Part V
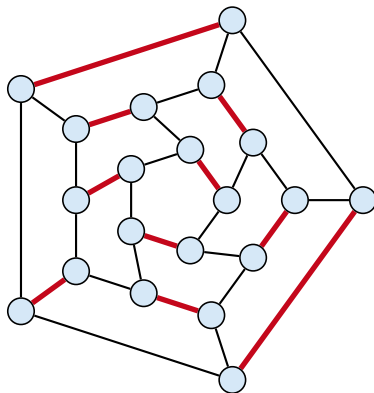
Matchings

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality

16 Bipartite Matching via Flows

Which flow algorithm to use?

▶ Generic augmenting path: $\mathcal{O}(m \operatorname{val}(f^*)) = \mathcal{O}(mn)$.

▶ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

▶ Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

17 Augmenting Paths for Matchings

Definitions.

▶ Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.

▶ For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.

▶ An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

Theorem 1
A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

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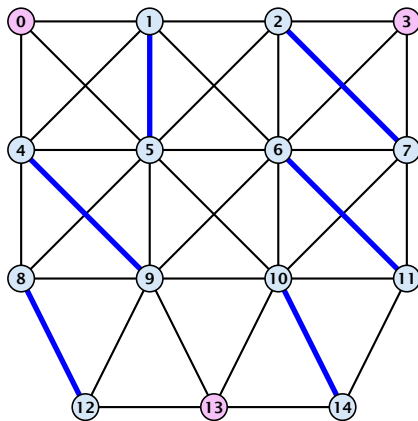# 17 Augmenting Paths for Matchings

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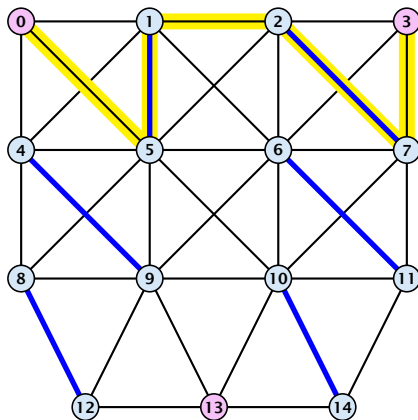- ▶ An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

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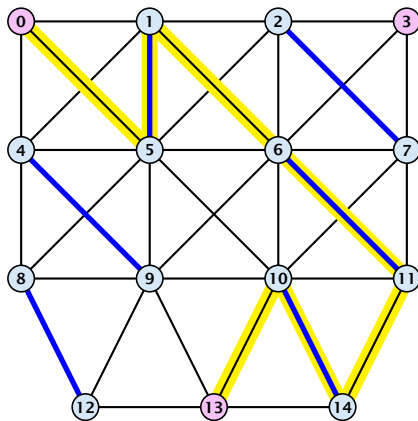
Augmenting Paths in Action
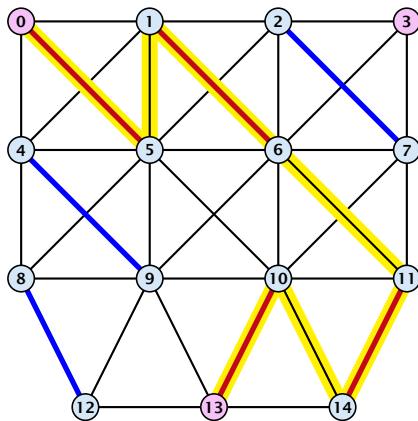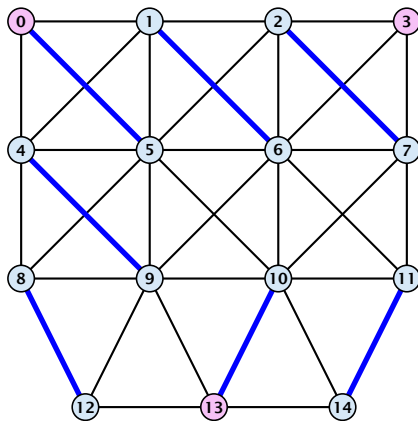
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17 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

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17 Augmenting Paths for Matchings

Algorithmic idea:

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 2

Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

The above theorem allows for an easier implementation of an augmenting path algorithm. Once we checked for augmenting paths starting from u we don't have to check for such paths in future rounds.
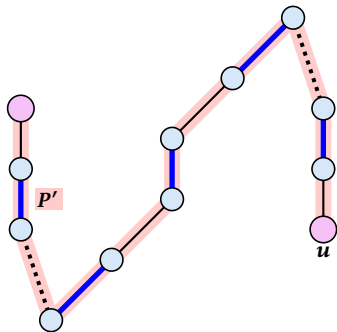
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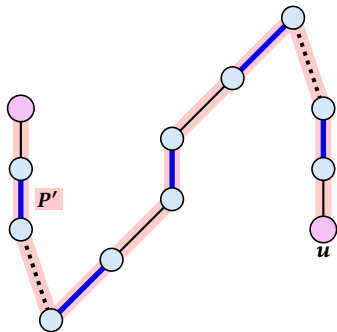# 17 Augmenting Paths for Matchings

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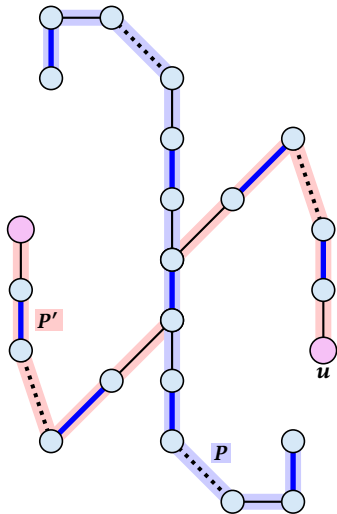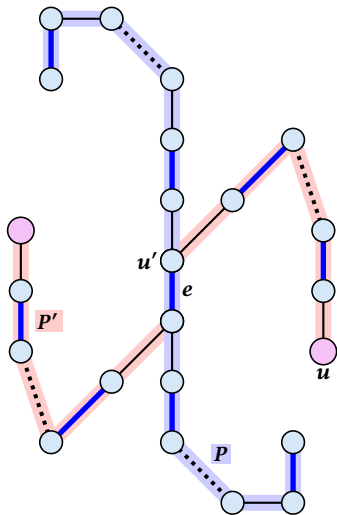
17 Augmenting Paths for Matchings

Proof

- ▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

- ▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).

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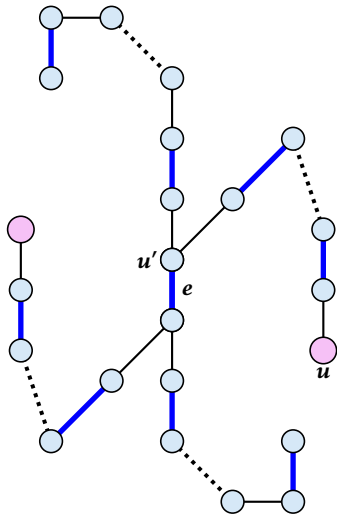
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▶ Let u' be the **first** node on P' that is in P, and let e be the matching edge from M' incident to u'.

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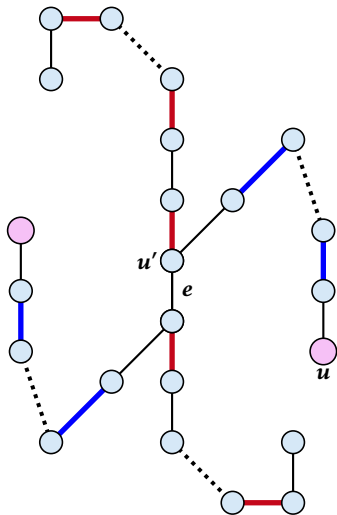
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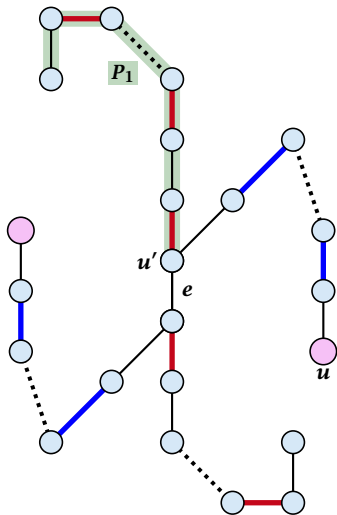# 17 Augmenting Paths for Matchings

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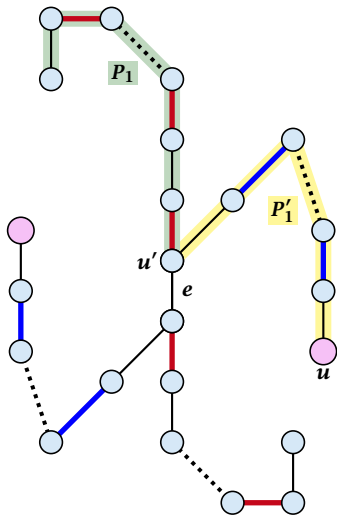
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▶ Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.

