Splay Trees

Disadvantage of balanced search trees:

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\+ after access, an element is moved to the root; splay(x)
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- only amortized guarantee
- read-operations change the tree

Splay Trees

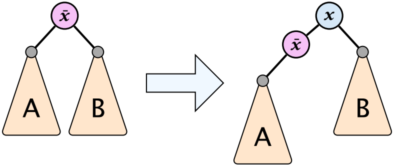
find(x)

- search for x according to a search tree
- let $\bar{x}$ be last element on search-path
- splay($\bar{x}$)

Splay Trees

insert(x)

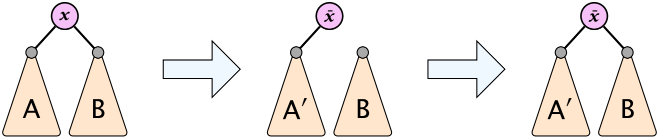
- search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)
- splay($\bar{x}$) moves $\bar{x}$ to the root
- insert x as new root



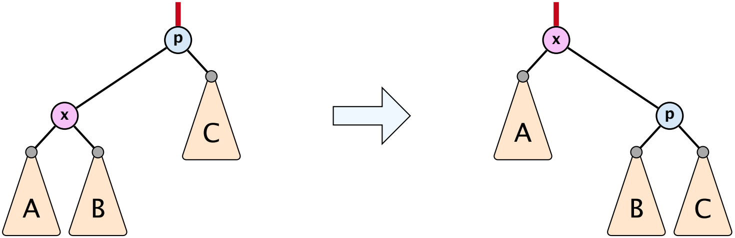
Splay Trees

delete(x)

- search for x; splay(x); remove x
- search largest element $\bar{x}$ in A
- splay($\bar{x}$) (on subtree A)
- connect root of B as right child of $\bar{x}$



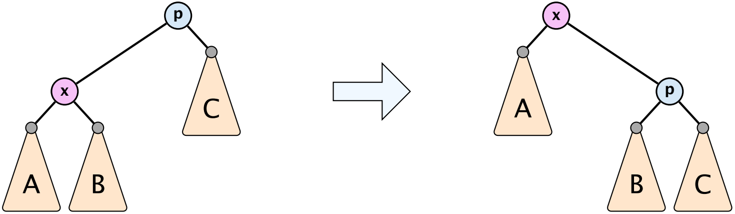
Move to Root



How to bring element to root?

- one (bad) option: moveToRoot(x)
- iteratively do rotation around parent of x until x is root
- if x is left child do right rotation otw. left rotation

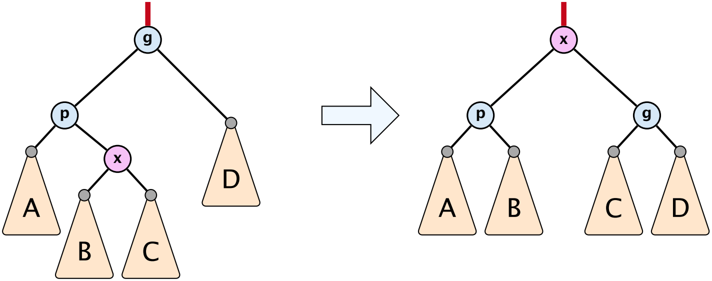
Splay: Zig Case



better option splay(x):

- zig case: if x is child of root do left rotation or right rotation around parent

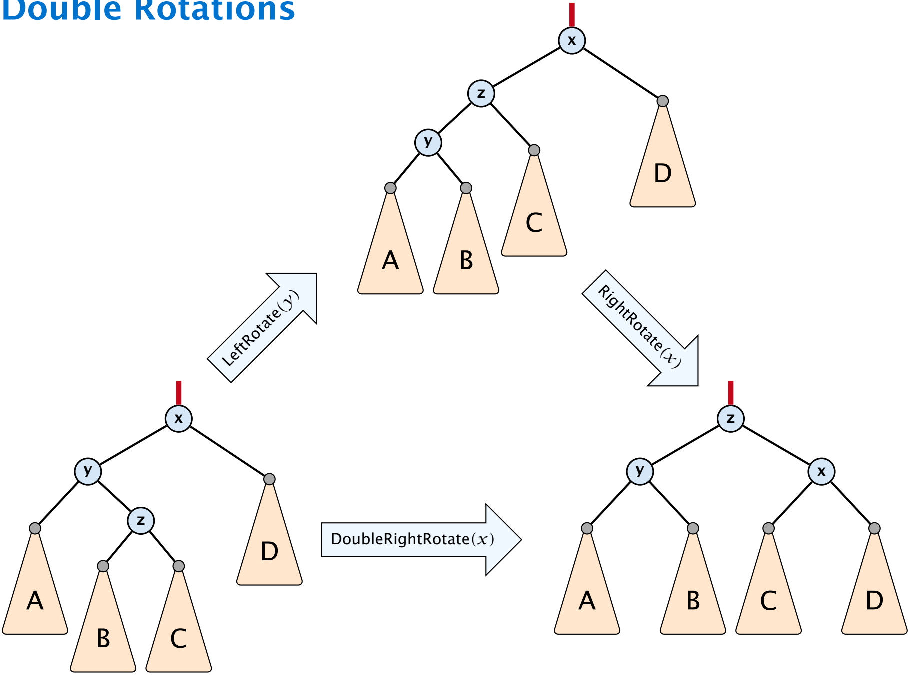
Splay: Zigzag Case



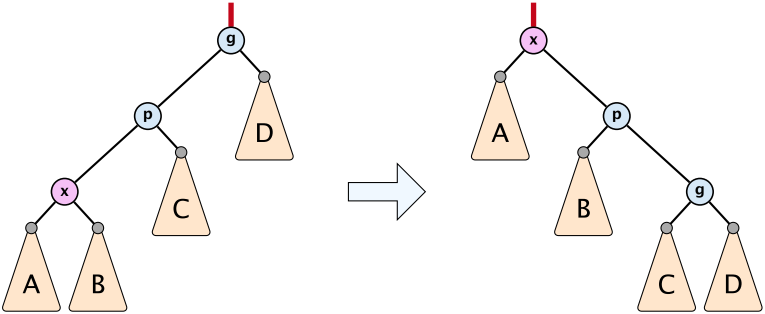
better option splay(x):

- zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



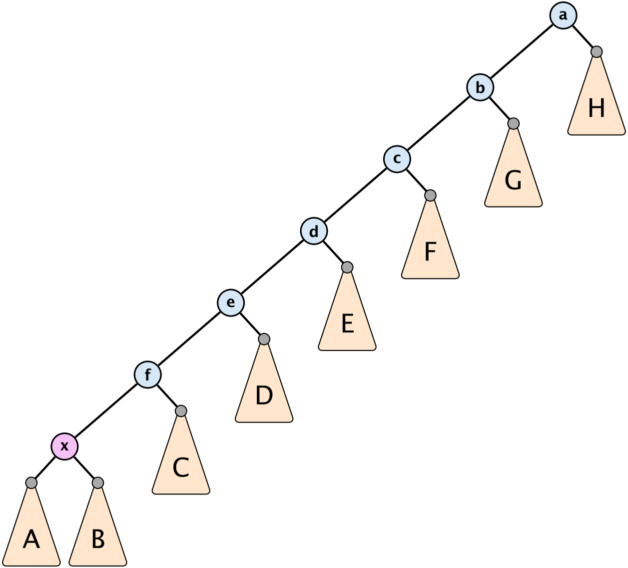
Splay: Zigzig Case



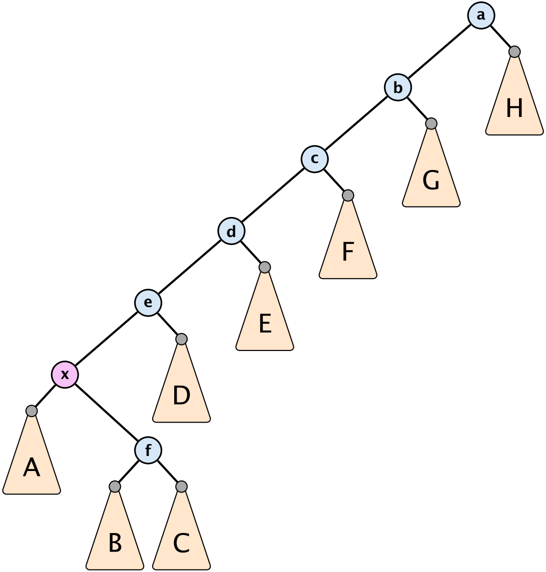
better option splay(x):

- zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root



Splay vs. Move to Root



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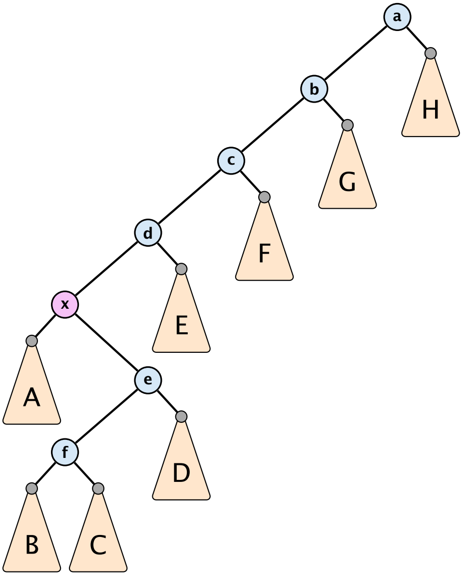

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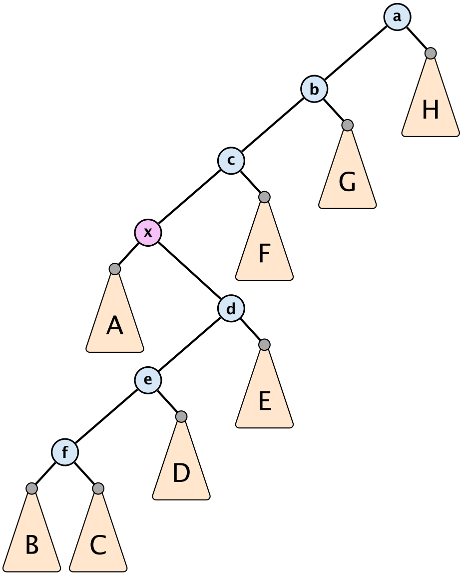

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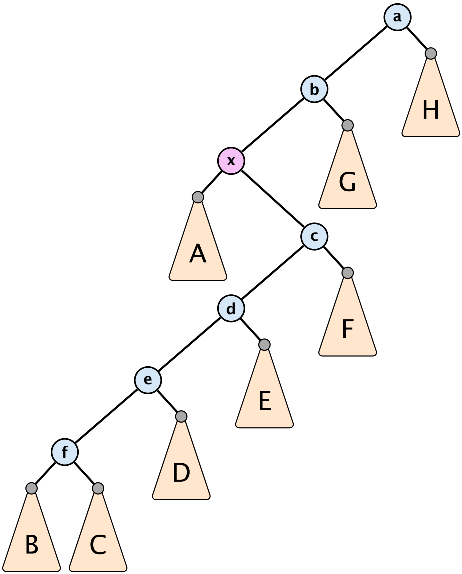

