7.2 Red Black Trees

Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

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2. All leaf nodes are black.
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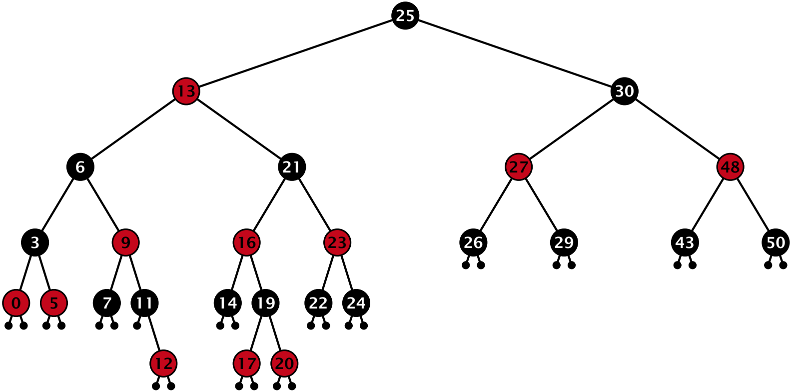
Red Black Trees: Example



7.2 Red Black Trees

Lemma 2

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 4

A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

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Proof of Lemma 4.

Induction on the height of v.

base case ($\text{height}(v) = 0$)

- If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.
- The black height of v is 0.
- The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

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Proof (cont.)

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- v has two children with strictly smaller height.
- These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
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Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \le n$.

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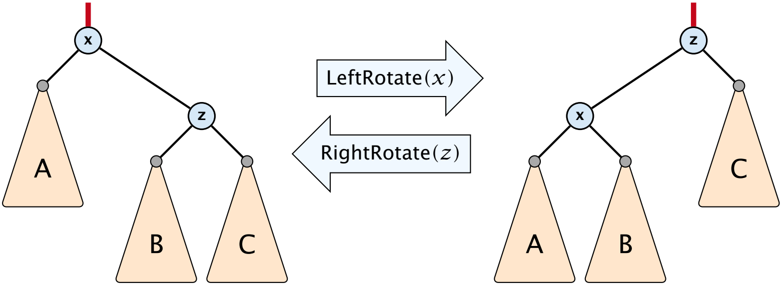
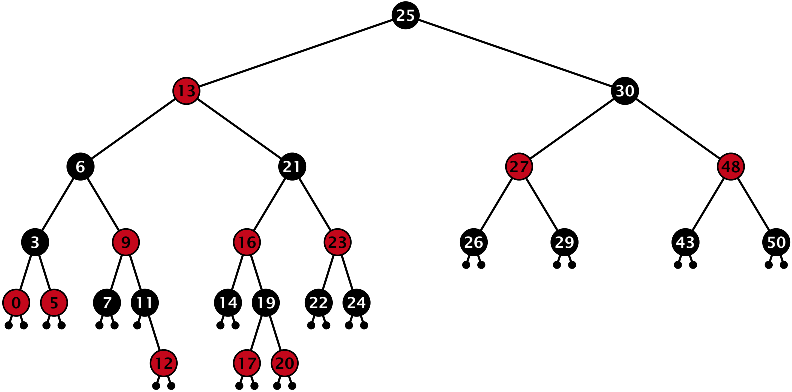
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

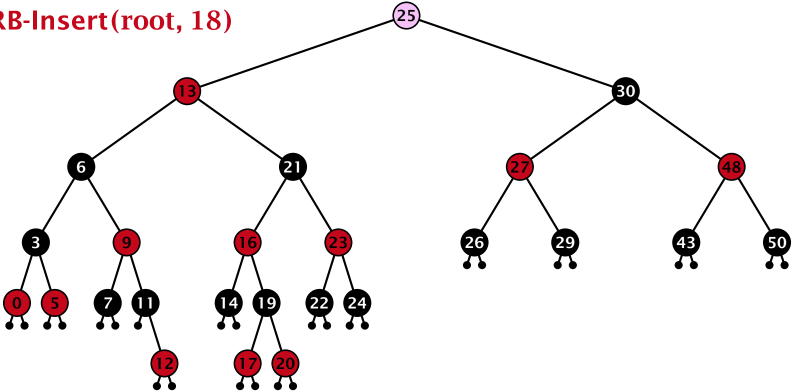


Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



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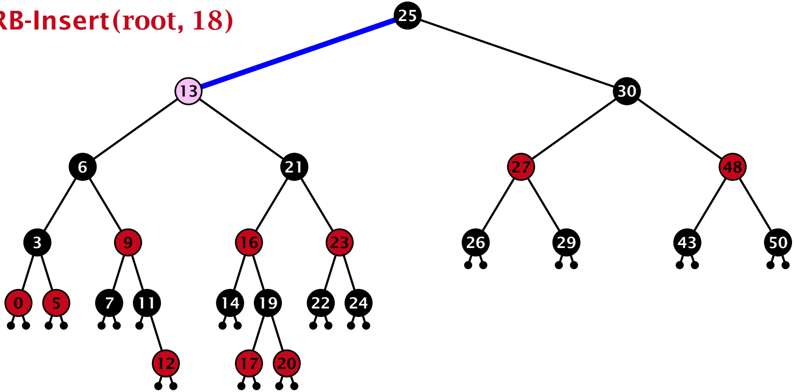

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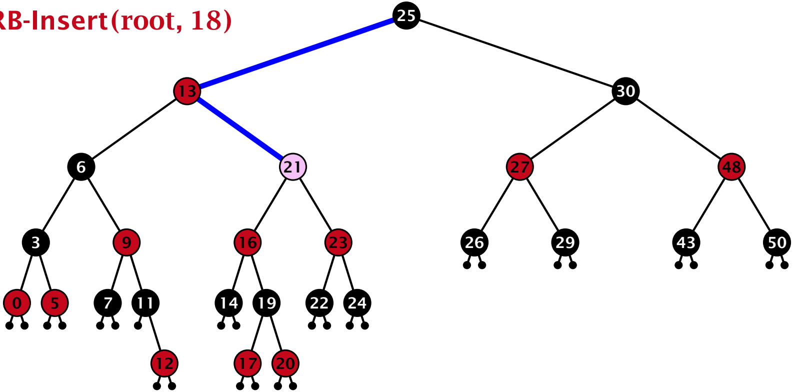

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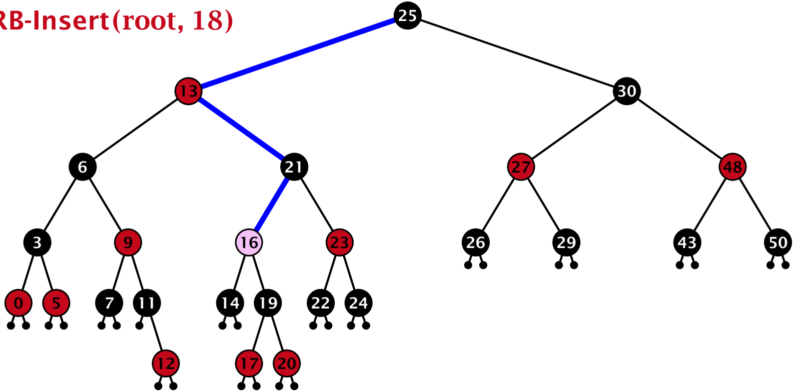

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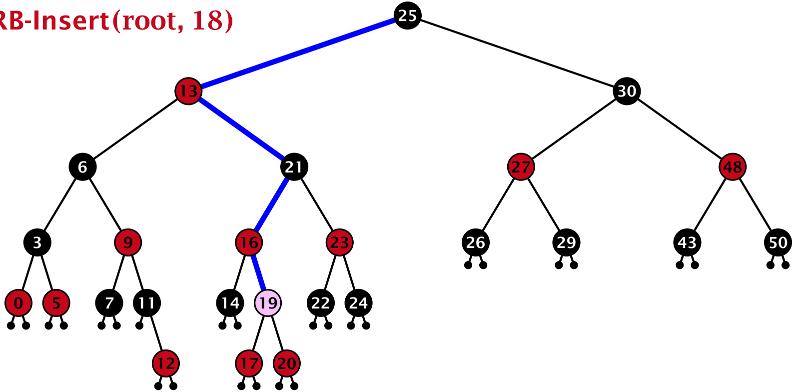