▶ u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P_1'.
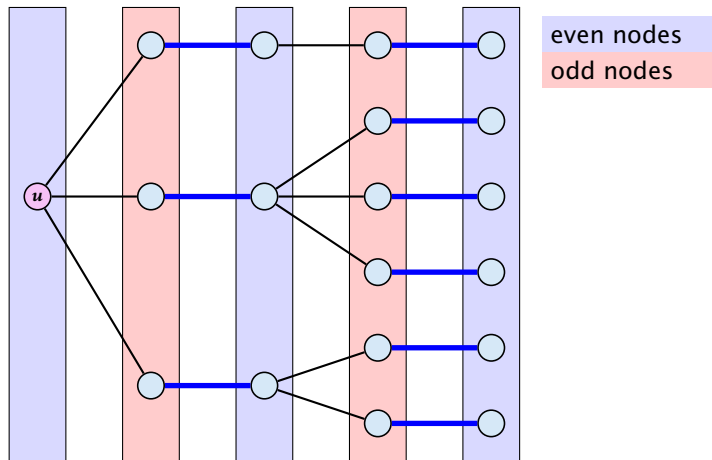
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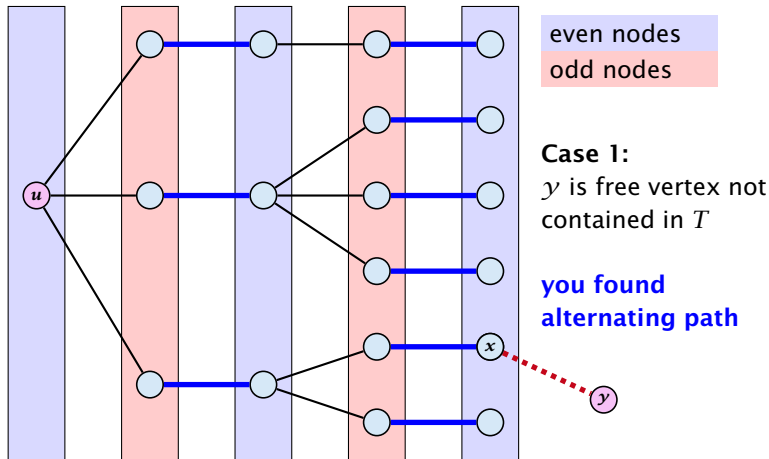
▶ $P_1 \circ P_1'$ is augmenting path in M (↯).

How to find an augmenting path?

Construct an alternating tree.

even nodes
odd nodes

How to find an augmenting path?

Construct an alternating tree.

even nodes
odd nodes

Case 1:
y is free vertex not
contained in T

**you found
alternating path**

How to find an augmenting path?

Construct an alternating tree.

even nodes
odd nodes

Case 2:
y is matched vertex
not in T; then
mate$[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.
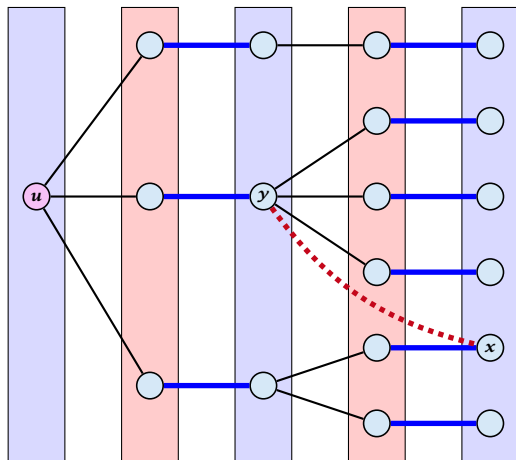

even nodes
odd nodes

Case 3:
y is already contained
in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.

even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

Algorithm 24 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i'] ← 0
 7:         Q ← ∅; Q. append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q. dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

graph $G = (S \cup S', E)$

$S = \{1, \ldots, n\}$

$S' = \{1', \ldots, n'\}$

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17:                         parent[y] ← x;
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start with an empty matching

Algorithm 24 BiMatch(G, $match$)

1: **for** $x \in V$ **do** $mate[x] \leftarrow 0$;
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3: **while** $free \geq 1$ **and** $r < n$ **do**
4:     $r \leftarrow r + 1$
5:     **if** $mate[r] = 0$ **then**
6:         **for** $i = 1$ **to** n **do** $parent[i'] \leftarrow 0$
7:         $Q \leftarrow \emptyset$; $Q.\text{append}(r)$; $aug \leftarrow$ false;
8:         **while** $aug =$ false **and** $Q \neq \emptyset$ **do**
9:             $x \leftarrow Q.\text{dequeue}()$;
10:             **for** $y \in A_x$ **do**
11:                 **if** $mate[y] = 0$ **then**
12:                     $\text{augm}(mate, parent, y)$;
13:                     $aug \leftarrow$ true;
14:                     $free \leftarrow free - 1$;
15:                 **else**
16:                     **if** $parent[y] = 0$ **then**
17:                         $parent[y] \leftarrow x$;
18:                         $Q.\text{enqueue}(mate[y])$;

free: number of unmatched nodes in S

r: root of current tree

Algorithm 24 BiMatch(G, *match*)

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15:                else
16:                    if parent[y] = 0 then
17:                        parent[y] ← x;
18:                        Q.enqueue(mate[y]);
```

as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

Algorithm 24 BiMatch(G, $match$)

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13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

r is the new node that we grow from.

Algorithm 24 BiMatch(G, *match*)

1: **for** $x \in V$ **do** *mate*$[x] \leftarrow 0$;
2: $r \leftarrow 0$; *free* $\leftarrow n$;
3: **while** *free* ≥ 1 **and** $r < n$ **do**
4:    $r \leftarrow r + 1$
5:    **if** *mate*$[r] = 0$ **then**
6:       **for** $i = 1$ **to** n **do** *parent*$[i'] \leftarrow 0$
7:       $Q \leftarrow \emptyset$; $Q.$append(r); *aug* $\leftarrow$ false;
8:       **while** *aug* = false **and** $Q \neq \emptyset$ **do**
9:          $x \leftarrow Q.$dequeue$()$;
10:          **for** $y \in A_x$ **do**
11:             **if** *mate*$[y] = 0$ **then**
12:                augm$(mate, parent, y)$;
13:                *aug* $\leftarrow$ true;
14:                *free* $\leftarrow$ *free* $- 1$;
15:             **else**
16:                **if** *parent*$[y] = 0$ **then**
17:                    *parent*$[y] \leftarrow x$;
18:                    $Q.$enqueue$(mate[y])$;

If r is free start tree construction

Algorithm 24 BiMatch(G, match)

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 1: for x ∈ V do mate[x] ← 0;
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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

Initialize an empty tree. Note that only nodes i' have parent pointers.

Algorithm 24 BiMatch(G, *match*)

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15:                else
16:                    if parent[y] = 0 then
17:                        parent[y] ← x;
18:                        Q.enqueue(mate[y]);
```

Q is a queue (BFS!!!).

aug is a Boolean that stores whether we already found an augmenting path.

Algorithm 24 BiMatch(*G*, *match*)

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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0 then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

as long as we did not augment and there are still unexamined leaves continue...

Algorithm 24 BiMatch(G, $match$)

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15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

take next unexamined leaf

Algorithm 24 BiMatch(G, *match*)

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4:    $r \leftarrow r + 1$
5:    **if** *mate*$[r] = 0$ **then**
6:       **for** $i = 1$ **to** n **do** *parent*$[i'] \leftarrow 0$
7:       $Q \leftarrow \emptyset$; $Q.$ append(r); *aug* $\leftarrow$ false;
8:       **while** *aug* = false **and** $Q \neq \emptyset$ **do**
9:          $x \leftarrow Q.$ dequeue$()$;
10:          **for** $y \in A_x$ **do**
11:             **if** *mate*$[y] = 0$ **then**
12:                augm(*mate*, *parent*, y);
13:                *aug* $\leftarrow$ true;
14:                *free* $\leftarrow$ *free* $- 1$;
15:             **else**
16:                **if** *parent*$[y] = 0$ **then**
17:                   *parent*$[y] \leftarrow x$;
18:                   $Q.$ enqueue(*mate*$[y]$);

if x has unmatched neighbour we found an augmenting path (note that $y \neq r$ because we are in a bipartite graph)

Algorithm 24 BiMatch(G, $match$)

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8:        **while** $aug =$ false **and** $Q \neq \emptyset$ **do**
9:           $x \leftarrow Q$. dequeue();
10:           **for** $y \in A_x$ **do**
11:              **if** $mate[y] = 0$ **then**
12:                 augm($mate, parent, y$);
13:                 $aug \leftarrow$ true;
14:                 $free \leftarrow free - 1$;
15:              **else**
16:                 **if** $parent[y] = 0$ **then**
17:                     $parent[y] \leftarrow x$;
18:                     Q. enqueue($mate[y]$);

do an augmentation...

Algorithm 24 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i'] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
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11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

setting *aug* = true ensures that the tree construction will not continue

Algorithm 24 BiMatch(G, *match*)

1: **for** $x \in V$ **do** $mate[x] \leftarrow 0$;
2: $r \leftarrow 0$; $free \leftarrow n$;
3: **while** $free \geq 1$ **and** $r < n$ **do**
4:    $r \leftarrow r + 1$
5:    **if** $mate[r] = 0$ **then**
6:       **for** $i = 1$ **to** n **do** $parent[i'] \leftarrow 0$
7:       $Q \leftarrow \emptyset$; Q. append(r); $aug \leftarrow$ false;
8:       **while** $aug =$ false **and** $Q \neq \emptyset$ **do**
9:          $x \leftarrow Q$. dequeue();
10:          **for** $y \in A_x$ **do**
11:             **if** $mate[y] = 0$ **then**
12:                augm(*mate*, *parent*, y);
13:                $aug \leftarrow$ true;
14:                $free \leftarrow free - 1$;
15:             **else**
16:                **if** $parent[y] = 0$ **then**
17:                   $parent[y] \leftarrow x$;
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reduce number of free nodes

Algorithm 24 BiMatch(G, $match$)

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1:  for x ∈ V do mate[x] ← 0;
2:  r ← 0; free ← n;
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8:          while aug = false and Q ≠ ∅  do
9:              x ← Q.dequeue();
10:             for y ∈ Aₓ do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
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14:                     free ← free − 1;
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16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
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```

if y is not in the tree yet

Algorithm 24 BiMatch(*G*, *match*)

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 1: for x ∈ V do mate[x] ← 0;
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17:                         parent[y] ← x;
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...put it into the tree

Algorithm 24 BiMatch(G, $match$)

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```

add its buddy to the set of unexamined leaves

18 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

▶ Input: undirected, bipartite graph $G = L \cup R, E$.

