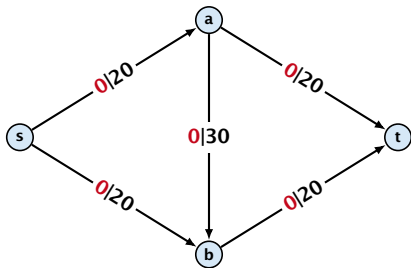


11 Augmenting Path Algorithms

Greedy-algorithm:

- ▶ start with $f(e) = 0$ everywhere
- ▶ find an s - t path with $f(e) < c(e)$ on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible

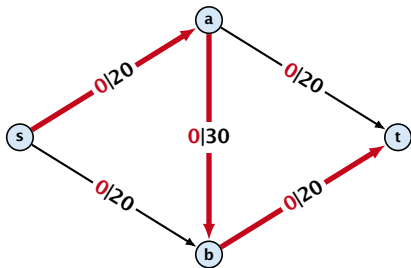


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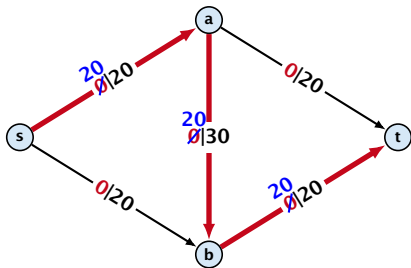


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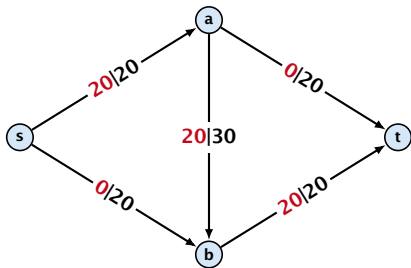


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flow value: 20

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

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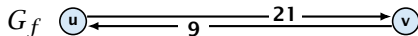
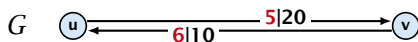
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Augmenting Path Algorithm

Definition 4

An **augmenting path** with respect to flow f , is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Augmenting Path Algorithm

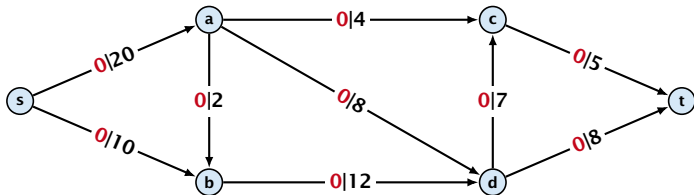
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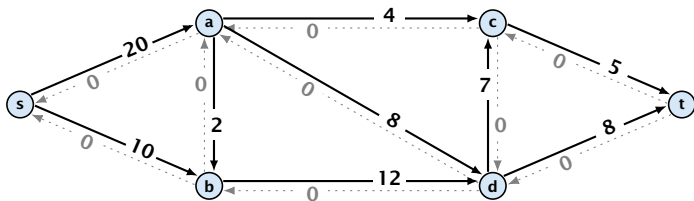
Algorithm 1 FordFulkerson($G = (V, E, c)$)

- 1: Initialize $f(e) \leftarrow 0$ for all edges.
- 2: **while** \exists augmenting path p in G_f **do**
- 3: augment as much flow along p as possible.

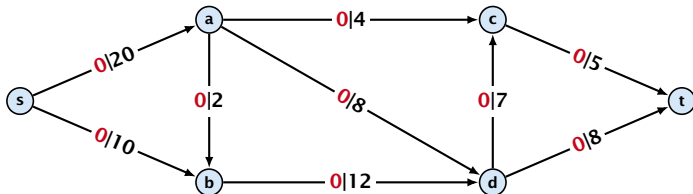
Augmenting Paths



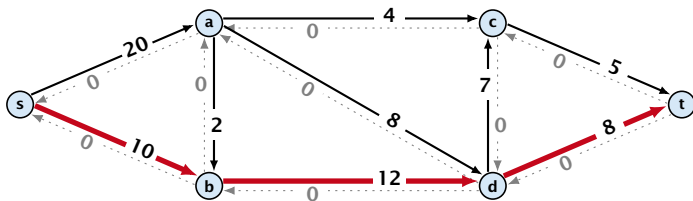
flow value: 0



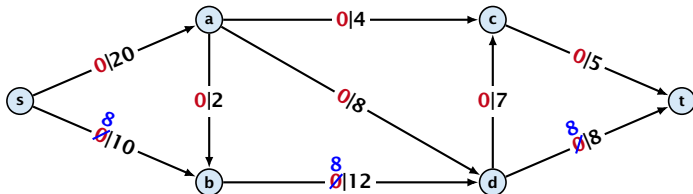
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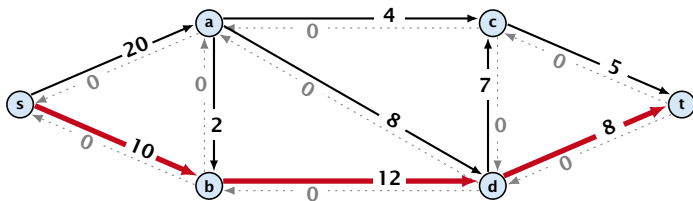
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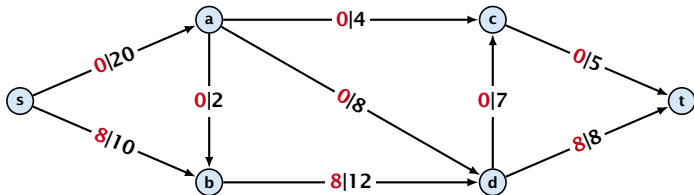
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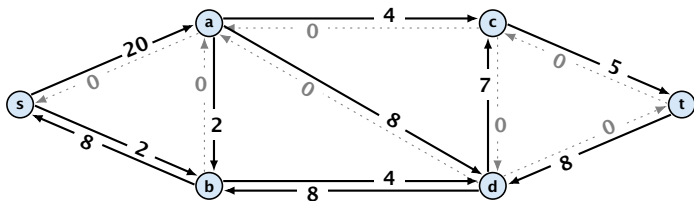
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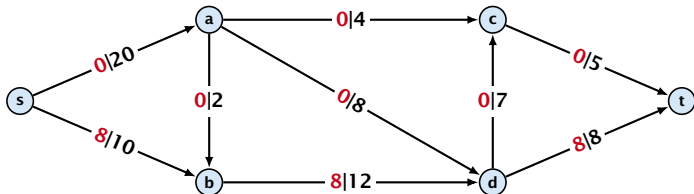
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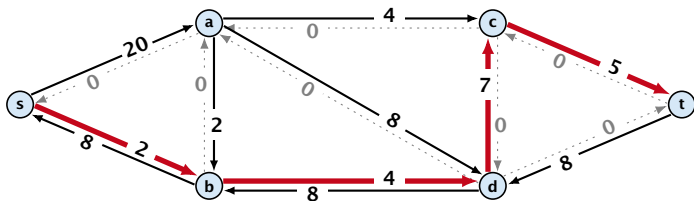
flow value: 8