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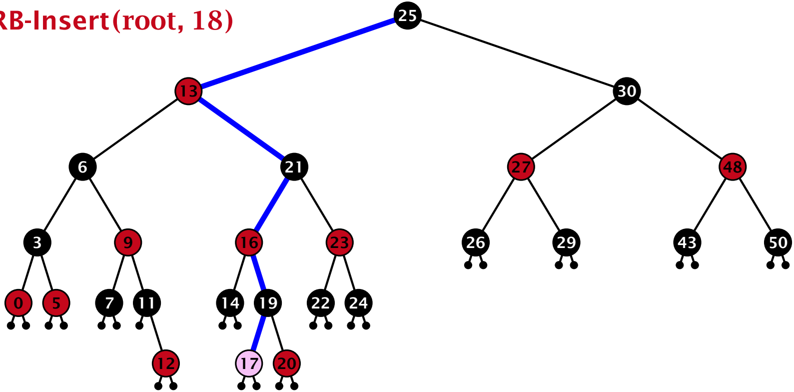

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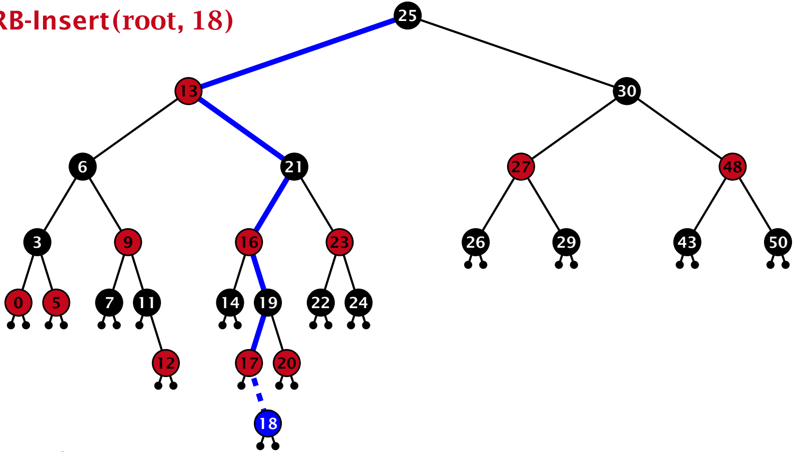

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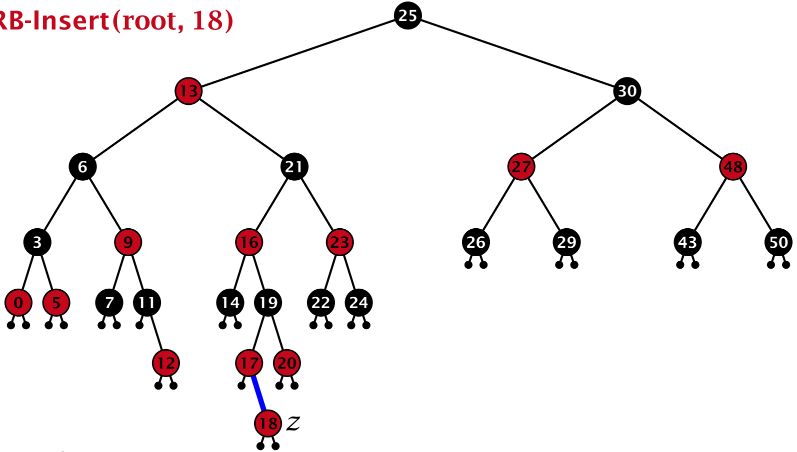

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Invariant of the fix-up algorithm:

- z is a red node
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- the only violation of red-black properties occurs at z and $\text{parent}[z]$
  - either both of them are red (most important case)
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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
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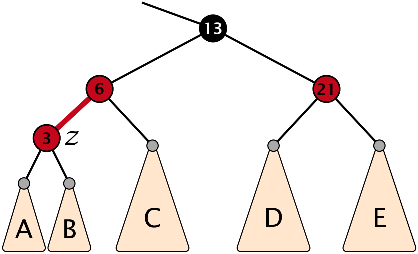
Case 1: Red Uncle



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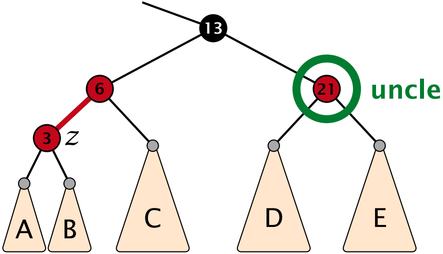

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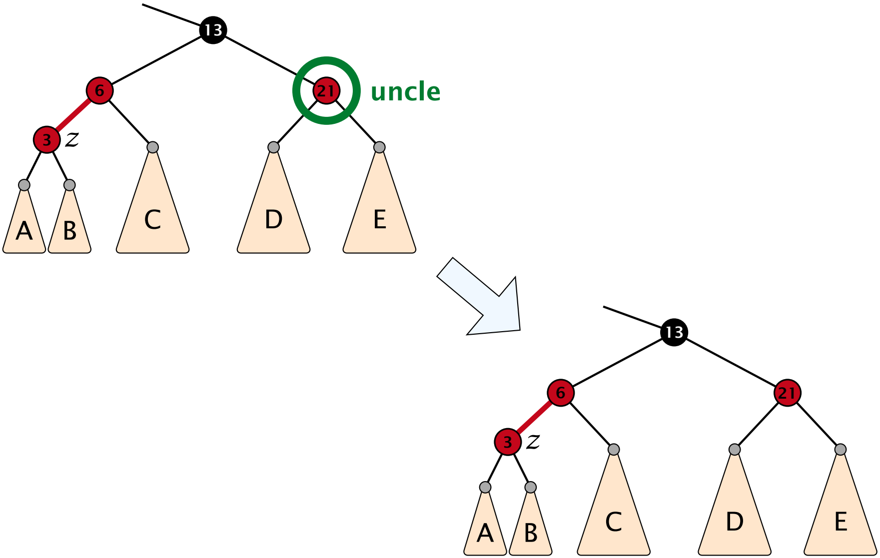

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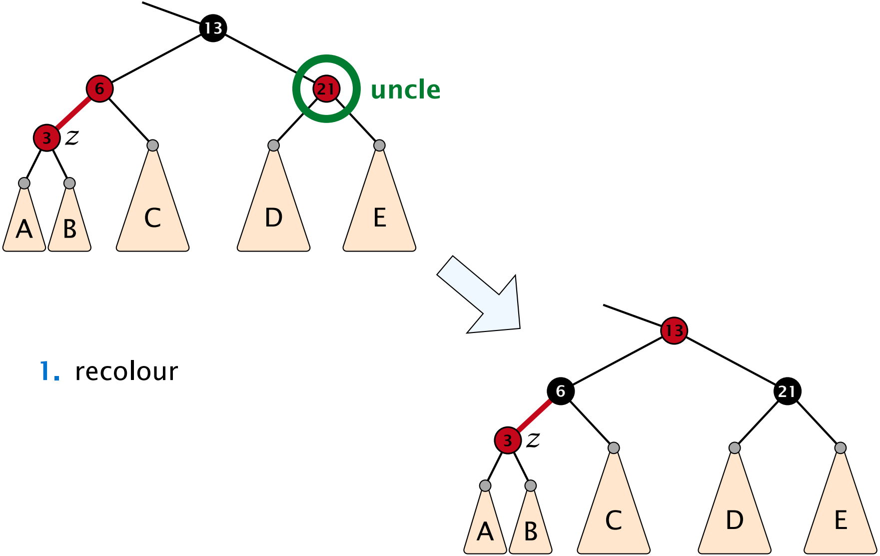

