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Problem Set 8
December 18, 2018

Advanced Algorithms

Due January 8, 2019 at 10:00

Note: You are welcome to submit in groups of two. If you wish to submit individually, you should solve Exercise 4 (dynamic table) and one of the other exercises.

Exercise 1 (Potential Method – 10 points)

1. Suppose we have a potential function Φ such that $\Phi_0 \neq 0$ and $\Phi_i \geq \Phi_0$ for all $i \geq 1$. Show that there exists a potential function Φ' such that $\Phi'_0 = 0$ and $\Phi'_i \geq 0$ for all i, and the total amortized costs using Φ' are the same as the total amortized costs using Φ.

2. In the class we studied increment operations on a binary counter. Suppose we wish not only to increment a binary counter but also to reset it to zero (i.e., make all bits in it 0). Show how to implement a binary counter as an array of bits so that any sequence of n INCREMENT and RESET operations takes time $O(n)$ on an initially zero counter. (Hint: keep a pointer to the high-order 1.)

Exercise 2 (Expensive powers of 2 – 10 points)

Given a data structure with costs c_i for the i'th operation where

$$c_i = \begin{cases} i \text{ for } i \in \{2^k : k \in \mathbb{N}\}, \\ 1 \text{ otherwise,} \end{cases}$$

determine the (constant) amortized costs for an operation using the potential method of analysis.

Exercise 3 (Amortized analysis of a ternary counter – 10 points)

We consider a ternary counter, i.e., a counter with three as its base and digits 0, 1, and 2. The counter starts at 0 and is incremented n times by 1. The cost for increasing the counter from $i-1$ to i is determined by the number of digits of the counter that has to be changed. Let $A(n)$ be the cost for successively increasing the counter from 0 to n. Show that $A(n)$ is linear in n and determine the minimal $c \in \mathbb{R}$ satisfying the following:

$$A(n) \leq cn \text{ for all } n \in \mathbb{N} \tag{1}$$

Hints:

1. Use amortized analysis to show that your $c \in \mathbb{R}$ fulfills inequality (1).

2. Show that $c - \varepsilon$ does not satisfy inequality (1) for any $\varepsilon > 0$.

Exercise 4 (Dynamic Table – 10 points)

Suppose that instead of contracting a table by halving its size when its load factor drops below 1/4, we contract it by multiplying its size by 2/3 when its load factor drops below 1/3. Using the potential function

$$\Phi(T) = |2 \cdot num[T] - size[T]|,$$

show that the amortized cost of a TABLE-DELETE that uses this strategy is bounded above by a constant.