▶ an edge $e = (\ell, r)$ has weight $w_e \geq 0$

▶ find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

Simplifying Assumptions (wlog [why?]):

▶ assume that $|L| = |R| = n$

▶ assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$

▶ can assume goal is to construct maximum weight perfect matching

Weighted Bipartite Matching

Theorem 3 (Halls Theorem)

A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S.

18 Weighted Bipartite Matching
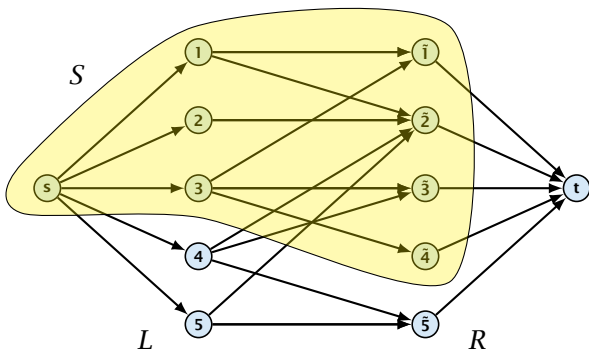
Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

- ▶ Let S denote a minimum cut and let $L_S \triangleq L \cap S$ and $R_S \triangleq R \cap S$ denote the portion of S inside L and R, respectively.
- ▶ Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.
- ▶ This gives $R_S \geq |\Gamma(L_S)|$.
- ▶ The size of the cut is $|L| - |L_S| + |R_S|$.
- ▶ Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

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Algorithm Outline

Idea:

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▶ Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

▶ Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.

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Reason:

▶ The weight of your matching M^* is

$$\sum_{(u,v) \in M^*} w_{(u,v)} = \sum_{(u,v) \in M^*} (x_u + x_v) = \sum_v x_v \ .$$

▶ Any other perfect matching M (in G, not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v) \in M} w_{(u,v)} \leq \sum_{(u,v) \in M} (x_u + x_v) = \sum_v x_v \ .$$

Algorithm Outline

What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Idea: reweight such that:

▶ the total weight assigned to nodes decreases

▶ the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

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Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

▶ Total node-weight decreases.

▶ Only edges from S to $R - \Gamma(S)$ decrease in their weight.

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Weighted Bipartite Matching

Edges not drawn have weight 0.

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Analysis

How many iterations do we need?

▶ One reweighting step increases the number of edges out of S by at least one.

▶ Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).

▶ This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.

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Analysis

- We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.

- This gives a polynomial running time.

How to find an augmenting path?

Construct an alternating tree.
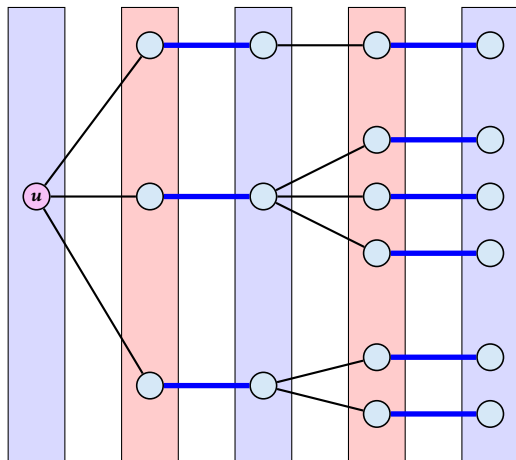
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How do we find S?

▶ Start on the left and compute an alternating tree, starting at any free node u.

▶ If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).

▶ The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.

▶ All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u. Hence, $|V_{odd}| = |\Gamma(V_{even})| < |V_{even}|$, and all odd vertices are saturated in the current matching.

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Analysis

- The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.

- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).

- An augmentation takes at most $\mathcal{O}(n)$ time.

- In total we obtain a running time of $\mathcal{O}(n^4)$.

- A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.

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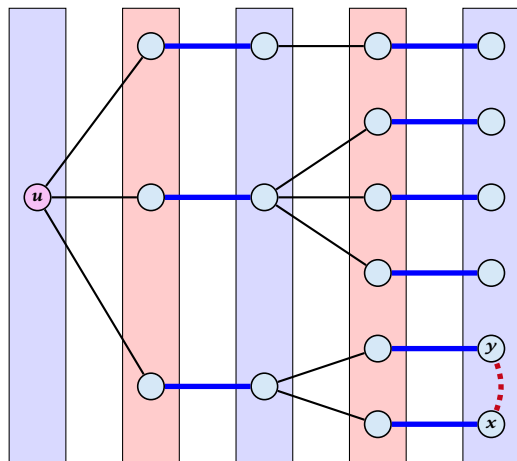
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even nodes
odd nodes

Case 4:
y is already contained
in T as an even vertex

can't ignore y

How to find an augmenting path?

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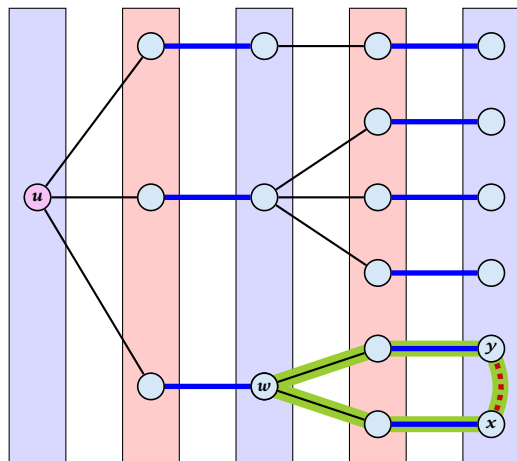
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odd nodes

Case 4:
y is already contained
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can't ignore y

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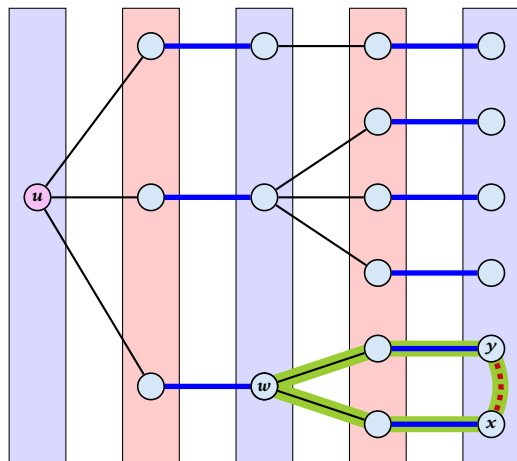
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The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w is called the stem of the blossom.

Flowers and Blossoms

Definition 4

A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

▶ A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).

▶ A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

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Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.

2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.

3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

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Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

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Flowers and Blossoms

Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

▶ Delete all vertices in B (and its incident edges) from G.

▶ Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

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Shrinking Blossoms

▶ Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.

▶ Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.

▶ Nodes that are connected in G to at least one node in B become connected to b in G'.