1. recolour

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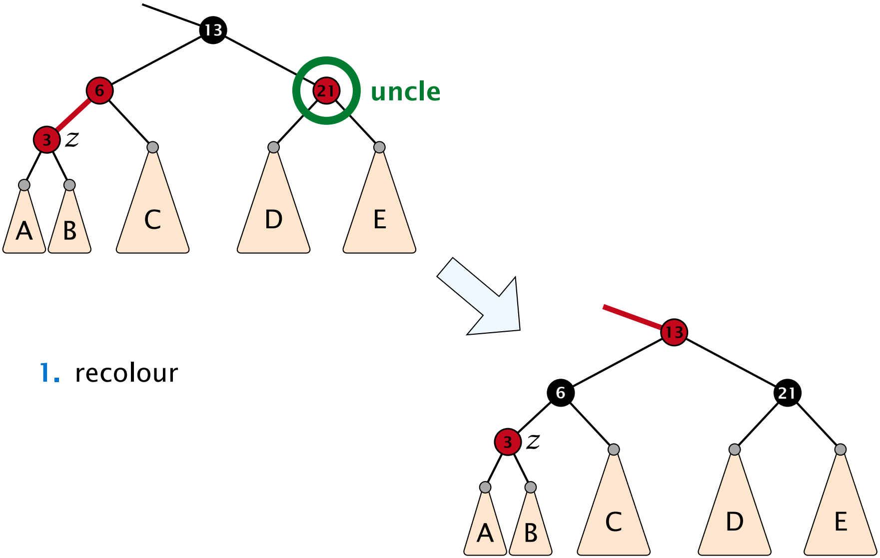

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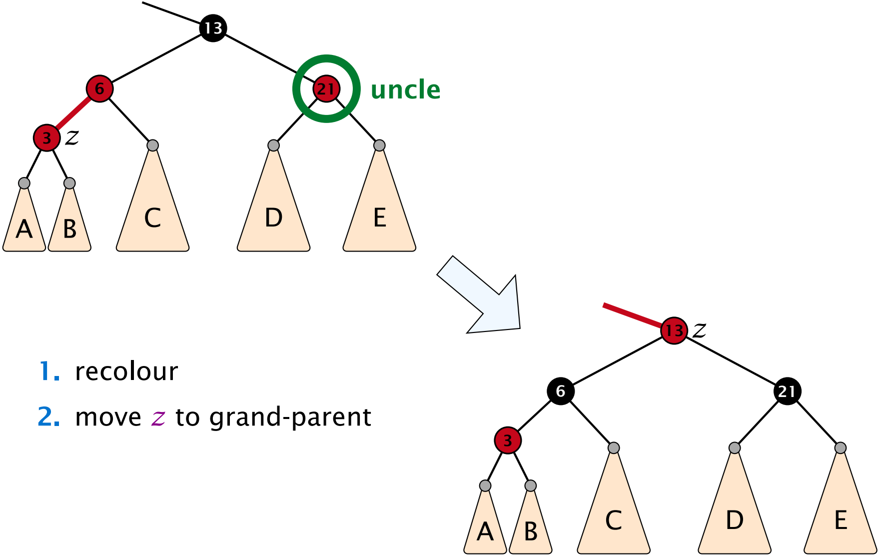

1. recolour
2. move z to grand-parent

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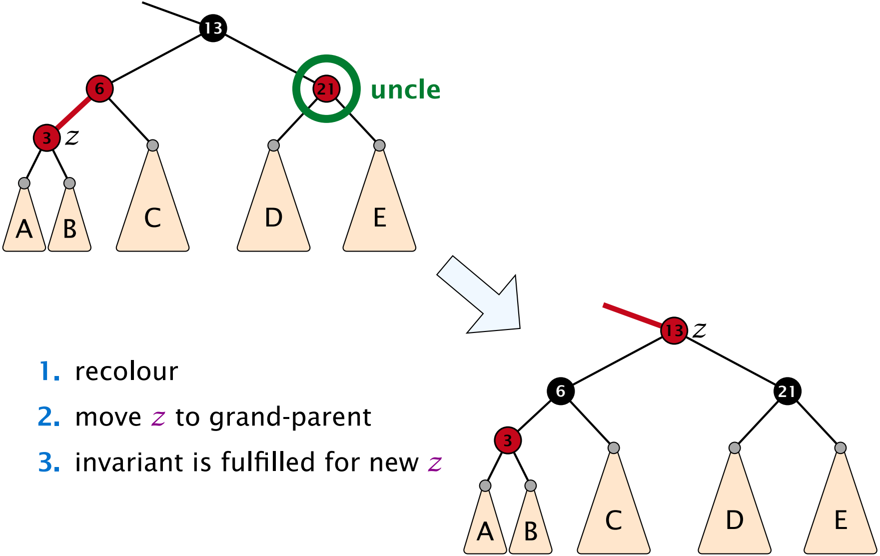

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3. invariant is fulfilled for new z

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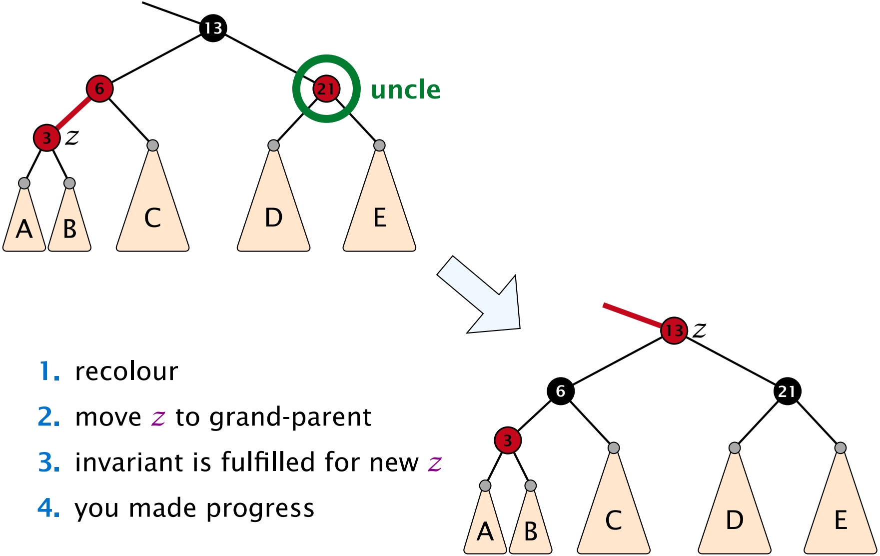

1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

Case 2b: Black uncle and z is left child

1. rotate around grandparent
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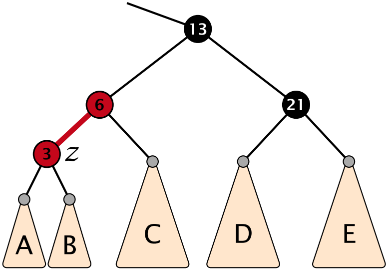
3. you have a red black tree



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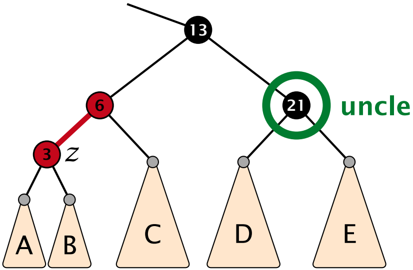

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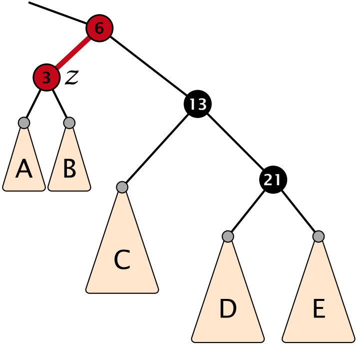

