WS 2021/22

Intro to Linear Programming

Harald Räcke

Fakultät für Informatik
TU München

http://www14.in.tum.de/lehre/2021WS/ea/

Winter Term 2021/22

Part I

Organizational Matters

Part I

Organizational Matters

- Modul: IN2003

Part I

Organizational Matters

- Modul: IN2003
- Name: "Efficient Algorithms and Data Structures"
  "Effiziente Algorithmen und Datenstrukturen"

Part I

Organizational Matters

- ▶ Modul: IN2003
- ▶ Name: "Efficient Algorithms and Data Structures"
          "Effiziente Algorithmen und Datenstrukturen"
- ▶ ECTS: 8 Credit points

Part I

Organizational Matters

▶ Modul: IN2003

▶ Name: "Efficient Algorithms and Data Structures"
       "Effiziente Algorithmen und Datenstrukturen"

▶ ECTS: 8 Credit points

▶ Lectures:
  ▶ 4 SWS
    Mon 10:00–12:00 (Room Interim2)
    Fri 10:00–12:00 (Room Interim2)

Part I

Organizational Matters

- Modul: IN2003
- Name: "Efficient Algorithms and Data Structures"
         "Effiziente Algorithmen und Datenstrukturen"
- ECTS: 8 Credit points
- Lectures:
  - 4 SWS
    Mon 10:00–12:00 (Room Interim2)
    Fri 10:00–12:00 (Room Interim2)
- Webpage: `http://www14.in.tum.de/lehre/2021WS/ea/`

► Required knowledge:

- ▶ Required knowledge:
  - ▶ IN0001, IN0003
    "Introduction to Informatics 1/2"
    "Einführung in die Informatik 1/2"

- ▶ Required knowledge:
  - ▶ IN0001, IN0003
    "Introduction to Informatics 1/2"
    "Einführung in die Informatik 1/2"
  - ▶ IN0007
    "Fundamentals of Algorithms and Data Structures"
    "Grundlagen: Algorithmen und Datenstrukturen" (GAD)

▶ Required knowledge:
  ▶ IN0001, IN0003
    "Introduction to Informatics 1/2"
    "Einführung in die Informatik 1/2"
  ▶ IN0007
    "Fundamentals of Algorithms and Data Structures"
    "Grundlagen: Algorithmen und Datenstrukturen" (GAD)
  ▶ IN0011
    "Basic Theoretic Informatics"
    "Einführung in die Theoretische Informatik" (THEO)

- ▶ Required knowledge:
  - ▶ IN0001, IN0003
    "Introduction to Informatics 1/2"
    "Einführung in die Informatik 1/2"
  - ▶ IN0007
    "Fundamentals of Algorithms and Data Structures"
    "Grundlagen: Algorithmen und Datenstrukturen" (GAD)
  - ▶ IN0011
    "Basic Theoretic Informatics"
    "Einführung in die Theoretische Informatik" (THEO)
  - ▶ IN0015
    "Discrete Structures"
    "Diskrete Strukturen" (DS)

- ▶ Required knowledge:
  - ▶ IN0001, IN0003
    "Introduction to Informatics 1/2"
    "Einführung in die Informatik 1/2"
  - ▶ IN0007
    "Fundamentals of Algorithms and Data Structures"
    "Grundlagen: Algorithmen und Datenstrukturen" (GAD)
  - ▶ IN0011
    "Basic Theoretic Informatics"
    "Einführung in die Theoretische Informatik" (THEO)
  - ▶ IN0015
    "Discrete Structures"
    "Diskrete Strukturen" (DS)
  - ▶ IN0018
    "Discrete Probability Theory"
    "Diskrete Wahrscheinlichkeitstheorie" (DWT)

The Lecturer

- ▶ Harald Räcke
- ▶ Email: raecke@in.tum.de
- ▶ Room: 03.09.044
- ▶ Office hours: (by appointment)

Tutorials

1	Monday,	12:00–14:00,	00.08.038	(Michael Laraia)
3	Monday,	14:00–16:00,	02.09.023	(Ruslan Zabrodin)
4	Tuesday,	10:00–12:00,	00.08.053	(Letian Shi)
5	Tuesday,	14:00–16:00,	00.08.038	(Arnor Kristmundsson)
6	Wednesday,	10:00–12:00,	03.11.018	(Abdelrahman Metwally)
2	Wednesday,	12:00–14:00,	online	(Arnor Kristmundsson)
8	Wednesday,	14:00–16:00,	online	(Abdelrahman Metwally)
9	Thursday,	16:00–18:00,	online	(Michael Laraia)
7	Friday,	12:00–14:00,	00.13.009A	(Ruslan Zabrodin)

Registration for Tutorials

Registration Period for Tutorial Sessions:

Saturday, 23 Oct– Tuesday, 26 Oct

via TUMonline; you have to choose at least 3 options...