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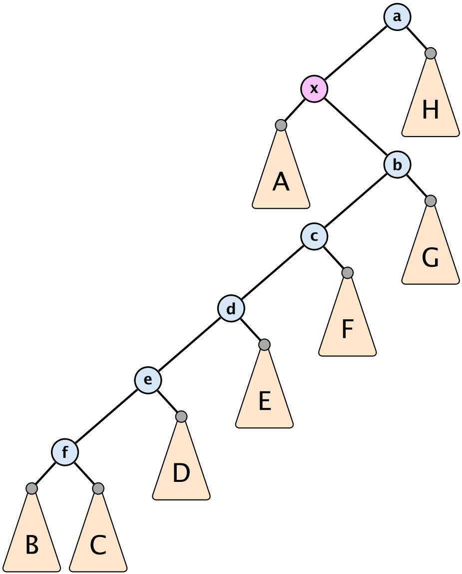

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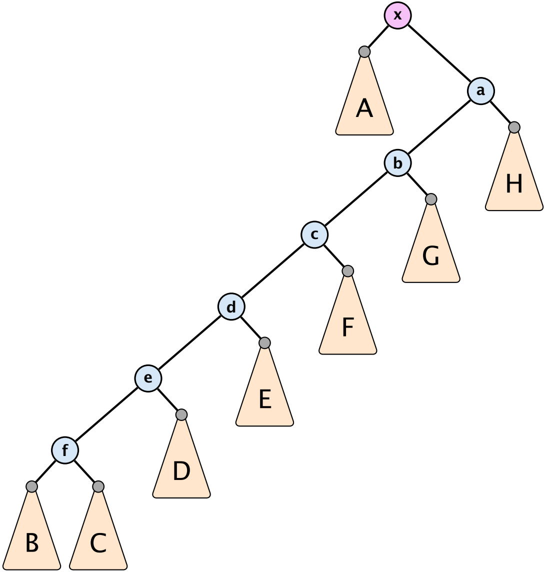

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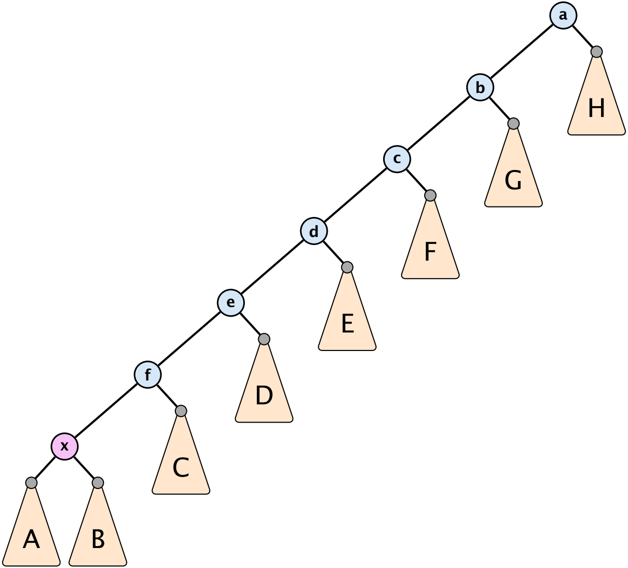

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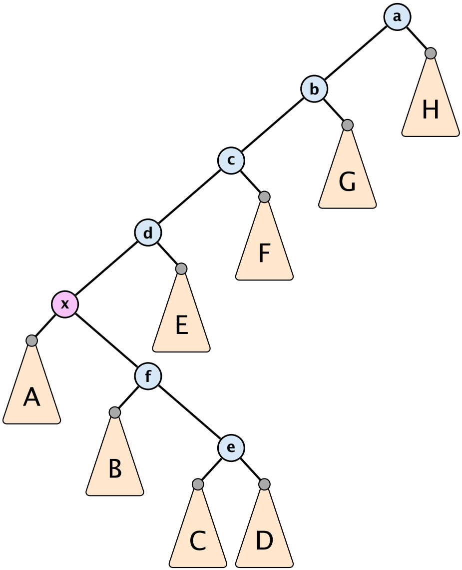

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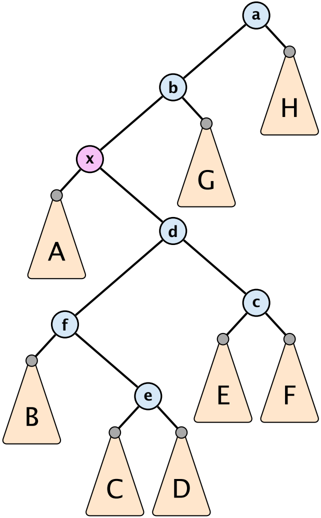

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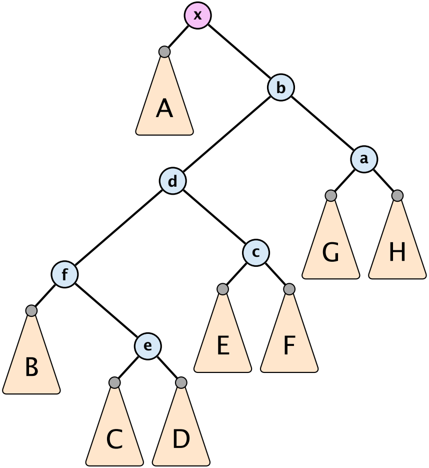

Static Optimality

Suppose we have a sequence of m find-operations. find(x) appears h_x times in this sequence.