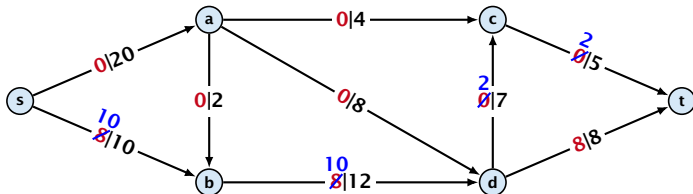
Augmenting Paths



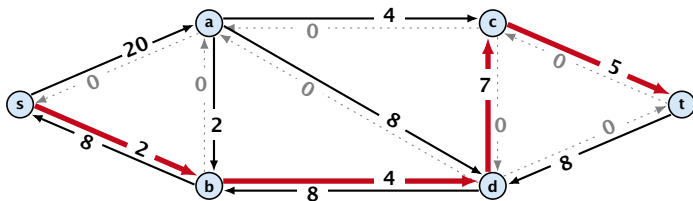
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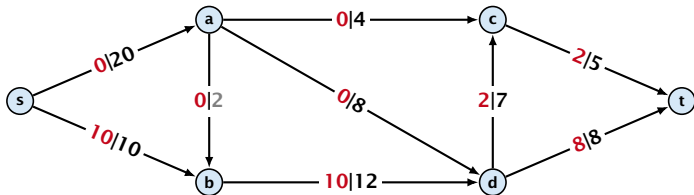
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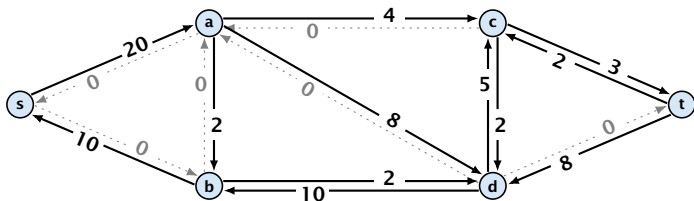
flow value: 8



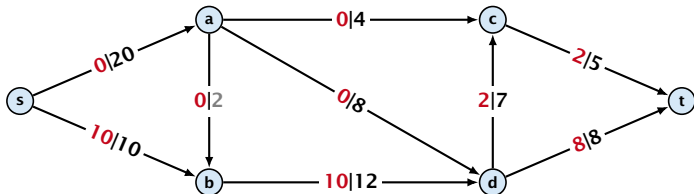
Augmenting Paths



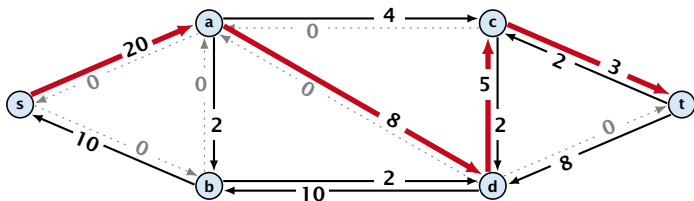
flow value: 10



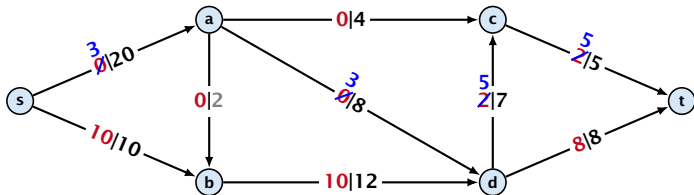
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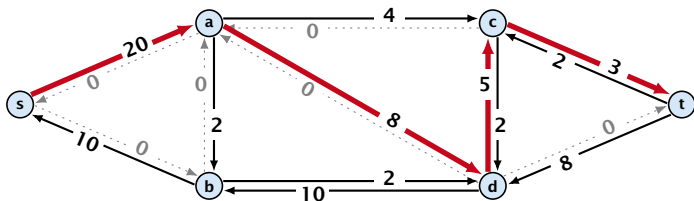
flow value: 10



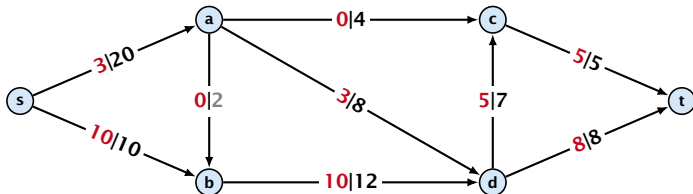
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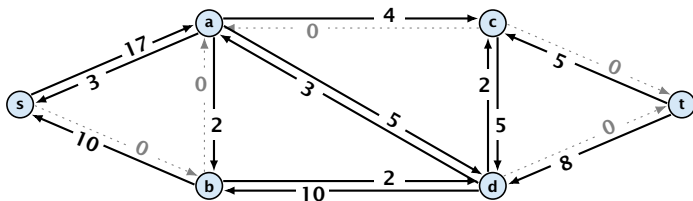
flow value: 10



Augmenting Paths



flow value: 13



Augmenting Path Algorithm

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A flow f is a maximum flow **iff** there are no augmenting paths.

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Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\text{val}(f) = \text{cap}(A, V \setminus A)$.



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Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\text{val}(f) = \text{cap}(A, V \setminus A)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

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This we already showed.

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If there were an augmenting path, we could improve the flow.
Contradiction.

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- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$\text{val}(f)$

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$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A .

Analysis

Assumption:

All capacities are integers between 1 and C .

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Invariant:

Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral throughout the algorithm.

