Mincost Flow

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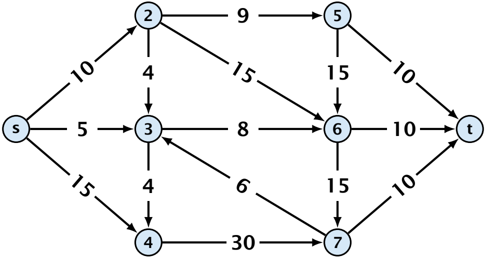
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow



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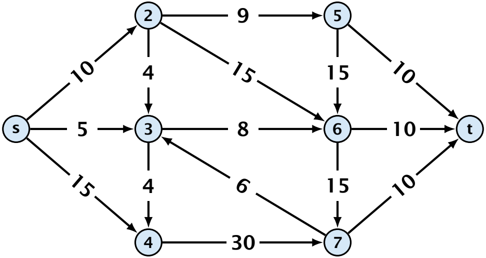

- Given a flow network for a standard maxflow problem.

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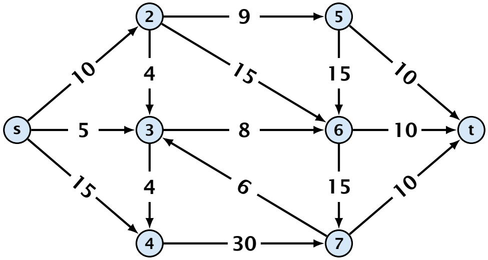

- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

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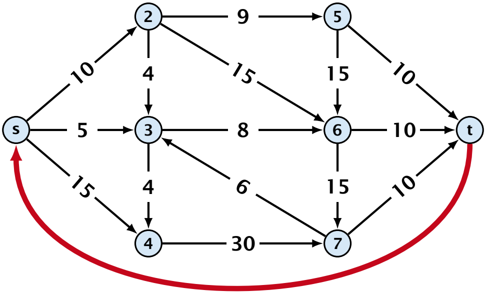

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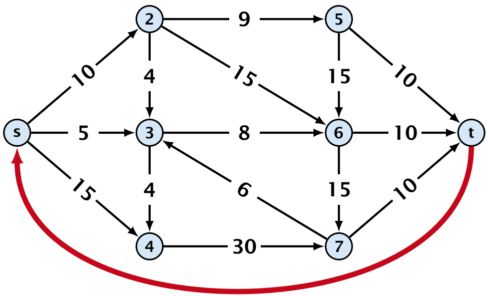

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- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

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Solve decision version of maxflow:

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- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value at least k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E:\;\; 0 \le f(e) \le u(e) \\ & \forall v \in V:\;\; f(v) = b(v) \end{array}$$

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A more general model?

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E:\ \ \ell(e) \le f(e) \le u(e) \\ & \forall v \in V:\ \ a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$
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Generalization

Differences

- Flow along an edge e may have non-zero lower bound $\ell(e)$.
- Flow along e may have negative upper bound $u(e)$.
- The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

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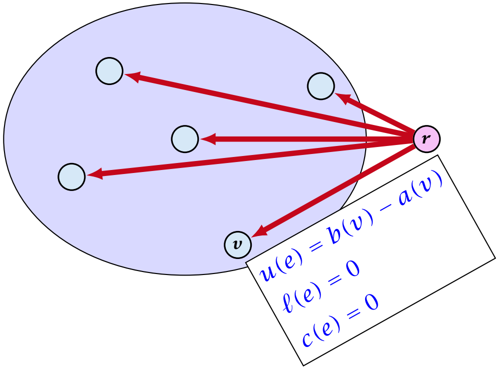

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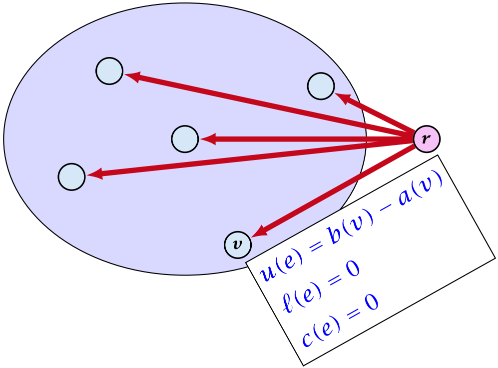