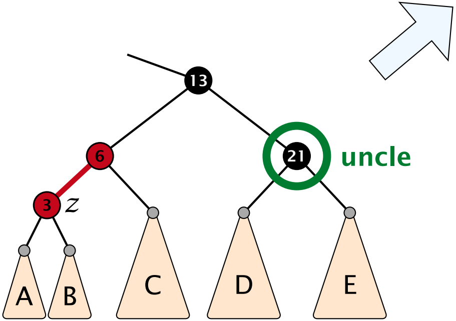

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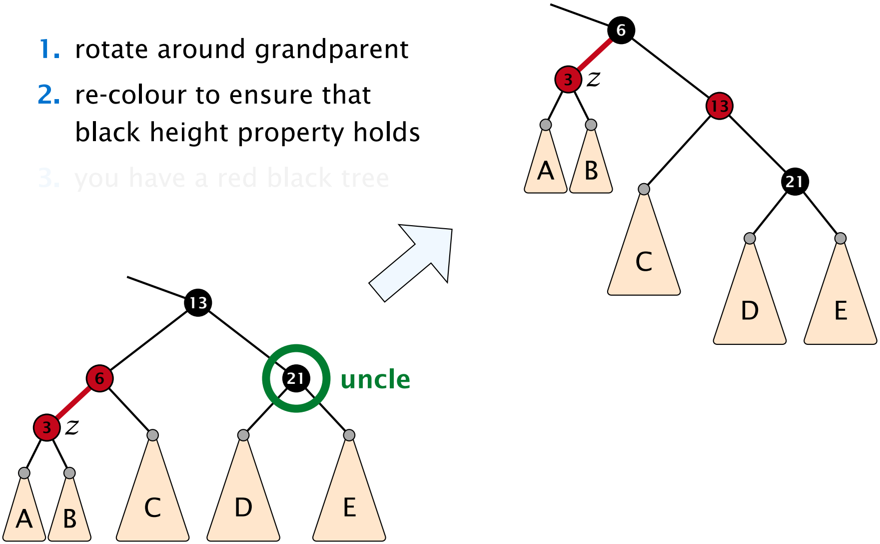

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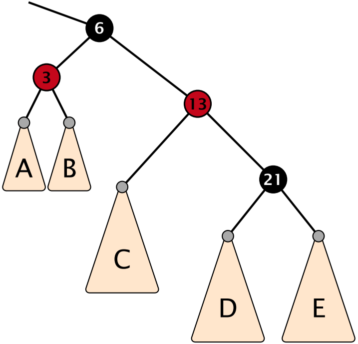

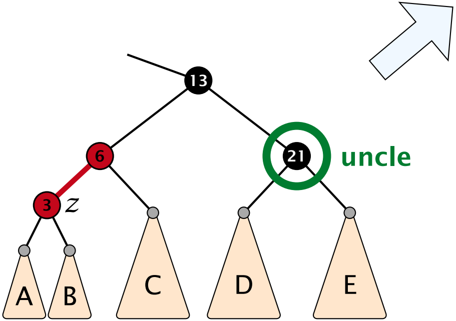

Case 2a: Black uncle and z is right child

1. rotate around parent
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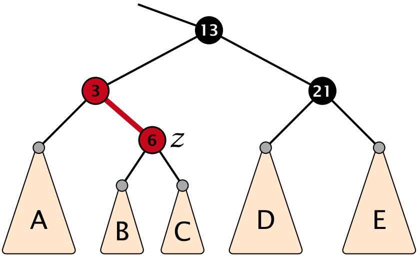
3. you have Case 2b.



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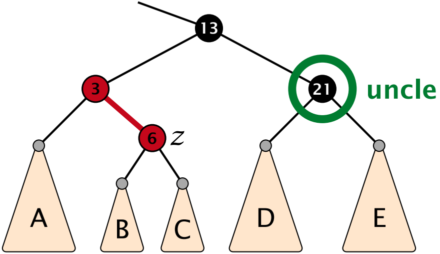

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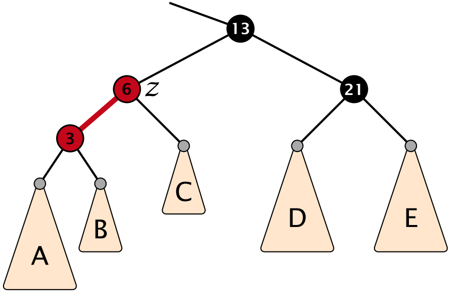

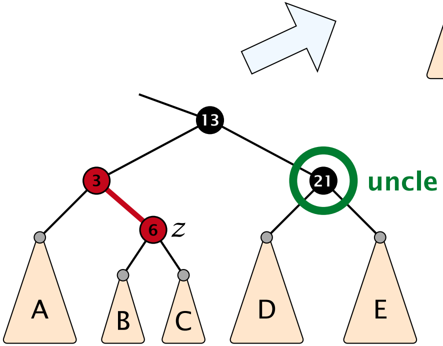

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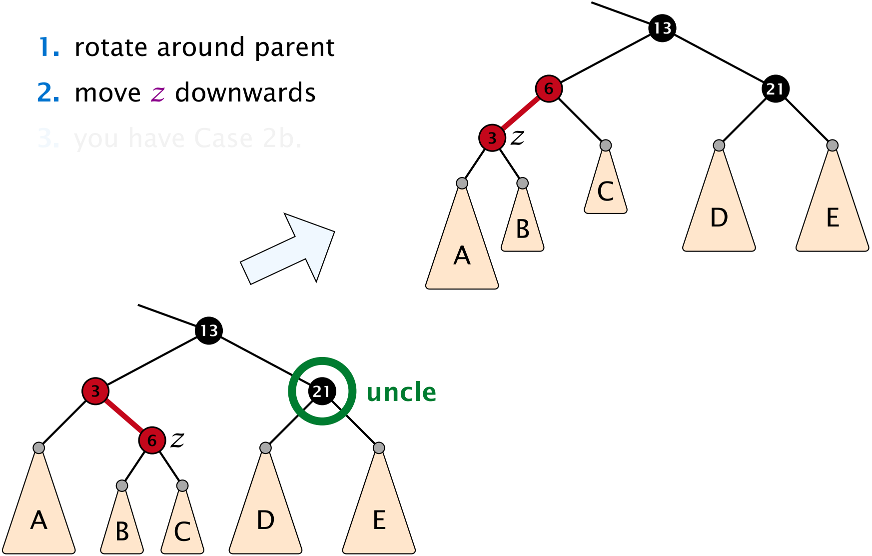

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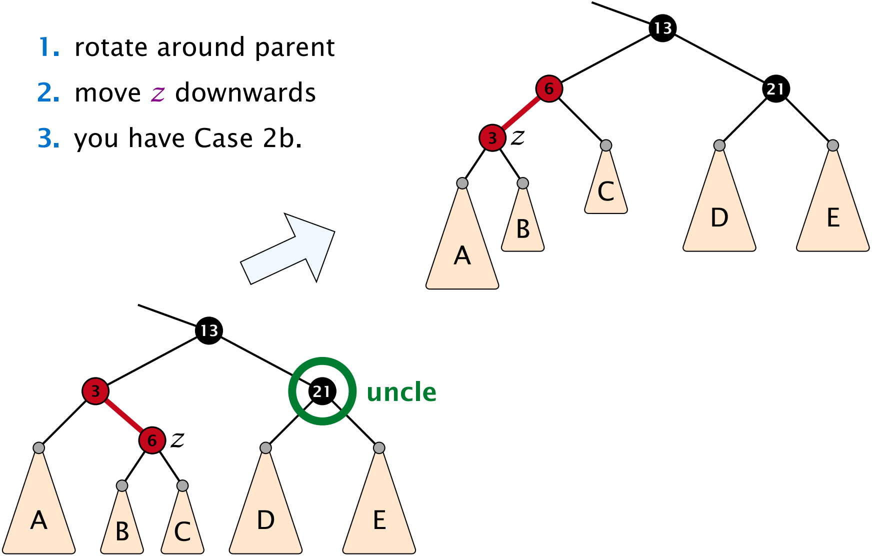

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a → Case 2b → red-black tree
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Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

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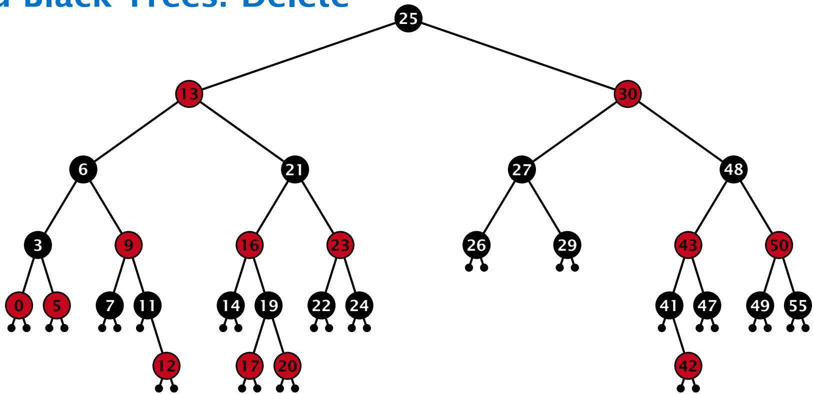

