6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} \frac{a_n}{n!} z^n.$$

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} \frac{a_n}{n!} z^n.$$

6.4 Generating Functions

Example 9

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

$$F(z) = 1 \,.$$

2. The generating function of the sequence $(1, 1, 1, \ldots)$ is

$$F(z) = \frac{1}{1 - z} \,.$$

6.4 Generating Functions

Example 9

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

$$F(z) = 1 .$$

2. The generating function of the sequence $(1, 1, 1, \ldots)$ is

$$F(z) = \frac{1}{1 - z} .$$

$$F(z) = \sum_{n \geq 0} 1 \cdot z^n$$

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\ge 0} a_n z^n$ and $g = \sum_{n\ge 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\ge 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\ge 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\ge0} a_n z^n$ and $g = \sum_{n\ge0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\ge0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\ge0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n) z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n) z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\ge 0} a_n z^n$ and $g = \sum_{n\ge 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\ge 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\ge 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

$a_n z^n \quad b_0 \cdot z^0 = a_n b_0 \cdot z^n$

$a_{n-1} z^{n-1} \quad b_1 \cdot z^1 = a_{n-1} \cdot b_1 \cdot z^n$

There are no convergence issues here.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\geq 0}^{\infty} z^n\right) = 1 \ .$$

This is well-defined.

6.4 Generating Functions

What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\geq 0}^{\infty} z^n\right) = 1 \ .$$

This is well-defined.

$$\sum_{n\geq 0} z^n - \underbrace{\sum_{n\geq 0} z^{n+1}}_{\sum_{n\geq 1} z^n \,=\, \sum_{n\geq 0} z^n - 1}$$

6.4 Generating Functions

What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\geq 0}^{\infty} z^n \right) = 1 \ .$$

This is well-defined.

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n\geq 0} z^n = \frac{1}{1-z} \ .$$

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n\geq 0} z^n = \frac{1}{1-z} \; .$$

$$\frac{0\cdot(1-z) - 1\cdot(-1)}{(1-z)^2}$$

We can compute the derivative:

$$\sum_{n\geq 1} nz^{n-1} = \frac{1}{(1-z)^2}$$

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n\geq 0} z^n = \frac{1}{1-z} \; .$$

We can compute the derivative:

$$\underbrace{\sum_{n\geq 1} nz^{n-1}}_{\sum_{n\geq 0}(n+1)z^n} = \frac{1}{(1-z)^2}$$

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n\geq 0} z^n = \frac{1}{1-z} \;.$$

We can compute the derivative:

$$\underbrace{\sum_{n\geq 1} nz^{n-1}}_{\sum_{n\geq 0}(n+1)z^n} = \frac{1}{(1-z)^2}$$

Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

6.4 Generating Functions

We can repeat this

6.4 Generating Functions

We can repeat this

$$\sum_{n\geq 0}(n+1)z^n = \frac{1}{(1-z)^2}\ .$$

6.4 Generating Functions

We can repeat this

$$\sum_{n\geq 0}(n+1)z^n = \frac{1}{(1-z)^2} \ .$$

Derivative:

$$\sum_{n\geq 1} n(n+1)z^{n-1} = \frac{2}{(1-z)^3}$$

6.4 Generating Functions

We can repeat this

$$\sum_{n\geq 0}(n+1)z^n = \frac{1}{(1-z)^2} \enspace .$$

Derivative:

$$\underbrace{\sum_{n\geq 1} n(n+1)z^{n-1}}_{\sum_{n\geq 0}(n+1)(n+2)z^n} = \frac{2}{(1-z)^3}$$

6.4 Generating Functions

We can repeat this

$$\sum_{n\geq 0}(n+1)z^n = \frac{1}{(1-z)^2} \ .$$

Derivative:

$$\underbrace{\sum_{n\geq 1} n(n+1)z^{n-1}}_{\sum_{n\geq 0}(n+1)(n+2)z^n} = \frac{2}{(1-z)^3}$$

Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k}$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} = \sum_{n\geq 0}(n+k)\cdot\ldots\cdot(n+1)z^n$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} = \sum_{n\geq 0}(n+k)\cdot\ldots\cdot(n+1)z^n$$

$$= \frac{k!}{(1-z)^{k+1}} \ .$$

$$\frac{1}{1-z}$$

(k-1)-st derivative $\frac{(k-1)!}{(1-z)^k}$

$$\frac{0 + k(1-z)^{k-1}(-1)\cdot(k-1)!}{(1-z)^{2k}} \quad k+1$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} = \sum_{n\geq 0}(n+k)\cdot\ldots\cdot(n+1)z^n$$

$$= \frac{k!}{(1-z)^{k+1}} \ .$$

Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}} \ .$$

$$\frac{(n+k)!}{k!\;n!} = \frac{(n+k)\cdot\ldots\cdot(n+1)}{k!}$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} = \sum_{n\geq 0}(n+k)\cdot\ldots\cdot(n+1)z^n$$
$$= \frac{k!}{(1-z)^{k+1}} \ .$$