Registration for Attending the Lecture

▶ For the following lectures we will do random checks of ca. 10% of the vaccination certificates.

Registration for Attending the Lecture

▶ For the following lectures we will do random checks of ca. 10% of the vaccination certificates.

▶ Then you have to sit one seat apart according to current Corona regulations.

Registration for Attending the Lecture

▶ For the following lectures we will do random checks of ca. 10% of the vaccination certificates.

▶ Then you have to sit one seat apart according to current Corona regulations.

▶ The number of seats reduces to roughly 140.

Registration for Attending the Lecture

▶ For the following lectures we will do random checks of ca. 10% of the vaccination certificates.

▶ Then you have to sit one seat apart according to current Corona regulations.

▶ The number of seats reduces to roughly 140.

▶ Therefore, you must register if you want to attend the lecture inside the lecture hall.

Registration for Attending the Lecture

- ▶ For the following lectures we will do random checks of ca. 10% of the vaccination certificates.

- ▶ Then you have to sit one seat apart according to current Corona regulations.

- ▶ The number of seats reduces to roughly 140.

- ▶ Therefore, you must register if you want to attend the lecture inside the lecture hall.

- ▶ This is done via Moodle.

Assignment sheets

In order to pass the module you need to pass an exam.

Assessment

Assignment Sheets:

▶ An assignment sheet is usually made available on Friday on the module webpage.

Assessment

Assignment Sheets:

- ▶ An assignment sheet is usually made available on Friday on the module webpage.
- ▶ Solutions have to be handed in in the following week before the lecture on Monday.

Assessment

Assignment Sheets:

▶ An assignment sheet is usually made available on Friday on the module webpage.

▶ Solutions have to be handed in in the following week before the lecture on Monday.

▶ Solutions are submitted electronically via Moodle.

Assessment

Assignment Sheets:

- ▶ An assignment sheet is usually made available on Friday on the module webpage.
- ▶ Solutions have to be handed in in the following week before the lecture on Monday.
- ▶ Solutions are submitted electronically via Moodle.
- ▶ Solutions have to be given in English.

Assessment

Assignment Sheets:

- ▶ An assignment sheet is usually made available on Friday on the module webpage.
- ▶ Solutions have to be handed in in the following week before the lecture on Monday.
- ▶ Solutions are submitted electronically via Moodle.
- ▶ Solutions have to be given in English.
- ▶ Solutions will be discussed in the tutorial of the week when the sheet has been handed in, i.e., sheet may not be corrected by this time.

Assessment

Assignment Sheets:

- ▶ An assignment sheet is usually made available on Friday on the module webpage.

- ▶ Solutions have to be handed in in the following week before the lecture on Monday.

- ▶ Solutions are submitted electronically via Moodle.

- ▶ Solutions have to be given in English.

- ▶ Solutions will be discussed in the tutorial of the week when the sheet has been handed in, i.e., sheet may not be corrected by this time.

- ▶ You should submit solutions in groups of up to **2** people.

Assessment

Assignment Sheets:

▶ Submissions must be handwritten by a member of the group. Please indicate who wrote the submission.

Assessment

Assignment Sheets:

▶ Submissions must be handwritten by a member of the group. Please indicate who wrote the submission.

▶ Don't forget name and student id number for each group member.

Assessment

Assignment can be used to improve you grade

Assessment

Assignment can be used to improve you grade

▶ If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10}\text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \le 4 \\ x & \text{otw.} \end{cases}$$

Assessment

Assignment can be used to improve you grade

▶ If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10}\text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \le 4 \\ x & \text{otw.} \end{cases}$$

▶ It will improve by 0.3 or 0.4, respectively.
Examples:

Assessment

Assignment can be used to improve you grade

▶ If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10} \text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \le 4 \\ x & \text{otw.} \end{cases}$$

▶ It will improve by 0.3 or 0.4, respectively.
  Examples:
  ▶ 3.3 → 3.0

Assessment

Assignment can be used to improve you grade

▶ If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10} \mathrm{round}\left(10\left(\frac{\mathrm{round}(3x)-1}{3}\right)\right) & 1 < x \le 4 \\ x & \text{otw.} \end{cases}$$