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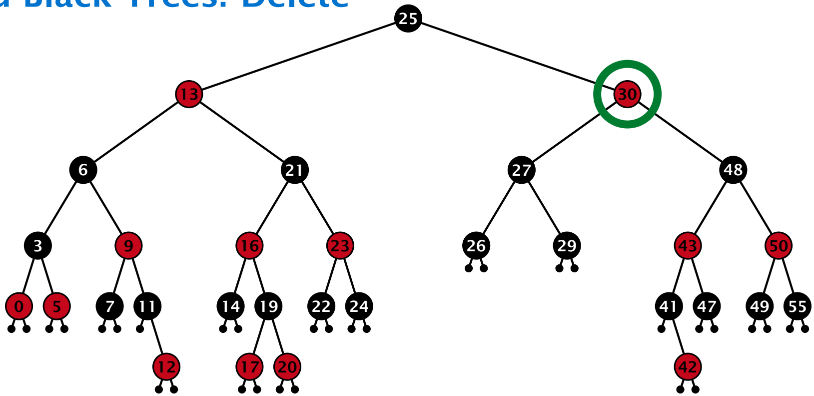

Case 3:

Element has two children

- do normal delete
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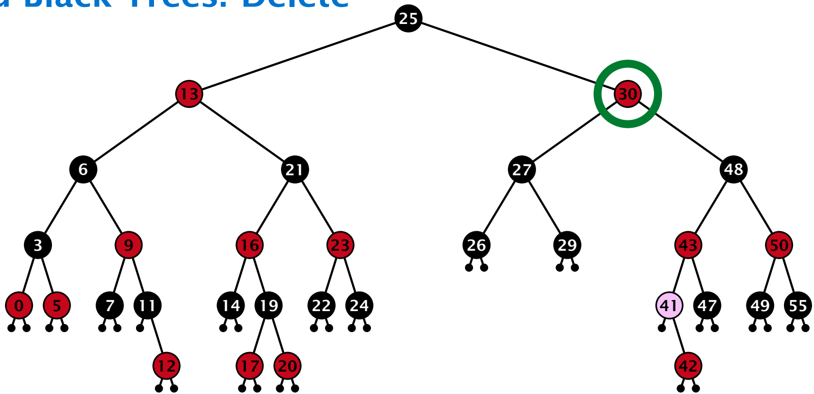

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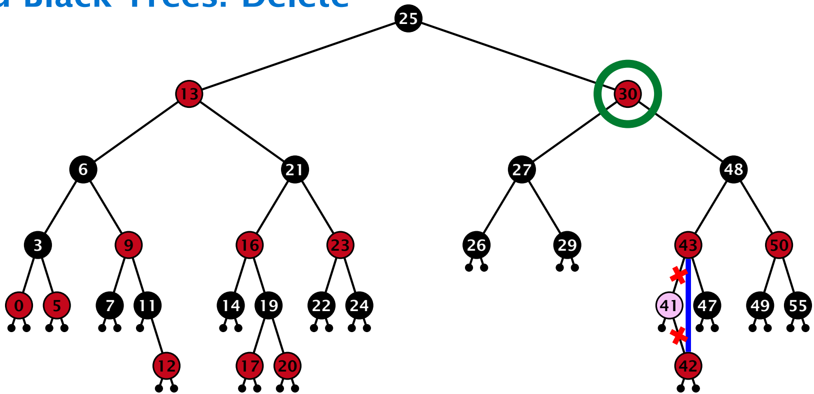

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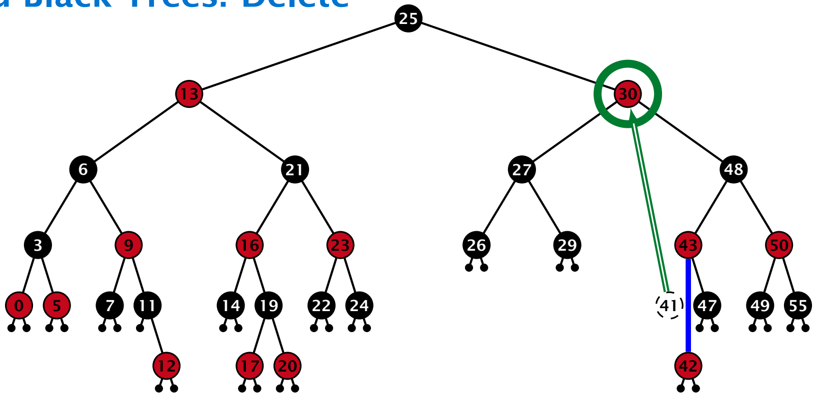

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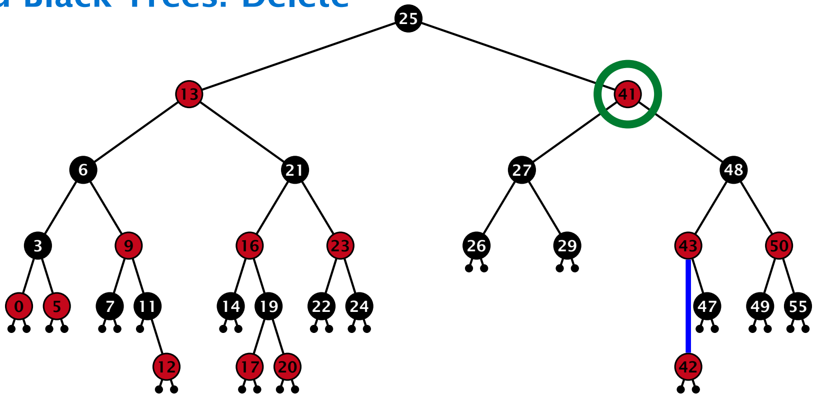

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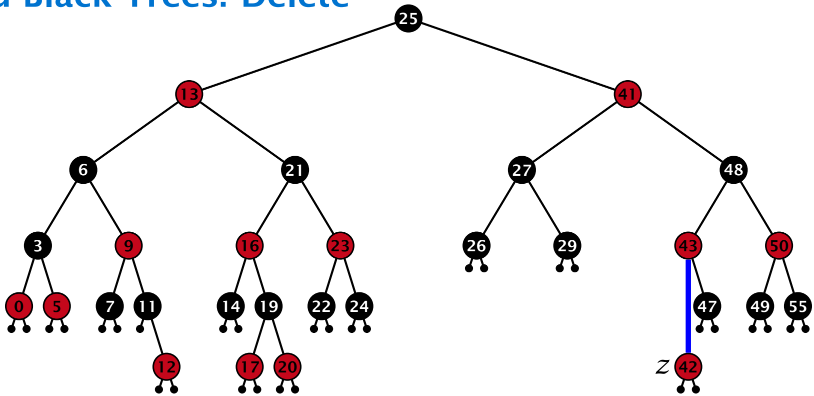

Delete:

- deleting black node messes up black-height property
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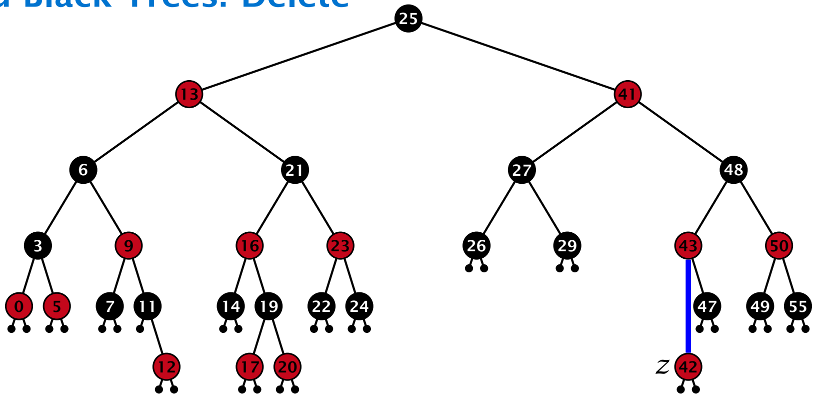

