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- Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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Abuse of notation

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3. People write e.g. $h(n) = f(n) + o(g(n))$ when they mean that there exists a function $z : \mathbb{N} \to \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) = f(n) + z(n)$.

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

Asymptotic Notation in Equations

How do we interpret an expression like:

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

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How do we interpret an expression like:

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"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$$

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{ f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n) \text{ with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \Big\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

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Lemma 1

Let f, g be functions with the property
$\exists n_0 > 0\, \forall n \geq n_0 : f(n) > 0$ *(the same for g). Then*

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*

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The expressions also hold for Ω. Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

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  Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

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Formal Definition

Let f, g denote functions from $\mathbb{N}^d$ to $\mathbb{R}_0^+$.

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Example 2

- $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n,m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n,m) = n - 1$

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