▶ It will improve by 0.3 or 0.4, respectively.
  Examples:
  ▶ 3.3 → 3.0
  ▶ 2.0 → 1.7

Assessment

Assignment can be used to improve you grade

- ▶ If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10}\text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \le 4 \\ x & \text{otw.} \end{cases}$$

- ▶ It will improve by 0.3 or 0.4, respectively.
  Examples:
    - ▶ 3.3 → 3.0
    - ▶ 2.0 → 1.7
    - ▶ 3.7 → 3.3

Assessment

Assignment can be used to improve you grade

- ▶ If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10}\text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \le 4 \\ x & \text{otw.} \end{cases}$$

- ▶ It will improve by 0.3 or 0.4, respectively.
  Examples:
  - ▶ 3.3 → 3.0
  - ▶ 2.0 → 1.7
  - ▶ 3.7 → 3.3
  - ▶ 1.0 → 1.0

Assessment

Assignment can be used to improve you grade

- ▶ If you obtain a bonus your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10}\text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \le 4 \\ x & \text{otw.} \end{cases}$$

- ▶ It will improve by 0.3 or 0.4, respectively.
  Examples:
  - ▶ 3.3 → 3.0
  - ▶ 2.0 → 1.7
  - ▶ 3.7 → 3.3
  - ▶ 1.0 → 1.0
  - ▶ > 4.0 no improvement

Assessment

Assignment can be used to improve you grade

Requirements for Bonus

- ▶ 50% of the points are achieved on submissions 2-8,
- ▶ 50% of the points are achieved on submissions 9-14,
- ▶ each group member has written at least 4 solutions.

1 Contents

- ▶ Foundations
  - ▶ Machine models
  - ▶ Efficiency measures
  - ▶ Asymptotic notation
  - ▶ Recursion

1 Contents

- ▶ Foundations
  - ▶ Machine models
  - ▶ Efficiency measures
  - ▶ Asymptotic notation
  - ▶ Recursion
- ▶ Higher Data Structures
  - ▶ Search trees
  - ▶ Hashing
  - ▶ Priority queues
  - ▶ Union/Find data structures

1 Contents

- ▶ Foundations
  - ▶ Machine models
  - ▶ Efficiency measures
  - ▶ Asymptotic notation
  - ▶ Recursion
- ▶ Higher Data Structures
  - ▶ Search trees
  - ▶ Hashing
  - ▶ Priority queues
  - ▶ Union/Find data structures
- ▶ Cuts/Flows

1 Contents

- ▶ Foundations
  - ▶ Machine models
  - ▶ Efficiency measures
  - ▶ Asymptotic notation
  - ▶ Recursion
- ▶ Higher Data Structures
  - ▶ Search trees
  - ▶ Hashing
  - ▶ Priority queues
  - ▶ Union/Find data structures
- ▶ Cuts/Flows
- ▶ Matchings

2 Literatur

📄 Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman:
The design and analysis of computer algorithms,
Addison-Wesley Publishing Company: Reading (MA), 1974

📄 Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest,
Clifford Stein:
Introduction to algorithms,
McGraw-Hill, 1990

📄 Michael T. Goodrich, Roberto Tamassia:
Algorithm design: Foundations, analysis, and internet examples,
John Wiley & Sons, 2002

2 Literatur

📑 Ronald L. Graham, Donald E. Knuth, Oren Patashnik:
Concrete Mathematics,
2. Auflage, Addison-Wesley, 1994

📑 Volker Heun:
Grundlegende Algorithmen: Einführung in den Entwurf und die Analyse effizienter Algorithmen,
2. Auflage, Vieweg, 2003

📑 Jon Kleinberg, Eva Tardos:
Algorithm Design,
Addison-Wesley, 2005

📑 Donald E. Knuth:
The art of computer programming. Vol. 1: Fundamental Algorithms,
3. Auflage, Addison-Wesley, 1997

2 Literatur

📄 Donald E. Knuth:
The art of computer programming. Vol. 3: Sorting and Searching,
3. Auflage, Addison-Wesley, 1997

📄 Christos H. Papadimitriou, Kenneth Steiglitz:
Combinatorial Optimization: Algorithms and Complexity,
Prentice Hall, 1982

📄 Uwe Schöning:
Algorithmik,
Spektrum Akademischer Verlag, 2001

📄 Steven S. Skiena:
The Algorithm Design Manual,
Springer, 1998

Part II

Foundations

3 Goals

▶ Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.

3 Goals

▶ Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.

▶ Learn how to analyze and judge the efficiency of algorithms.

3 Goals

▶ Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.

▶ Learn how to analyze and judge the efficiency of algorithms.