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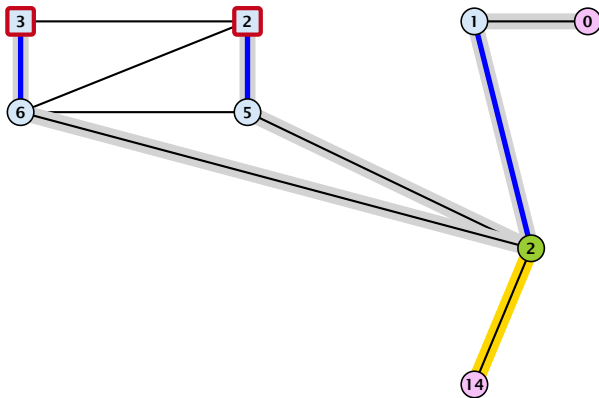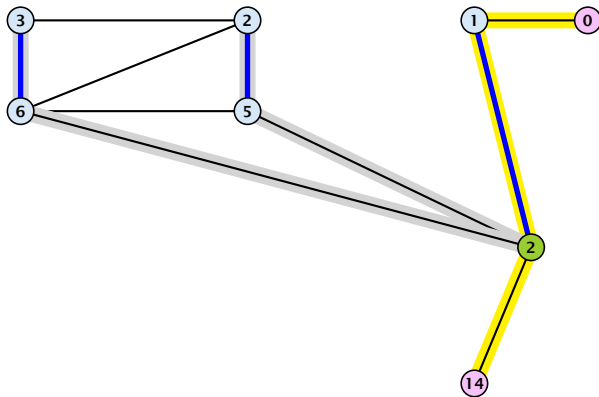at least one node in B become
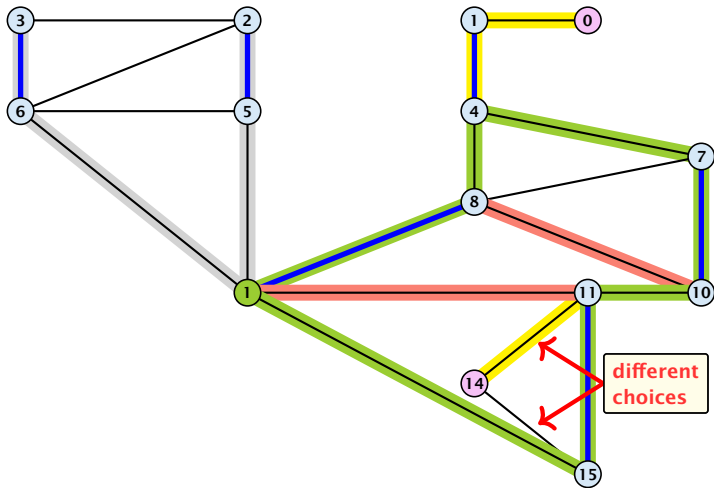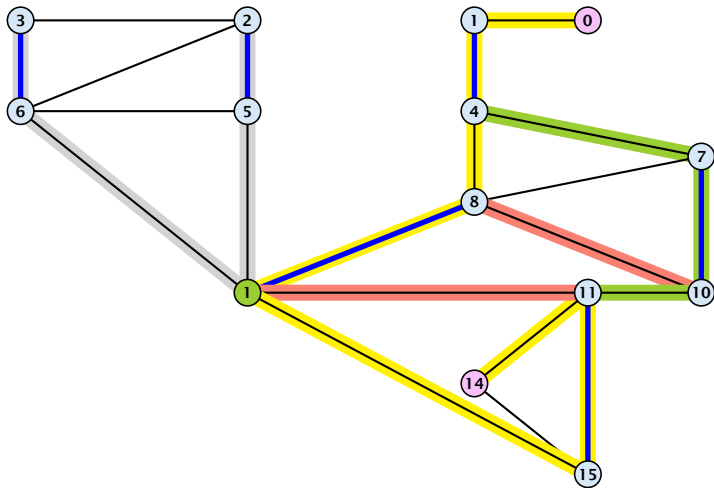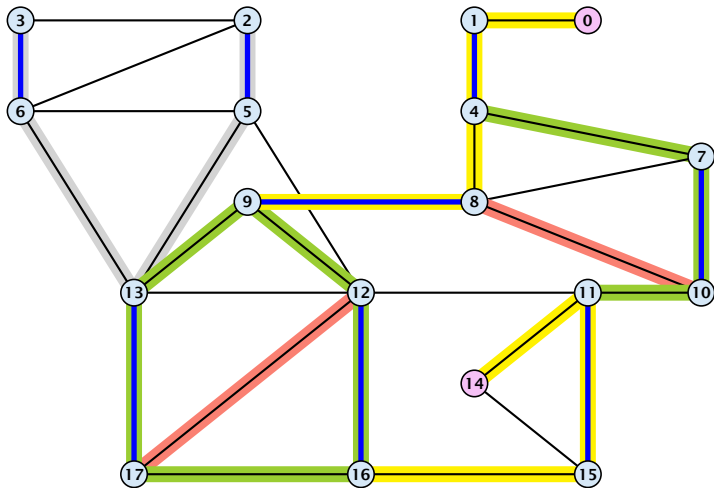connected to b in G'.

Example: Blossom Algorithm

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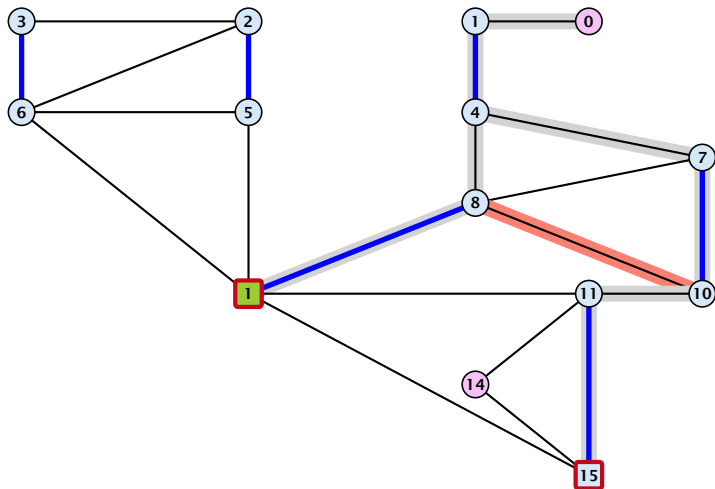
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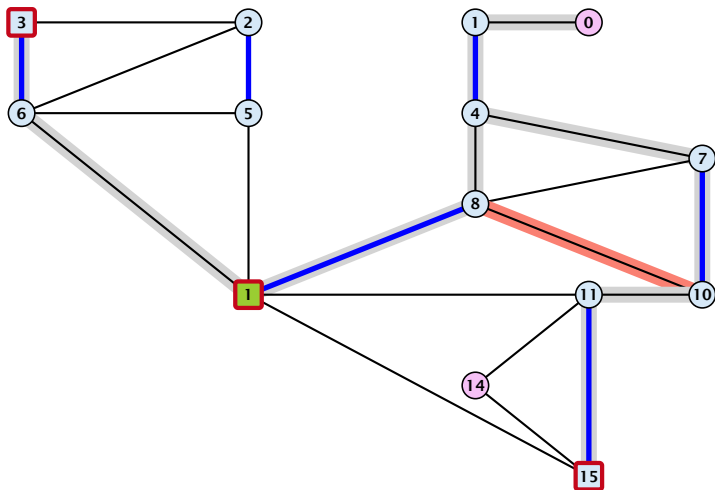
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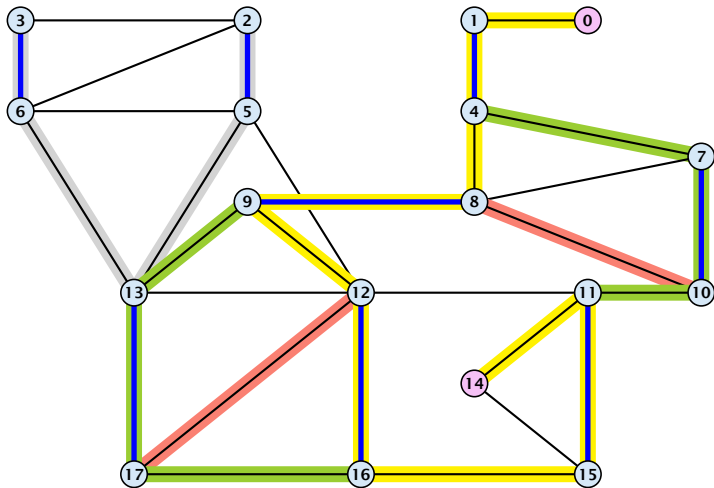
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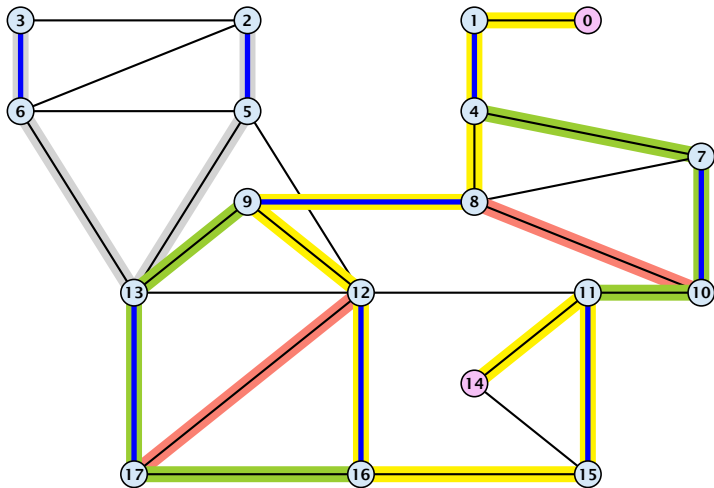
Example: Blossom Algorithm

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Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 5

If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

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If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

Correctness

Proof.

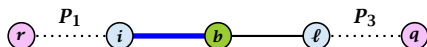If P' does not contain b it is also an augmenting path in G.

Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

Case 1: non-empty stem

▶ Next suppose that the stem is non-empty.
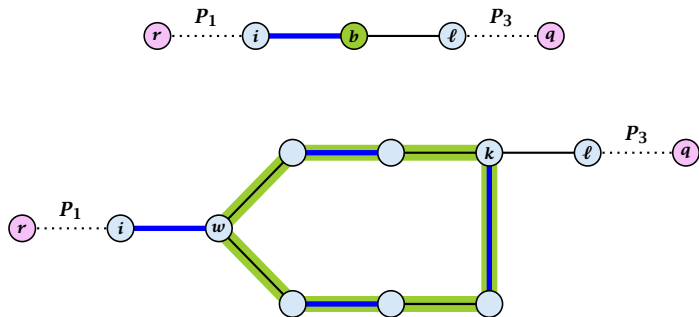
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Correctness

▶ After the expansion ℓ must be incident to some node in the blossom. Let this node be k.

▶ If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.

▶ $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

▶ If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

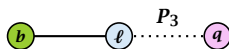# Correctness

Proof.

Case 2: empty stem

▶ If the stem is empty then after expanding the blossom, $w = r$.

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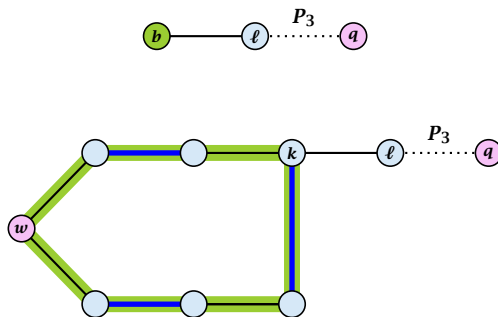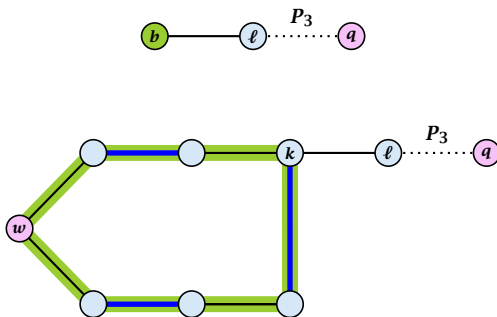▶ If the stem is empty then after expanding the blossom,
  $w = r$.