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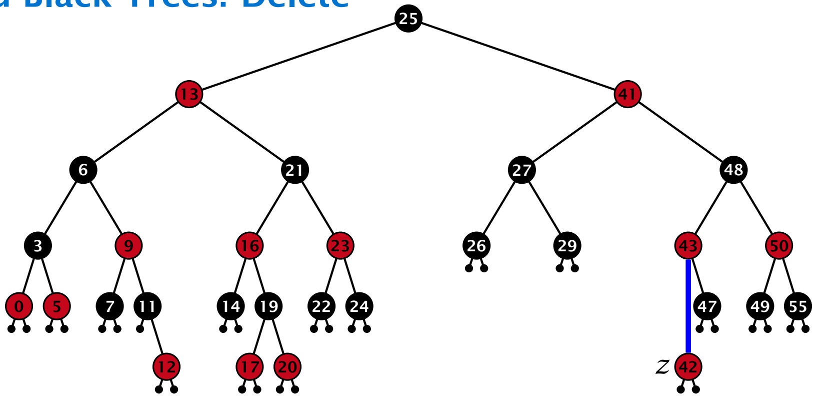

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Invariant of the fix-up algorithm

- the node z is black
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Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

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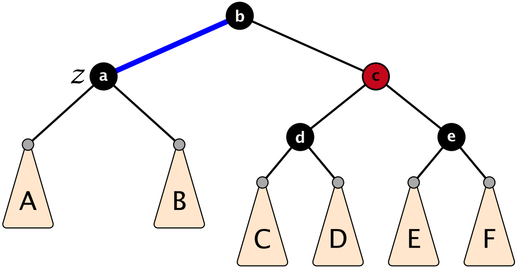

1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4

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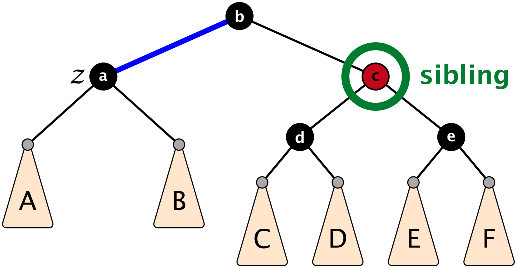

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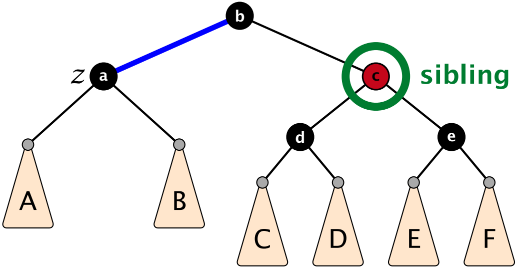

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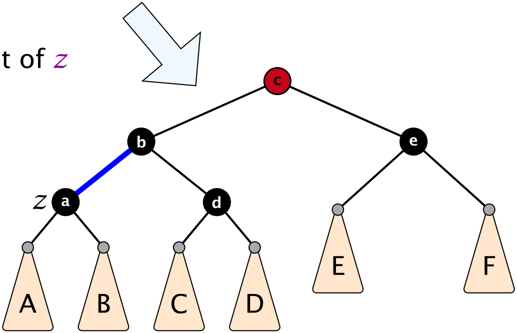

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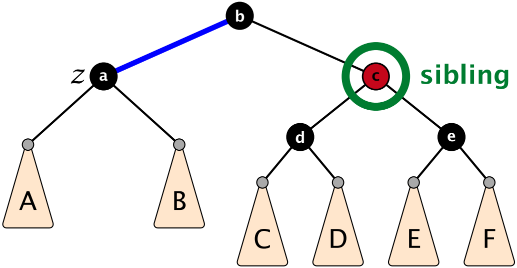

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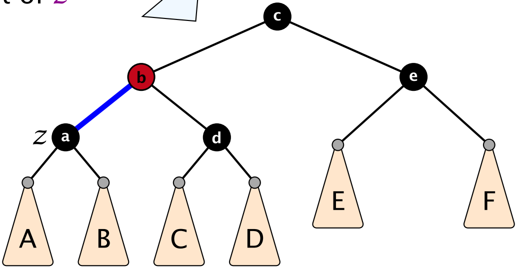

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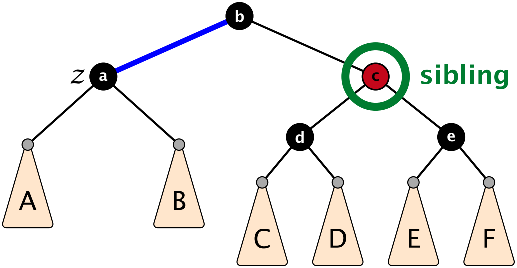

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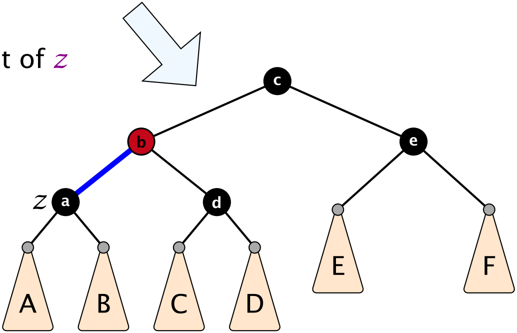

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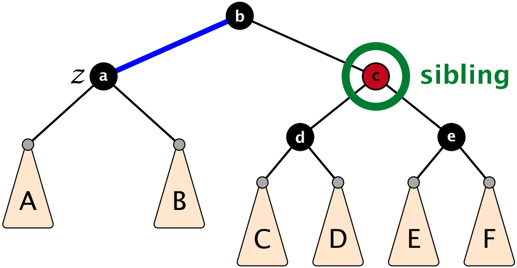

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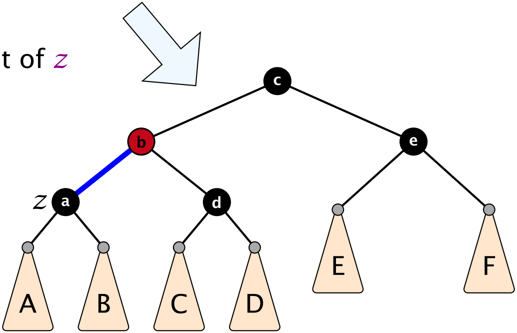

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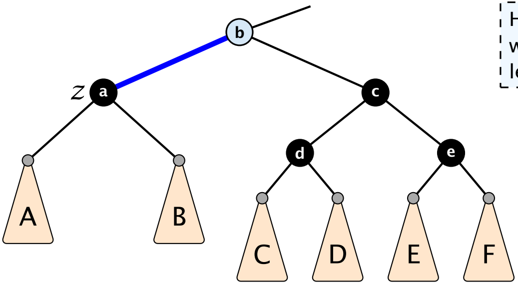

Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
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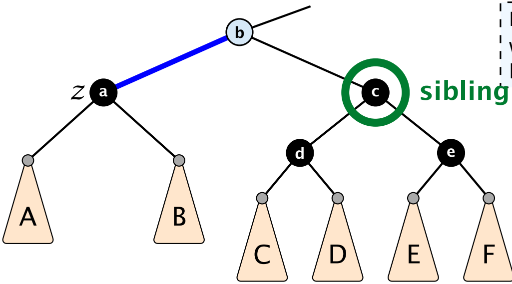

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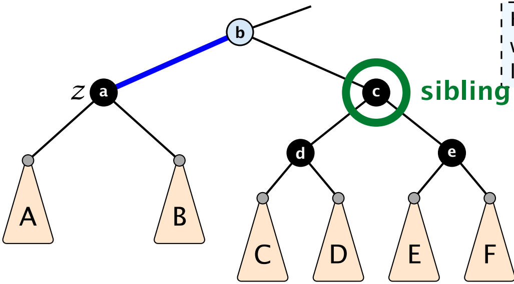