▶ Learn how to design efficient algorithms.

4 Modelling Issues

What do you measure?

▶ Memory requirement

4 Modelling Issues

What do you measure?

▶ Memory requirement

▶ Running time

4 Modelling Issues

What do you measure?

- ▶ Memory requirement
- ▶ Running time
- ▶ Number of comparisons

4 Modelling Issues

What do you measure?

- ▶ Memory requirement
- ▶ Running time
- ▶ Number of comparisons
- ▶ Number of multiplications

4 Modelling Issues

What do you measure?

- ▶ Memory requirement
- ▶ Running time
- ▶ Number of comparisons
- ▶ Number of multiplications
- ▶ Number of hard-disc accesses

4 Modelling Issues

What do you measure?

▶ Memory requirement

▶ Running time

▶ Number of comparisons

▶ Number of multiplications

▶ Number of hard-disc accesses

▶ Program size

4 Modelling Issues

What do you measure?

- ▶ Memory requirement
- ▶ Running time
- ▶ Number of comparisons
- ▶ Number of multiplications
- ▶ Number of hard-disc accesses
- ▶ Program size
- ▶ Power consumption

4 Modelling Issues

What do you measure?

▶ Memory requirement
▶ Running time
▶ Number of comparisons
▶ Number of multiplications
▶ Number of hard-disc accesses
▶ Program size
▶ Power consumption
▶ . . .

4 Modelling Issues

How do you measure?

4 Modelling Issues

How do you measure?

▶ Implementing and testing on representative inputs
  ▶ How do you choose your inputs?
  ▶ May be very time-consuming.
  ▶ Very reliable results if done correctly.
  ▶ Results only hold for a specific machine and for a specific set of inputs.

4 Modelling Issues

How do you measure?

- Implementing and testing on representative inputs
    - How do you choose your inputs?
    - May be very time-consuming.
    - Very reliable results if done correctly.
    - Results only hold for a specific machine and for a specific set of inputs.

- Theoretical analysis in a specific model of computation.
    - Gives asymptotic bounds like "this algorithm always runs in time $\mathcal{O}(n^2)$".
    - Typically focuses on the worst case.
    - Can give lower bounds like "any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case".

4 Modelling Issues

Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

4 Modelling Issues

Input length
The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

4 Modelling Issues

Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

▶ the size of the input (number of bits)

4 Modelling Issues

Input length
The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

- ▶ the size of the input (number of bits)
- ▶ the number of arguments

4 Modelling Issues

Input length
The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

▶ the size of the input (number of bits)

▶ the number of arguments

Example 1

Suppose n numbers from the interval $\{1, \ldots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

Model of Computation

How to measure performance

Model of Computation

How to measure performance

1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), . . .

Model of Computation

How to measure performance

1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), . . .

2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, . . .

Model of Computation

How to measure performance

1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), . . .

2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, . . .

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

Turing Machine

▶ Very simple model of computation.
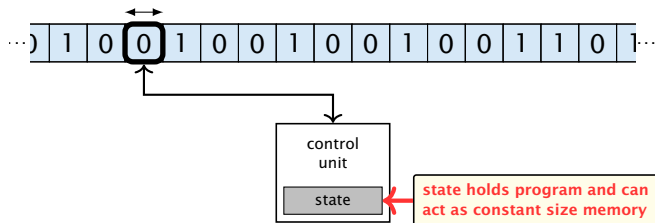
Turing Machine

- ▶ Very simple model of computation.
- ▶ Only the "current" memory location can be altered.
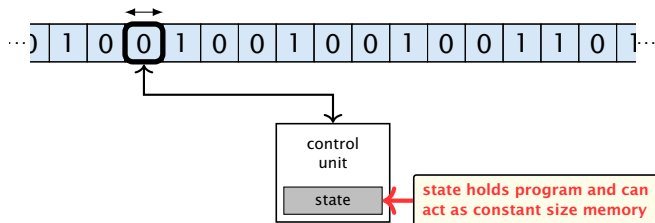
Turing Machine

- ▶ Very simple model of computation.
- ▶ Only the "current" memory location can be altered.
- ▶ Very good model for discussing computabiliy, or polynomial vs. exponential time.
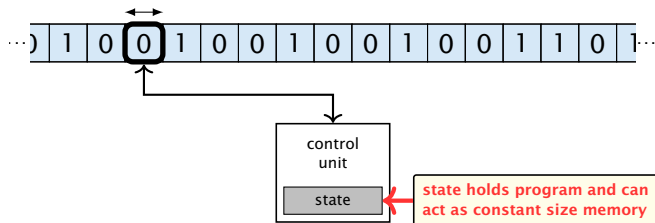
Turing Machine