▶ The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 6

If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Correctness

Proof.

▶ If P does not contain a node from B there is nothing to prove.

▶ We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.

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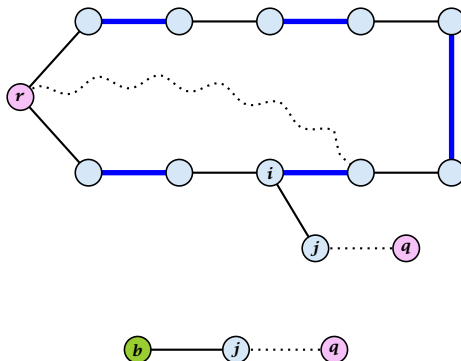
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Correctness

Illustration for Case 1:

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

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Algorithm 25 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \emptyset$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

Search for an augmenting path
starting at r.

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$A(i)$ contains neighbours of node i.

We create a copy $\bar{A}(i)$ so that we later can shrink blossoms.

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found is just a Boolean that allows
to abort the search process...

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In the beginning no node is in the tree.

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Put the root in the tree.

list could also be a set or a stack.

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As long as there are nodes with
unexamined neighbours...

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...examine the next one

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If you found augmenting path
abort and start from next root.

Algorithm 26 examine($i, found$)

1: **for** all $j \in \bar{A}(i)$ **do**
2:     **if** j is even **then** contract(i, j) and **return**
3:     **if** j is unmatched **then**
4:         $q \leftarrow j$;
5:         pred(q) $\leftarrow i$;
6:         $found \leftarrow$ true;
7:         **return**
8:     **if** j is matched and unlabeled **then**
9:         pred(j) $\leftarrow i$;
10:        pred(mate(j)) $\leftarrow j$;
11:       add mate(j) to $list$

Examine the neighbours of a node i

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For all neighbours j do...

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You have found a blossom...

Algorithm 26 examine(i, *found*)
1: **for** all $j \in \bar{A}(i)$ **do**
2:      **if** j is even **then** contract(i, j) and **return**
3:      **if** j is unmatched **then**
4:        $q \leftarrow j$;
5:        $\mathrm{pred}(q) \leftarrow i$;
6:        *found* $\leftarrow$ true;
7:        **return**
8:      **if** j is matched and unlabeled **then**
9:        $\mathrm{pred}(j) \leftarrow i$;
10:        $\mathrm{pred}(\mathrm{mate}(j)) \leftarrow j$;
11:        add $\mathrm{mate}(j)$ to $list$

> You have found a free node which
> gives you an augmenting path.

Algorithm 26 examine(i, *found*)

1: **for** all $j \in \bar{A}(i)$ **do**
2:     **if** j is even **then** contract(i, j) and **return**
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8:     **if** j is matched and unlabeled **then**
9:         pred(j) $\leftarrow i$;
10:        pred(mate(j)) $\leftarrow j$;
11:        add mate(j) to *list*

If you find a matched node that is not
in the tree you grow...

Algorithm 26 examine(i, *found*)

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3:      **if** j is unmatched **then**
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8:      **if** j is matched and unlabeled **then**
9:          pred(j) $\leftarrow i$;
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11:          add mate(j) to *list*

mate(j) is a new node from which you can grow further.

Algorithm 27 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Contract blossom identified by
nodes i and j

Algorithm 27 contract(i, j)

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Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

Algorithm 27 contract(i, j)

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6: delete nodes in B from the graph

Identify all neighbours of b.

Time: $\mathcal{O}(m)$ (how?)

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 6: delete nodes in B from the graph

> b will be an even node, and it has
> unexamined neighbours.

Algorithm 27 contract(i, j)

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Every node that was adjacent to a node
in B is now adjacent to b

Algorithm 27 contract(i, j)

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Only for making a blossom
expansion easier.

Algorithm 27 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
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5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only delete links from nodes not in *B* to *B*.

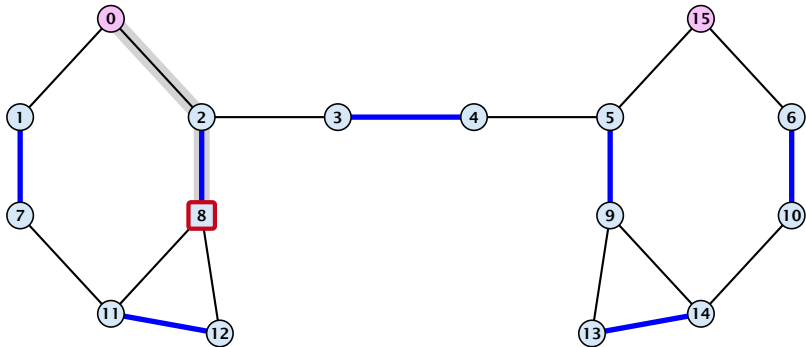When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

Analysis

▶ A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

▶ The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.

▶ There are at most n contractions as each contraction reduces the number of vertices.

▶ The expansion can trivially be done in the same time as needed for all contractions.

▶ An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.

▶ In total the running time is at most

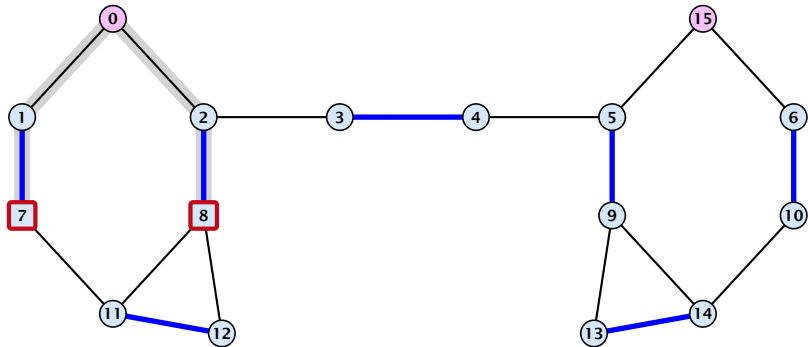$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

Analysis

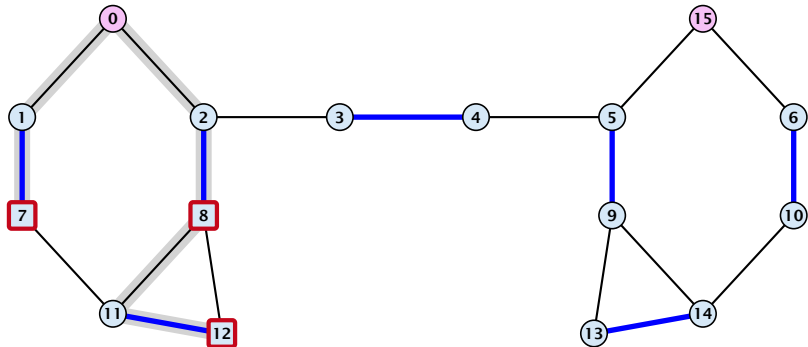▶ A contraction operation can be performed in time $\mathcal{O}(m)$.
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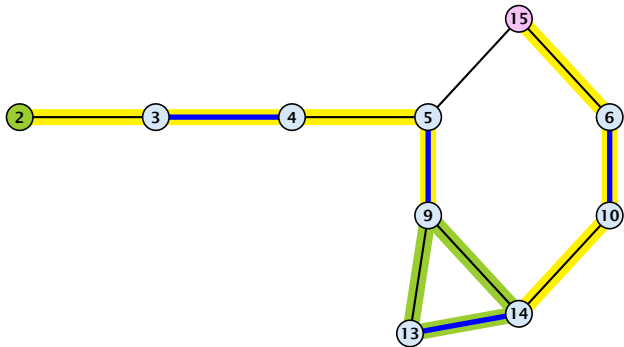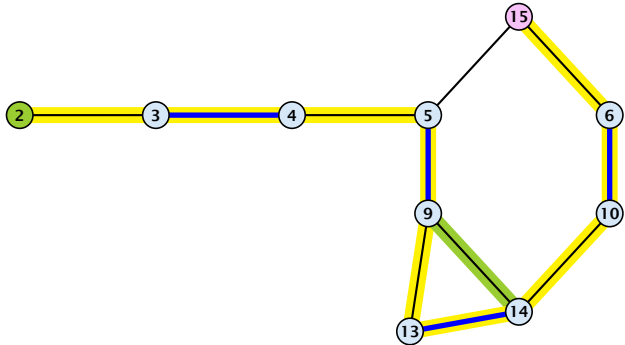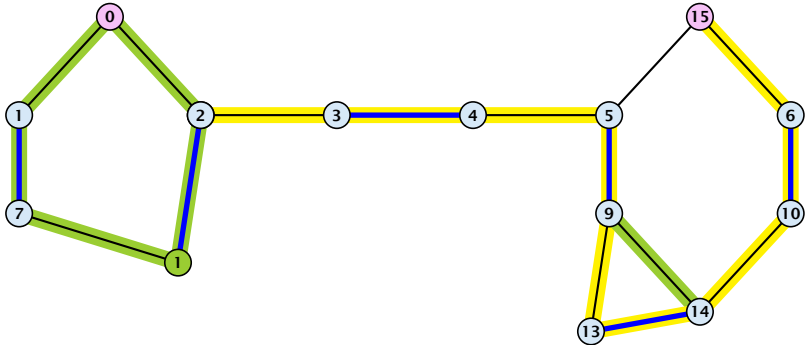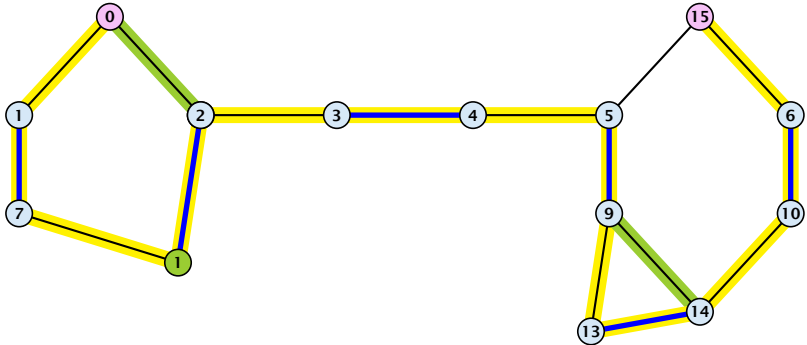$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

