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$$\left(\frac{1}{b^i}\right)^{\log_b a} = \frac{1}{b^{i \cdot \log_b a}}$$

$$\frac{1}{a^i} = \frac{1}{(b^{\log_b a})^i}$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

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$n = b^\ell \Rightarrow \ell = \log_b n$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$\begin{array}{ccc} & \parallel & \\ \log_b b^\ell & - & \log_b b^i \\ \parallel & & \parallel \\ \ell & - & i \end{array}$$

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$$\sum_{i=1}^{\ell} i = \frac{\ell(\ell+1)}{2}$$

$$\sum_{i=1}^{\ell} i^2 \approx \frac{1}{3} \ell^3$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \end{aligned}$$



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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) = a^i$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

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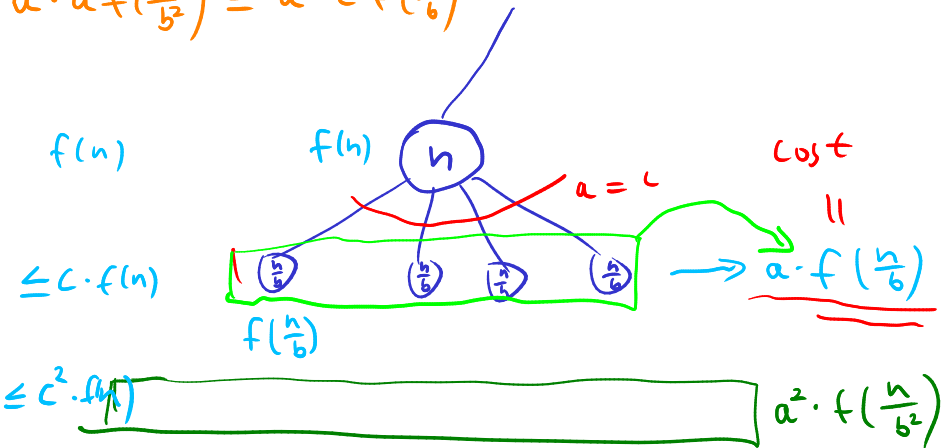
$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

$$a \cdot a f\left(\frac{n}{b^2}\right) \leq a \cdot c f\left(\frac{n}{b}\right)$$



$$\sum_{i=0}^{\infty} c^i \cdot f(n) = f(n) \cdot \underbrace{\sum_{i=0}^{\infty} c^i}_{= f(n) \cdot \frac{c^{k+1} - 1}{c - 1}}$$

$$\boxed{Q \rightarrow \infty} \leq f(n) \cdot \frac{1}{1-c}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

## Example: Multiplying Two Integers

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The diagram shows two 9-bit integers,  $A$  and  $B$ , aligned for addition. Integer  $A$  is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer  $B$  is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of  $B$ . A vertical light blue box highlights the rightmost bit of  $A$  (the least significant bit) and the bit of  $B$  directly below it, indicating the first step of the addition process.



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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline 0 \end{array} \begin{array}{l} A \\ B \end{array}$$

The diagram shows the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. A vertical box highlights the rightmost bit of the result, which is 0. A small '1' is written below the line under the 8th bit position, indicating a carry.

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The diagram illustrates the addition of two integers, A and B, using a register of constant size. The numbers are represented as binary strings. A vertical box highlights the last two bits of the numbers and the resulting sum. The carry bits are shown as small '1's below the lines.

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$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 1 & 1 & & \\ & & & & & & & & 0 & 0 \end{array}$$

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The diagram illustrates the addition of two integers, A and B, using a register of constant size. The integers are represented as binary strings: A = 110110101 and B = 100010011. The addition is performed bit-by-bit from right to left, with carry bits (indicated by small '1's) being passed to the next higher bit. The result of the addition is shown as 000, indicating that the carry bits are zero.