- ▶ Very simple model of computation.
- ▶ Only the "current" memory location can be altered.
- ▶ Very good model for discussing computabiliy, or polynomial vs. exponential time.
- ▶ Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.
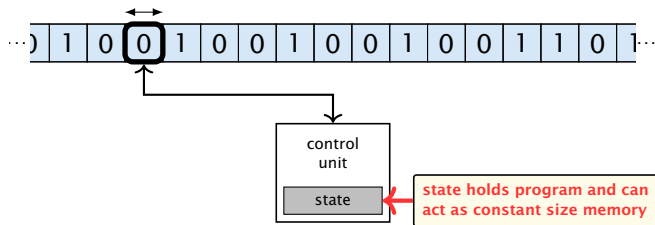
Turing Machine

▶ Very simple model of computation.

▶ Only the "current" memory location can be altered.

▶ Very good model for discussing computabiliy, or polynomial vs. exponential time.

▶ Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.

⟹ **Not a good model for developing efficient algorithms.**
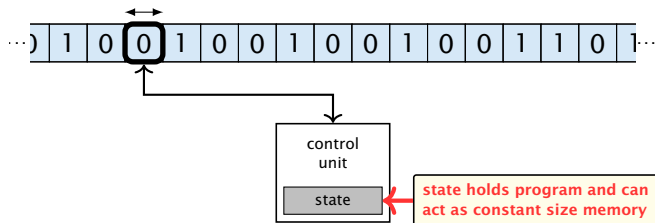
Random Access Machine (RAM)

▶ Input tape and output tape (sequences of zeros and ones; unbounded length).
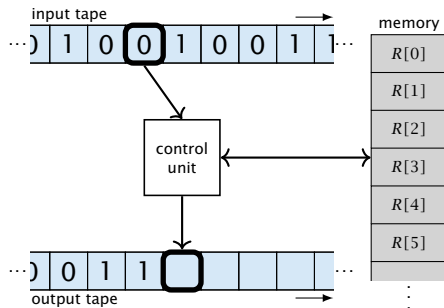
Random Access Machine (RAM)

- ▶ Input tape and output tape (sequences of zeros and ones; unbounded length).
- ▶ Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \ldots$.
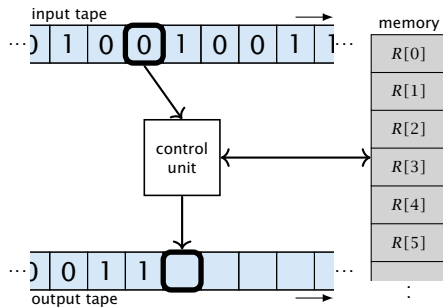
Random Access Machine (RAM)

▶ Input tape and output tape (sequences of zeros and ones; unbounded length).

▶ Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \dots$.

▶ Registers hold integers.
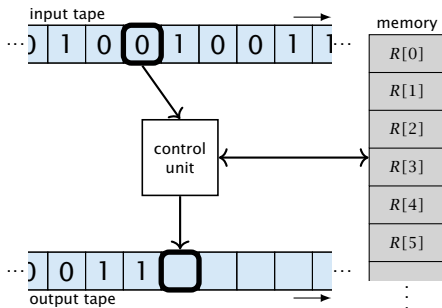
Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \dots$.
- Registers hold integers.
- Indirect addressing.
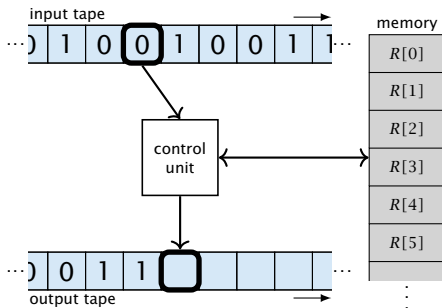
Random Access Machine (RAM)

Operations

- ▶ input operations (input tape → $R[i]$)

Random Access Machine (RAM)

Operations

▶ input operations (input tape → $R[i]$)

  ▶ READ i

Random Access Machine (RAM)

Operations

- input operations (input tape $\rightarrow R[i]$)
    - READ i
- output operations ($R[i] \rightarrow$ output tape)

Random Access Machine (RAM)

Operations
- input operations (input tape $\rightarrow R[i]$)
    - READ i
- output operations ($R[i] \rightarrow$ output tape)
    - WRITE i

Random Access Machine (RAM)

