6 Recurrences

Algorithm 2 mergesort(list L)

$n \leftarrow \text{size}(L)$
if $n \leq 1$ **return** L
$L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
$L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
$\text{mergesort}(L_1)$
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$L \leftarrow \text{merge}(L_1, L_2)$
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This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

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How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

For this we need to solve the recurrence.

Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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Informal way:

Assume that instead we have

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Informal way:
Assume that instead we have

$$T(n) \le \begin{cases} 2T(\frac{n}{2}) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d.

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\begin{aligned}
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Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

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  Hence, statement is true if we choose $d \geq c$.

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We consider the following recurrence instead of the original one:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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We also make a guess of $T(n) \leq dn \log n$ and get

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$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1}$

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for a suitable choice of d.

6.2 Master Theorem

Lemma 1

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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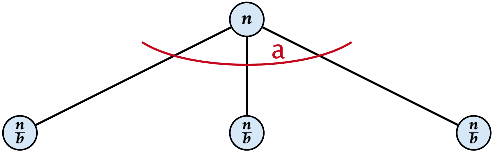

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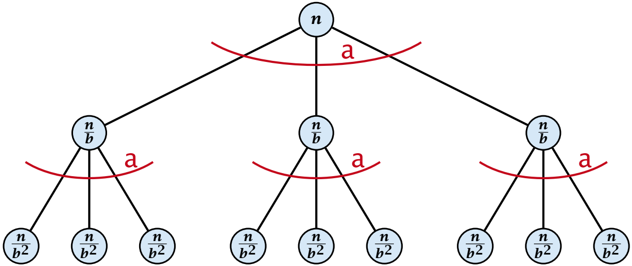

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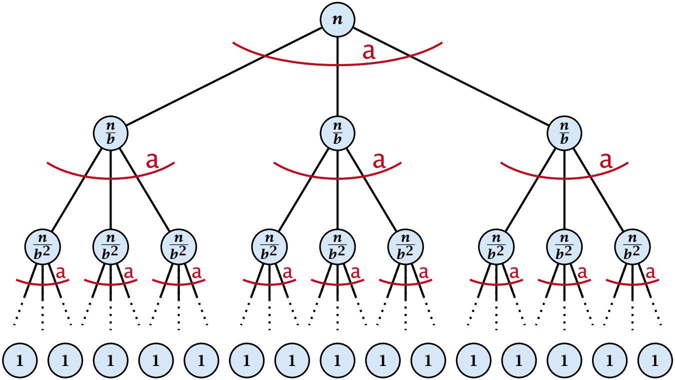

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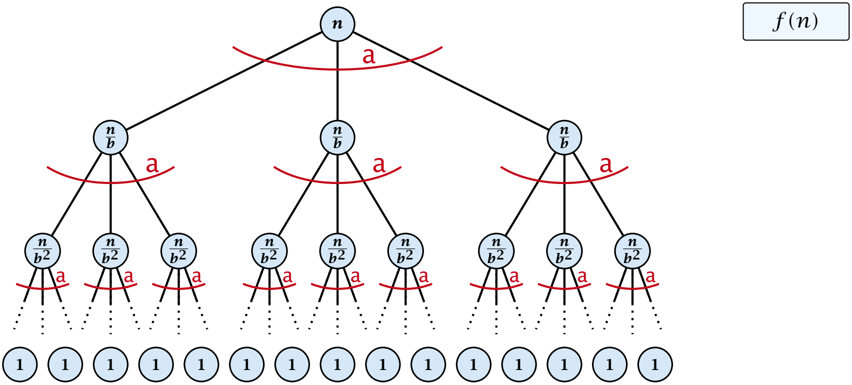

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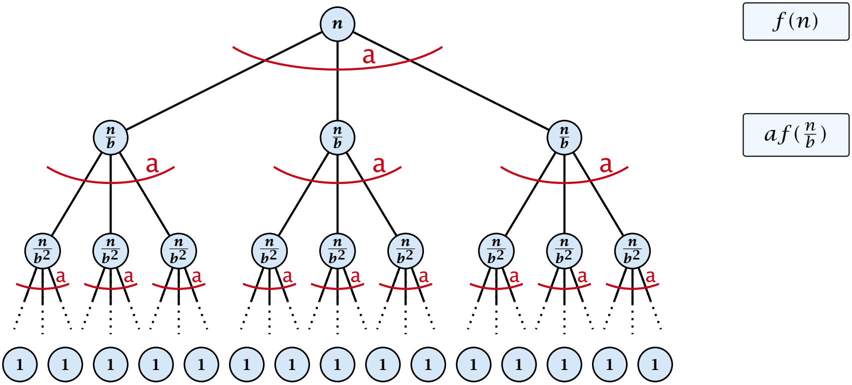

