

## 4 Modelling Issues

### What do you measure?

- ▶ Memory requirement
- ▶ Running time
- ▶ Number of comparisons
- ▶ Number of multiplications
- ▶ Number of hard-disc accesses
- ▶ Program size
- ▶ Power consumption
- ▶ ...

## 4 Modelling Issues

### How do you measure?

- ▶ Implementing and testing on representative inputs
  - ▶ How do you choose your inputs?
  - ▶ May be very time-consuming.
  - ▶ Very reliable results if done correctly.
  - ▶ Results only hold for a specific machine and for a specific set of inputs.
  
- ▶ Theoretical analysis in a specific **model of computation**.
  - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time  $\mathcal{O}(n^2)$ ”.
  - ▶ Typically focuses on the **worst case**.
  - ▶ Can give lower bounds like “any comparison-based sorting algorithm needs at least  $\Omega(n \log n)$  comparisons in the worst case”.

## 4 Modelling Issues

### Input length

The theoretical bounds are usually given by a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  that maps the **input length** to the running time (or storage space, comparisons, multiplications, program size etc.).

The **input length** may e.g. be

- ▶ the size of the input (number of bits)
- ▶ the number of arguments

### Example 1

Suppose  $n$  numbers from the interval  $\{1, \dots, N\}$  have to be sorted. In this case we usually say that the input length is  $n$  instead of e.g.  $n \log N$ , which would be the number of bits required to encode the input.

## How to measure performance

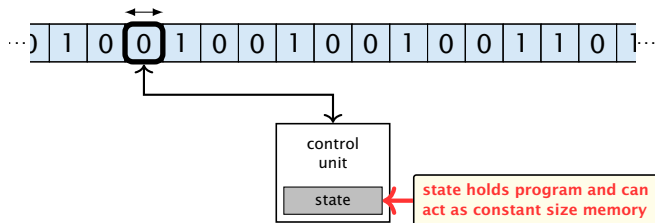
1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

# Turing Machine

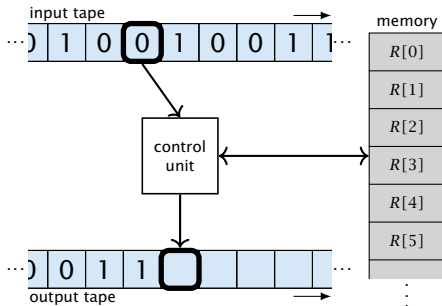
- ▶ Very simple model of computation.
- ▶ Only the “current” memory location can be altered.
- ▶ Very good model for discussing computability, or polynomial vs. exponential time.
- ▶ Some simple problems like recognizing whether input is of the form  $xx$ , where  $x$  is a string, have quadratic lower bound.

⇒ **Not a good model for developing efficient algorithms.**



# Random Access Machine (RAM)

- ▶ Input tape and output tape (sequences of zeros and ones; unbounded length).
- ▶ Memory unit: infinite but countable number of registers  $R[0], R[1], R[2], \dots$
- ▶ Registers hold integers.
- ▶ Indirect addressing.



Note that in the picture on the right the tapes are one-directional, and that a READ- or WRITE-operation always advances its tape.

# Random Access Machine (RAM)

## Operations

- ▶ input operations (input tape  $\rightarrow R[i]$ )
  - ▶ READ  $i$
- ▶ output operations ( $R[i] \rightarrow$  output tape)
  - ▶ WRITE  $i$
- ▶ register-register transfers
  - ▶  $R[j] := R[i]$
  - ▶  $R[j] := 4$
- ▶ **indirect** addressing
  - ▶  $R[j] := R[R[i]]$   
loads the content of the  $R[i]$ -th register into the  $j$ -th register
  - ▶  $R[R[i]] := R[j]$   
loads the content of the  $j$ -th into the  $R[i]$ -th register

# Random Access Machine (RAM)

## Operations

- ▶ branching (including loops) based on comparisons
  - ▶ jump  $x$   
jumps to position  $x$  in the program;  
sets instruction counter to  $x$ ;  
reads the next operation to perform from register  $R[x]$
  - ▶ jumpz  $x R[i]$   
jump to  $x$  if  $R[i] = 0$   
if not the instruction counter is increased by 1;
  - ▶ jumpi  $i$   
jump to  $R[i]$  (indirect jump);
- ▶ arithmetic instructions:  $+$ ,  $-$ ,  $\times$ ,  $/$ 
  - ▶  $R[i] := R[j] + R[k];$   
 $R[i] := -R[k];$

The jump-directives are very close to the jump-instructions contained in the assembler language of real machines.



# Model of Computation

- ▶ **uniform** cost model  
Every operation takes time 1.
- ▶ **logarithmic** cost model  
The cost depends on the content of memory cells:
  - ▶ The time for a step is equal to the largest operand involved;
  - ▶ The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

**Bounded word RAM model:** cost is uniform but the largest value stored in a register may not exceed  $2^w$ , where usually  $w = \log_2 n$ .

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size  $n$  must be at least  $\log_2 n$  as otherwise the computer could either not store the problem instance or not address all its memory.

# 4 Modelling Issues

## Example 2

### Algorithm 1 RepeatedSquaring( $n$ )

```
1:  $r \leftarrow 2$ ;  
2: for  $i = 1 \rightarrow n$  do  
3:    $r \leftarrow r^2$   
4: return  $r$ 
```

- ▶ running time (for Line 3):
  - ▶ uniform model:  $n$  steps
  - ▶ logarithmic model:  
 $2 + 3 + 5 + \dots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$
- ▶ space requirement:
  - ▶ uniform model:  $\mathcal{O}(1)$
  - ▶ logarithmic model:  $\mathcal{O}(2^n)$

There are **different types of complexity bounds**:

- ▶ **best-case** complexity:

$$C_{bc}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

- ▶ **worst-case** complexity:

$$C_{wc}(n) := \max\{C(x) \mid |x| = n\}$$

Usually moderately easy to analyze; sometimes too pessimistic.

- ▶ **average case** complexity:

$$C_{avg}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x)$$

more general: probability measure  $\mu$

$\mu$  is a probability distribution over inputs of length  $n$ .

$$C_{avg}(n) := \sum_{x \in I_n} \mu(x) \cdot C(x)$$

|        |                                |
|--------|--------------------------------|
| $C(x)$ | cost of instance $x$           |
| $ x $  | input length of instance $x$   |
| $I_n$  | set of instances of length $n$ |

There are **different types of complexity bounds**:

▶ **amortized** complexity:

The average cost of data structure operations over a worst case sequence of operations.

▶ **randomized** complexity:

The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input  $x$ . Then take the worst-case over all  $x$  with  $|x| = n$ .

$\mu$  is a probability distribution over inputs of length  $n$ .

|        |                                |
|--------|--------------------------------|
| $C(x)$ | cost of instance $x$           |
| $ x $  | input length of instance $x$   |
| $I_n$  | set of instances of length $n$ |

# 4 Modelling Issues

## Bibliography

- [MS08] Kurt Mehlhorn, Peter Sanders:  
*Algorithms and Data Structures — The Basic Toolbox*,  
Springer, 2008
- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
*Introduction to algorithms (3rd ed.)*,  
McGraw-Hill, 2009

Chapter 2.1 and 2.2 of [MS08] and Chapter 2 of [CLRS90] are relevant for this section.