Operations

- ▶ input operations (input tape → $R[i]$)
  - ▶ READ i
- ▶ output operations ($R[i]$ → output tape)
  - ▶ WRITE i
- ▶ register-register transfers

Random Access Machine (RAM)

Operations

- ▶ input operations (input tape $\rightarrow R[i]$)
  - ▶ READ i
- ▶ output operations ($R[i] \rightarrow$ output tape)
  - ▶ WRITE i
- ▶ register-register transfers
  - ▶ $R[j]$:= $R[i]$

Random Access Machine (RAM)

Operations

▶ input operations (input tape $\rightarrow R[i]$)
  ▶ READ i
▶ output operations ($R[i] \rightarrow$ output tape)
  ▶ WRITE i
▶ register-register transfers
  ▶ $R[j]$:= $R[i]$
  ▶ $R[j]$:= 4

Random Access Machine (RAM)

Operations

▶ input operations (input tape $\rightarrow R[i]$)
  ▶ READ i
▶ output operations ($R[i] \rightarrow$ output tape)
  ▶ WRITE i
▶ register-register transfers
  ▶ $R[j]$:= $R[i]$
  ▶ $R[j]$:= 4
▶ **indirect** addressing

Random Access Machine (RAM)

Operations

- ▶ input operations (input tape → $R[i]$)
    - ▶ READ i
- ▶ output operations ($R[i]$ → output tape)
    - ▶ WRITE i
- ▶ register-register transfers
    - ▶ $R[j]$:= $R[i]$
    - ▶ $R[j]$:= 4
- ▶ **indirect** addressing
    - ▶ $R[j]$:= $R[R[i]]$
      loads the content of the $R[i]$-th register into the j-th register

Random Access Machine (RAM)

Operations

▶ input operations (input tape → $R[i]$)
  ▶ READ i
▶ output operations ($R[i]$ → output tape)
  ▶ WRITE i
▶ register-register transfers
  ▶ $R[j]$:= $R[i]$
  ▶ $R[j]$:= 4
▶ **indirect** addressing
  ▶ $R[j]$:= $R[R[i]]$
    loads the content of the $R[i]$-th register into the j-th register
  ▶ $R[R[i]]$:= $R[j]$
    loads the content of the j-th into the $R[i]$-th register

Random Access Machine (RAM)

Operations

- ▶ branching (including loops) based on comparisons

Random Access Machine (RAM)

Operations

▶ branching (including loops) based on comparisons

▶ jump x
jumps to position x in the program;
sets instruction counter to x;
reads the next operation to perform from register $R[x]$

Random Access Machine (RAM)

Operations

▶ branching (including loops) based on comparisons

    ▶ jump x
       jumps to position x in the program;
       sets instruction counter to x;
       reads the next operation to perform from register $R[x]$

    ▶ jumpz x $R[i]$
       jump to x if $R[i] = 0$
       if not the instruction counter is increased by 1;

Random Access Machine (RAM)

Operations

▶ branching (including loops) based on comparisons

    ▶ jump x

    jumps to position x in the program;

    sets instruction counter to x;

    reads the next operation to perform from register $R[x]$

    ▶ jumpz x $R[i]$

    jump to x if $R[i] = 0$

    if not the instruction counter is increased by 1;

    ▶ jumpi i

    jump to $R[i]$ (indirect jump);

Random Access Machine (RAM)

Operations

▶ branching (including loops) based on comparisons
  ▶ jump x
    jumps to position x in the program;
    sets instruction counter to x;
    reads the next operation to perform from register $R[x]$
  ▶ jumpz x $R[i]$
    jump to x if $R[i] = 0$
    if not the instruction counter is increased by 1;
  ▶ jumpi i
    jump to $R[i]$ (indirect jump);
▶ arithmetic instructions: $+$, $-$, $\times$, $/$

Random Access Machine (RAM)

Operations

▶ branching (including loops) based on comparisons

- ▶ jump x
  jumps to position x in the program;
  sets instruction counter to x;
  reads the next operation to perform from register $R[x]$
- ▶ jumpz x $R[i]$
  jump to x if $R[i] = 0$
  if not the instruction counter is increased by 1;
- ▶ jumpi i
  jump to $R[i]$ (indirect jump);

▶ arithmetic instructions: $+$, $-$, $\times$, $/$

- ▶ $R[i] := R[j] + R[k]$;
  $R[i] := -R[k]$;

Model of Computation

- **uniform** cost model
  Every operation takes time 1.

Model of Computation

- ▶ uniform cost model
  Every operation takes time 1.

