6.2 Master Theorem

Lemma 1

Let $a \geq 1, b \geq 1$ *and* $\epsilon > 0$ *denote constants. Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \enspace .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ *then* $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ *then* $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ *and for sufficiently large* n $af(\frac{n}{b}) \leq cf(n)$ *for some constant* $c < 1$ *then* $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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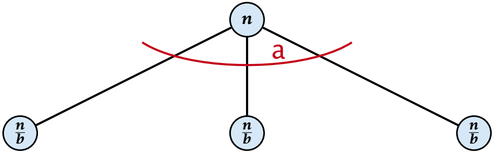

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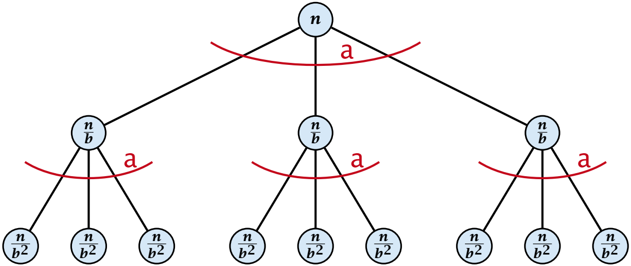

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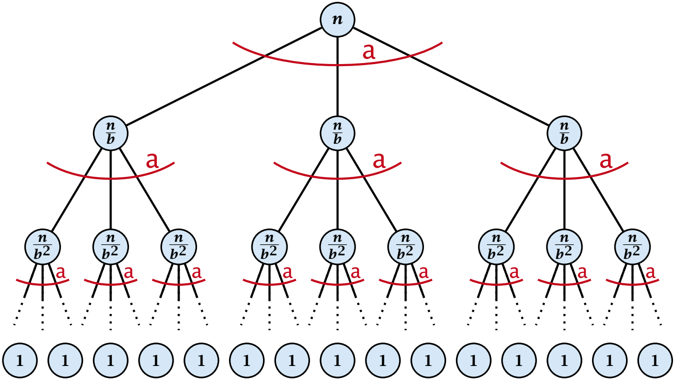

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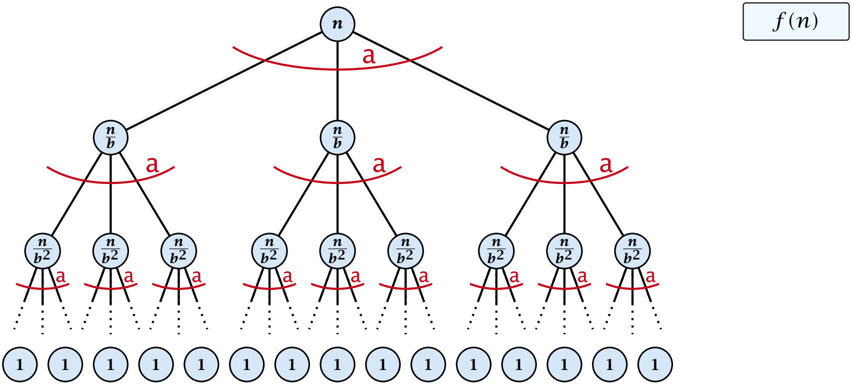

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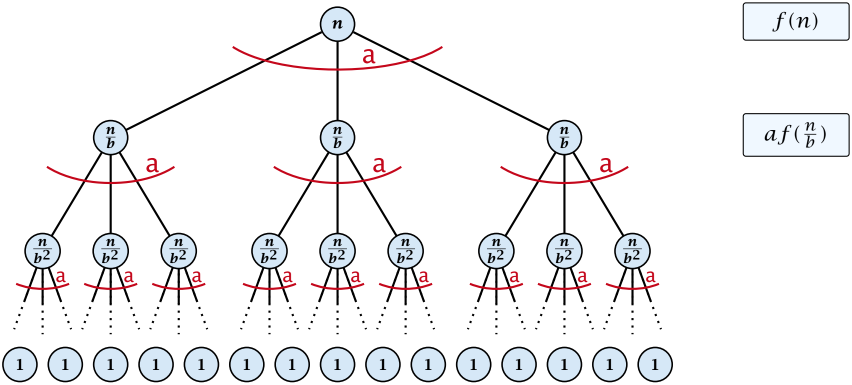