Hence:

$$\sum_{n\geq 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}} \ .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

6.4 Generating Functions

$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0} (n+1)z^n - \sum_{n\geq 0} z^n$$

6.4 Generating Functions

$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n$$
$$= \frac{1}{(1-z)^2} - \frac{1}{1-z}$$

6.4 Generating Functions

$$
\begin{aligned}
\sum_{n\geq 0} nz^n &= \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n \\
&= \frac{1}{(1-z)^2} - \frac{1}{1-z} \\
&= \frac{z}{(1-z)^2}
\end{aligned}
$$

6.4 Generating Functions

$$\begin{aligned}\sum_{n\geq 0} nz^n &= \sum_{n\geq 0}(n+1)z^n - \sum_{n\geq 0} z^n \\ &= \frac{1}{(1-z)^2} - \frac{1}{1-z} \\ &= \frac{z}{(1-z)^2}\end{aligned}$$

The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z) = \sum_{n \geq 0} a_n z^n$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \end{aligned}$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\ &= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \end{aligned}$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \\
&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\ &= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \\ &= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n \\ &= zA(z) + \sum_{n\geq 0} z^n \end{aligned}$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} z^n \\
&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n \\
&= zA(z) + \sum_{n\geq 0} z^n \\
&= zA(z) + \frac{1}{1-z}
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Solving for $A(z)$ gives

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Solving for $A(z)$ gives

$$A(z) = \frac{1}{(1-z)^2}$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2}$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n\geq 0} (n+1) z^n$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n\geq 0}(n+1)z^n$$

Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
n	$\frac{z}{(1-z)^2}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
n	$\frac{z}{(1-z)^2}$
a^n	$\frac{1}{1-az}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
n	$\frac{z}{(1-z)^2}$
a^n	$\frac{1}{1-az}$
n^2	$\frac{z(1+z)}{(1-z)^3}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\frac{1}{1-z}$
$n+1$	$\frac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\frac{1}{(1-z)^{k+1}}$
n	$\frac{z}{(1-z)^2}$
a^n	$\frac{1}{1-az}$
n^2	$\frac{z(1+z)}{(1-z)^3}$
$\frac{1}{n!}$	e^z

Some Generating Functions

n-th sequence element	*generating function*

Some Generating Functions

n-th sequence element	*generating function*
cf_n	cF

Some Generating Functions

n-th sequence element	generating function
cf_n	cF
$f_n + g_n$	$F + G$

Some Generating Functions

n-th sequence element	*generating function*
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$

Some Generating Functions

$$\sum_{n\ge 0} f_n z^{n+k} = \sum_{n\ge k} f_{n-k} \cdot z^n = \sum_{n\ge 0} g_n \cdot z^n$$

n-th sequence element	*generating function*
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
f_{n-k} $(n \ge k)$; 0 otw.	$z^k F$

$f_0\ f_1\ f_2\ f_3\ f_4$

k

$0\ 0\ 0\ 0\ f_0\ f_1\ f_2$

$g_k\ g_{k+1}$

Some Generating Functions

n-th sequence element	*generating function*
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$
1 1 1 1 1 1 1 1	1 1
n-th element 1 2 3 4 5 ..	→
index 0 1 2 3 4	n+1

Some Generating Functions

$z\left(\sum_{n\ge 0} f_n z^n\right)' = z\sum_{n\ge 1} n\cdot f_n\cdot z^{n-1}$

$= \sum_{n\ge 1} n f_n\cdot z^n$

$= \sum_{n\ge 0} (n f_n) z^n$

n-th sequence element	generating function
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k}\ \ (n \ge k);\ \ 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\frac{F(z)}{1-z}$
nf_n	$z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$

Some Generating Functions

n-th sequence element	*generating function*
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k} \quad (n \geq k);\quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\frac{F(z)}{1-z}$
nf_n	$z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$
$c^n f_n$	$F(cz)$

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\ge 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
5. Write $f(z)$ as a formal power series.
   Techniques:

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
5. Write $f(z)$ as a formal power series.
   Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
5. Write $f(z)$ as a formal power series.
   Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)
   - lookup in tables

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n\geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
5. Write $f(z)$ as a formal power series.
   Techniques:
   - partial fraction decomposition (Partialbruchzerlegung)
   - lookup in tables
6. The coefficients of the resulting power series are the a_n.

