Technische Universität München
Fakultät für Informatik
Lehrstuhl für Algorithmen und Komplexität
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Winter term 2018/19
Assignment 6
November 19, 2018

Efficient Algorithms and Data Structures I

*Deadline: November 26, 10:15 am in the **Efficient Algorithms** mailbox.*

Homework 1 (4 Points)

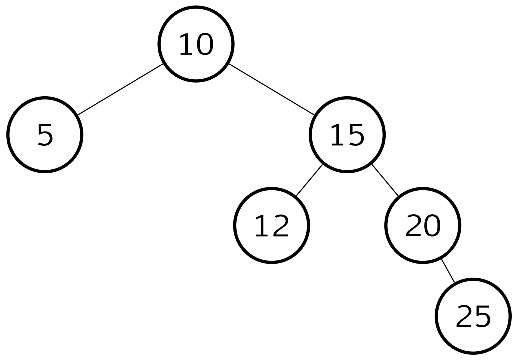
(a) If we insert a node in a red-black tree and then immediately delete the same node, is the resulting tree always identical to the original tree? Explain your answer.

(b) For each $n = 2^k$, describe a red-black tree on n keys that realizes the smallest possible ratio of red internal nodes to black internal nodes.

(c) Prove or disprove: The longest simple path from a node x in a red-black tree to a descendant leaf has length at most twice that of the shortest simple path from node x to a descendant leaf.

Homework 2 (4 Points)

The teddy bear Alan is asked to decorate the Christmas tree for a major shopping center in the city center of Munich. This year, he chooses to use the glass baubles to form a nice splay tree. Initially, he obtains the following tree:



However, during the month of December, several changes must be made to the tree, as some baubles must be inspected more closely and others are added to the tree.
Show what Alan's tree looks like after each change. The tree must remain a splay tree. Always carry out each operation on the result of the previous operation.

1. Search 12

2. Search 25

3. Insert 22

Homework 3 (6 Points)

(a) ~~Let Φ be the potential function used to analyze splay trees, i.e. $\Phi(T) = \sum_{v \in T} r(v)$. For any $n > 2$, construct a tree T minimizing Φ. Prove your claim.~~

(b) Show that splaying roughly halves the depth of every node on the search path. More precisely, let d be the depth of some node on the search path before the splaying operation and d' its depth after the splaying operation. Show that $d' \le \lceil d/2 \rceil + 3$ holds.

Homework 4 (6 Points)

Suppose that, for the purpose of the analysis, we assign some weight $w(x)$ to any node x of a given splay tree. Let $s(x) = \sum_{y \in T_x} w(y)$ be the sum of all node weights in the subtree of x and the rank $r(x) = \log(s(x))$.
Define the *improved potential function* of the splay tree T as $\Phi_w(T) = \sum_{x \in T} r(x)$.
In this exercise, we show that a clever choice of weights w helps prove interesting properties of splay trees. The simple choice $w(x) = 1$ gives the analysis seen in the lecture.

(a) Let the *amortized cost* of operation i with cost c_i be $\hat{c}_i = c_i + \Phi_w(T_i) - \Phi_w(T_{i-1})$. Show that any sequence of m operations on the tree T_0 has a cost of

$$\sum_i \hat{c}_i - \Phi_w(T_m) + \Phi_w(T_0) \ ,$$

where T_i is the tree after operation i.

(b) Let $W = \sum_T w(x)$. Show that the rank $r(x)$ decreases at most by $\log(W) - \log w(x)$ during any sequence of search operations.

(c) Suppose that item x is accessed $q(x) > 0$ times during some sequence of m search operations on tree T, i.e. $m = \sum_x q(x)$. Choose weights $w(x)$ to show that the total cost of the sequence is at most

$$\mathcal{O}\left(m + \sum_{x \in T} q(x) \log(m/q(x)) \right) \ .$$

Hint: Adapt the amortized analysis from the lecture using your own choice of weights. The relevant slide is Slide 183.

Tutorial Exercise 1

(a) Describe how to implement a queue using two stacks and $O(1)$ additional memory, so that the amortized time for any ENQUEUE or DEQUEUE operation is $O(1)$. The only access you have to the stacks is through the standard subroutines PUSH and POP.

(b) A quack is a data structure combining properties of both stacks and queues. It can be viewed as a list of elements written left to right such that 3 operations are possible:

  (i) QPUSH: add a new item to the left end of the list

  (ii) QPOP: remove the item on the left end of the list

  (iii) QPULL: remove the item on the right end of the list

  Implement a quack using 3 stacks and $O(1)$ additional memory, so that the amortized time for any QPUSH, QPOP, or QPULL operation is $O(1)$. Again, you are only allowed to access the stacks through the standard functions PUSH and POP.

Tutorial Exercise 2

Programmer Paul is given a magical closed-source library called `Priorizer`. A `Priorizer` supports the operations INSERT and EXTRACT-MAX. INSERT allows to insert an integer. EXTRACT-MAX extracts the largest integer from the `Priorizer` and returns NULL, if the `Priorizer` is empty. Both operations run in constant time.

(a) Argue briefly how to use a `Priorizer` to obtain a data structure that supports INSERT and EXTRACT-MIN in constant time. EXTRACT-MIN should extract the smallest integer from the data structure.

(b) Paul's supervisor Sunny needs a data structure that supports the operations INSERT, EXTRACT-MAX and EXTRACT-MIN,. She requests that Paul makes all three operations run in amortized constant time.

  Help Paul by designing the requested data structure. Your implementation should use two `Priorizers`. Additionally, you may use a linked list and a constant amount of additional storage (e.g., to store an integer).

  Describe precisely how the operations are implemented. Then analyze their running times using a suitable potential function. You do not need to prove that your implementation works correctly.

Hint: Use one `Priorizer` for "large" elements, the other one for "small" elements. What can you do if one of the `Priorizers` runs empty?

```
Proudest [work]?  It's hard to choose.  I
like the self-adjusting search tree data
structure that Danny Sleator and I developed.
That's a nice one.
                                - R. E. Tarjan
```