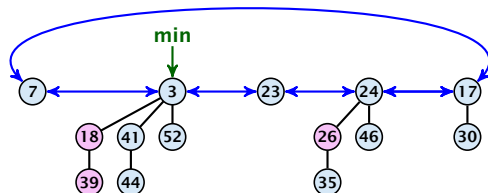


## 8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



## 8.3 Fibonacci Heaps

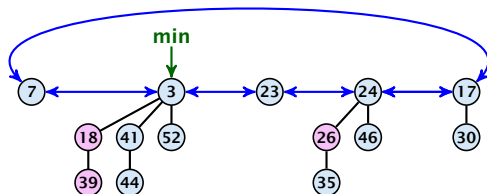
### Additional implementation details:

- ▶ Every node  $x$  stores its degree in a field  $x.degree$ . Note that this can be updated in constant time when adding a child to  $x$ .
- ▶ Every node stores a boolean value  $x.marked$  that specifies whether  $x$  is **marked** or not.

## 8.3 Fibonacci Heaps

### The potential function:

- ▶  $t(S)$  denotes the number of trees in the heap.
- ▶  $m(S)$  denotes the number of marked nodes.
- ▶ We use the potential function  $\Phi(S) = t(S) + 2m(S)$ .



The potential is  $\Phi(S) = 5 + 2 \cdot 3 = 11$ .

## 8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use  $c$  to denote the amount of work that a unit of potential can pay for.

## 8.3 Fibonacci Heaps

### S. minimum()

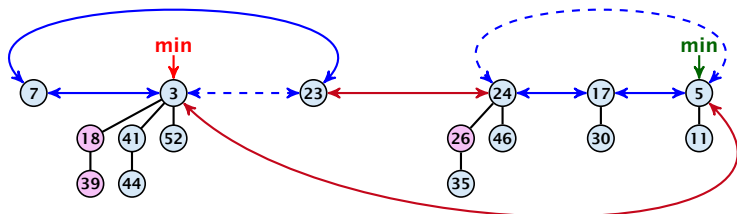
- ▶ Access through the min-pointer.
- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Amortized cost  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

### S. merge( $S'$ )

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer

- In the figure below the dashed edges are replaced by red edges.
- The minimum of the left heap becomes the new minimum of the merged heap.



### Running time:

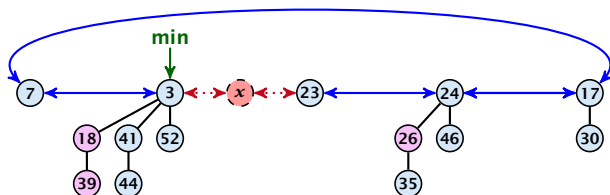
- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Hence, amortized cost is  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

$x$  is inserted next to the min-pointer as this is our entry point into the root-list.

### S. insert( $x$ )

- ▶ Create a new tree containing  $x$ .
- ▶ Insert  $x$  into the root-list.
- ▶ Update min-pointer, if necessary.



### Running time:

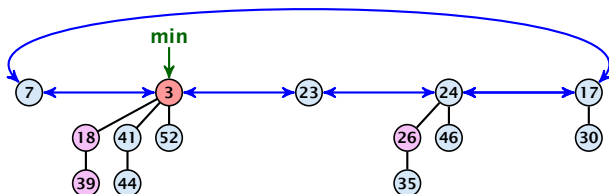
- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ Change in potential is  $+1$ .
- ▶ Amortized cost is  $c + \mathcal{O}(1) = \mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

$D(\min)$  is the number of children of the node that stores the minimum.

### S. delete-min( $x$ )

- ▶ Delete minimum; add child-trees to heap;  
time:  $D(\min) \cdot \mathcal{O}(1)$ .
- ▶ Update min-pointer; time:  $(t + D(\min)) \cdot \mathcal{O}(1)$ .



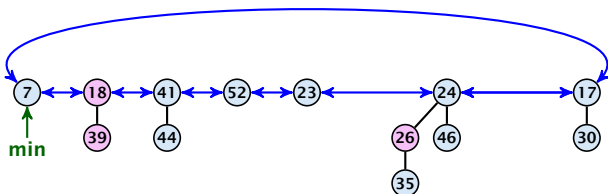


## 8.3 Fibonacci Heaps

$D(\min)$  is the number of children of the node that stores the minimum.