The Recursion Tree

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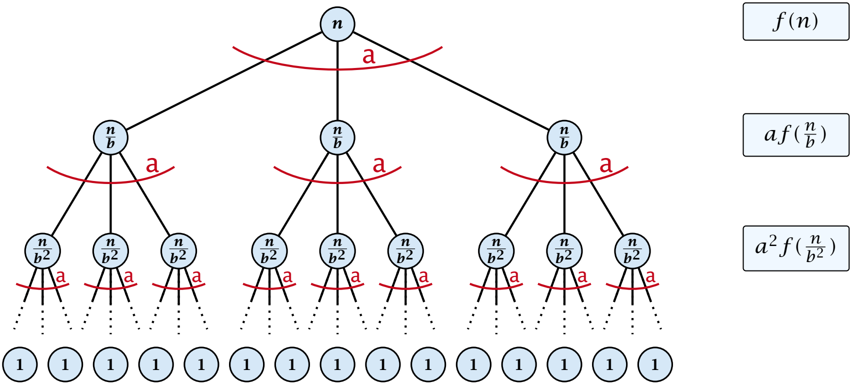

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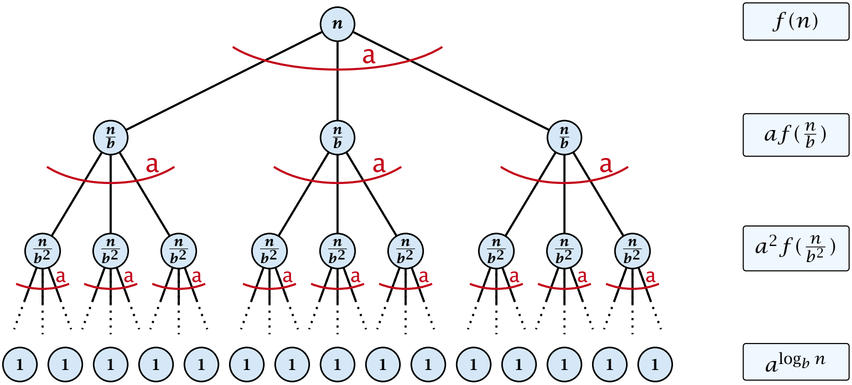

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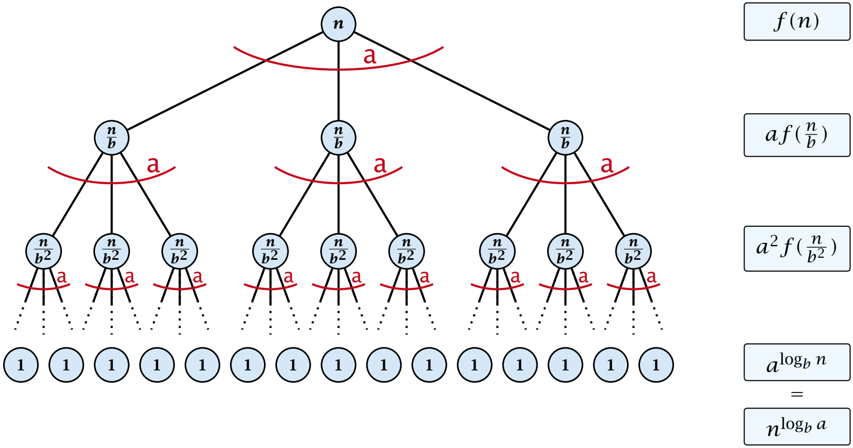

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \le c n^{\log_b a - \epsilon}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}
\end{aligned}
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$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}}$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad = cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i
\end{aligned}
$$

$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$

$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}}$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\ \boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\ \boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1) \end{aligned}$$

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\end{aligned}
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$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$

$$
= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i
$$

$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}}$

$$
\begin{aligned}
&= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^\epsilon - 1) \\
&= cn^{\log_b a - \epsilon} (n^\epsilon - 1) / (b^\epsilon - 1)
\end{aligned}
$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a-\epsilon}$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
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\boxed{b^{-i(\log_b a-\epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} \quad &= cn^{\log_b a-\epsilon} \sum_{i=0}^{\log_b n-1} (b^\epsilon)^i \\
\boxed{\textstyle\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a-\epsilon}(b^{\epsilon\log_b n} - 1)/(b^\epsilon - 1) \\
&= cn^{\log_b a-\epsilon}(n^\epsilon - 1)/(b^\epsilon - 1) \\
&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}$$