The cost of a static search tree T is:

$$\mathrm{cost}(T) = m + \sum_x h_x \,\mathrm{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\mathrm{cost}(T_{\mathrm{min}}))$, where T_{min} is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- the cost for accessing element x is $1 + \mathrm{depth}(x)$;
- after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\mathrm{cost}(A, S))$, for processing S.

Lemma 1

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ for all operations.

Amortized Analysis

Definition 2

A data structure with operations $\mathrm{op}_1(), \dots, \mathrm{op}_k()$ has amortized running times $t_1, \dots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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$$\sum_{i=1}^{k} c_i$$

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$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0)$$

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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\,\mathbf{push}()$
- $S.\,\mathbf{pop}()$
- $S.\,\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

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- $S.\textbf{pop}()$
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- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- $S.\textbf{push}()$: cost 1.
- $S.\textbf{pop}()$: cost 1.
- $S.\textbf{multipop}(k)$: cost $\min\{\text{size}, k\} = k$.

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Use potential function $\Phi(S) =$ number of elements on the stack.

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$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \leq 0 \ .$$

- $\boldsymbol{S.\mathbf{multipop}(k)}$: cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 \ .$$

Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

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Actual cost:

- Changing bit from 0 to 1: cost 1.
- Changing bit from 1 to 0: cost 1.
- Increment: cost is $k+1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k=1$).

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$$\hat{C}_{0\to 1} = C_{0\to 1} + \Delta\Phi = 1 + 1 \le 2 \ .$$

- Changing bit from 1 to 0:

$$\hat{C}_{1\to 0} = C_{1\to 0} + \Delta\Phi = 1 - 1 \le 0 \ .$$

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Amortized cost:

- Changing bit from 0 to 1:

$$\hat{C}_{0\to1} = C_{0\to1} + \Delta\Phi = 1 + 1 \le 2 \enspace .$$

- Changing bit from 1 to 0:

$$\hat{C}_{1\to0} = C_{1\to0} + \Delta\Phi = 1 - 1 \le 0 \enspace .$$

- Increment: Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \to 0)$-operations, and one $(0 \to 1)$-operation.

  Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \le 2$.

Splay Trees

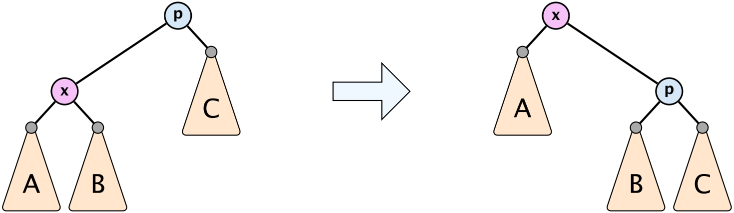
potential function for splay trees:

- size $\mathrm{s}(x) = |T_x|$
- rank $\mathrm{r}(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

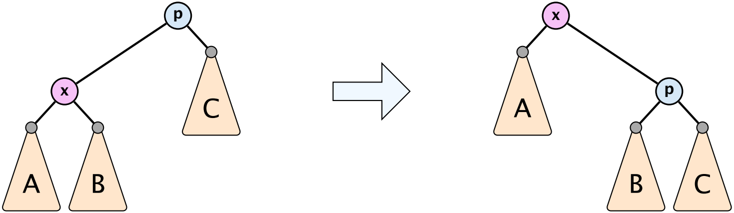
The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



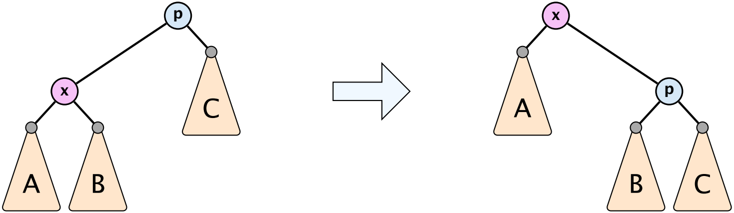
$$\Delta\Phi =$$

Splay: Zig Case



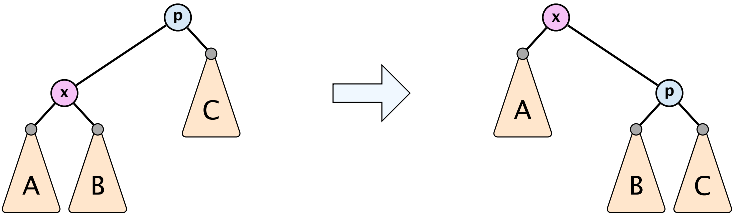
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case



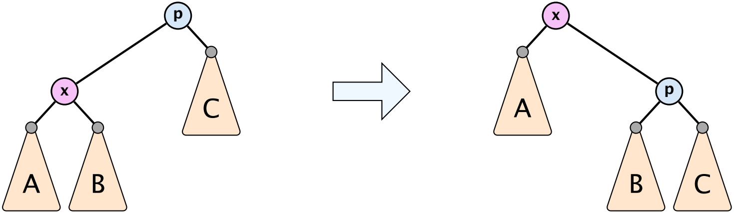
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p)\\ &= r'(p) - r(x)\end{aligned}$$

Splay: Zig Case



$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\
&= r'(p) - r(x) \\
&\le r'(x) - r(x)
\end{aligned}
$$