### S. delete-min( $x$ )

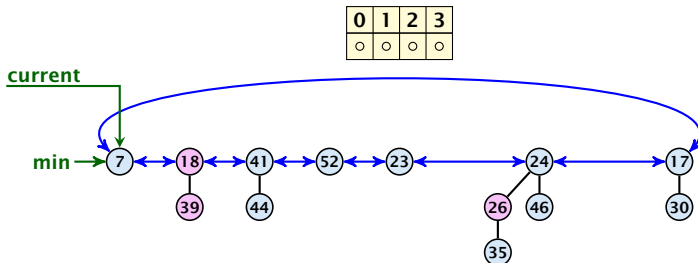
- ▶ Delete minimum; add child-trees to heap; time:  $D(\min) \cdot \mathcal{O}(1)$ .
- ▶ Update min-pointer; time:  $(t + D(\min)) \cdot \mathcal{O}(1)$ .



- ▶ Consolidate root-list so that no roots have the same degree. Time  $t \cdot \mathcal{O}(1)$  (see next slide).

## 8.3 Fibonacci Heaps

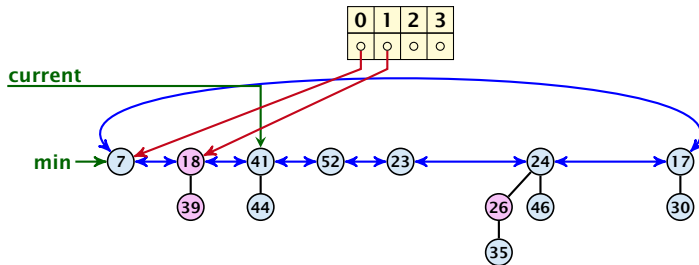
Consolidate:



During the consolidation we traverse the root list. Whenever we discover two trees that have the same degree we merge these trees. In order to efficiently check whether two trees have the same degree, we use an array that contains for every degree value  $d$  a pointer to a tree left of the current pointer whose root has degree  $d$  (if such a tree exist).

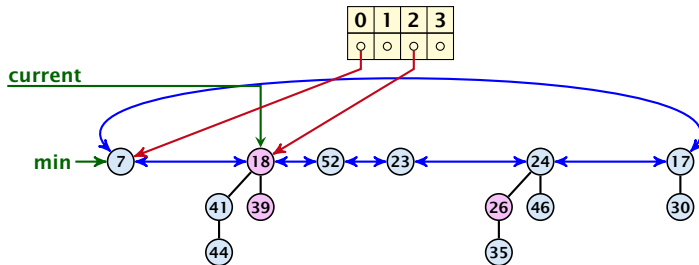
## 8.3 Fibonacci Heaps

Consolidate:



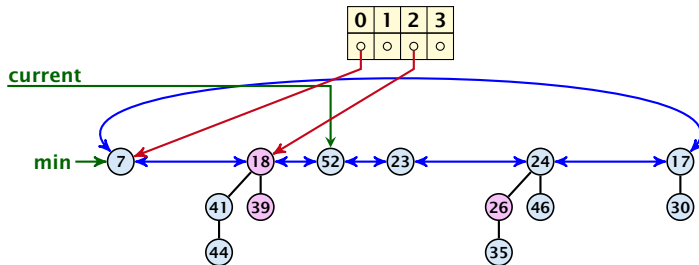
## 8.3 Fibonacci Heaps

Consolidate:



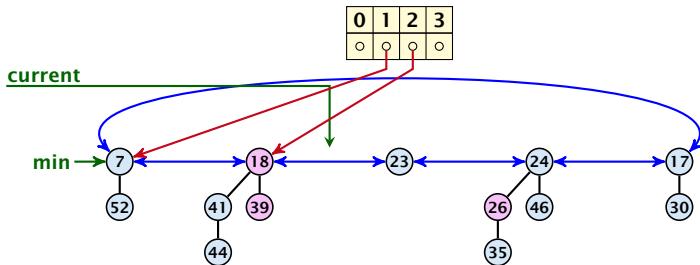
## 8.3 Fibonacci Heaps

Consolidate:



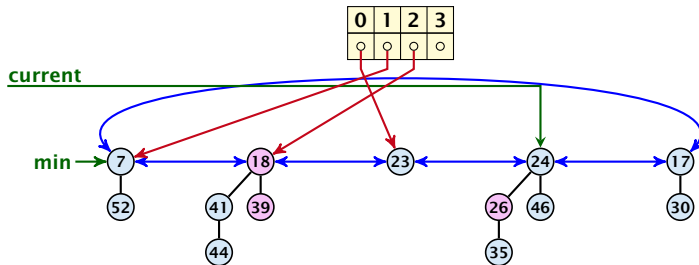
## 8.3 Fibonacci Heaps

Consolidate:



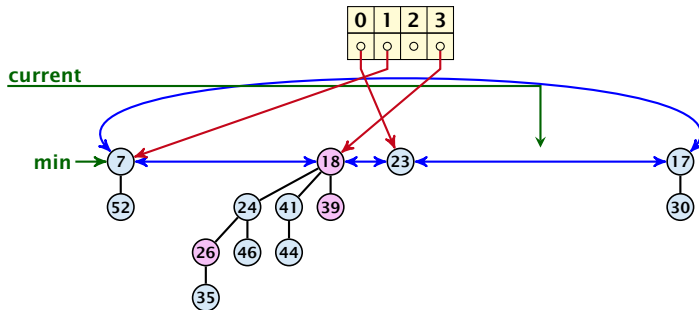
## 8.3 Fibonacci Heaps

Consolidate:



## 8.3 Fibonacci Heaps

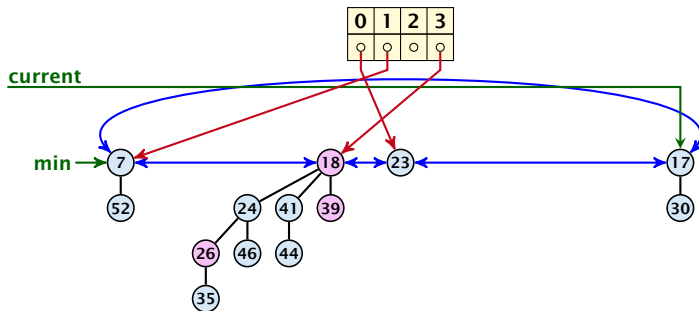
Consolidate:





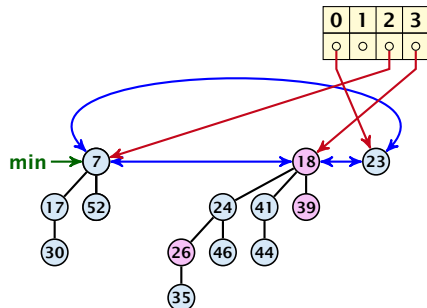
## 8.3 Fibonacci Heaps

Consolidate:



## 8.3 Fibonacci Heaps

Consolidate:



## 8.3 Fibonacci Heaps

$t$  and  $t'$  denote the number of trees before and after the `delete-min()` operation, respectively.  
 $D_n$  is an upper bound on the degree (i.e., number of children) of a tree node.

### Actual cost for `delete-min()`

- ▶ At most  $D_n + t$  elements in root-list before consolidate.
- ▶ Actual cost for a `delete-min` is at most  $\mathcal{O}(1) \cdot (D_n + t)$ .  
Hence, there exists  $c_1$  s.t. actual cost is at most  $c_1 \cdot (D_n + t)$ .

### Amortized cost for `delete-min()`

- ▶  $t' \leq D_n + 1$  as degrees are different after consolidating.
- ▶ Therefore  $\Delta\Phi \leq D_n + 1 - t$ ;
- ▶ We can pay  $c \cdot (t - D_n - 1)$  from the potential decrease.
- ▶ The amortized cost is

$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)\end{aligned}$$

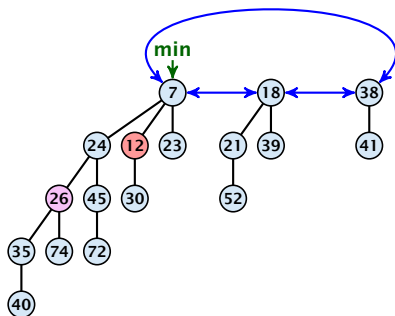
for  $c \geq c_1$  .

## 8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then  
 $D_n \leq \log n$ .

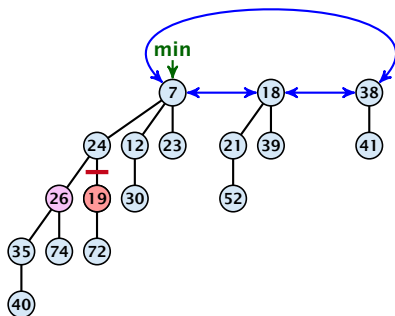
## Fibonacci Heaps: decrease-key(handle $h, v$ )



### Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by  $h$ . Nothing else to do.

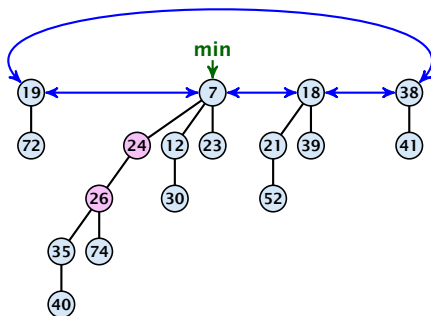
## Fibonacci Heaps: decrease-key(handle $h, v$ )



### Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element  $x$  reference by  $h$ .
- ▶ If the heap-property is violated, cut the parent edge of  $x$ , and make  $x$  into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of  $x$  (unless it's a root).