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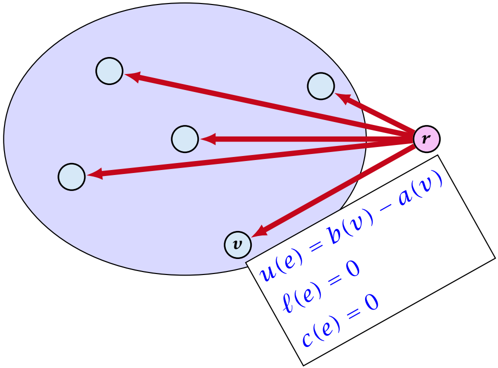

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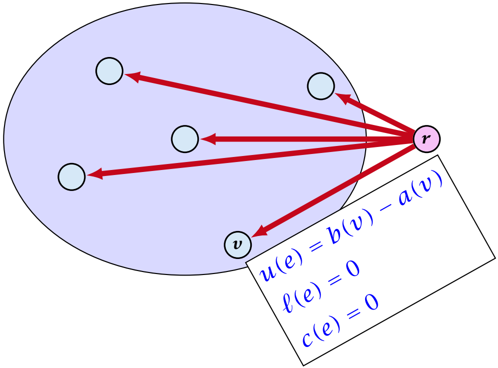

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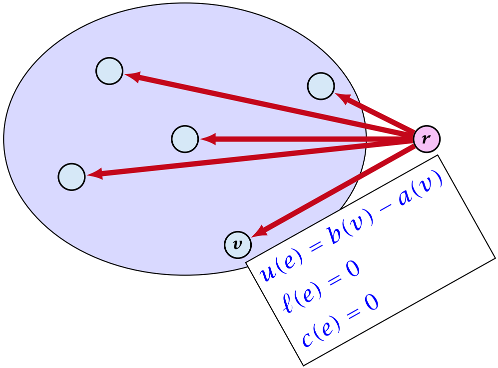

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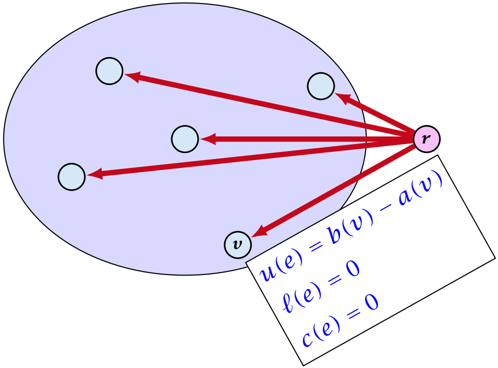

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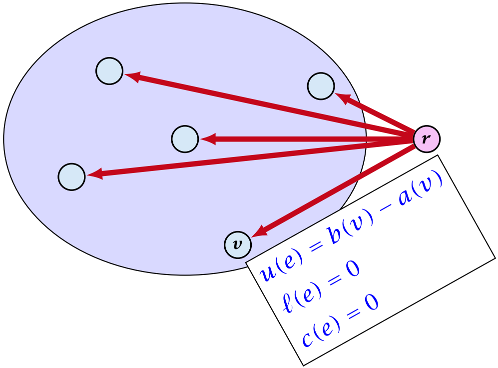

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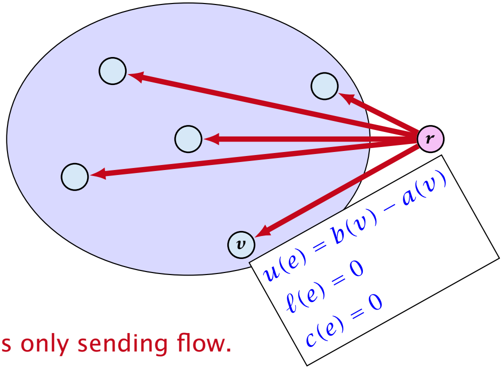
$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction II