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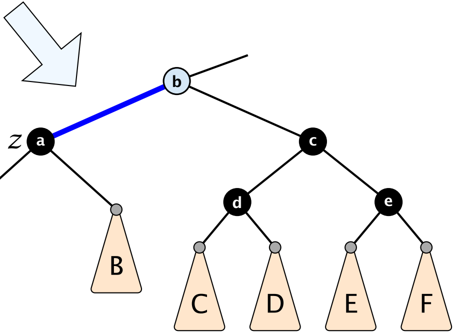

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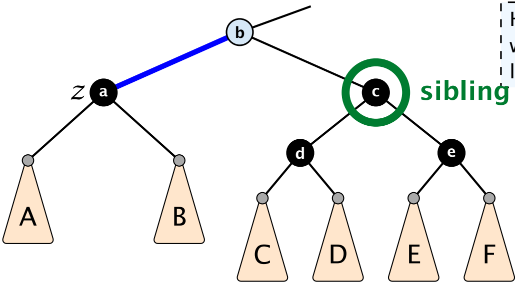

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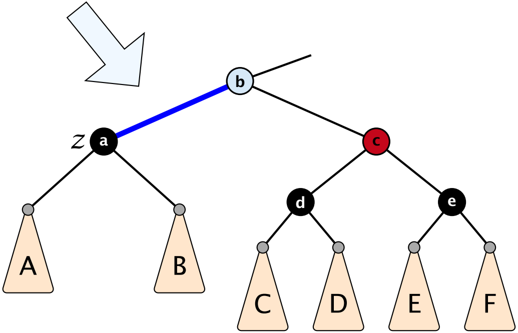

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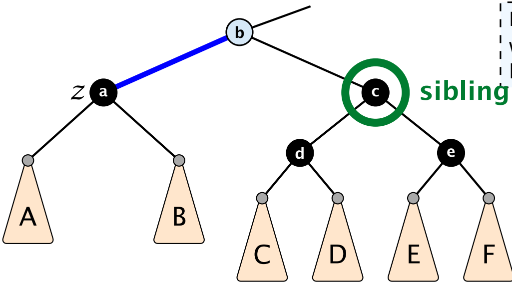

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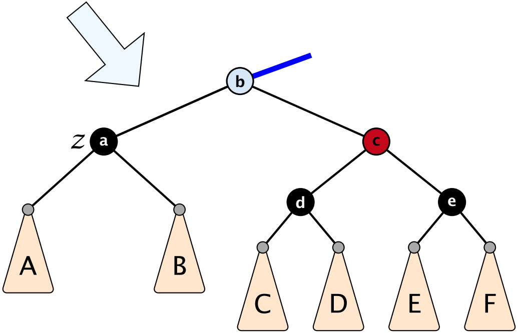

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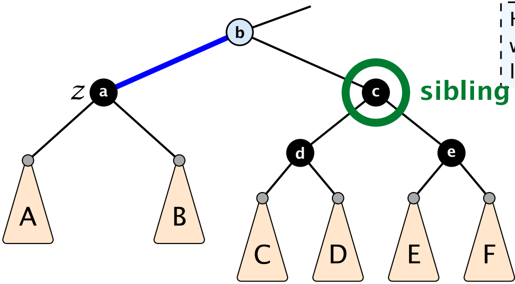

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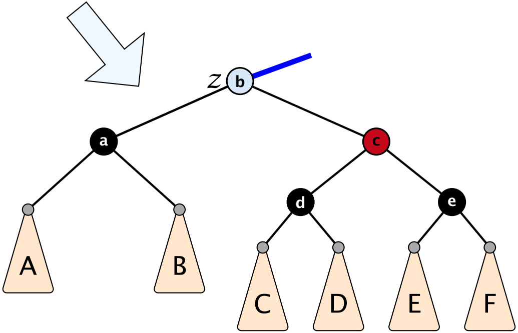

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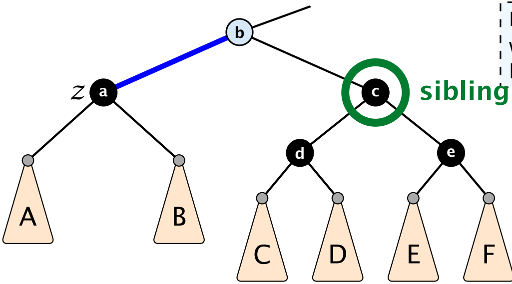
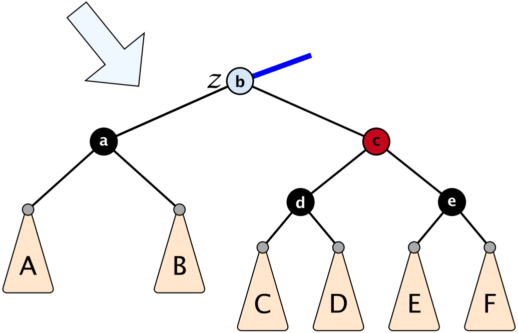


Here b is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done



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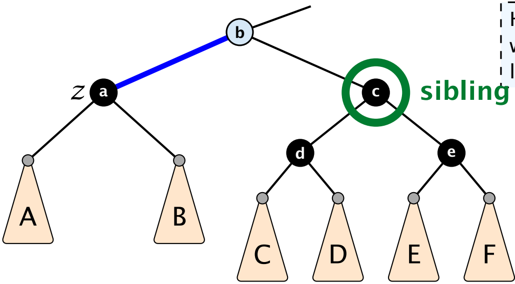

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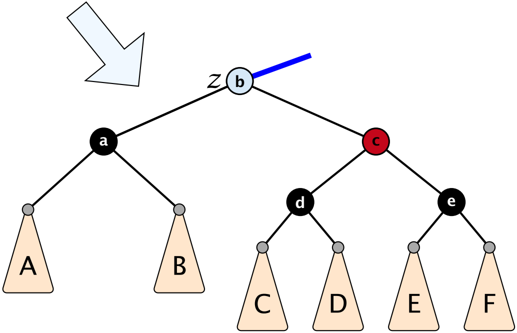

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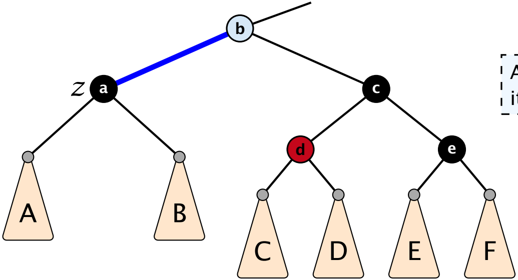
1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



Again the blue color of b indicates that it can either be black or red.

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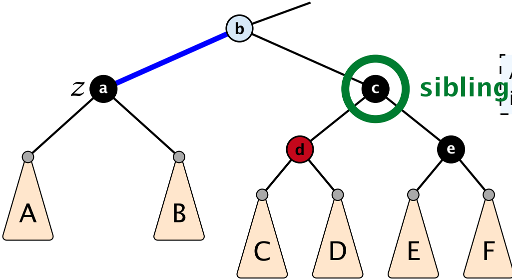

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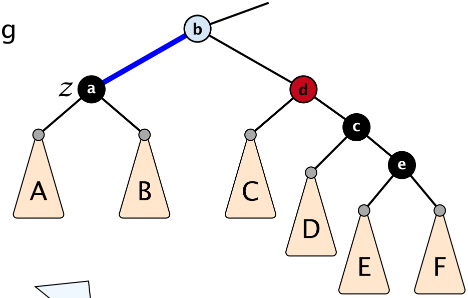

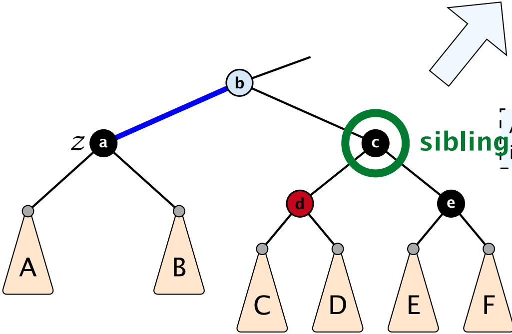

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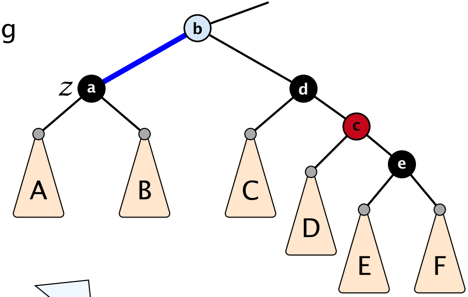

