There is a close relation between red-black trees and (2, 4)-trees:
(2, 4)-trees and red black trees

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Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same (2, 4)-tree.
There is a close relation between red-black trees and \((2, 4)\)-trees:
(2, 4)-trees and red black trees

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![Diagram of (2, 4)-trees and red black trees](image-url)
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