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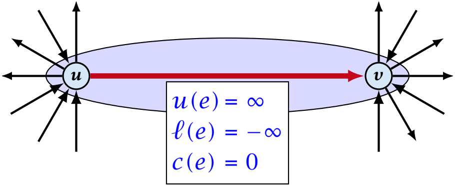

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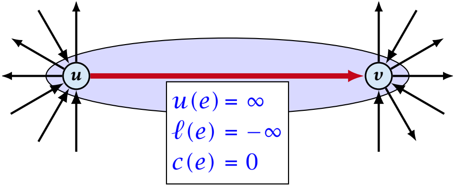

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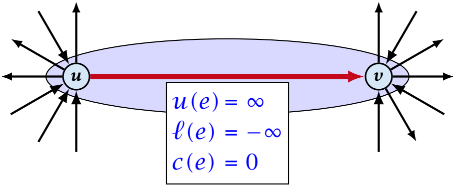

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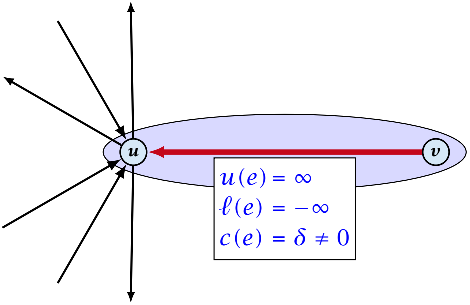
If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:



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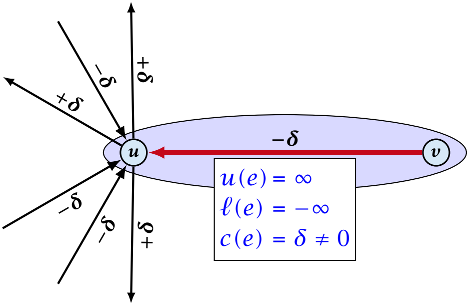

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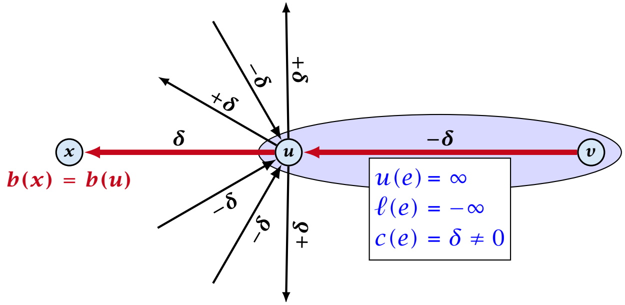

Additionally we set $b(u) = 0$.

Reduction III

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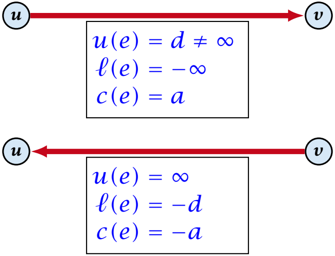

Replace the edge by an edge in opposite direction.

Reduction IV

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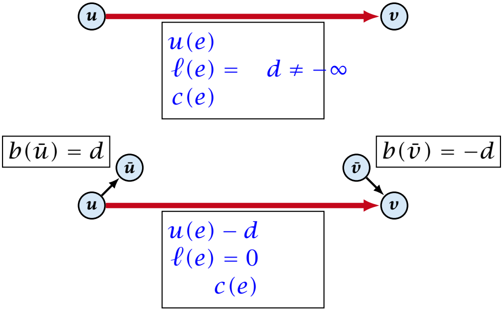

The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.