Example: $a_n = 2a_{n-1}, a_0 = 1$

1. Set up generating function:

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n\geq 1} a_n z^n$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n\geq 1} a_n z^n$$

2. Plug in:

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\ge 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n\ge 1} a_n z^n$$

2. Plug in:

$$A(z) = 1 + \sum_{n\ge 1} (2a_{n-1}) z^n$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n\geq 1}(2a_{n-1})z^n$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n\geq 1}(2a_{n-1})z^n$$

$$= 1 + 2z\sum_{n\geq 1}a_{n-1}z^{n-1}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\ &= 1 + 2z\sum_{n\geq 1} a_{n-1}z^{n-1} \\ &= 1 + 2z\sum_{n\geq 0} a_n z^n \end{aligned}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\ &= 1 + 2z\sum_{n\geq 1}a_{n-1}z^{n-1} \\ &= 1 + 2z\sum_{n\geq 0}a_n z^n \\ &= 1 + 2z\cdot A(z) \end{aligned}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$\begin{aligned} A(z) &= 1 + \sum_{n\geq 1}(2a_{n-1})z^n \\ &= 1 + 2z\sum_{n\geq 1}a_{n-1}z^{n-1} \\ &= 1 + 2z\sum_{n\geq 0}a_n z^n \\ &= 1 + 2z \cdot A(z) \end{aligned}$$

4. Solve for $A(z)$.

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$
\begin{aligned}
A(z) &= 1 + \sum_{n\geq 1} (2a_{n-1})z^n \\
&= 1 + 2z \sum_{n\geq 1} a_{n-1}z^{n-1} \\
&= 1 + 2z \sum_{n\geq 0} a_n z^n \\
&= 1 + 2z \cdot A(z)
\end{aligned}
$$

4. Solve for $A(z)$.

$$
A(z) = \frac{1}{1-2z}
$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$A(z) = \frac{1}{1-2z}$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{1-2z}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$A(z) = \sum_{n\geq 0} a_n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} a_n z^n \end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} a_n z^n \\ &= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned} A(z) &= \sum_{n\ge 0} a_n z^n \\ &= a_0 + \sum_{n\ge 1} a_n z^n \\ &= 1 + \sum_{n\ge 1} (3a_{n-1} + n) z^n \\ &= 1 + 3z \sum_{n\ge 1} a_{n-1} z^{n-1} + \sum_{n\ge 1} n z^n \end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} a_n z^n \\
&= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \\
&= 1 + 3z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} n z^n \\
&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n
\end{aligned}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$
\begin{aligned}
A(z) &= \sum_{n\geq 0} a_n z^n \\
&= a_0 + \sum_{n\geq 1} a_n z^n \\
&= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \\
&= 1 + 3z \sum_{n\geq 1} a_{n-1} z^{n-1} + \sum_{n\geq 1} n z^n \\
&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
\end{aligned}
$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

$$\frac{1}{1-3z} + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \stackrel{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

$$\frac{A}{(x-\lambda_1)} + \frac{B}{(x-\lambda_2)} + \frac{C}{(x-\lambda_2)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \stackrel{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

This gives

$$z^2 - z + 1 = A(1-z)^2 + B(1-3z)(1-z) + C(1-3z)$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \stackrel{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

This gives

$$z^2 - z + 1 = A(1-z)^2 + B(1-3z)(1-z) + C(1-3z)$$
$$= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z)$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \stackrel{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \\ &= (A+3B)z^2 + (-2A-4B-3C)z + (A+B+C) \end{aligned}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

Example: $a_n = 3a_{n-1} + n,\ a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

$$= \frac{7}{4} \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n+1) z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$
\begin{aligned}
A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\
&= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1) z^n \\
&= \sum_{n \geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \Big) z^n
\end{aligned}
$$

Example: $a_n = 3a_{n-1} + n,\ a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$
\begin{aligned}
A(z) &= \frac{7}{4}\cdot\frac{1}{1-3z} - \frac{1}{4}\cdot\frac{1}{1-z} - \frac{1}{2}\cdot\frac{1}{(1-z)^2} \\
&= \frac{7}{4}\cdot\sum_{n\ge 0} 3^n z^n - \frac{1}{4}\cdot\sum_{n\ge 0} z^n - \frac{1}{2}\cdot\sum_{n\ge 0}(n+1)z^n \\
&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n \\
&= \sum_{n\ge 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n
\end{aligned}
$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$\begin{aligned} A(z) &= \frac{7}{4}\cdot\frac{1}{1-3z} - \frac{1}{4}\cdot\frac{1}{1-z} - \frac{1}{2}\cdot\frac{1}{(1-z)^2} \\ &= \frac{7}{4}\cdot\sum_{n\geq 0} 3^n z^n - \frac{1}{4}\cdot\sum_{n\geq 0} z^n - \frac{1}{2}\cdot\sum_{n\geq 0}(n+1)z^n \\ &= \sum_{n\geq 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\Big)z^n \\ &= \sum_{n\geq 0}\Big(\frac{7}{4}\cdot 3^n - \frac{1}{2}n - \frac{3}{4}\Big)z^n \end{aligned}$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.