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&= cn^{\log_b a-\epsilon}(n^\epsilon - 1)/(b^\epsilon - 1) \\
&= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)
\end{aligned}
$$

Hence,

$$
T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)}
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\end{aligned}
$$

Hence,

$$
T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b (a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}
$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

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&= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1 \\
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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \mathbf{\Omega}(n^{\log_b a} \log_b n)$$

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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k
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Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

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&= cn^{\log_b a} \sum_{i=0}^{\ell - 1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \qquad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \tfrac{1}{k}\ell^{k+1}
\end{aligned}
$$

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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \qquad &= c n^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= c n^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\
&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}
$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

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\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})
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\end{aligned}
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$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$

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\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}} \quad \leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$$

Hence,

$$T(n) \leq \mathcal{O}(f(n)) \qquad \boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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$$
\begin{array}{ccccccccc|c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \boxed{1} & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \boxed{1} & B \\
\hline
\end{array}
$$

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & {}_{1}\,1 & & B \\
\hline
 & & & & & & & & 0 & &
\end{array}$$

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1 & 0 & 0 & 0 & 1 & 0 & 0_{1} & 1_{1} & 1 & & B \\
\hline
 & & & & & & & 0 & 0 & &
\end{array}
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$$\begin{array}{cccccccccc} 1&1&0&1&1&0&1&0&1&\quad A\\ 1&0&0&0&1&0&0_1&1_1&1&\quad B\\ \hline &&&&&&&0&0& \end{array}$$

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1&1&0&1&1&0&1&0&1&\quad& A\\
1&0&0&0&1&0&0&1&1&& B\\
 & & & & &{}_1&{}_1&{}_1& && \\
\hline
 & & & & & &0&0&0&&
\end{array}
$$

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1 & 0 & 0 & 0 & 1 & 0_{1} & 0_{1} & 1_{1} & 1 & & B \\
\hline
 & & & & & \boxed{} & 0 & 0 & 0 & &
\end{array}$$

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\begin{array}{ccccccccccc}
1 & 1 & 0 & 1 & 1 & \boxed{0} & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & & & \boxed{1} & 0 & 0 & 0 & &
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & & {}_{0} & {}_{1} & {}_{1} & {}_{1} & & & \\
\hline
 & & & & & 1 & 0 & 0 & 0 & &
\end{array}
$$

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & & 0 & 1 & 0 & 0 & 0 & &
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 & & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & {}_1 & {}_1 & {}_0 & {}_1 & {}_1 & {}_1 & & & \\
\hline
 & & & \boxed{0} & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \quad & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & \quad & B \\
 & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}$$

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
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$$

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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For this we first need to be able to add two integers A and B:

$$
\begin{array}{cccccccccc}
\boxed{1} & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
{}_0 & {}_0 & {}_1 & {}_1 & {}_0 & {}_1 & {}_1 & {}_1 & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}
$$

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\begin{array}{cccccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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For this we first need to be able to add two integers A and B:

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

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\begin{array}{cccccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
\boxed{1} & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
1	0	1	1	0	0	1	0	0	0	

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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$$\begin{array}{c} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \end{array}$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\begin{array}{r} 1\;0\;0\;0\;1 \times 1\;0\;1\;\boxed{1} \\ \hline \end{array}$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & \boxed{1} \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
$$

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\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
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\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & 0
\end{array}
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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0
\end{array}
$$

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$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{r}
1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\
\hline
1\ 0\ 0\ 0\ 1 \\
1\ 0\ 0\ 0\ 1\ 0 \\
0\ 0\ 0\ 0\ 0\ 0\ 0 \\
1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\
\hline
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrr}
 & & 1 & 0 & 0 & 0 & 1 & \times \; 1 \; 0 \; 1 \; 1 \\
\hline
 & & & & 1 & 0 & 0 & 0 \; 1 \\
 & & & 1 & 0 & 0 & 0 & 1 \; 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 \; 0 \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \; 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \; 1
\end{array}
$$

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.