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1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
					1	1	1		
						0	0	0	

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The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line. The bit at position 5 (from the right) is 1, and the bits at positions 6, 7, and 8 are 0, 0, and 0 respectively. A vertical box highlights the bit at position 5, which is 1. This bit is the result of the carry from the addition of the bits at positions 4 and 5 of A and B. The carry from the addition of the bits at positions 5 and 6 of A and B is 0, and the carry from the addition of the bits at positions 6 and 7 of A and B is 1. The carry from the addition of the bits at positions 7 and 8 of A and B is 1. The carry from the addition of the bits at positions 8 and 9 of A and B is 1.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$



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The diagram illustrates the addition of two integers, A and B, using a register of constant size. The numbers are represented as binary strings. A vertical box highlights the current bit position being processed, which is the 5th bit from the right (the 4th bit from the left). The carry bits are shown as small subscripts below the digits.

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1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

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The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line: 0, 0, 1, 0, 0, 0. A vertical light blue box highlights the 4th bit (index 3) of both A and B, and the 4th bit of the result, which is 0. This bit is the result of adding the 4th bits of A and B (1 + 0) plus the carry-in from the 3rd bit (1).



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1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
		1	0	0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A vertical box highlights the third bit of A (0) and the second bit of B (0), which are being added together. Below the horizontal line, the result of this addition is shown as a 1 in the second column and a 0 in the third column. Small subscripts are present below the bits of B: 0 under the first bit, 1 under the second bit, 1 under the third bit, 0 under the fourth bit, 1 under the fifth bit, 1 under the sixth bit, and 1 under the seventh bit.

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<hr/>									
		1	0	0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line: 1 0 0 1 0 0 0. A vertical blue box highlights the first two bits of the input (1 1) and the first bit of the result (1). Small subscripts are placed below the bits of B: 0, 1, 1, 0, 1, 1, 1.

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	<hr/>									
	1	1	0	0	1	0	0	0		

The diagram shows the addition of two 9-bit integers, A and B. The result is a 10-bit integer. The first two bits of the result (11) are highlighted in a light blue box, indicating a carry-out from the most significant bit of the input integers.

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	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	



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	0	1	1	0	0	1	0	0	0	



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	<hr/>									
1	0	1	1	0	0	1	0	0	0	

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two  $n$ -bit integers can be added in time  $\mathcal{O}(n)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

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Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \end{array}$$



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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} \phantom{1\ 0\ 0\ 0\ 1} 0 \\ \phantom{1\ 0\ 0\ 0\ 1} \phantom{1\ 0\ 0\ 0\ 1} \phantom{0} 0 \end{array}$$

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**Time requirement:**

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**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .

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**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .
- ▶ Adding  $m$  numbers of length  $\leq 2n$ :  
 $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$ .

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

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$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

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$$\begin{array}{|c|c|} \hline B_1 & B_0 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline A_1 & A_0 \\ \hline \end{array}$$

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
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```

$\mathcal{O}(1)$

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$\mathcal{O}(n)$

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$\mathcal{O}(n)$

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$\mathcal{O}(1)$

$\mathcal{O}(1)$

$\mathcal{O}(n)$

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$2T(\frac{n}{2}) + \mathcal{O}(n)$

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3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
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5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
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## Example: Multiplying Two Integers

### Algorithm 3 $\text{mult}(A, B)$

1: <b>if</b> $ A  =  B  = 1$ <b>then</b>	$\mathcal{O}(1)$
2: <b>return</b> $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split $A$ into $A_0$ and $A_1$	$\mathcal{O}(n)$
4: split $B$ into $B_0$ and $B_1$	$\mathcal{O}(n)$
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6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T\left(\frac{n}{2}\right) + \mathcal{O}(n)$
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8: <b>return</b> $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

# Example: Multiplying Two Integers

**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2:  $f(n) = \Theta(n^{\log_b a} \log^k n)$        $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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## Example: Multiplying Two Integers

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In our case  $a = 4$ ,  $b = 2$ , and  $f(n) = \Theta(n)$ . Hence, we are in Case 1, since  $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$ .

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⇒ Not better than the “school method”.

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Hence,

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1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
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$\mathcal{O}(1)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

$T(\frac{n}{2})$

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$\mathcal{O}(n)$   $T(\frac{n}{2} + 1)$

$T(\frac{n}{2})$

$T(\frac{n}{2})$

$T(\frac{n}{2}) + \mathcal{O}(n)$



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## Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

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Again we are in Case 1. We get a running time of  $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$ .

A huge improvement over the "school method".

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