Residual Graph

Version A:

The residual graph G' for a mincost flow is just a copy of the graph $\boxed{G.}$

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - \boxed{f(e)}$ and $u(e') = u(e) - f(e)$.

$\ell(e)=5$ $\qquad$ $u(e)=10$ $\qquad$ $\bigcirc \xrightarrow{e} \bigcirc$

$f(e)=7$

$\ell(e)=-2$ $\qquad$ $u(e)=3$

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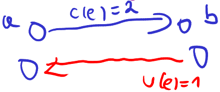

Version B:

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

feasibility

- capacity constraints (residual graph)
- net flows

Suppose I have flow f and G_f

$f+g$ feasible $\iff$ g feasible in G_f

$\Updownarrow$ $\Updownarrow$

$\forall e \quad \ell'(e) \le g(e) \le u'(e)$

$\Updownarrow$

$\ell(e) \le f(e)+g(e) \le u(e) \iff \forall e \quad \ell(e)-f(e) \le g(e) \le u(e)-f(e)$

$$f(v) = b(v)$$

$$\|$$

$$\sum_{e \in IN(v)} f(e) - \sum_{e \in OUT(v)} f(e)$$

$$(f+g)(v)$$

$$\|$$

$$\sum_{e \in IN(v)} \big(f(e) + g(e)\big) - \sum_{e \in OUT(v)} \big(f(e) + g(e)\big) = f(v) + g(v)$$

14 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

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A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \le u(e)$ for every edge of G.

$\mathrm{cost}(f+g)$

$$= \sum_{e} (f+g)(e) \cdot c(e)$$

$$= \sum_{e} f(e) \cdot c(e) + \sum_{e} g(e) \cdot c(e)$$

$$= \mathrm{cost}(f) + \mathrm{cost}(g)$$

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f *does not have a feasible circulation of negative cost.*

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Then $f + g$ is a feasible flow with cost $\mathrm{cost}(f) + \mathrm{cost}(g) < \mathrm{cost}(f)$. Hence, f is not minimum cost.

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- $\Leftarrow$ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

$f^* - f$ has net flow 0 $\Rightarrow$ circulation

$\mathrm{cost}(f^* - f) = \mathrm{cost}(f^*) - \mathrm{cost}(f) < 0$

take an edge

$$(f^* - f)(e) = f^*(e) - f(e)$$

$$l'(e) = l(e) - f(e) \qquad u'(e) = u(e) - f(e)$$

$$l(e) - \cancel{f(e)} \leq f^*(e) - \cancel{f(e)} \leq u(e) - \cancel{f(e)}$$

✓

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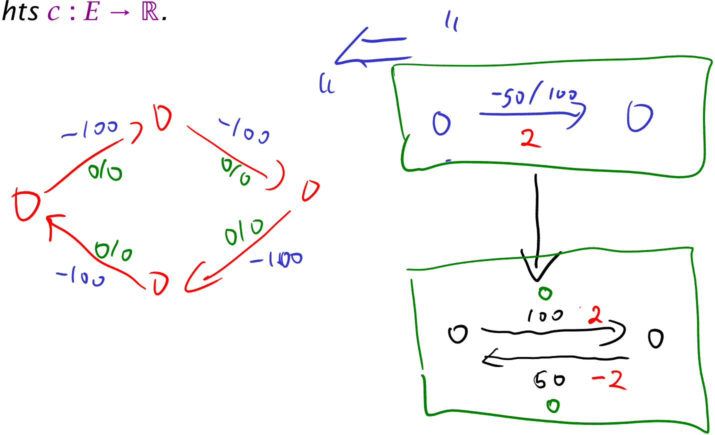
⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.



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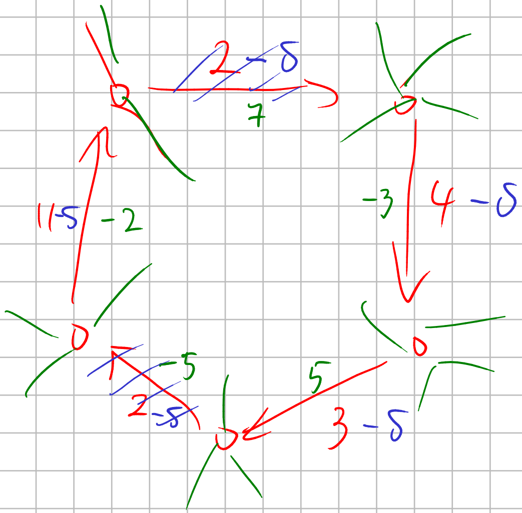

$\delta = 2$

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- You still have a circulation with negative cost.