Lemma 7

The algorithm terminates in at most $\text{val}(f^*) \leq nC$ iterations, where f^* denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.

Lemma 7

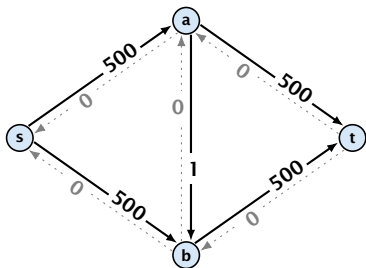
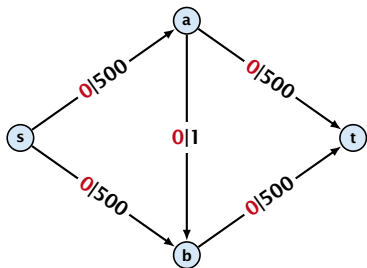
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Theorem 8

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.

A Bad Input

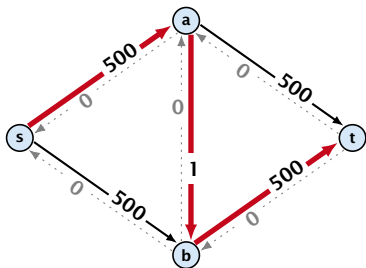
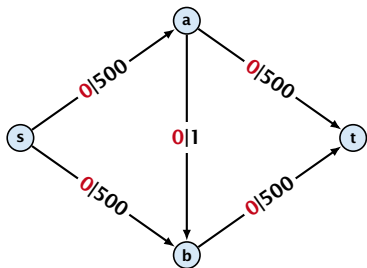
Problem: The running time may not be polynomial



flow value: 0

A Bad Input

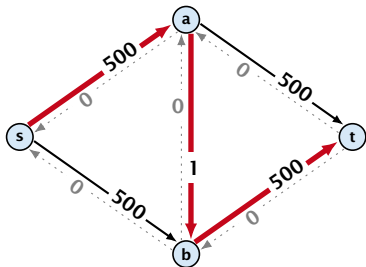
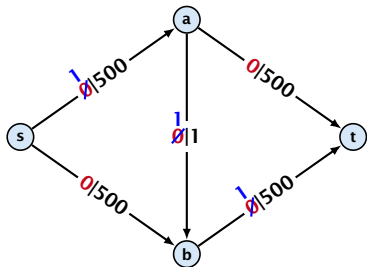
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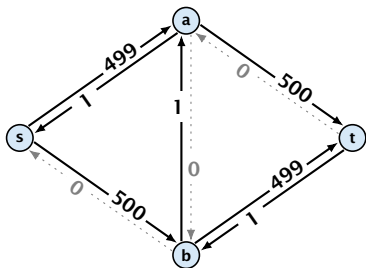
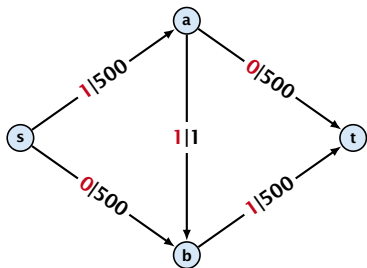
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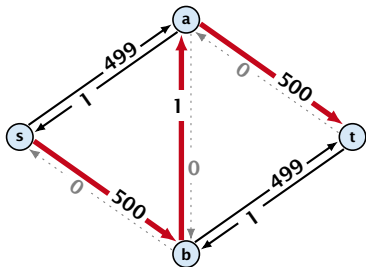
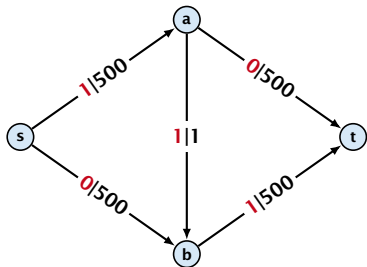
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flow value: 1

A Bad Input

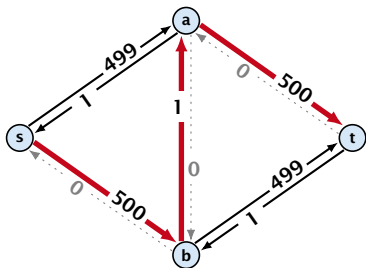
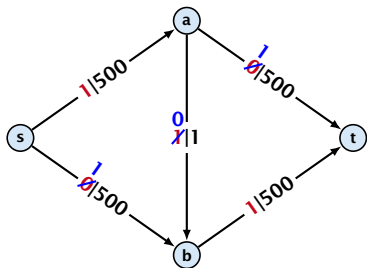
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flow value: 1

A Bad Input

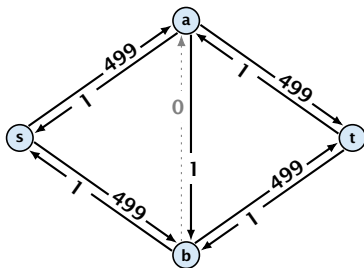
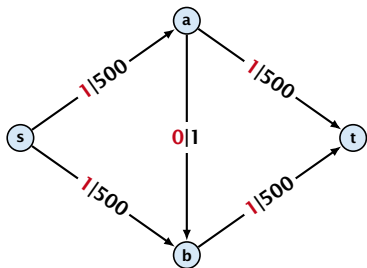
Problem: The running time may not be polynomial



flow value: 1

A Bad Input

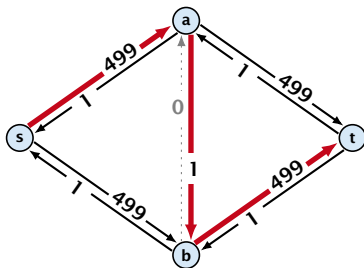
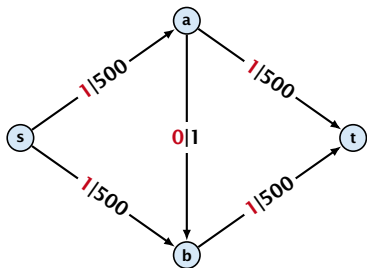
Problem: The running time may not be polynomial