- ▶ logarithmic cost model
  The cost depends on the content of memory cells:

Model of Computation

- ▶ uniform cost model
  Every operation takes time 1.

- ▶ logarithmic cost model
  The cost depends on the content of memory cells:
  - ▶ The time for a step is equal to the largest operand involved;

Model of Computation

▶ **uniform** cost model
Every operation takes time 1.

▶ **logarithmic** cost model
The cost depends on the content of memory cells:

  ▶ The time for a step is equal to the largest operand involved;
  ▶ The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

Model of Computation

▶ uniform cost model
Every operation takes time 1.

▶ logarithmic cost model
The cost depends on the content of memory cells:

▶ The time for a step is equal to the largest operand involved;
▶ The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)
1: $r \leftarrow 2$;
2: **for** $i = 1 \rightarrow n$ **do**
3: $\qquad r \leftarrow r^2$
4: **return** r

4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

1: $r \leftarrow 2;$
2: **for** $i = 1 \rightarrow n$ **do**
3:     $r \leftarrow r^2$
4: **return** r

▶ running time (for Line 3):

4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)
1: $r \leftarrow 2$;
2: **for** $i = 1 \rightarrow n$ **do**
3: $\qquad r \leftarrow r^2$
4: **return** r

▶ running time (for Line 3):
  ▶ uniform model: n steps

4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

1: $r \leftarrow 2$;
2: **for** $i = 1 \to n$ **do**
3: $\quad\quad r \leftarrow r^2$
4: **return** r

▶ running time (for Line 3):
  ▶ uniform model: n steps
  ▶ logarithmic model:
    $2 + 3 + 5 + \cdots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$

4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)
1: $r \leftarrow 2$;
2: **for** $i = 1 \to n$ **do**
3:      $r \leftarrow r^2$
4: **return** r

▶ running time (for Line 3):
  ▶ uniform model: n steps
  ▶ logarithmic model:
    $2 + 3 + 5 + \cdots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$
▶ space requirement:

4 Modelling Issues

Example 2

> **Algorithm 1** RepeatedSquaring(n)
> ---
> 1: $r \leftarrow 2$;
> 2: **for** $i = 1 \rightarrow n$ **do**
> 3: $r \leftarrow r^2$
> 4: **return** r

▶ running time (for Line 3):
  ▶ uniform model: n steps
  ▶ logarithmic model:
    $2 + 3 + 5 + \cdots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$
▶ space requirement:
  ▶ uniform model: $\mathcal{O}(1)$

4 Modelling Issues

Example 2

> **Algorithm 1** RepeatedSquaring(n)
> ___
> 1: $r \leftarrow 2$;
> 2: **for** $i = 1 \to n$ **do**
> 3: $r \leftarrow r^2$
> 4: **return** r

▶ running time (for Line 3):
- ▶ uniform model: n steps
- ▶ logarithmic model:
  $2 + 3 + 5 + \cdots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$

▶ space requirement:
- ▶ uniform model: $\mathcal{O}(1)$
- ▶ logarithmic model: $\mathcal{O}(2^n)$

There are different types of complexity bounds:

▶ best-case complexity:

$$C_{\mathrm{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

There are different types of complexity bounds:

▶ best-case complexity:

$$C_{\mathrm{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

▶ worst-case complexity:

$$C_{\mathrm{wc}}(n) := \max\{C(x) \mid |x| = n\}$$

Usually moderately easy to analyze; sometimes too pessimistic.

There are different types of complexity bounds:

▶ best-case complexity:

$$C_{\mathrm{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

▶ worst-case complexity:

$$C_{\mathrm{wc}}(n) := \max\{C(x) \mid |x| = n\}$$

Usually moderately easy to analyze; sometimes too pessimistic.

▶ average case complexity:

$$C_{\mathrm{avg}}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x)$$

There are different types of complexity bounds:

▶ best-case complexity:

$$C_{\text{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

▶ worst-case complexity:

$$C_{\text{wc}}(n) := \max\{C(x) \mid |x| = n\}$$

Usually moderately easy to analyze; sometimes too pessimistic.

▶ average case complexity:

$$C_{\text{avg}}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x)$$

more general: probability measure μ

$$C_{\text{avg}}(n) := \sum_{x \in I_n} \mu(x) \cdot C(x)$$

There are different types of complexity bounds:

- ▶ amortized complexity:
  The average cost of data structure operations over a worst case sequence of operations.

There are different types of complexity bounds:

▶ amortized complexity:
  The average cost of data structure operations over a worst case sequence of operations.

