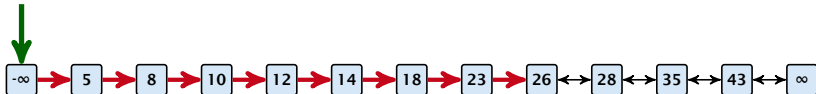


## 7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

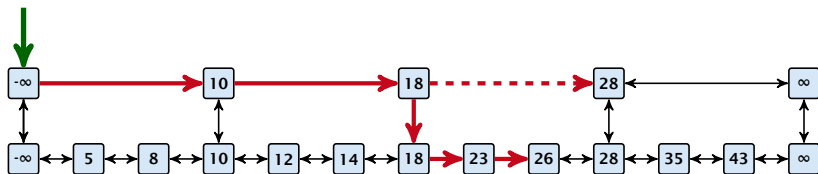
- ▶ time for search  $\Theta(n)$
- ▶ time for insert  $\Theta(n)$  (dominated by searching the item)
- ▶ time for delete  $\Theta(1)$  if we are given a handle to the object, otw.  $\Theta(n)$



## 7.5 Skip Lists

How can we improve the search-operation?

**Add an express lane:**



Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).

Worst case search time:  $|L_1| + \frac{|L_0|}{|L_1|}$  (ignoring additive constants)

Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

## 7.5 Skip Lists

Add more express lanes. Lane  $L_i$  contains roughly every  $\frac{L_{i-1}}{L_i}$ -th item from list  $L_{i-1}$ .

**Search(x) ( $k + 1$  lists  $L_0, \dots, L_k$ )**

- ▶ Find the largest item in list  $L_k$  that is smaller than  $x$ . At most  $|L_k| + 2$  steps.
- ▶ Find the largest item in list  $L_{k-1}$  that is smaller than  $x$ . At most  $\lceil \frac{|L_{k-1}|}{|L_k|+1} \rceil + 2$  steps.
- ▶ Find the largest item in list  $L_{k-2}$  that is smaller than  $x$ . At most  $\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \rceil + 2$  steps.
- ▶ ...
- ▶ At most  $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$  steps.

## 7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e.,  $\frac{|L_{i-1}|}{|L_i|} = r$ , and, hence,  $L_k \approx r^{-k}n$ .

Worst case running time is:  $\mathcal{O}(r^{-k}n + kr)$ .

Choose  $r = n^{\frac{1}{k+1}}$ . Then

$$\begin{aligned}r^{-k}n + kr &= \left(n^{\frac{1}{k+1}}\right)^{-k}n + kn^{\frac{1}{k+1}} \\ &= n^{1-\frac{k}{k+1}} + kn^{\frac{1}{k+1}} \\ &= (k+1)n^{\frac{1}{k+1}}.\end{aligned}$$

Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.

## 7.5 Skip Lists

### How to do insert and delete?

- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

**Use randomization instead!**

## 7.5 Skip Lists

### Insert:

- ▶ A search operation gives you the insert position for element  $x$  in every list.
- ▶ Flip a coin until it shows head, and record the number  $t \in \{1, 2, \dots\}$  of trials needed.
- ▶ Insert  $x$  into lists  $L_0, \dots, L_{t-1}$ .

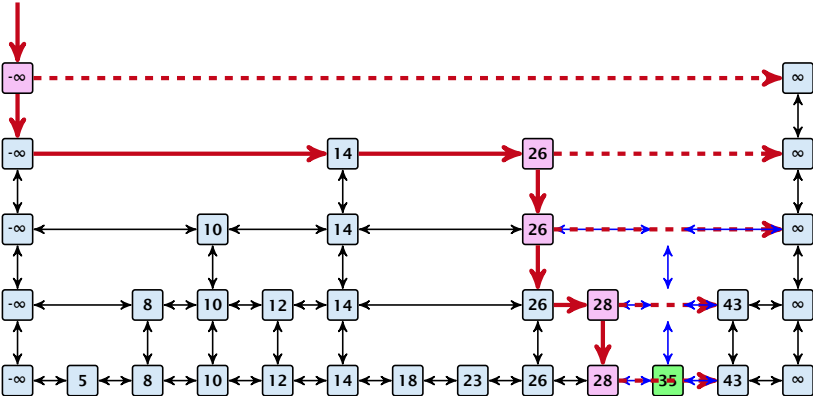
### Delete:

- ▶ You get all predecessors via backward pointers.
- ▶ Delete  $x$  in all lists it actually appears in.