flow value: 2

A Bad Input

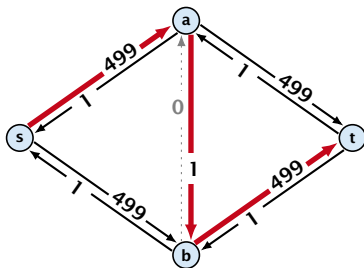
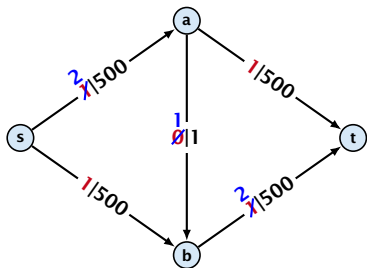
Problem: The running time may not be polynomial



flow value: 2

A Bad Input

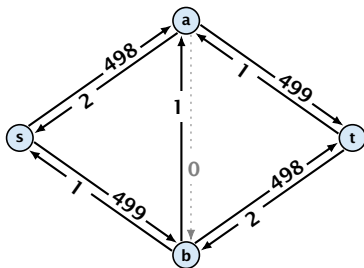
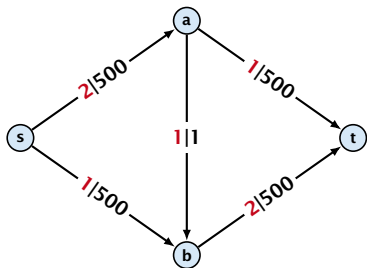
Problem: The running time may not be polynomial



flow value: 2

A Bad Input

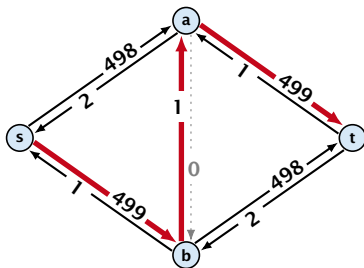
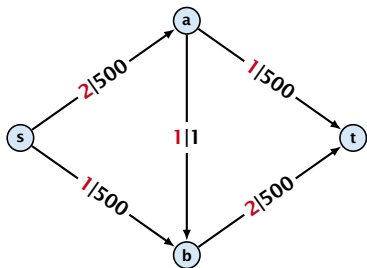
Problem: The running time may not be polynomial



flow value: 3

A Bad Input

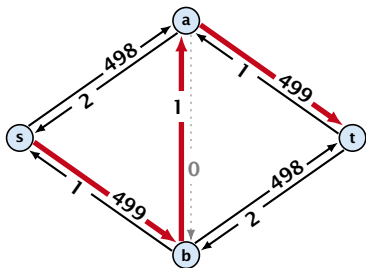
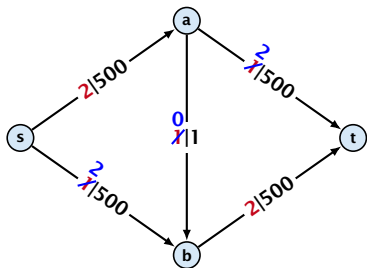
Problem: The running time may not be polynomial



flow value: 3

A Bad Input

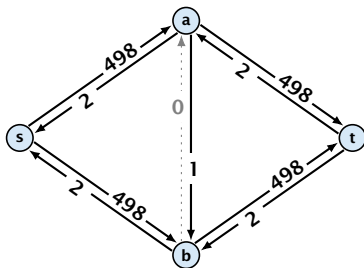
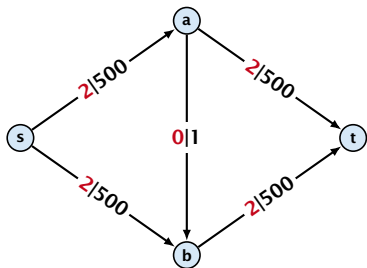
Problem: The running time may not be polynomial



flow value: 3

A Bad Input

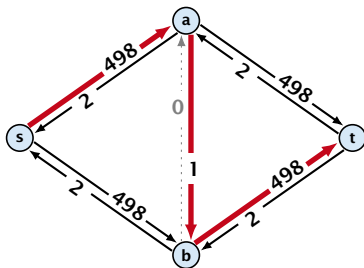
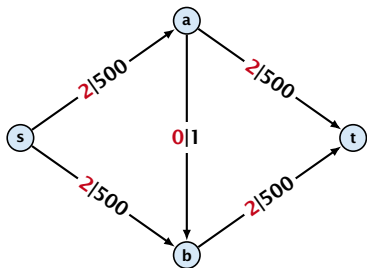
Problem: The running time may not be polynomial



flow value: 4

A Bad Input

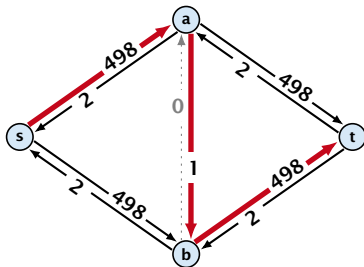
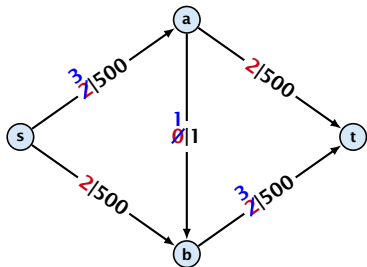
Problem: The running time may not be polynomial



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A Bad Input

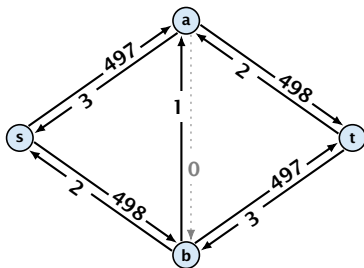
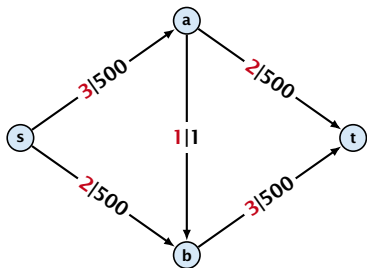
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flow value: 4