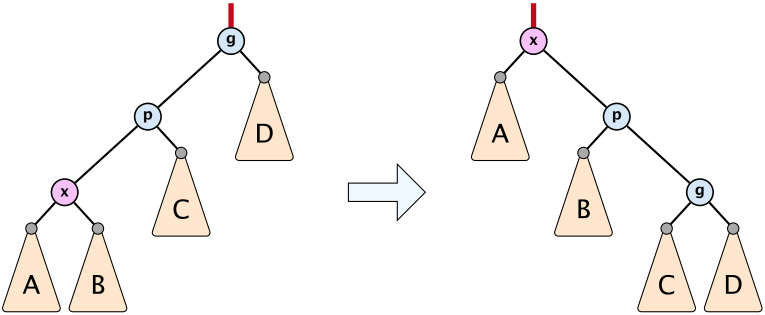
Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p)\\ &= r'(p) - r(x)\\ &\le r'(x) - r(x)\end{aligned}$$

$$\text{cost}_{\text{zig}} \le 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case



$\Delta\Phi =$

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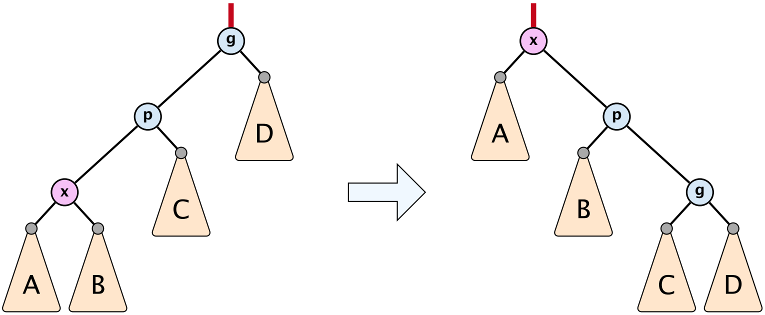
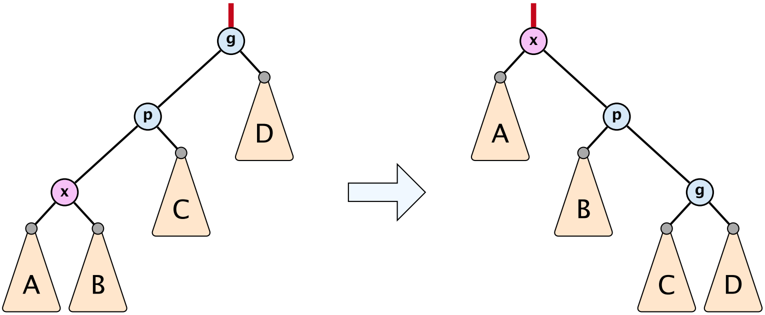


$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzig Case



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\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p)
\end{aligned}
$$

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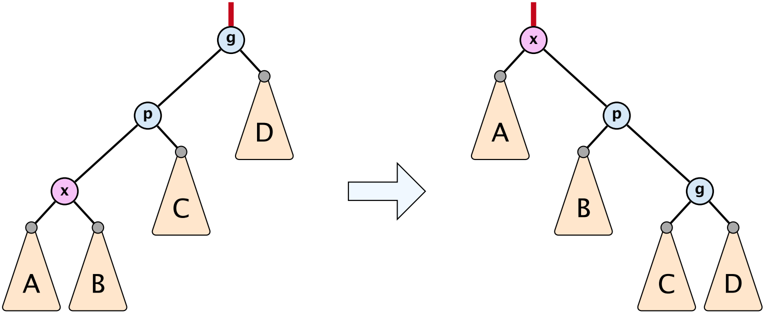

$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\le r'(x) + r'(g) - r(x) - r(x)\end{aligned}$$

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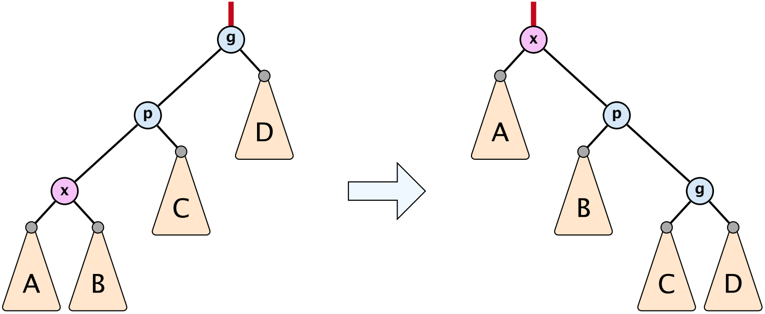

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\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p) \\
&\le r'(x) + r'(g) - r(x) - r(x) \\
&= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)
\end{aligned}
$$

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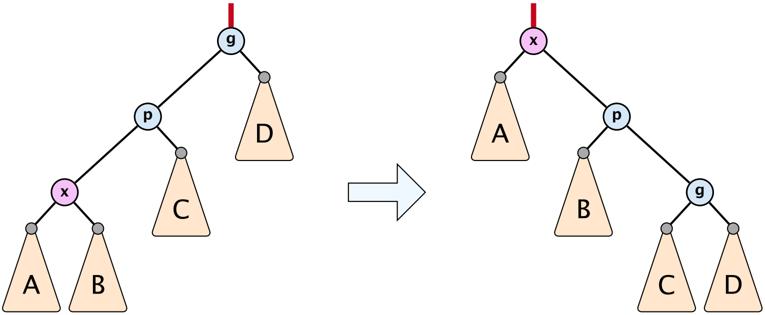

$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\le r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\end{aligned}$$

