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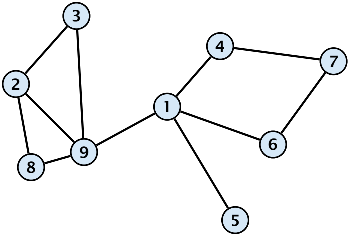

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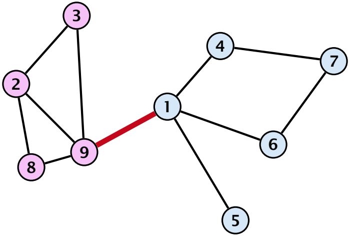

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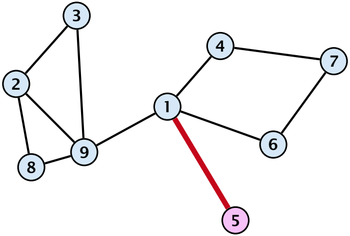

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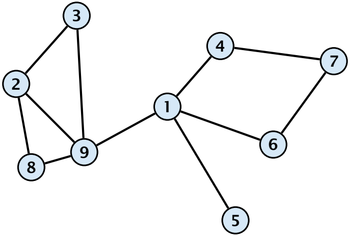

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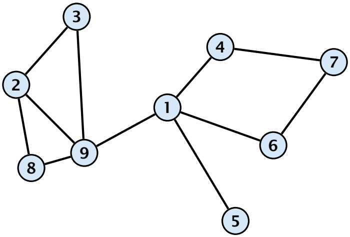

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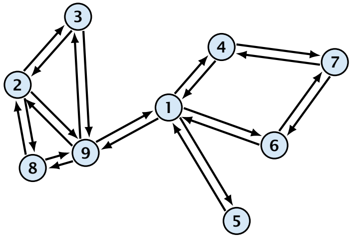

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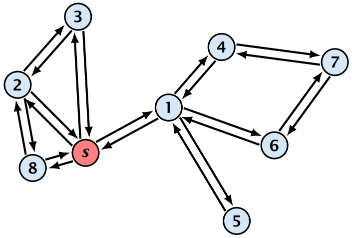

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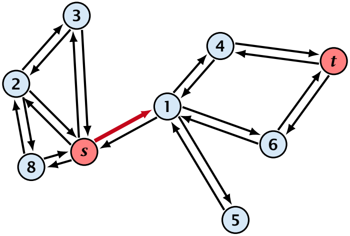

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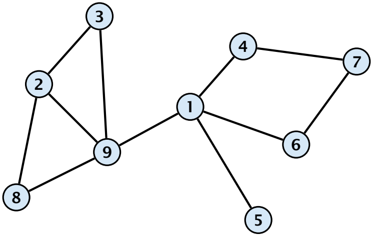

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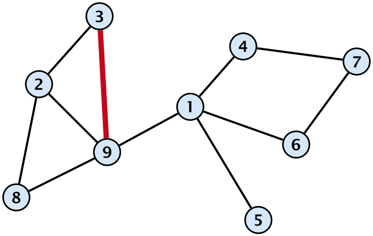

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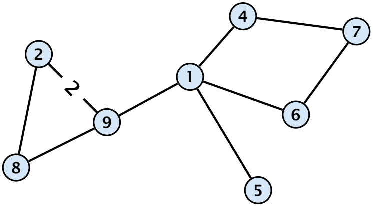

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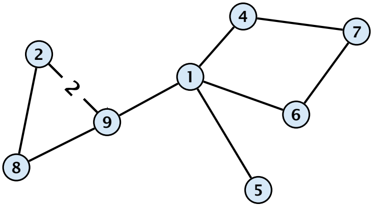

- Edge-contractions do no decrease the size of the mincut.

Edge Contractions

We can perform an edge-contraction in time $\mathcal{O}(n)$.

Randomized Mincut Algorithm

Algorithm 1 KargerMincut($G = (V, E, c)$)

for $i = 1 \to n - 2$ **do**
    choose $e \in E$ randomly with probability $c(e)/c(E)$
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- What is the probability that this algorithm returns a mincut?