A Bad Input

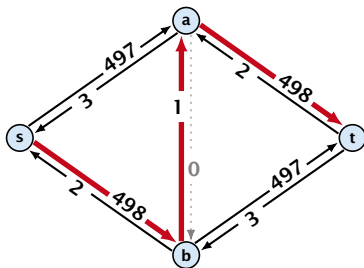
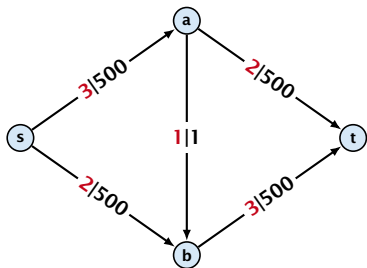
Problem: The running time may not be polynomial



flow value: 5

A Bad Input

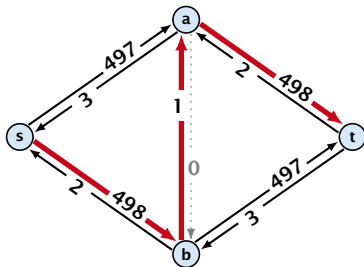
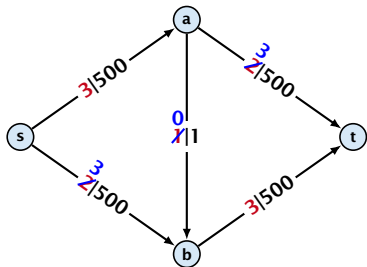
Problem: The running time may not be polynomial



flow value: 5

A Bad Input

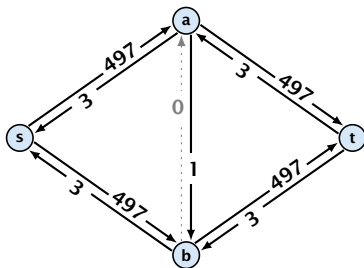
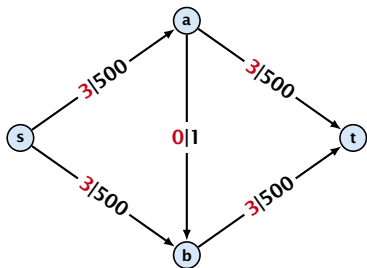
Problem: The running time may not be polynomial



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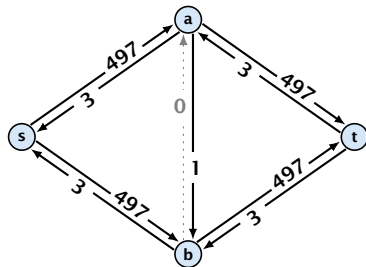
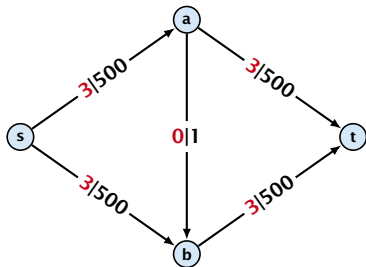
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Problem: The running time may not be polynomial



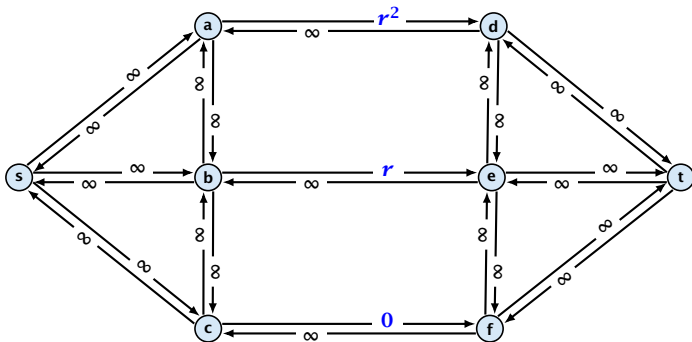
flow value: 6

Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Pathological Input

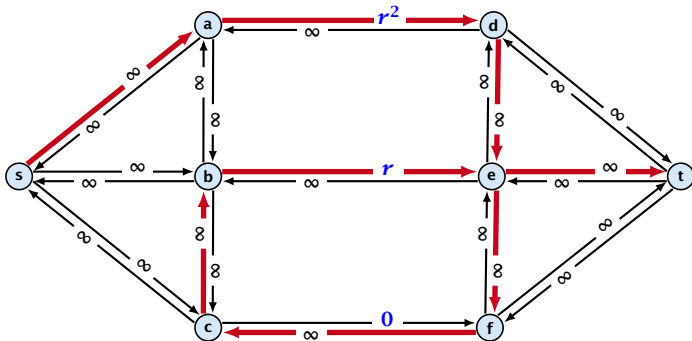
Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



flow value: 0

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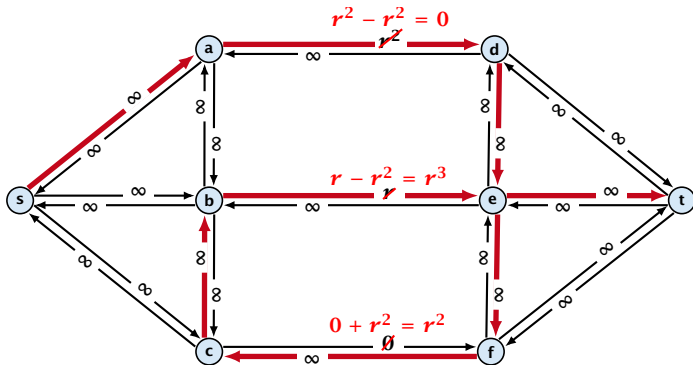
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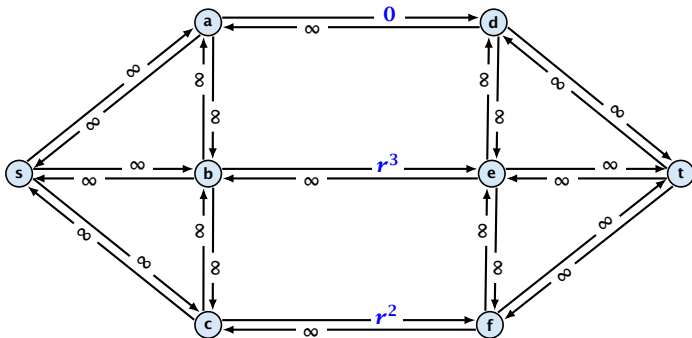
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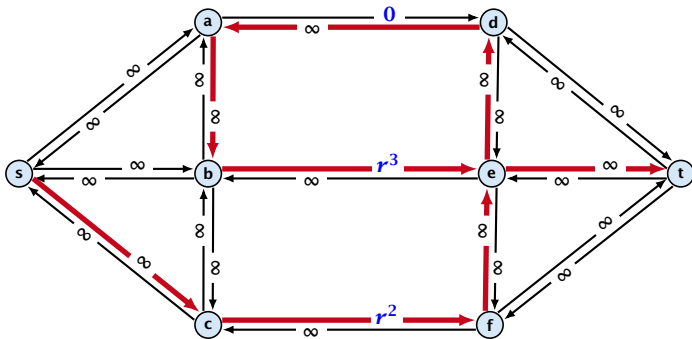
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flow value: r^2