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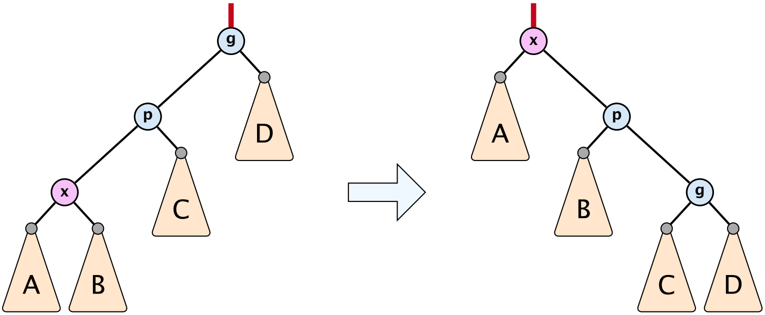

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\end{aligned}
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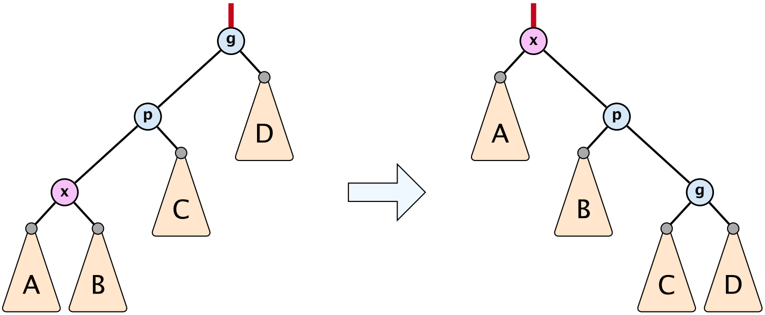

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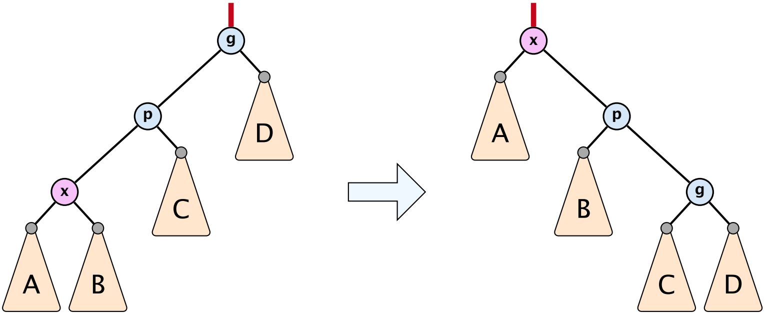
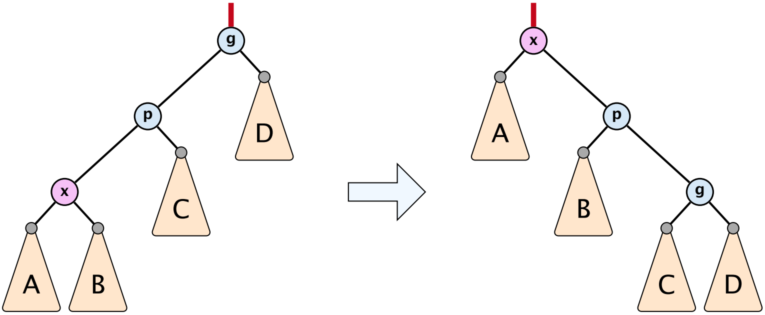
&= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\
&\le -2 + 3(r'(x) - r(x)) \quad \Rightarrow \mathrm{cost}_{\mathrm{zigzig}} \le 3(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzig Case



$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

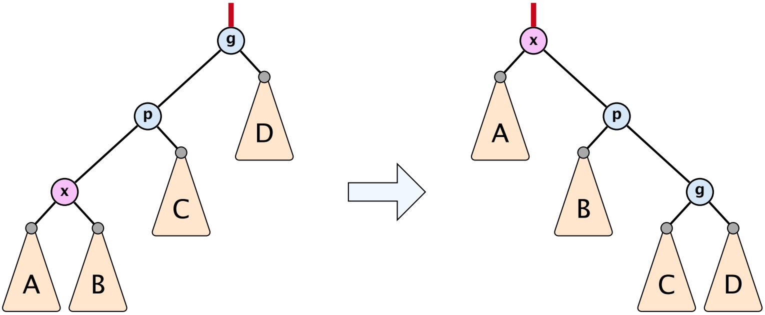
Splay: Zigzig Case



$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

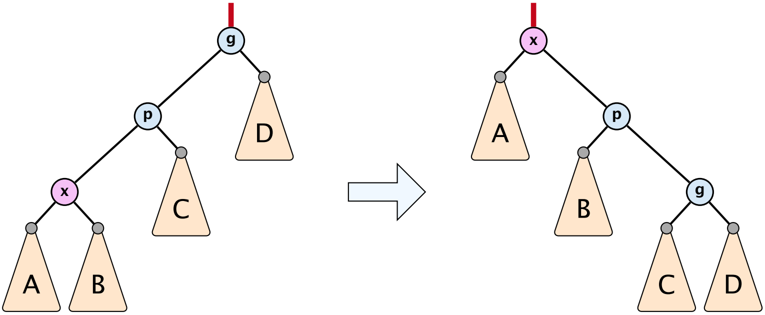
$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

Splay: Zigzig Case



$$
\begin{aligned}
&\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)\\
&\qquad\qquad = \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)\\
&\qquad\qquad = \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)
\end{aligned}
$$