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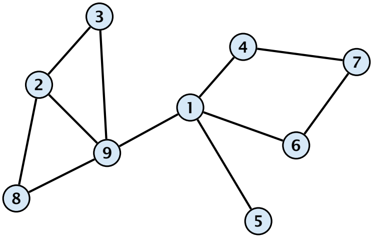

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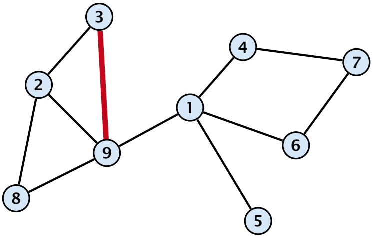

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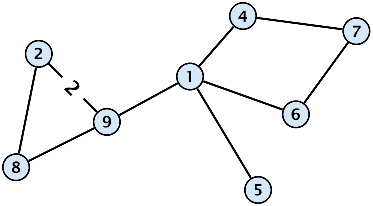

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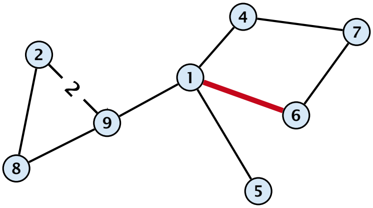

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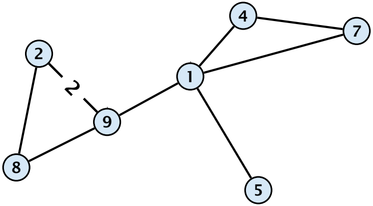

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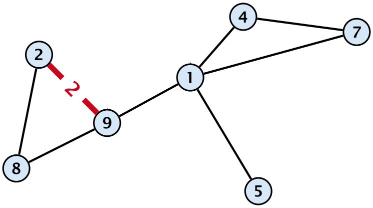

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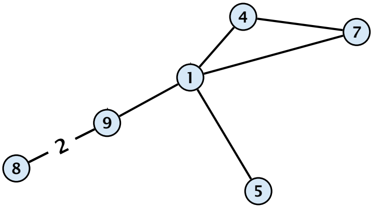

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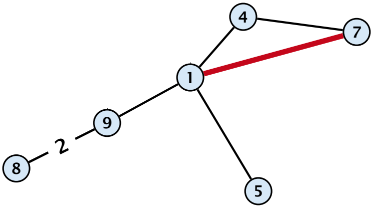

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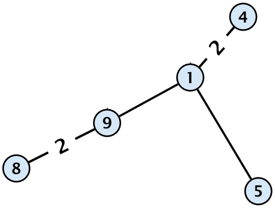

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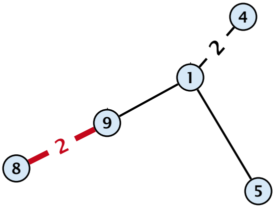

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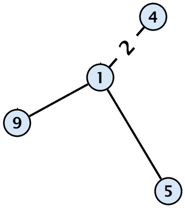

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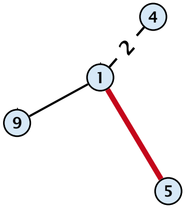

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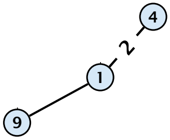

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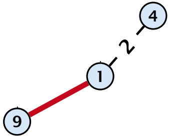

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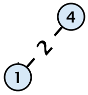

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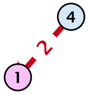

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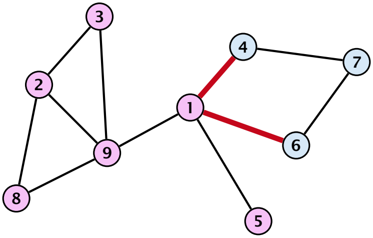

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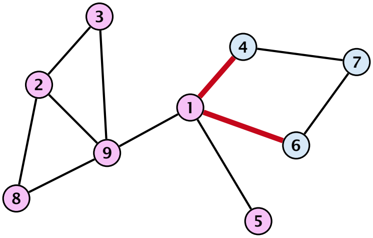

What is the probability that this algorithm returns a mincut?

Analysis

What is the probability that a given mincut A is still possible after round i?

- It is still possible to obtain cut A in the end if so far no edge in $(A, V \setminus A)$ has been contracted.

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- Hence, the probability of choosing an edge from the cut is at most $\min / c(E) \leq 2/(n - i + 1)$.

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$$\prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} = \frac{t(t-1)}{n(n-1)} \enspace .$$

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Choosing $t = 2$ gives that with probability $1/\binom{n}{2}$ the algorithm computes a mincut.