Example: Blossom Algorithm

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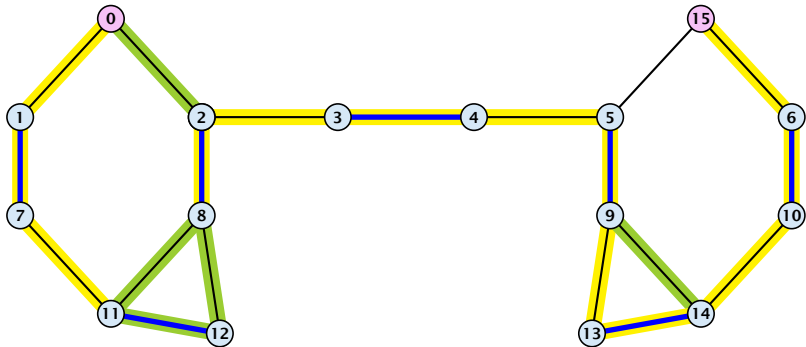# Example: Blossom Algorithm

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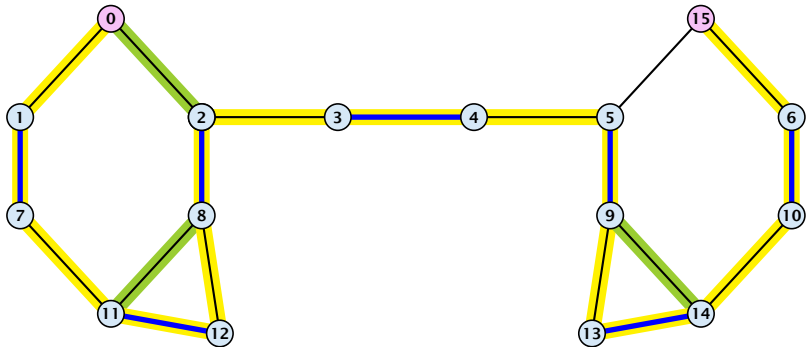
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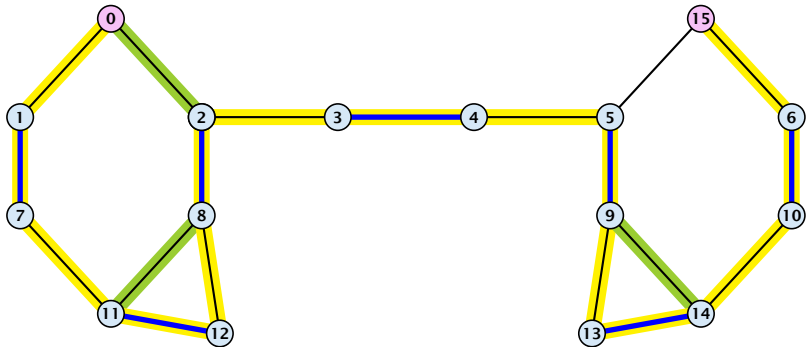
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A Fast Matching Algorithm

Algorithm 28 Bimatch-Hopcroft-Karp(G)

1: $M \leftarrow \emptyset$
2: **repeat**
3:       let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be maximal set of
4:       vertex-disjoint, shortest augmenting path w.r.t. M.
5:       $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
6: **until** $\mathcal{P} = \emptyset$
7: **return** M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis Hopcroft-Karp

Lemma 7

Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

Analysis Hopcroft-Karp

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Proof:

- ▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- ▶ Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- ▶ The connected components of G are cycles and paths.
- ▶ The graph contains $k \cong |M^*| - |M|$ more red edges than blue edges.
- ▶ Hence, there are at least k components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. M.

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Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ *vertex-disjoint augmenting path w.r.t. M.*

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Analysis Hopcroft-Karp

▶ Let $P_1, \ldots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).

▶ $M' \cong M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.

▶ Let P be an augmenting path in M'.

Lemma 8

The set $A \cong M \oplus (M' \oplus P) = (P_1 \cup \cdots \cup P_k) \oplus P$ contains at least $(k+1)\ell$ edges.

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Proof.

- The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.

- Hence, the set contains at least $k + 1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.

- Each of these paths is of length at least ℓ.

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▶ Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.

▶ This edge is not contained in A.

▶ Hence, $|A| \leq k\ell + |P| - 1$.

▶ The lower bound on $|A|$ gives $(k + 1)\ell \leq |A| \leq k\ell + |P| - 1$, and hence $|P| \geq \ell + 1$.

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Analysis Hopcroft-Karp

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Proof.
The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

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Analysis Hopcroft-Karp

Lemma 10

The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

Proof.

▶ After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.

▶ Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.

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Analysis Hopcroft-Karp

Lemma 11

One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

construct a "level graph" G':

- ▶ construct Level 0 that includes all free vertices on left side L
- ▶ construct Level 1 containing all neighbors of Level 0
- ▶ construct Level 2 containing matching neighbors of Level 1
- ▶ construct Level 3 containing all neighbors of Level 2
- ▶ . . .
- ▶ stop when a level (apart from Level 0) contains a free vertex

can be done in time $\mathcal{O}(m)$ by a modified BFS

Analysis Hopcroft-Karp

- ▶ a shortest augmenting path must go from Level 0 to the last layer constructed
- ▶ it can only use edges between layers
- ▶ construct a maximal set of vertex disjoint augmenting path connecting the layers
- ▶ for this, go forward until you either reach a free vertex or you reach a "dead end" v
- ▶ if you reach a free vertex delete the augmenting path and all incident edges from the graph
- ▶ if you reach a dead end backtrack and delete v together with its incident edges