Splay: Zigzig Case



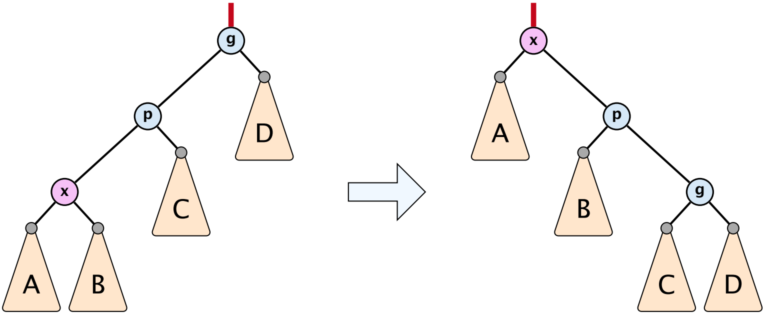
$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case



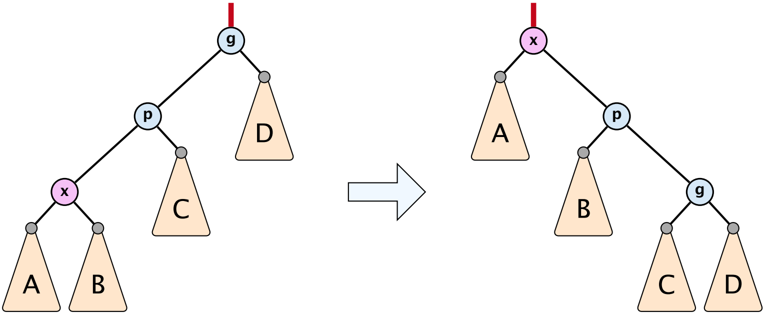
$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

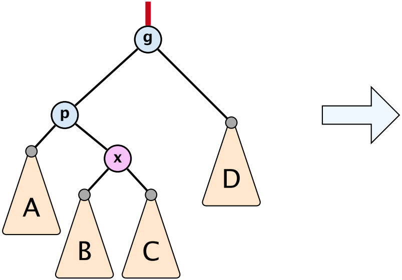
$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big)$$

Splay: Zigzig Case



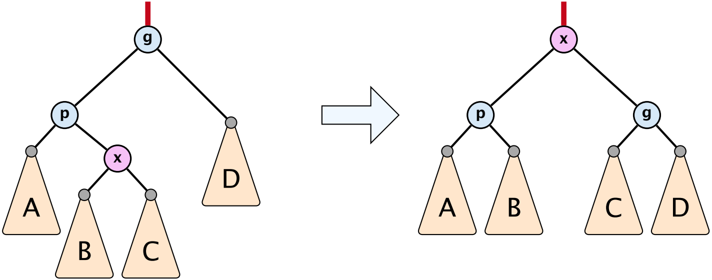
$$
\begin{aligned}
\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big) \\
&= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big) \\
&= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big) \\
&\le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big) = -1
\end{aligned}
$$

Splay: Zigzag Case



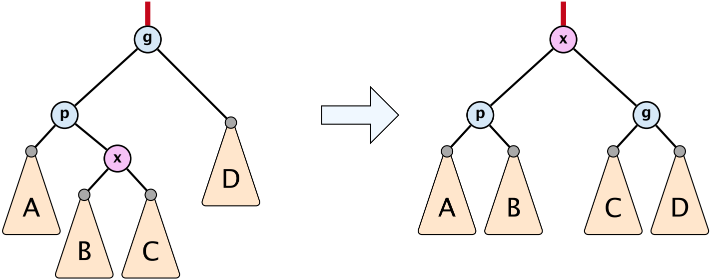
$\Delta\Phi =$

Splay: Zigzag Case



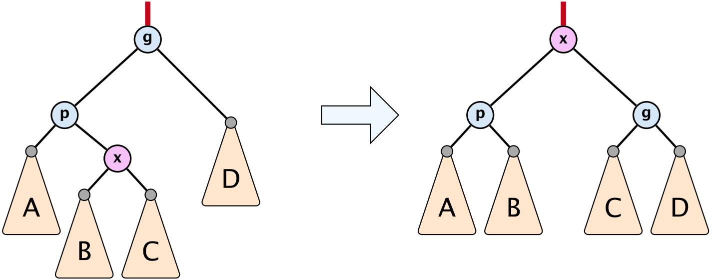
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzag Case



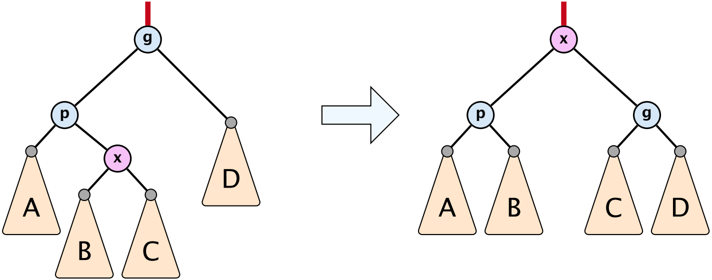
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p)\end{aligned}$$

Splay: Zigzag Case



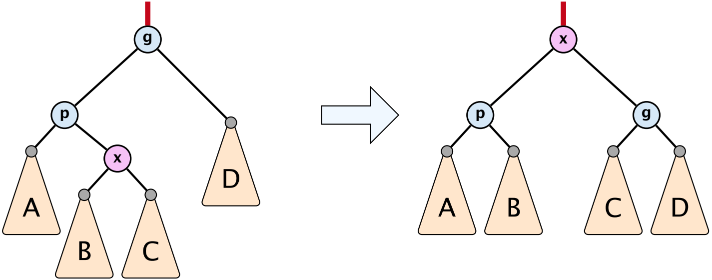
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x)\end{aligned}$$