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$$
\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

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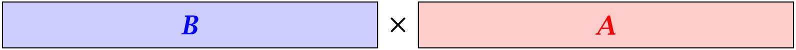

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{b_{n-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_0}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \;\; b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \;\; a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{B_1 \quad\mid\quad B_0} \times \boxed{A_1 \quad\mid\quad A_0}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

B_1	B_0	$\times$	A_1	A_0

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $|A| = |B| = 1$ **then**
    return $a_0 \cdot b_0$
split A into A_0 and A_1
split B into B_0 and B_1
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
$Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $|A| = |B| = 1$ **then** $\quad \mathcal{O}(1)$
  return $a_0 \cdot b_0$
split A into A_0 and A_1
split B into B_0 and B_1
$Z_2 \leftarrow \text{mult}(A_1, B_1)$
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Example: Multiplying Two Integers

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We get the following recurrence:

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Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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$\Longrightarrow$ Not better then the "school method".

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A huge improvement over the “school method”.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

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Note that we ignore boundary conditions for the moment.

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- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

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- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

6.3 The Characteristic Polynomial

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Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \ldots \;\middle|\; \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a vector space.

The Homogenous Case

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

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This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i.

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Lemma 2
Assume that the characteristic polynomial has k distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form

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Proof.

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i' s$ such that these conditions are met:

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$$\alpha_1 \cdot \lambda_1 \quad + \quad \alpha_2 \cdot \lambda_2 \quad + \quad \cdots \quad + \quad \alpha_k \cdot \lambda_k \quad = \quad T[1]$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\begin{array}{ccccccccc} \alpha_1 \cdot \lambda_1 & + & \alpha_2 \cdot \lambda_2 & + & \cdots & + & \alpha_k \cdot \lambda_k & = & T[1] \\ \alpha_1 \cdot \lambda_1^2 & + & \alpha_2 \cdot \lambda_2^2 & + & \cdots & + & \alpha_k \cdot \lambda_k^2 & = & T[2] \end{array}$$

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Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^{k} (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

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Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

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Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

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Hence,

$$c_0 \underbrace{n\lambda_i^{n}}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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Doing this again gives

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We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

The Homogeneous Case

Lemma 3
Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

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The characteristic polynomial is

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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Hence, the solution is of the form

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)^n+\beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

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$T[0] = 0$ gives $\alpha + \beta = 0$.

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Hence, the solution is of the form

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1\Longrightarrow\alpha-\beta=\frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \quad ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

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The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

The Inhomogeneous Case

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

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and so on...

6.4 Generating Functions

Definition 4 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

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$$F(z) := \sum_{n\geq 0} a_n z^n \;;$$

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- exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} \frac{a_n}{n!} z^n \ .$$

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There are no convergence issues here.

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We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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Then, it is important to think about convergence/convergence radius etc.

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It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

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This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

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$$\begin{aligned}\sum_{n\geq k} n(n-1)\cdot\ldots\cdot(n-k+1)z^{n-k} &= \sum_{n\geq 0}(n+k)\cdot\ldots\cdot(n+1)z^n \\ &= \frac{k!}{(1-z)^{k+1}}\ .\end{aligned}$$

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Hence:

$$\sum_{n\ge 0}\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}}\ .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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&= \frac{z}{(1-z)^2}
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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

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&= z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} z^n \\
&= zA(z) + \sum_{n\geq 0} z^n \\
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Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

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n^2	$\frac{z(1+z)}{(1-z)^3}$
$\frac{1}{n!}$	e^z

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$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$

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$f_{n-k} \quad (n \geq k); \quad 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\frac{F(z)}{1-z}$

Some Generating Functions

n-th sequence element	*generating function*
cf_n	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
f_{n-k} $(n \geq k)$; 0 otw.	$z^k F$
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6. The coefficients of the resulting power series are the a_n.

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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
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$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

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$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \end{aligned}$$

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   This leads to the following conditions:

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This leads to the following conditions:

$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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$$= \frac{7}{4} \cdot \sum_{n\geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\geq 0} z^n - \frac{1}{2} \cdot \sum_{n\geq 0} (n+1) z^n$$

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6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 6

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= 2 \\
f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.
\end{aligned}
$$

6.5 Transformation of the Recurrence

Example 6

$$f_0 = 1$$
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f_n &= 2^{F_n}
\end{aligned}
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Example 7

$$f_1 = 1$$

$$f_n = 3f_{\frac{n}{2}} + n\text{; for } n = 2^k,\ k \geq 1\ ;$$

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Then:

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6 Recurrences

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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k
\end{aligned}
$$

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We get

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\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i
\end{aligned}
$$

6 Recurrences

We get

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&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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