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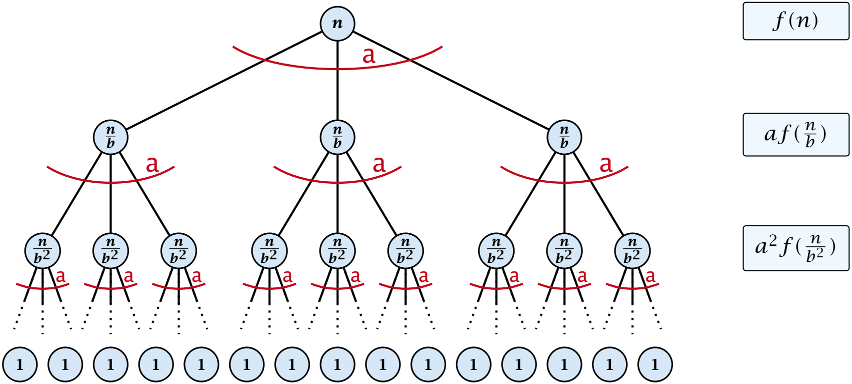

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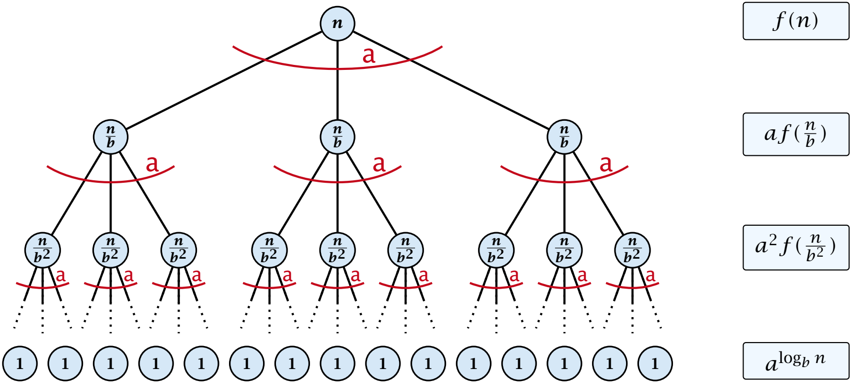

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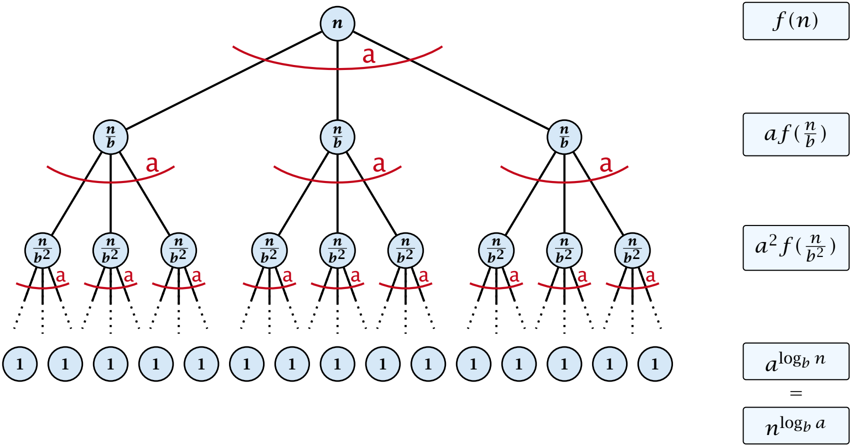

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\le c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a-\epsilon} \end{aligned}$$

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\end{aligned}
$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}}$$

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&= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} \left(b^{\epsilon}\right)^i
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\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\
\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} \quad &= cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)
\end{aligned}
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&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}
$$

Hence,

$$T(n) \le \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)}$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \\
\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad &= cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i \\
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&= \frac{c}{b^{\epsilon} - 1} n^{\log_b a}(n^{\epsilon} - 1)/(n^{\epsilon})
\end{aligned}$$

Hence,

$$T(n) \le \left(\frac{c}{b^{\epsilon} - 1} + 1\right) n^{\log_b (a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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&= cn^{\log_b a} \sum_{i=0}^{\log_b n-1} 1
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Hence,

$$
T(n) = \mathcal{O}(n^{\log_b a} \log_b n)
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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}$$

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$$

Hence,

$$
T(n) = \Omega(n^{\log_b a} \log_b n)
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Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

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T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
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&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k
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$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n-1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\ \boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \end{aligned}$$

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\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
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&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \tfrac{1}{k}\ell^{k+1}
\end{aligned}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

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&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\
&= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\
&\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}
\end{aligned}
$$

$$\boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a+\epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
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$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$$

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&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
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Hence,

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n-1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$\boxed{\Rightarrow T(n) = \Theta(f(n)).}$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \boxed{1} & A \\
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\hline
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\begin{array}{ccccccccc|c|c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1_{1} & 1 & & B \\
\hline
 & & & & & & & & 0 & &
\end{array}
$$