Splay: Zigzag Case



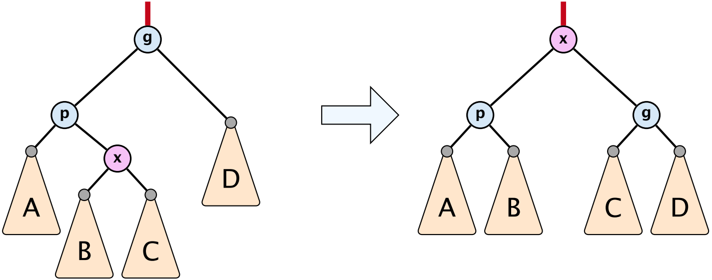
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)\\ &= r'(p) + r'(g) - r(x) - r(p)\\ &\le r'(p) + r'(g) - r(x) - r(x)\\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)\end{aligned}$$

Splay: Zigzag Case



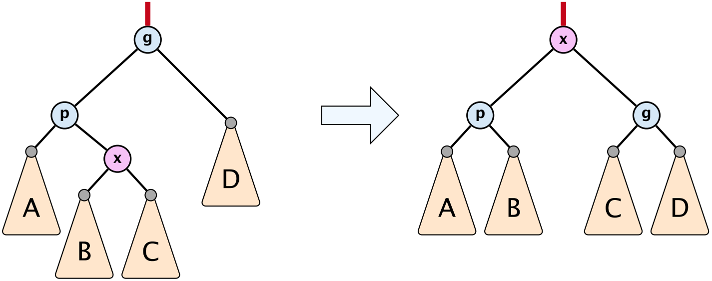
$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p) \\
&\le r'(p) + r'(g) - r(x) - r(x) \\
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\
&\le -2 + 2(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzag Case



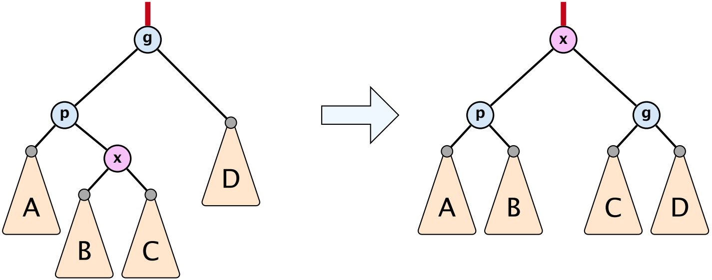
$$
\begin{aligned}
\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\
&= r'(p) + r'(g) - r(x) - r(p) \\
&\leq r'(p) + r'(g) - r(x) - r(x) \\
&= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\
&\leq -2 + 2(r'(x) - r(x)) \qquad \Rightarrow \mathrm{cost}_{\mathsf{zigzag}} \leq 3(r'(x) - r(x))
\end{aligned}
$$

Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

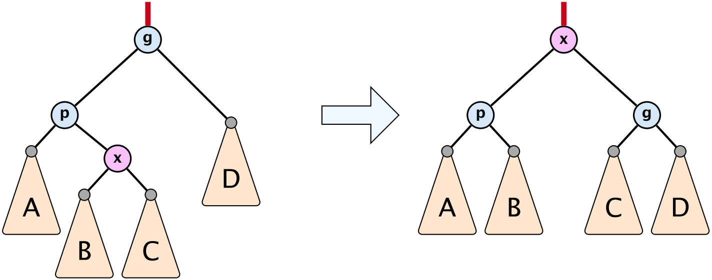
Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

Splay: Zigzag Case

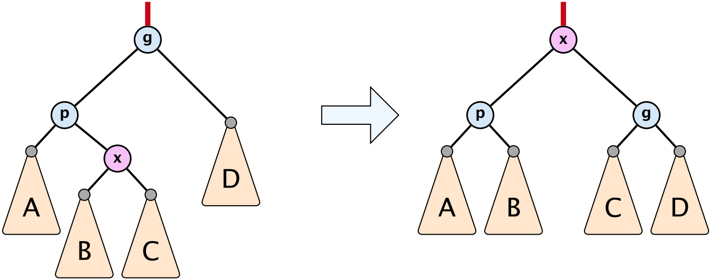


$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

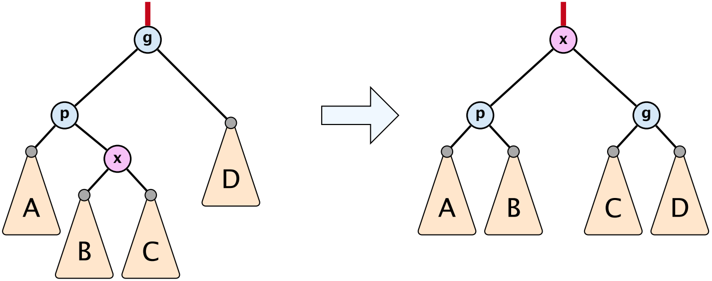
$$\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right)$$

Splay: Zigzag Case



$$
\begin{aligned}
&\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big) \\
&\qquad\qquad = \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big) \\
&\qquad\qquad \le \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \le \log\left(\frac{1}{2}\right)
\end{aligned}
$$

Splay: Zigzag Case



$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$\leq \log\left(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1$$

Amortized cost of the whole splay operation:

$$
\begin{aligned}
&\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
&= 2 + 3(r(\text{root}) - r_0(x)) \\
&\leq \mathcal{O}(\log n)
\end{aligned}
$$