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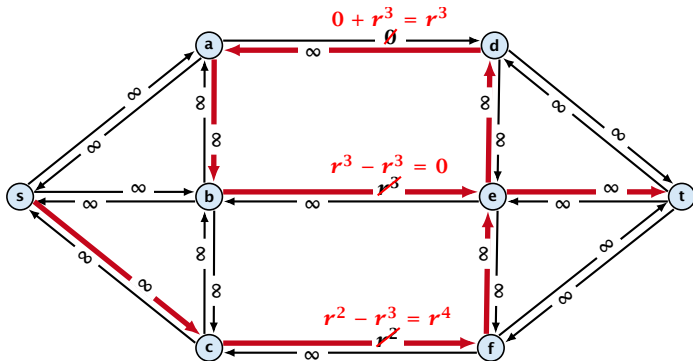
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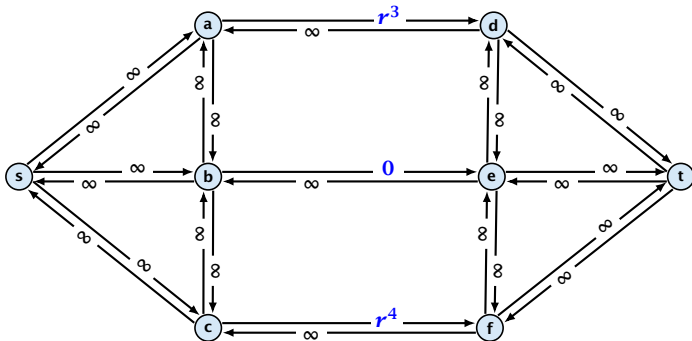
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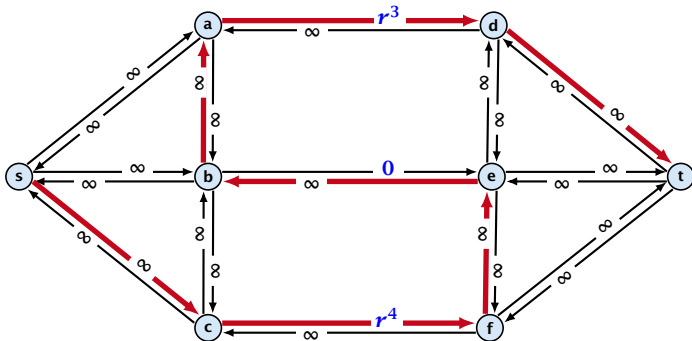
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flow value: $r^2 + r^3$

A Pathological Input

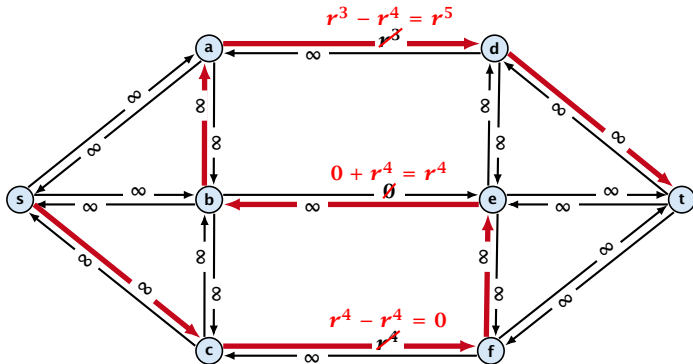
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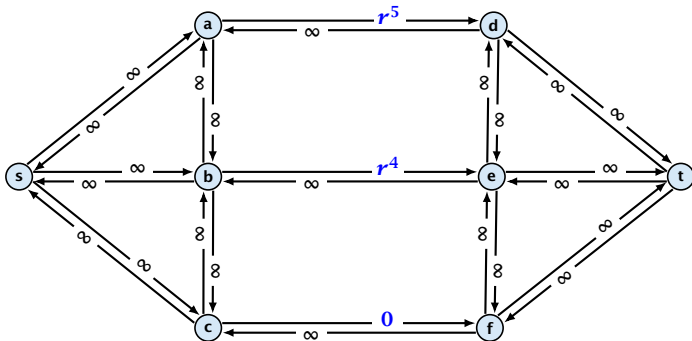
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flow value: $r^2 + r^3 + r^4$

Running time may be infinite!!!

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- ▶ Choose the shortest augmenting path.

Overview: Shortest Augmenting Paths

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Lemma 9

The length of the shortest augmenting path never decreases.

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After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

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Proof.

- ▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- ▶ $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.



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Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f (along non-zero edges).

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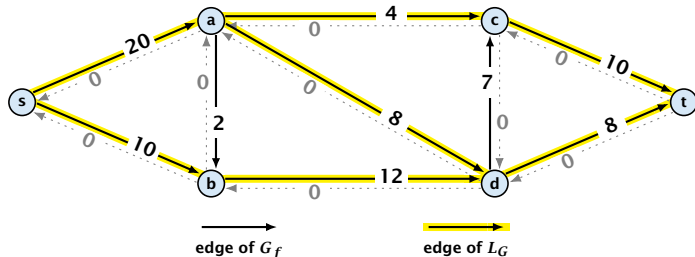
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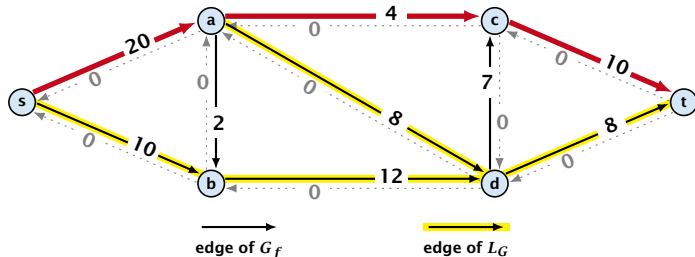