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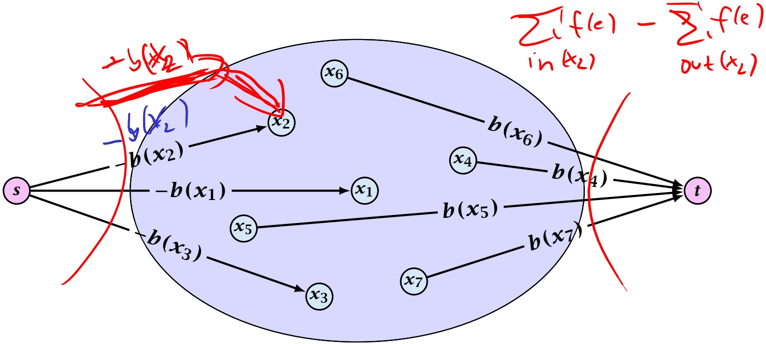
- Find directed cycle only using edges that have non-zero flow.
- If this cycle has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

14 Mincost Flow

Algorithm 23 CycleCanceling$(G = (V, E), c, u, b)$

```
establish a feasible flow f in G
while G_f contains negative cycle do
    use Bellman-Ford to find a negative circuit Z
    δ ← min{u_f(e) | e ∈ Z}
    augment δ units along Z and update G_f
```

How do we find the initial feasible flow?

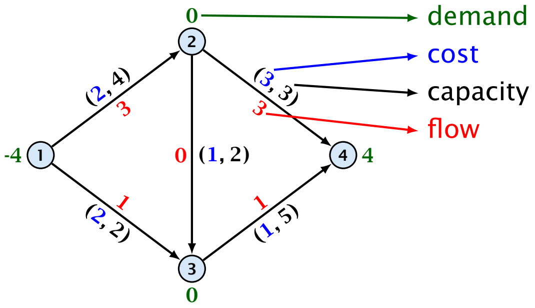


- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$\Leftarrow$

$$\boxed{\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v)\ .}$$

14 Mincost Flow



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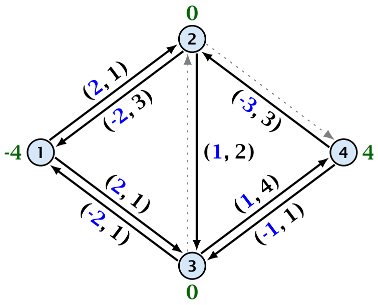

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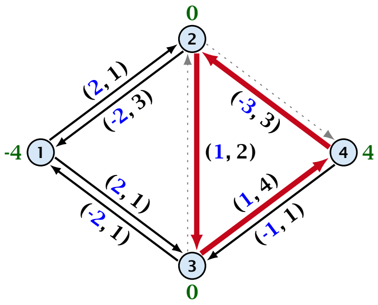
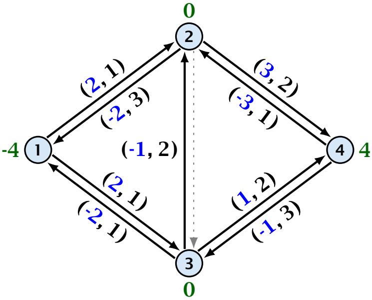
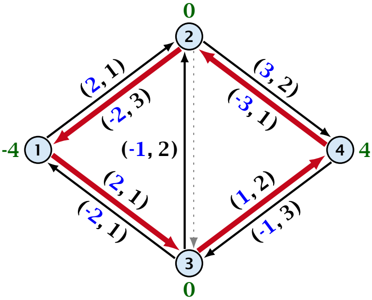



0
2
(2, 1)
(-2, 3)
(3, 2)
(-3, 1)
-4
1
(-1, 2)
4
4
(2, 1)
(-2, 1)
(1, 2)
(-1, 3)
3
0

14 Mincost Flow



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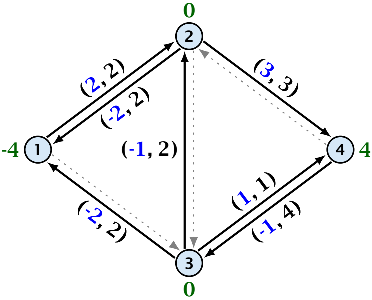

Lemma 87

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \le C$ and $|u(e)| \le U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \dots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

14 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$
$c : E \to \mathbb{R}$;

Lemma 88 (without proof)

A general mincost flow problem can be solved in polynomial time.