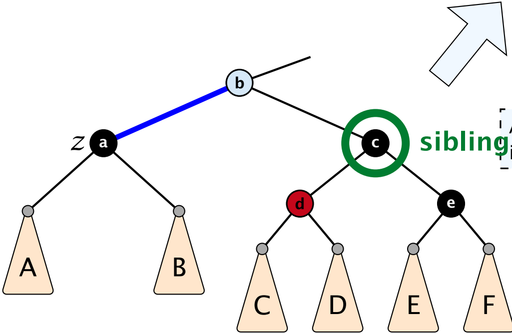

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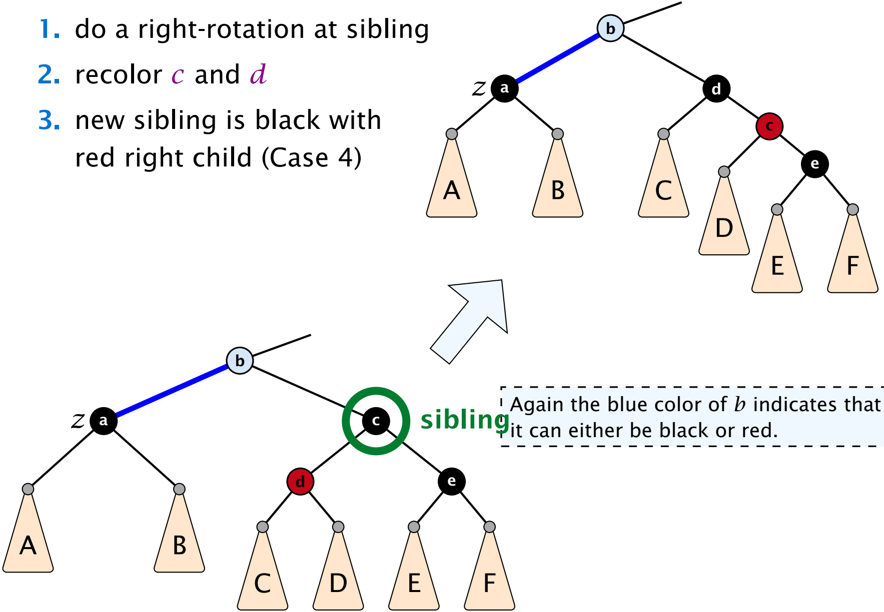

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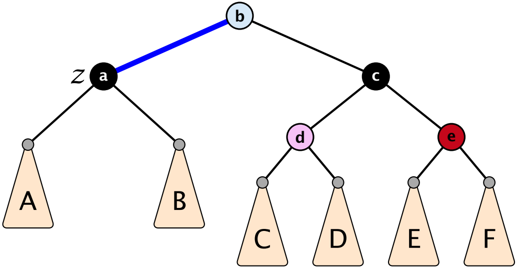

- Here b and d are either red or black but have possibly different colors.
- We recolor c by giving it the color of b.

1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree

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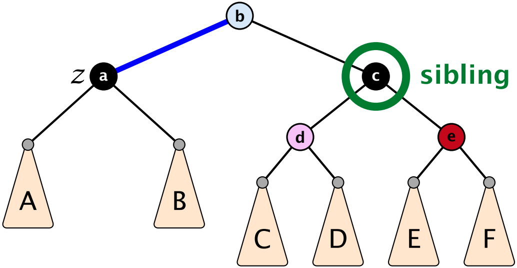

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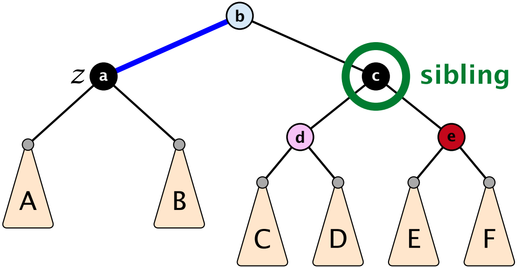

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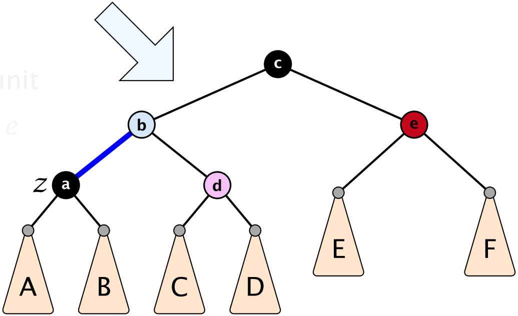

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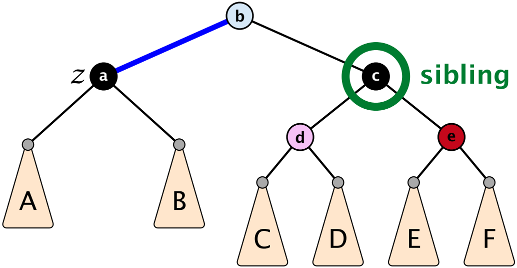

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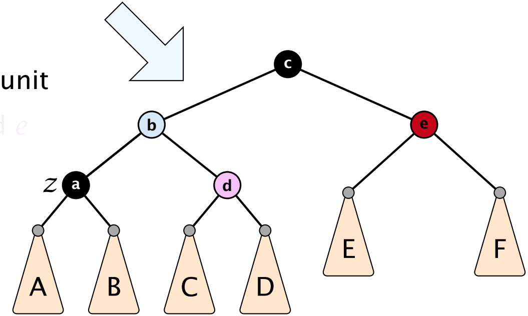

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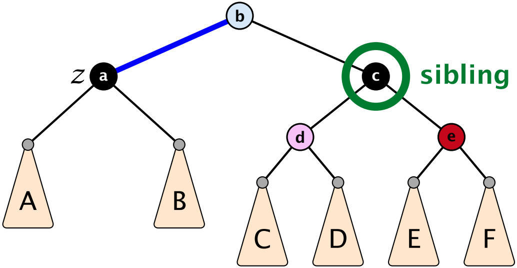

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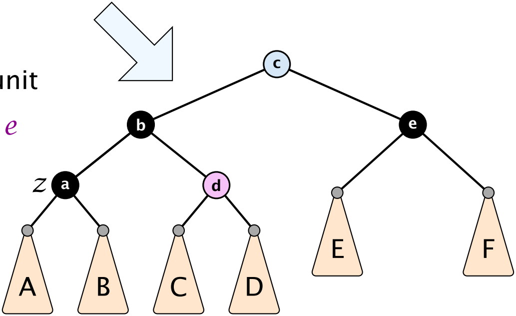

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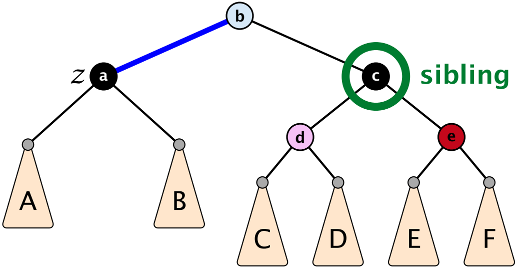

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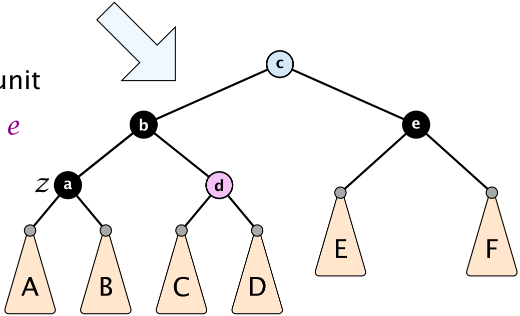

Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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