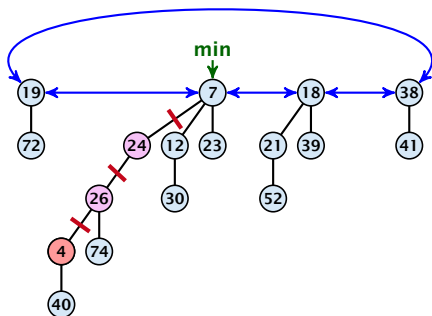
## Fibonacci Heaps: decrease-key(handle $h, v$ )



### Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element  $x$  reference by  $h$ .
- ▶ If the heap-property is violated, cut the parent edge of  $x$ , and make  $x$  into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of  $x$  (unless it's a root).

## Fibonacci Heaps: decrease-key(handle $h, v$ )

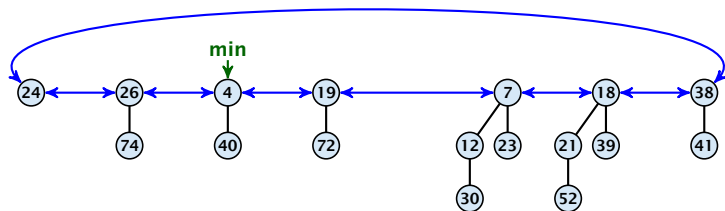


### Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element  $x$  reference by  $h$ .
- ▶ Cut the parent edge of  $x$ , and make  $x$  into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.



## Fibonacci Heaps: decrease-key(handle $h, v$ )



### Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element  $x$  reference by  $h$ .
- ▶ Cut the parent edge of  $x$ , and make  $x$  into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

# Fibonacci Heaps: decrease-key(handle $h, v$ )

## Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element  $x$  reference by  $h$ .
- ▶ Cut the parent edge of  $x$ , and make  $x$  into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

Marking a node can be viewed as a first step towards becoming a root. The first time  $x$  loses a child it is marked; the second time it loses a child it is made into a root.

# Fibonacci Heaps: decrease-key(handle $h, v$ )

## Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of  $\ell$  cuts.
- ▶ Hence, cost is at most  $c_2 \cdot (\ell + 1)$ , for some constant  $c_2$ .

## Amortized cost:

- ▶  $t' = t + \ell$ , as every cut creates one new root.
- ▶  $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$ , since all but the first cut unmarks a node; the last cut may mark a node.

- ▶  $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$

- ▶ Amortized cost is at most

$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$

if  $c \geq c_2$ .

$t$  and  $t'$ : number of trees before and after operation.

$m$  and  $m'$ : number of marked nodes before and after operation.

# Delete node

***H. delete( $x$ ):***

- ▶ decrease value of  $x$  to  $-\infty$ .
- ▶ delete-min.

**Amortized cost:  $\mathcal{O}(D_n)$**

- ▶  $\mathcal{O}(1)$  for decrease-key.
- ▶  $\mathcal{O}(D_n)$  for delete-min.

## 8.3 Fibonacci Heaps

### Lemma 1

Let  $x$  be a node with degree  $k$  and let  $y_1, \dots, y_k$  denote the children of  $x$  in the order that they were linked to  $x$ . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

The marking process is very important for the proof of this lemma. It ensures that a node can have lost at most one child since the last time it became a non-root node. When losing a first child the node gets marked; when losing the second child it is cut from the parent and made into a root.

## 8.3 Fibonacci Heaps

### Proof

- ▶ When  $y_i$  was linked to  $x$ , at least  $y_1, \dots, y_{i-1}$  were already linked to  $x$ .
- ▶ Hence, at this time  $\text{degree}(x) \geq i - 1$ , and therefore also  $\text{degree}(y_i) \geq i - 1$  as the algorithm links nodes of equal degree only.
- ▶ Since, then  $y_i$  has lost at most one child.
- ▶ Therefore,  $\text{degree}(y_i) \geq i - 2$ .

## 8.3 Fibonacci Heaps

- ▶ Let  $s_k$  be the minimum possible size of a sub-tree rooted at a node of degree  $k$  that can occur in a Fibonacci heap.
- ▶  $s_k$  monotonically increases with  $k$
- ▶  $s_0 = 1$  and  $s_1 = 2$ .

Let  $x$  be a degree  $k$  node of size  $s_k$  and let  $y_1, \dots, y_k$  be its children.

$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

## 8.3 Fibonacci Heaps

$\phi = \frac{1}{2}(1 + \sqrt{5})$  denotes the *golden ratio*.  
Note that  $\phi^2 = 1 + \phi$ .

### Definition 2

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

### Facts:

1.  $F_k \geq \phi^k$ .
2. For  $k \geq 2$ :  $F_k = 2 + \sum_{i=0}^{k-2} F_i$ .

The above facts can be easily proved by induction. From this it follows that  $s_k \geq F_k \geq \phi^k$ , which gives that the maximum degree in a Fibonacci heap is logarithmic.



$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$