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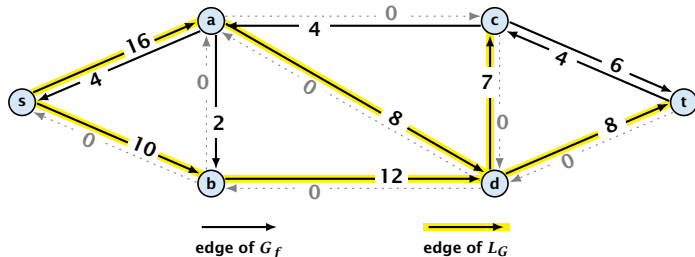


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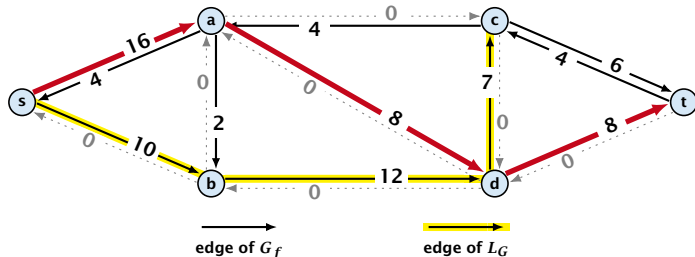


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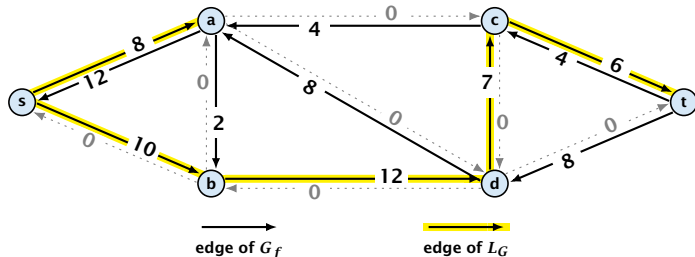


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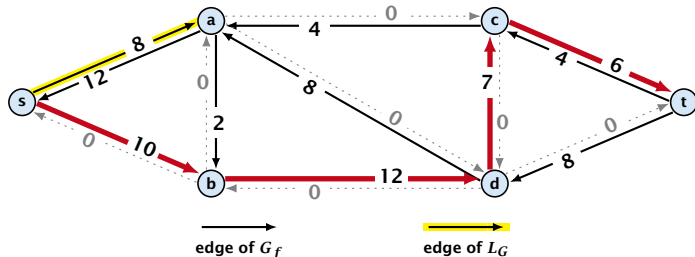


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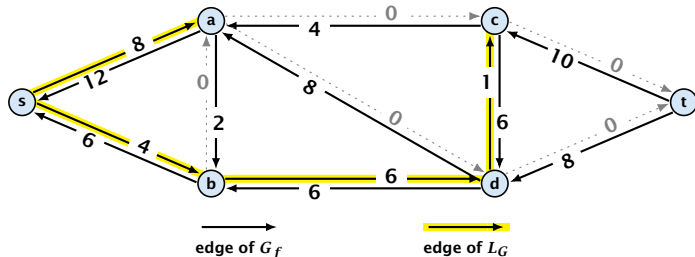


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In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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The length of the shortest augmenting path never decreases.

