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Problem Set 8
December 18, 2017

# Online and Approximation Algorithms

*Due January 8, 2018 at 10:00*

## Exercise 1 (Path Game – 10 points)

Consider the following 2-player game. There is a graph $G = (V, E)$ and the game takes place in alternating turns. In each turn, a player picks an edge $e \in E$ which has not been chosen by any player before, so that the selected edges form a single path. The first player who is unable to choose such an edge loses the game.
Show that, if the starting player is given a perfect matching $M$ of $G$, there exists a winning strategy for him.

## Exercise 2 (Randomized Matching – 10 points)

Consider the following randomized online algorithm for the maximum matching problem on bipartite graphs. Whenever a new vertex $v \in V$ arrives, match $v$ with a vertex $u \in U$ chosen uniformly at random among the currently unmatched neighbors of $v$. Show that the competitive ratio of this algorithm cannot be better than $\frac{1}{2}$.

*Hint: Consider a bipartite graph $G = (U \cup V, E)$ such that $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$. The vertices $u_i$ and $v_j$ are connected if and only if either $1 \le i, j \le \frac{n}{2}$, or $i + j = n + 1$.*

## Exercise 3 (Ranking – 10 points)

Let $G = (U \cup V, E)$ be a bipartite graph. Prove that the *Ranking* algorithm fulfills the following property.
When fixing a permutation $\pi$ on $U$, the following methods produce the same matching:

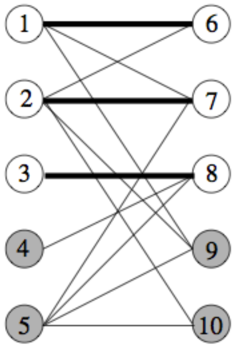
**Method 1** Nodes of $V$ arrive online and each node $v \in V$ is matched to an adjacent node $u \in U$ that has the lowest rank according to $\pi$.

**Method 2** Nodes in $V = \{v_1, ..., v_{|V|}\}$ are known in advance and nodes in $U$ arrive in an online fashion according to $\pi$. Every node $u \in U$ is matched to an adjacent node $v \in V$ with the lowest index number.

## Exercise 4 (Augmenting Paths – 10 points)

Consider a bipartite graph $G = (U \cup V, E)$ and a matching $M$ of $G$. A *simple path* of $G$ is a collection of edges $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$ where all $v_i$'s are distinct. Such a path can be also represented as $v_0, v_1, \dots, v_k$. An *alternating path* of $G$ with respect to $M$ is a simple path which alternates between edges in $M$ and edges in $E - M$. An *augmenting*

*path* of $G$ with respect to $M$ is an alternating path in which the first and last vertices are unmatched (i.e. they are not the endpoint of any edge in $M$).
For example, in the following graph all paths 4,8,3 and 6,1,7,2 as well as 5,7,2,6,1,9 are alternating with respect to the matching $M = \{(1,6),(2,7),(3,8)\}$. However, only the last one is augmenting.




- Show that a matching is maximum if and only if there are no augmenting paths w.r.t. it.