Analysis Hopcroft-Karp
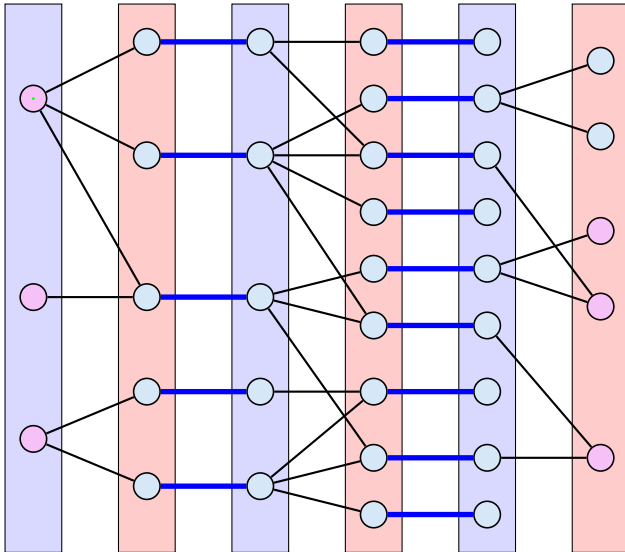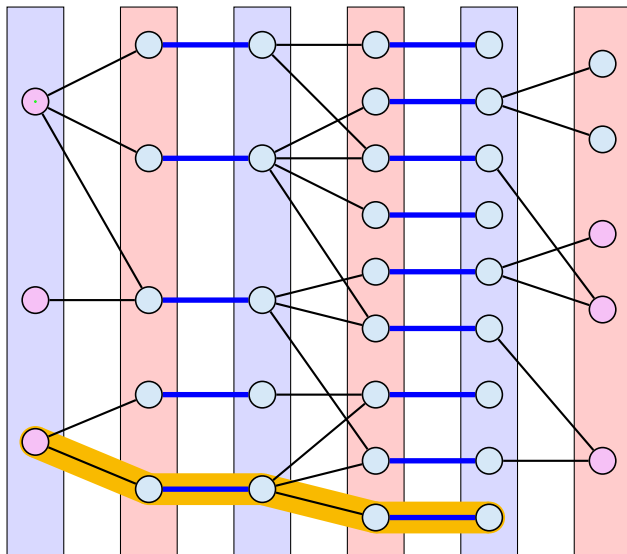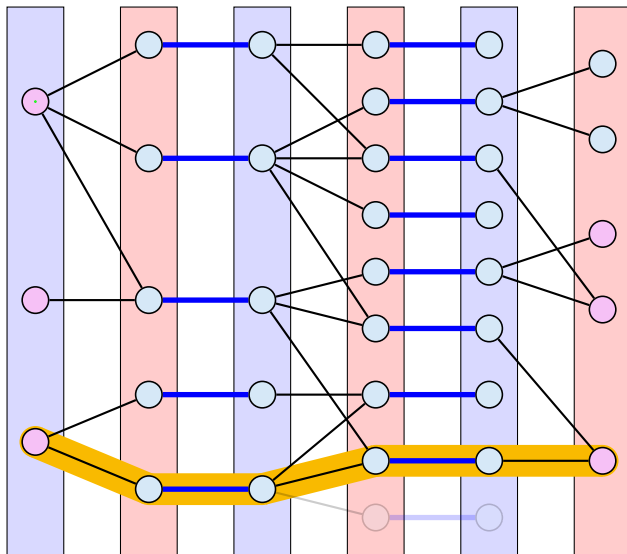
Analysis Hopcroft-Karp
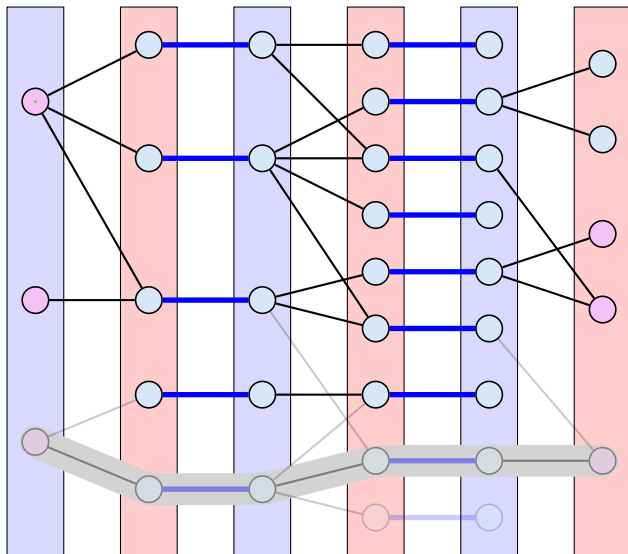
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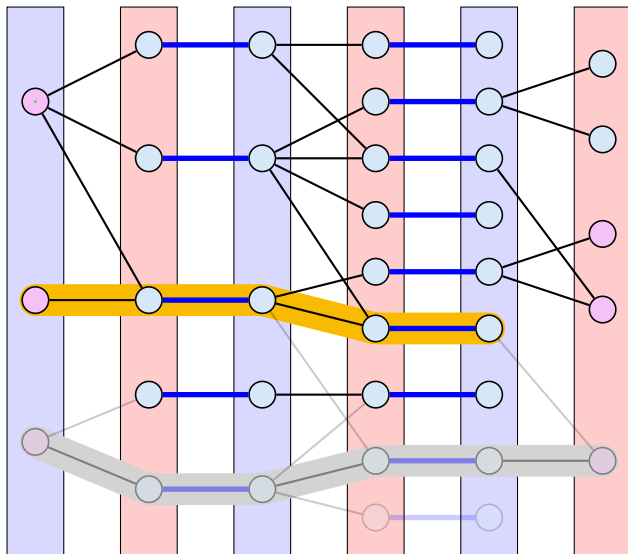
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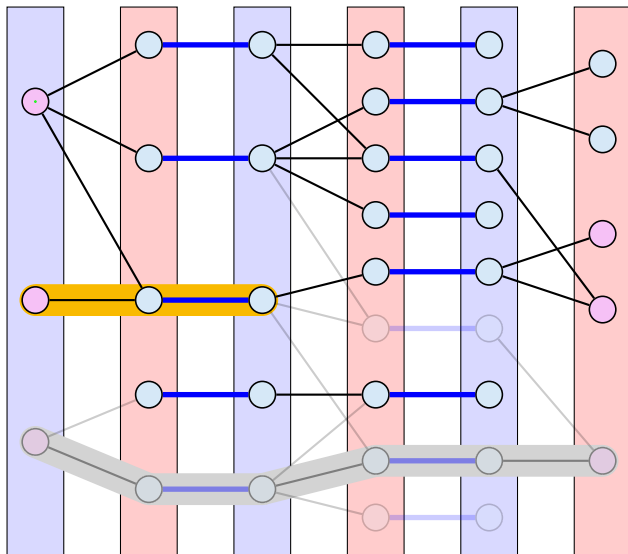
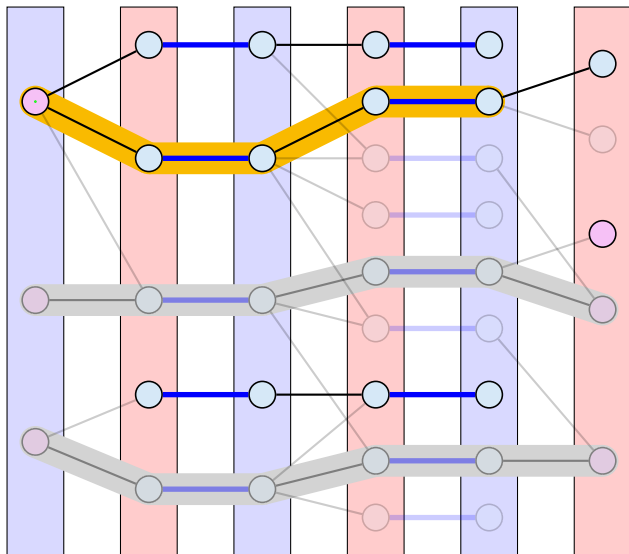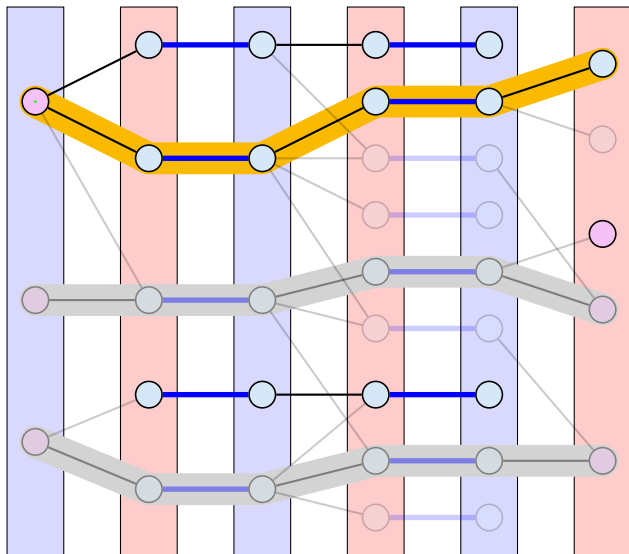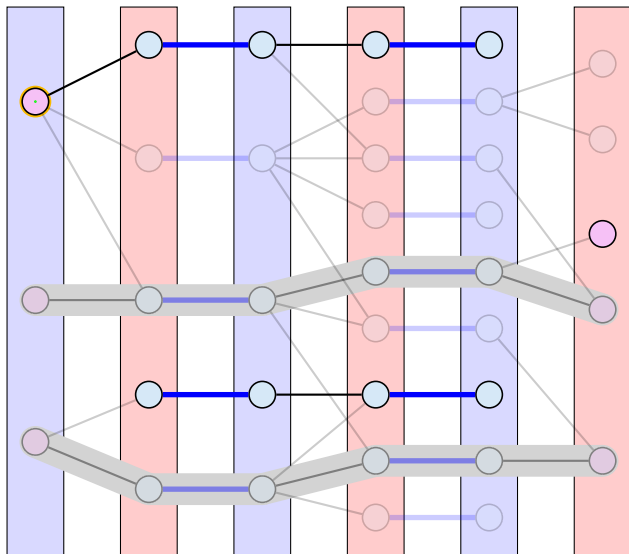# Analysis Hopcroft-Karp
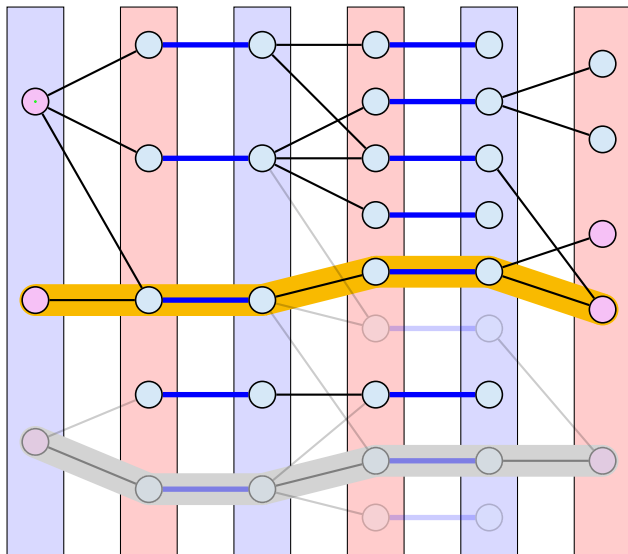
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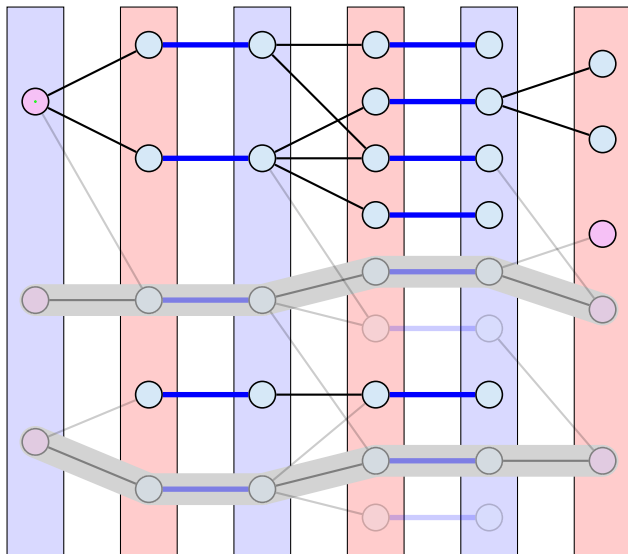
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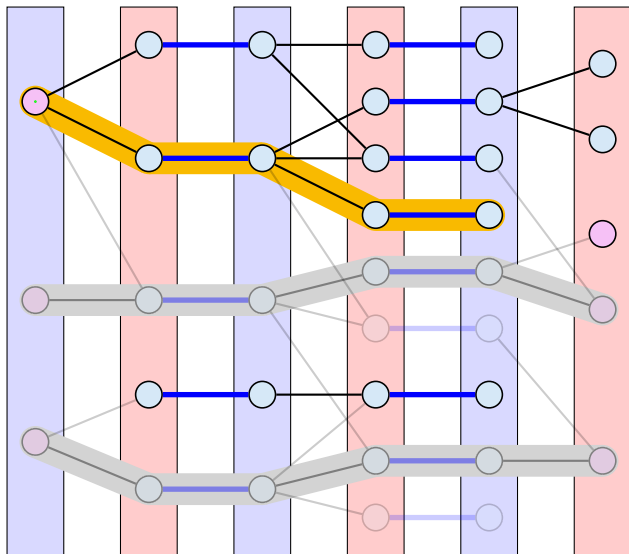
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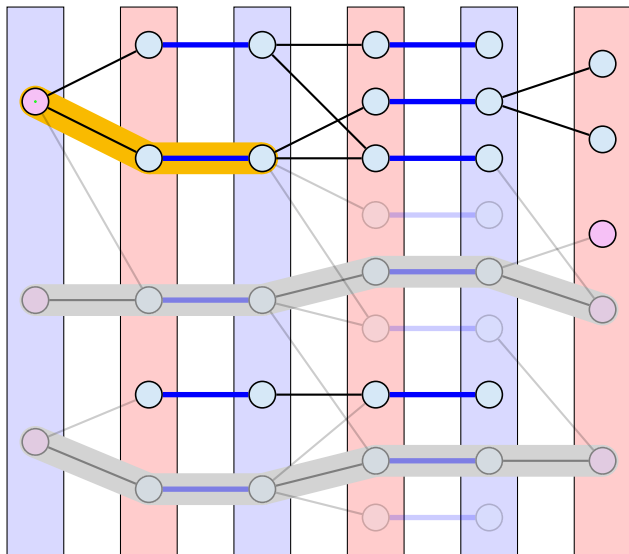
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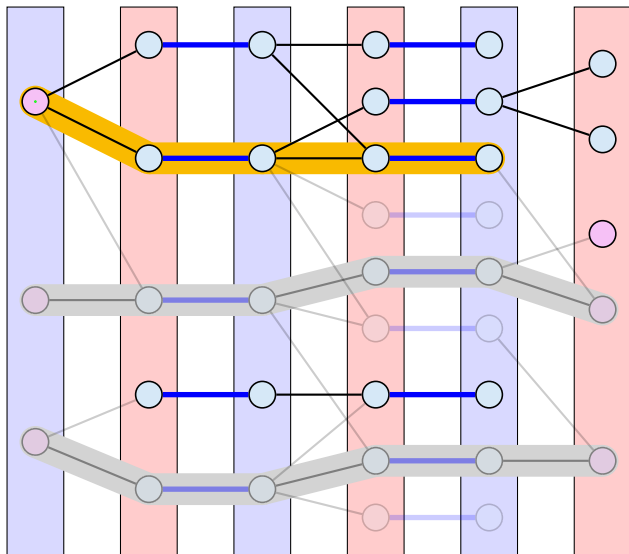
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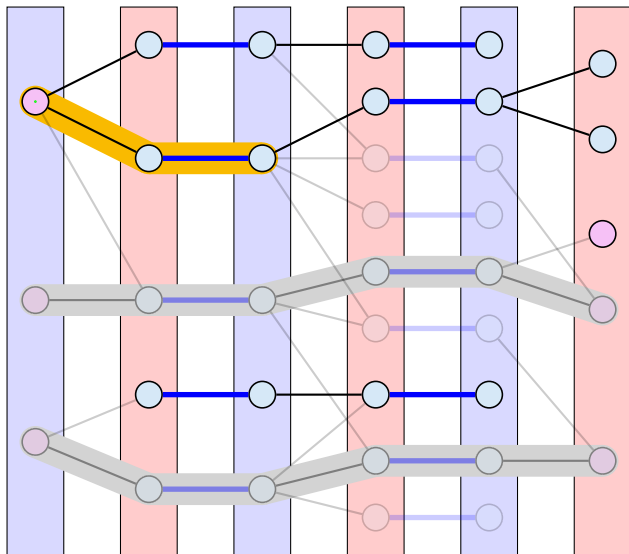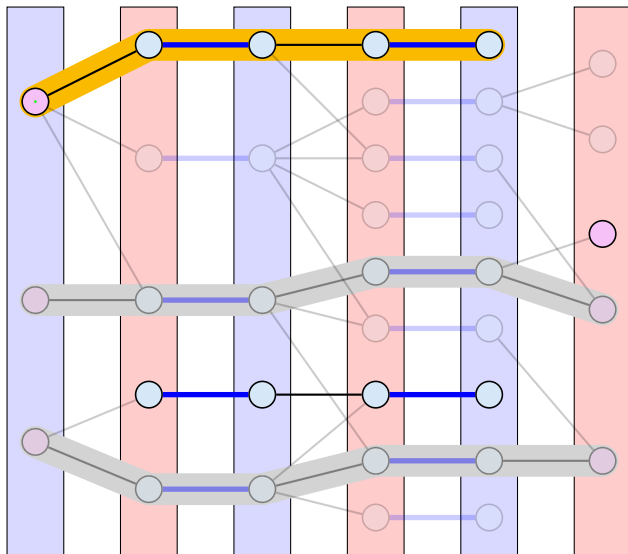
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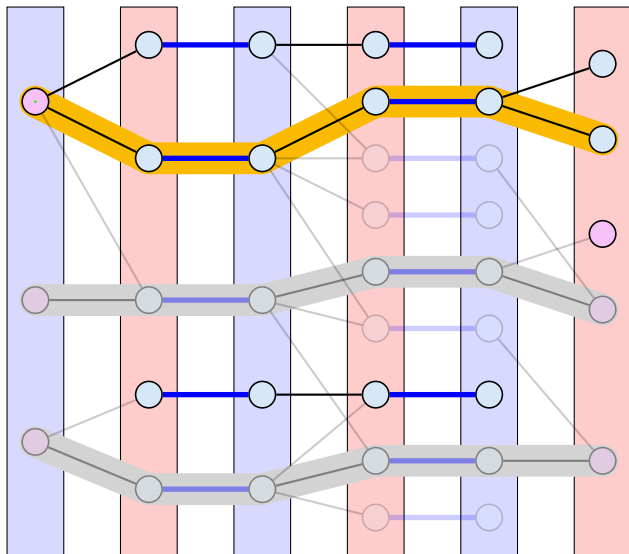
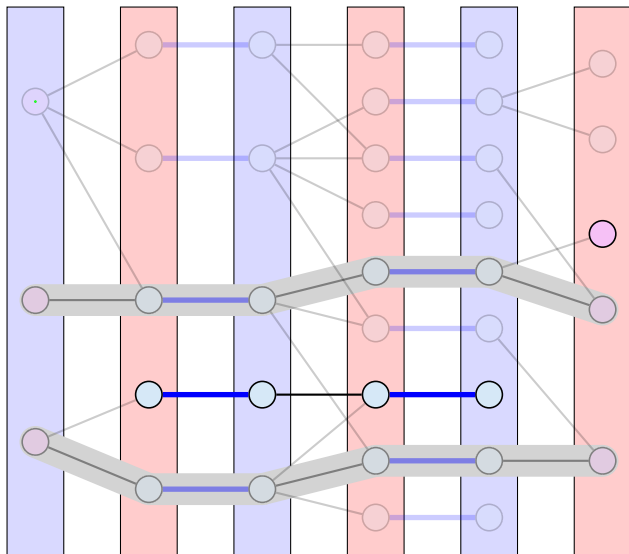# Analysis Hopcroft-Karp