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Theorem 7

The randomized mincut algorithm computes an optimal cut with high probability. The total running time is $\mathcal{O}(n^4 \log n)$.

Improved Algorithm

Algorithm 2 RecursiveMincut$(G = (V, E, c))$

for $i = 1 \rightarrow n - n/\sqrt{2}$ **do**
  choose $e \in E$ randomly with probability $c(e)/c(E)$
  $G \leftarrow G/e$
if $|V| = 2$ **return** cut-value;
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- This gives $T(n) = \mathcal{O}(n^2 \log n)$.

Probability of Success

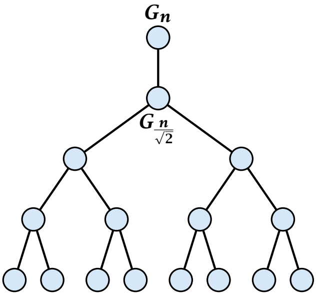
The probability of not contracting an edge from the mincut during one iteration through the for-loop is at least

$$\frac{t(t-1)}{n(n-1)} \geq \frac{t^2}{n^2} = \frac{1}{2} \ ,$$

as $t = \frac{n}{\sqrt{2}}$.

Probability of Success

recursion tree



size of rest graph

n

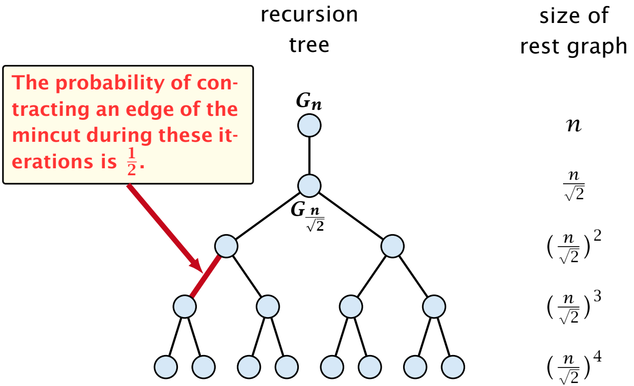
$\frac{n}{\sqrt{2}}$

$\left(\frac{n}{\sqrt{2}}\right)^2$

$\left(\frac{n}{\sqrt{2}}\right)^3$

$\left(\frac{n}{\sqrt{2}}\right)^4$

Probability of Success



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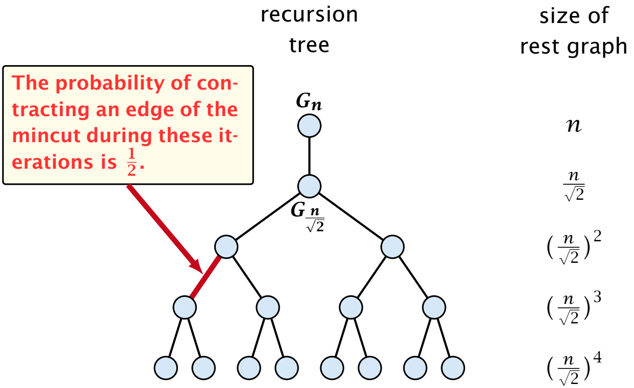

We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability $\frac{1}{2}$. If in the end you have a path from the root to at least one leaf node you are successful.

Probability of Success

Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

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Lemma 8

The probability that an edge e is alive is at least $\frac{1}{h(e)+1}$.

Probability of Success

Proof.

- An edge e with $h(e) = 1$ is alive if and only if it is not deleted. Hence, it is alive with proability at least $\frac{1}{2}$.

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- Let p_d be the probability that an edge e with $h(e) = d$ is alive. For $d > 1$ this happens for edge $e = \{c, p\}$ if it is not deleted **and** if one of the child-edges connecting to c is alive.

Probability of Success

Proof.

- An edge e with $h(e) = 1$ is alive if and only if it is not deleted. Hence, it is alive with proability at least $\frac{1}{2}$.
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$$p_d = \frac{1}{2}\left(2p_{d-1} - p_{d-1}^2\right) \quad \boxed{\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]}$$

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Lemma 9

One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ *and has a success probability of* $\Omega(\frac{1}{\log n})$.

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Doing $\Theta(\log^2 n)$ *runs gives that the algorithm succeeds with high probability. The total running time is* $\mathcal{O}(n^2 \log^3 n)$.