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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & & & & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0_{1} & 0_{1} & 1_{1} & 1 & & B \\
\hline
 & & & & & & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & & & \boxed{1} & 0 & 0 & 0 & &
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 & & & & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
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 & & & & & 1 & 0 & 0 & 0 & &
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 & & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & & & 0 & 0 & 1 & 0 & 0 & 0 & &
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1 & 1 & \boxed{0} & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & & {}_1 & {}_1 & {}_0 & {}_1 & {}_1 & {}_1 & & & \\
\hline
 & & & 0 & 0 & 1 & 0 & 0 & 0 & &
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1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
	0	1	1	0	1	1	1		
		1	0	0	1	0	0	0	

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1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
1_{0} & 0_{0} & 0_{1} & 0_{1} & 1_{0} & 0_{1} & 0_{1} & 1_{1} & 1 & & B \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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 & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

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\begin{array}{cccccccccccc}
 & \boxed{1} & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
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$$\begin{array}{c|ccccccccc l}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \quad A\\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & \quad B\\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
\hline
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}$$

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For this we first need to be able to add two integers A and B:

$$\begin{array}{cccccccccccl}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
{\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
\boxed{1} & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}$$

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 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\
 {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & & \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

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$$\begin{array}{c} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \end{array}$$

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\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1
\end{array}
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\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & 0
\end{array}
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$$
\begin{array}{r}
1\ 0\ 0\ 0\ 1 \times 1\ 0\ \boxed{1}\ 1 \\
\hline
1\ 0\ 0\ 0\ 1 \\
1\ 0\ 0\ 0\ 1\ 0
\end{array}
$$

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\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
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\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

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$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

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\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0
\end{array}
$$

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$$
\begin{array}{r}
1\ 0\ 0\ 0\ 1 \times \boxed{1}\ 0\ 1\ 1 \\
\hline
1\ 0\ 0\ 0\ 1 \\
1\ 0\ 0\ 0\ 1\ 0 \\
0\ 0\ 0\ 0\ 0\ 0\ 0 \\
1\ 0\ 0\ 0\ 1\ 0\ 0\ 0
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{rrrrrrrrrr}
 & 1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.

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Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

Time requirement:

- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

$$\boxed{B} \times \boxed{A}$$

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b_{n-1}	$\cdots$	b_0	$\times$	a_{n-1}	$\cdots$	a_0

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$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \;\; b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \;\; a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

if $|A| = |B| = 1$ **then**
  return $a_0 \cdot b_0$
split A into A_0 and A_1
split B into B_0 and B_1
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
$Z_1 \leftarrow \mathrm{mult}(A_1, B_0) + \mathrm{mult}(A_0, B_1)$
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return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

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```
if |A| = |B| = 1 then                              O(1)
    return a_0 · b_0
split A into A_0 and A_1
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Z_2 ← mult(A_1, B_1)
Z_1 ← mult(A_1, B_0) + mult(A_0, B_1)
Z_0 ← mult(A_0, B_0)
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
```

- **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
  - **return** $a_0 \cdot b_0$
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- $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
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- $Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
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return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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$\Longrightarrow$ Not better then the "school method".

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We can use the following identity to compute Z_1:

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$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0 \end{aligned}$$

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Hence,

Algorithm 4 mult(A, B)

if $|A| = |B| = 1$ **then**
    return $a_0 \cdot b_0$
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$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
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Hence,

Algorithm 4 mult(A, B)

```
if |A| = |B| = 1 then                                   O(1)
    return a_0 · b_0
split A into A_0 and A_1
split B into B_0 and B_1
Z_2 ← mult(A_1, B_1)
Z_0 ← mult(A_0, B_0)
Z_1 ← mult(A_0 + A_1, B_0 + B_1) − Z_2 − Z_0
return Z_2 · 2^n + Z_1 · 2^(n/2) + Z_0
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We can use the following identity to compute Z_1:

$$\begin{aligned} Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1)\cdot(B_0 + B_1) - \overbrace{A_1B_1}^{=\,Z_2} - \overbrace{A_0B_0}^{=\,Z_0} \end{aligned}$$

Hence,

Algorithm 4 mult(A, B)

if $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
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$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
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    return $a_0 \cdot b_0$ $\mathcal{O}(1)$
split A into A_0 and A_1 $\mathcal{O}(n)$
split B into B_0 and B_1
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$
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\end{aligned}
$$

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if $\vert A\vert = \vert B\vert = 1$ **then**	$\mathcal{O}(1)$
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split B into B_0 and B_1	$\mathcal{O}(n)$
$Z_2 \leftarrow \mathrm{mult}(A_1, B_1)$	$T(\frac{n}{2})$
$Z_0 \leftarrow \mathrm{mult}(A_0, B_0)$	$T(\frac{n}{2})$
$Z_1 \leftarrow \mathrm{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \enspace .$$

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- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

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- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\quad T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".