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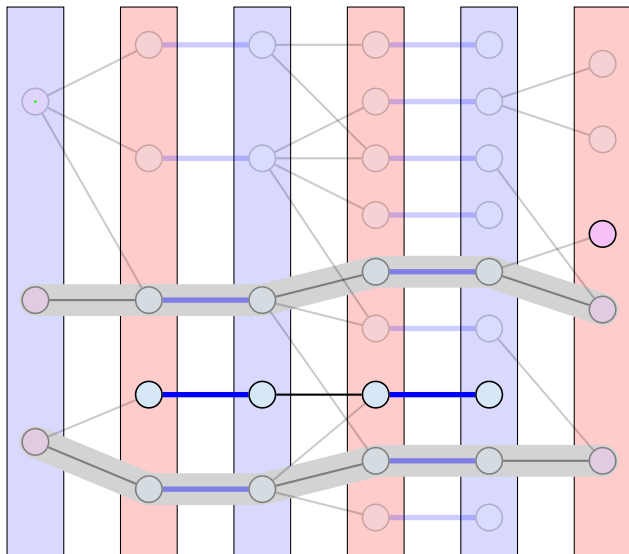
Analysis Hopcroft-Karp

Analysis Hopcroft-Karp

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Analysis Hopcroft-Karp

Analysis: Shortest Augmenting Path for Flows

cost for searches during a phase is $\mathcal{O}(mn)$

▶ a search (successful or unsuccessful) takes time $\mathcal{O}(n)$

▶ a search deletes at least one edge from the level graph

there are at most n phases

Time: $\mathcal{O}(mn^2)$.

Analysis for Unit-capacity Simple Networks

cost for searches during a phase is $\mathcal{O}(m)$

▶ an edge/vertex is traversed at most twice

need at most $\mathcal{O}(\sqrt{n})$ phases

▶ after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph

▶ hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$.