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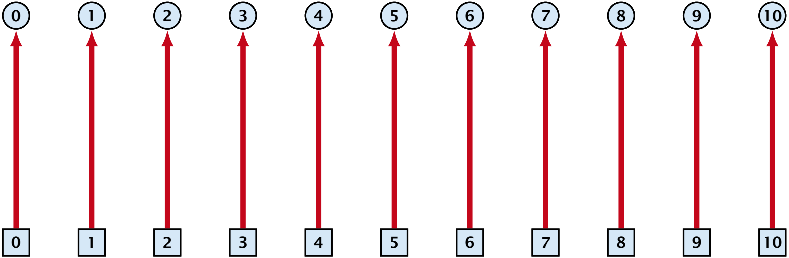
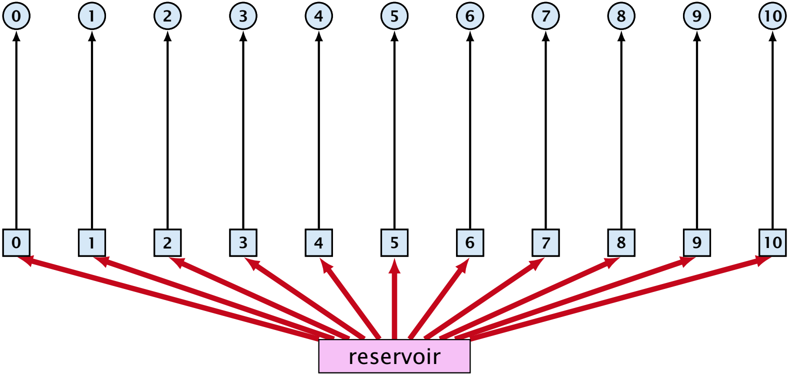
- Minimize cost.



day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



buy edges:

upper bound: $u(e_i) = \infty$;
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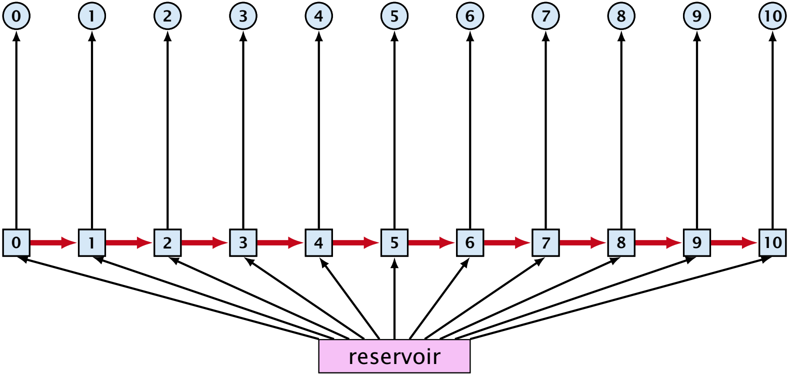
cost: $c(e) = p$



forward edges:

upper bound: $u(e_i) = \infty$;
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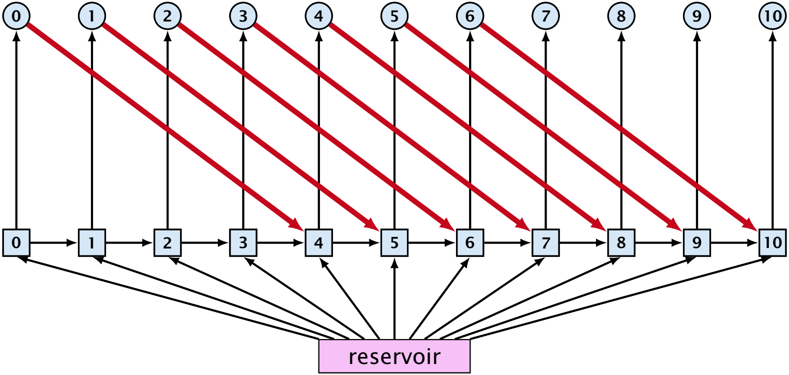

slow edges:

upper bound: $u(e_i) = \infty$;
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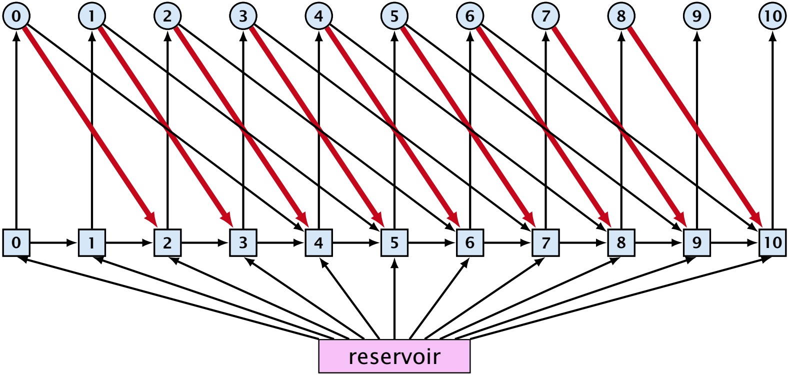
cost: $c(e) = s$



fast edges:

upper bound: $u(e_i) = \infty$;
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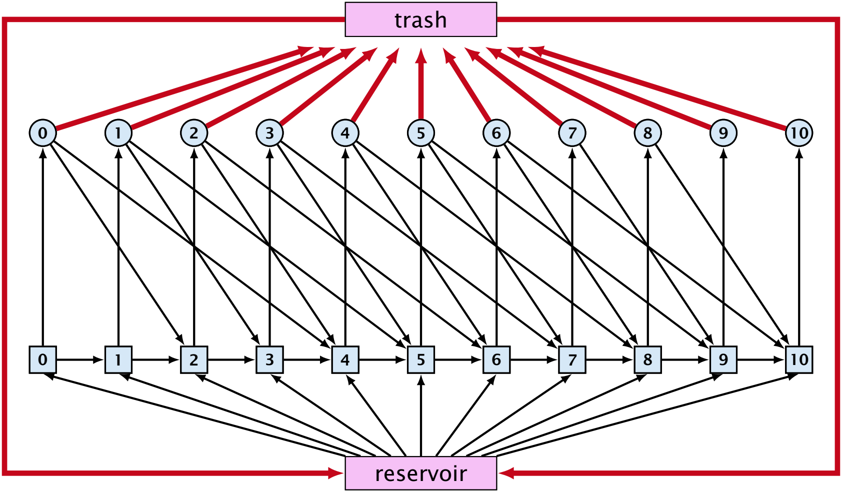
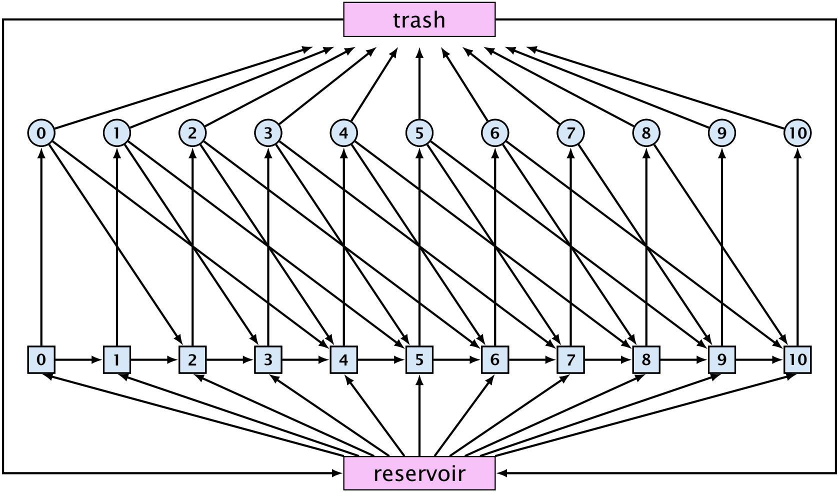
cost: $c(e) = f$



trash edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$


trash
0
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5
6
7
8
9
10
reservoir

Residual Graph

Version A:
The residual graph G' for a mincost flow is just a copy of the graph G.

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

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Version A:
The residual graph G' for a mincost flow is just a copy of the graph G.

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

Version B:
The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

14 Mincost Flow

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A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \le u(e)$ for every edge of G.