**The time for both operations is dominated by the search time.**

# 7.5 Skip Lists

Insert (35):



# High Probability

## Definition 1 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

Here the  $\mathcal{O}$ -notation hides a constant that may depend on  $\alpha$ .



# High Probability

Suppose there are **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

Then the probability that all  $E_i$  hold is at least

$$\begin{aligned}\Pr[E_1 \wedge \dots \wedge E_\ell] &= 1 - \Pr[\bar{E}_1 \vee \dots \vee \bar{E}_\ell] \\ &\geq 1 - n^c \cdot n^{-\alpha} \\ &= 1 - n^{c-\alpha} .\end{aligned}$$

This means  $\Pr[E_1 \wedge \dots \wedge E_\ell]$  holds with high probability.

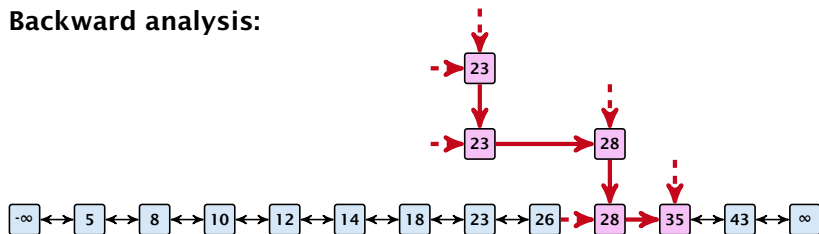
## 7.5 Skip Lists

### Lemma 2

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*

## 7.5 Skip Lists

Backward analysis:



At each point the path goes up with probability  $1/2$  and left with probability  $1/2$ .

We show that w.h.p:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

## 7.5 Skip Lists

### Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \dots \cdot (n-k+1)}{k \cdot \dots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\begin{aligned} \binom{n}{k} &= \frac{n \cdot \dots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!} \\ &= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{n}{k}\right)^k \cdot \sum_{i \geq 0} \frac{k^i}{i!} = \left(\frac{en}{k}\right)^k \end{aligned}$$

## 7.5 Skip Lists

Let  $E_{z,k}$  denote the event that a search path is of length  $z$  (number of edges) but does not visit a list above  $L_k$ .

In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

## 7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing  $k = \gamma \log n$  with  $\gamma \geq 1$  and  $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

$$\leq \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^k n^{-\alpha}$$

now choosing  $\beta = 6\alpha$  gives

$$\leq \left(\frac{42\alpha}{64\alpha}\right)^k n^{-\alpha} \leq n^{-\alpha}$$

for  $\alpha \geq 1$ .

## 7.5 Skip Lists

So far we fixed  $k = \gamma \log n$ ,  $\gamma \geq 1$ , and  $z = 7\alpha\gamma \log n$ ,  $\alpha \geq 1$ .

This means that a search path of length  $\Omega(\log n)$  visits a list on a level  $\Omega(\log n)$ , w.h.p.

Let  $A_{k+1}$  denote the event that the list  $L_{k+1}$  is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} .$$

For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

This means, the search requires at most  $z$  steps, w. h. p.

# Skip Lists

## Bibliography

[GT98] Michael T. Goodrich, Roberto Tamassia  
*Data Structures and Algorithms in JAVA*,  
John Wiley, 1998

Skip lists are covered in Chapter 7.5 of [GT98].