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&= c n^{\log_b a} \log_b n
\end{aligned}
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$$\left(\frac{1}{b^i}\right)^{\log_b a} = \frac{1}{b^{i \cdot \log_b a}}$$

$$\| $$

$$\frac{1}{a^i} = \frac{1}{\left(b^{\log_b a}\right)^i}$$

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Hence,

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Hence,

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$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

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$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

$$= c n^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$\parallel$$

$$\log_b b^\ell - \log_b b^i$$

$$\parallel \qquad\qquad \parallel$$

$$\ell \quad - \quad i$$

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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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&= c n^{\log_b a} \sum_{i=0}^{\ell - 1} (\ell - i)^k \\
&= c n^{\log_b a} \sum_{i=1}^{\ell} i^k
\end{aligned}
$$

$$\sum_{i=1}^{\ell} i = \frac{\ell(\ell+1)}{2}$$

$$\sum_{i=1}^{\ell} i^{2\,(k)} \approx \frac{1}{3}\ell^3$$

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&= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \frac{1}{k}\ell^{k+1}
\end{aligned}
$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

$= a^i$

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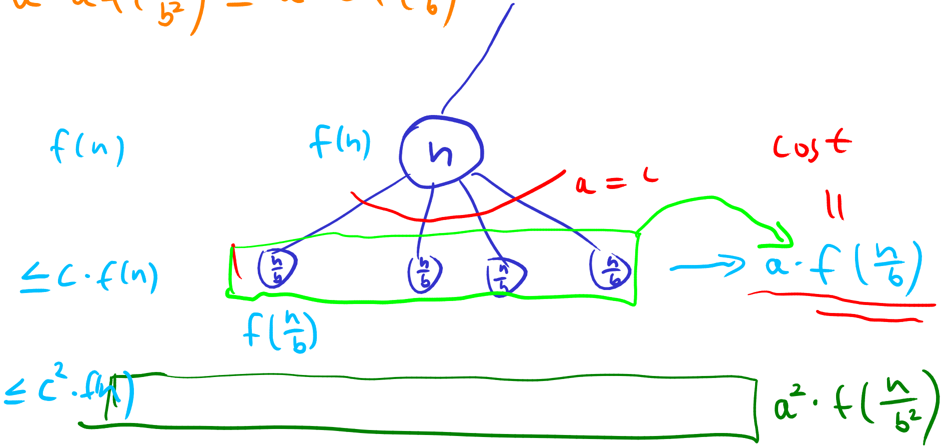
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\
&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

$$a \cdot a f\left(\frac{n}{b^2}\right) \leq a \cdot c f\left(\frac{n}{b}\right)$$



$$\sum_{i=0}^{\ell} c^i \cdot f(n) = f(n) \cdot \underbrace{\sum_{i=0}^{\ell} c^i}$$

$$= f(n) \cdot \frac{c^{\ell+1}-1}{c-1}$$

$$\boxed{\ell \to \infty} \leq f(n) \cdot \frac{1}{1-c}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \boxed{\mathcal{O}(n^{\log_b a})} \end{aligned}$$

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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
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\end{aligned}
$$

$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$$

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\begin{aligned}
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&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$\Rightarrow T(n) = \Theta(f(n))$.

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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 & & & & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
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 & & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & 1 & 1 & 0 & 1 & 1 & 1 & & & \\
\hline
 & & & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
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\hline
 & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
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 & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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For this we first need to be able to add two integers A and B:

$$
\begin{array}{c|cccccccc|l}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
{\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}
$$

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Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

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For this we first need to be able to add two integers A and B:

$$
\begin{array}{c|ccccccccc|c}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
\hline
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}
$$

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For this we first need to be able to add two integers A and B:

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
1	0	1	1	0	0	1	0	0	0	

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For this we first need to be able to add two integers A and B:

$$
\begin{array}{cccccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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$$\underline{1\;0\;0\;0\;1 \times 1\;0\;1\;1}$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\begin{array}{c} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ \boxed{1} \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & \boxed{1} \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
$$

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\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & 0
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
 & 1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{r}
10001 \times 1011 \\
\hline
10001 \\
100010 \\
0000000 \\
10001000 \\
\hline
10111011
\end{array}
$$

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
\end{array}
$$

$$
\begin{array}{rrrrrrrr}
 & & & 1 & 0 & 0 & 0 & 1 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

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$$B \times A$$

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Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{b_{n-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \;\; b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \;\; a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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Suppose that integers A and B are of length $n = 2^k$, for some k.

B_1	B_0	$\times$	A_1	A_0

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B_1	B_0	$\times$	A_1	A_0

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

```
1: if |A| = |B| = 1 then
2:     return a_0 · b_0
3: split A into A_0 and A_1
4: split B into B_0 and B_1
5: Z_2 ← mult(A_1, B_1)
6: Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
7: Z_0 ← mult(A_0, B_0)
8: return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
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```

$\mathcal{O}(1)$

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```
1: if |A| = |B| = 1 then                              O(1)
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```

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

1: **if** $|A| = |B| = 1$ **then** — $\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$ — $\mathcal{O}(1)$
3: split A into A_0 and A_1 — $\mathcal{O}(n)$
4: split B into B_0 and B_1 — $\mathcal{O}(n)$
5: $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
6: $Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$
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Algorithm 3 mult(A, B)

```
if |A| = |B| = 1 then                                  O(1)
    return a_0 · b_0                                   O(1)
split A into A_0 and A_1                               O(n)
split B into B_0 and B_1                               O(n)
Z_2 ← mult(A_1, B_1)                                   T(n/2)
Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
Z_0 ← mult(A_0, B_0)
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
```

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5: $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$ — $T(\frac{n}{2})$
6: $Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$
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Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $\lvert A\rvert = \lvert B\rvert = 1$ **then**	$\mathcal{O}(1)$
return $a_0 \cdot b_0$	$\mathcal{O}(1)$
split A into A_0 and A_1	$\mathcal{O}(n)$
split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$	
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

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split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
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return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

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$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
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$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$	$T(\frac{n}{2})$
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \;.$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Example: Multiplying Two Integers

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- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\quad T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\qquad T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

Example: Multiplying Two Integers

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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

$\Longrightarrow$ Not better then the "school method".

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Hence,

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1: **if** $|A| = |B| = 1$ **then**
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split A into A_0 and A_1	$\mathcal{O}(n)$
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$Z_2 \leftarrow \text{mult}(A_1, B_1)$	
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We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.

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