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  Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

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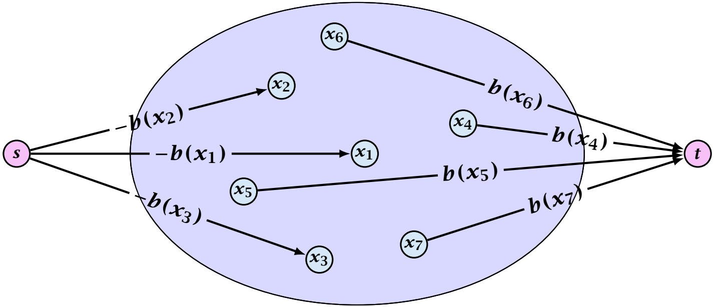
- You still have a circulation with negative cost.
- Repeat.

14 Mincost Flow

Algorithm 23 CycleCanceling$(G = (V, E), c, u, b)$

establish a feasible flow f in G
while G_f contains negative cycle **do**
    use Bellman-Ford to find a negative circuit Z
    $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
    augment δ units along Z and update G_f

How do we find the initial feasible flow?



- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \ .$$

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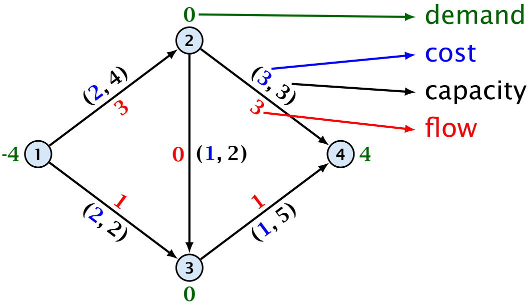

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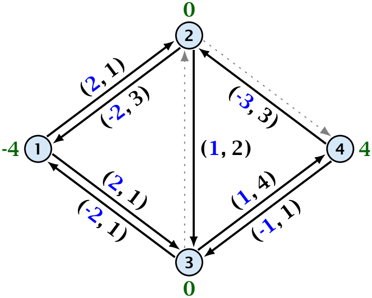

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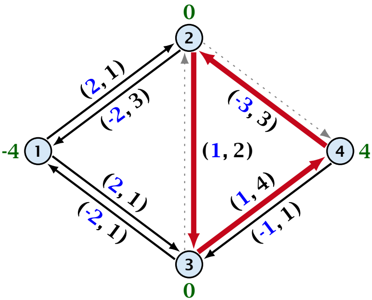

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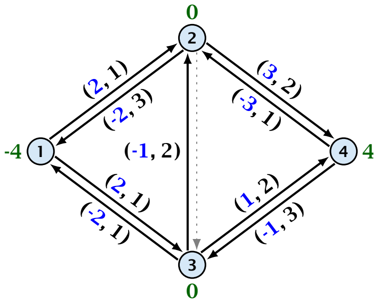

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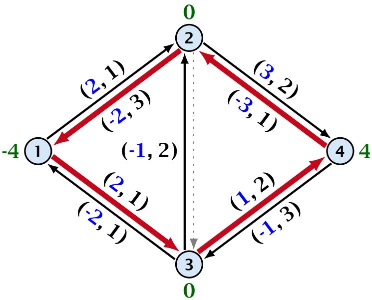

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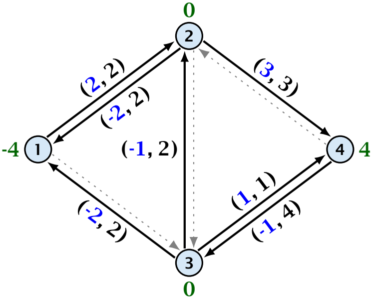

Lemma 8

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, *for integer capacities and costs, when for all edges* e, $|c(e)| \le C$ *and* $|u(e)| \le U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \ldots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

14 Mincost Flow

A general mincost flow problem is of the following form:

$$
\begin{array}{ll}
\text{min} & \sum_e c(e)f(e) \\
\text{s.t.} & \forall e \in E:\ \ \ell(e) \le f(e) \le u(e) \\
 & \forall v \in V:\ \ a(v) \le f(v) \le b(v)
\end{array}
$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$ $c : E \to \mathbb{R}$;

Lemma 9 (without proof)

A general mincost flow problem can be solved in polynomial time.