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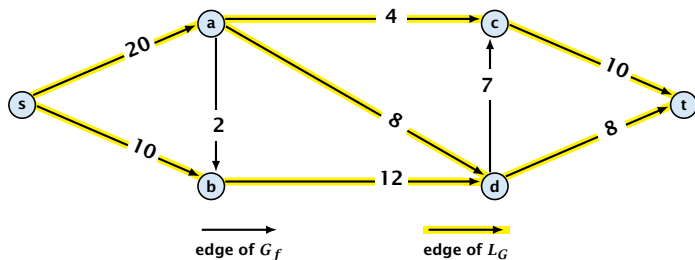
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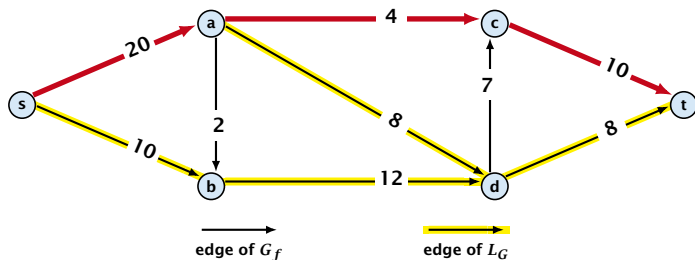
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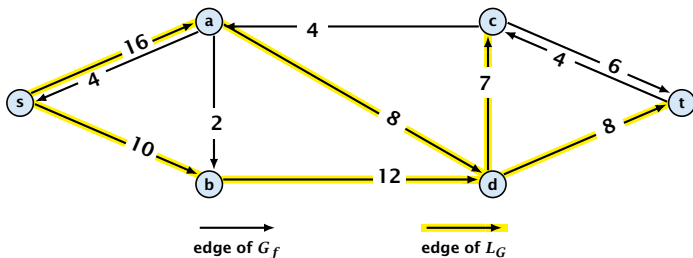
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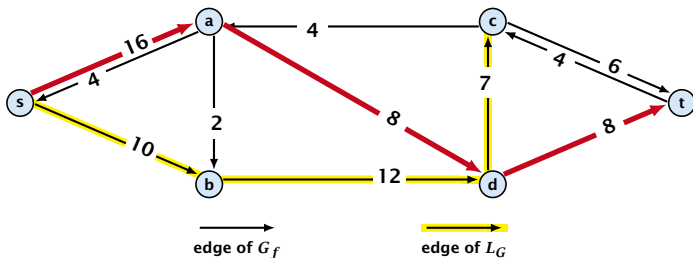
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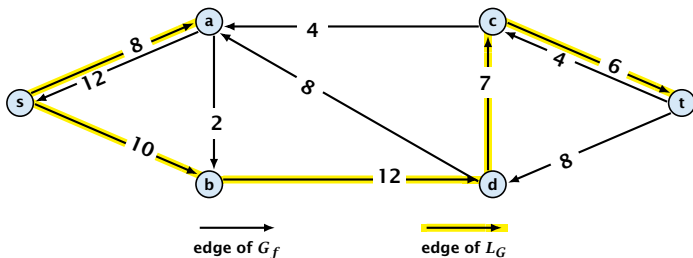
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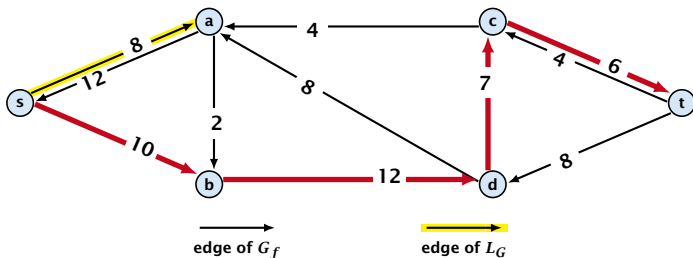
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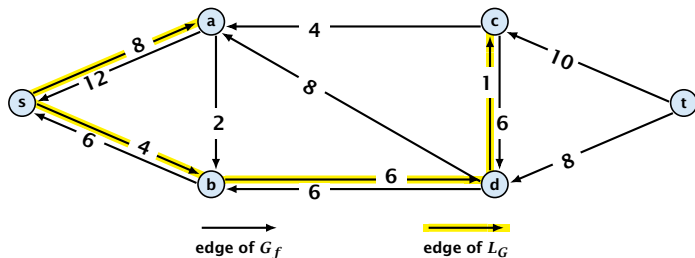
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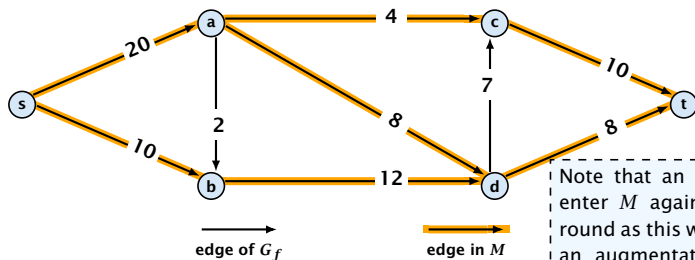
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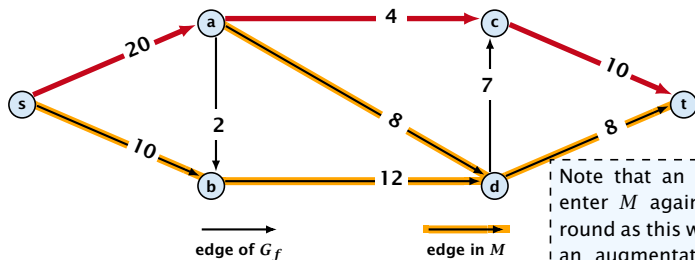
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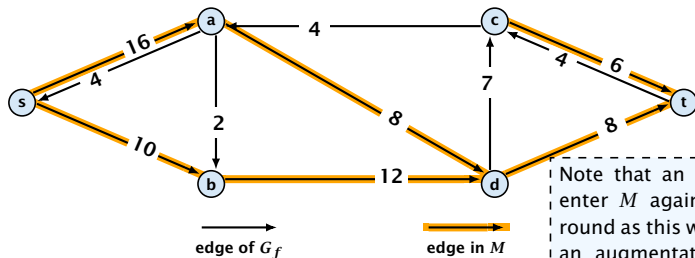
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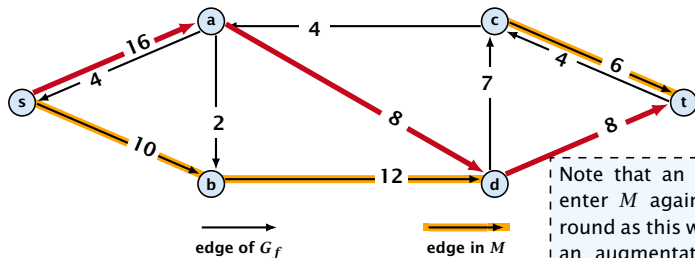
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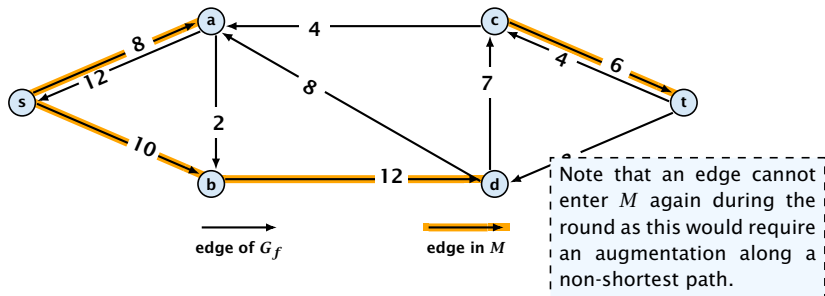
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Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

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When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

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However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

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We maintain a subset M of the edges of G_f with the guarantee that a shortest $s-t$ path using only edges from M is a shortest augmenting path.

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Note that M is not the set of edges of the level graph but a subset of level-graph edges.

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The total cost for performing an augmentation **during** a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in M for the next search.

Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing M for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in M and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation **during** a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in M for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

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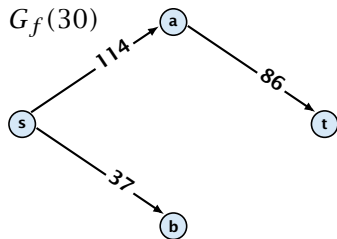
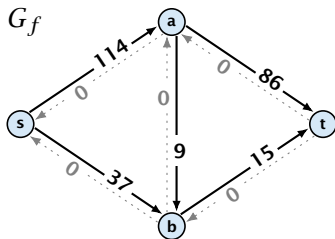
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Algorithm 1 $\text{maxflow}(G, s, t, c)$

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
```

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- ▶ this means we have a maximum flow.

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- ▶ There must exist an s - t cut in $G_f(\Delta)$ of zero capacity.
- ▶ In G_f this cut can have capacity at most $m\Delta$.
- ▶ This gives me an upper bound on the flow that I can still add.

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Theorem 17

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.