▶ randomized complexity:
  The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x. Then take the worst-case over all x with $|x| = n$.

5 Asymptotic Notation

We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

5 Asymptotic Notation

We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

▶ We are usually interested in the running times for large values of n. Then constant additive terms do not play an important role.

5 Asymptotic Notation

We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

- ▶ We are usually interested in the running times for large values of n. Then constant additive terms do not play an important role.

- ▶ An exact analysis (e.g. *exactly* counting the number of operations in a RAM) may be hard, but wouldn't lead to more precise results as the computational model is already quite a distance from reality.

5 Asymptotic Notation

We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

▶ We are usually interested in the running times for large values of n. Then constant additive terms do not play an important role.

▶ An exact analysis (e.g. *exactly* counting the number of operations in a RAM) may be hard, but wouldn't lead to more precise results as the computational model is already quite a distance from reality.

▶ A linear speed-up (i.e., by a constant factor) is always possible by e.g. implementing the algorithm on a faster machine.

5 Asymptotic Notation

We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

- ▶ We are usually interested in the running times for large values of n. Then constant additive terms do not play an important role.

- ▶ An exact analysis (e.g. *exactly* counting the number of operations in a RAM) may be hard, but wouldn't lead to more precise results as the computational model is already quite a distance from reality.

- ▶ A linear speed-up (i.e., by a constant factor) is always possible by e.g. implementing the algorithm on a faster machine.

- ▶ Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0 \ \exists n_0 \in \mathbb{N}_0 \ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
  (set of functions that asymptotically grow not faster than f)

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
(set of functions that asymptotically grow not faster than f)

▶ $\Omega(f) = \{g \mid \exists c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
(set of functions that asymptotically grow not slower than f)

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
(set of functions that asymptotically grow not faster than f)

▶ $\Omega(f) = \{g \mid \exists c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
(set of functions that asymptotically grow not slower than f)

▶ $\Theta(f) = \Omega(f) \cap \mathcal{O}(f)$
(functions that asymptotically have the same growth as f)

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

▶ $\mathcal{O}(f) = \{g \mid \exists c > 0\; \exists n_0 \in \mathbb{N}_0\; \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
(set of functions that asymptotically grow not faster than f)

▶ $\Omega(f) = \{g \mid \exists c > 0\; \exists n_0 \in \mathbb{N}_0\; \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
(set of functions that asymptotically grow not slower than f)

▶ $\Theta(f) = \Omega(f) \cap \mathcal{O}(f)$
(functions that asymptotically have the same growth as f)

▶ $o(f) = \{g \mid \forall c > 0\; \exists n_0 \in \mathbb{N}_0\; \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
(set of functions that asymptotically grow slower than f)

Asymptotic Notation

Formal Definition

Let f, g denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
  (set of functions that asymptotically grow not faster than f)

▶ $\Omega(f) = \{g \mid \exists c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
  (set of functions that asymptotically grow not slower than f)

▶ $\Theta(f) = \Omega(f) \cap \mathcal{O}(f)$
  (functions that asymptotically have the same growth as f)

▶ $o(f) = \{g \mid \forall c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
  (set of functions that asymptotically grow slower than f)

▶ $\omega(f) = \{g \mid \forall c > 0 \; \exists n_0 \in \mathbb{N}_0 \; \forall n \geq n_0 : [g(n) \geq c \cdot f(n)]\}$
  (set of functions that asymptotically grow faster than f)

Asymptotic Notation

There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

▶ $g \in \mathcal{O}(f)$: $\quad 0 \leq \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

Asymptotic Notation

There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

▶ $g \in \mathcal{O}(f)$: $\quad 0 \le \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

▶ $g \in \Omega(f)$: $\quad 0 < \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} \le \infty$

Asymptotic Notation

There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

▶ $g \in \mathcal{O}(f)$: $\quad 0 \leq \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

▶ $g \in \Omega(f)$: $\quad 0 < \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} \leq \infty$

▶ $g \in \Theta(f)$: $\quad 0 < \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

Asymptotic Notation

There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

▶ $g \in \mathcal{O}(f)$: $\quad 0 \le \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

▶ $g \in \Omega(f)$: $\quad 0 < \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} \le \infty$

▶ $g \in \Theta(f)$: $\quad 0 < \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

▶ $g \in o(f)$: $\quad \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} = 0$

Asymptotic Notation

There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

▶ $g \in \mathcal{O}(f)$: $0 \le \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

▶ $g \in \Omega(f)$: $0 < \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} \le \infty$

▶ $g \in \Theta(f)$: $0 < \lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} < \infty$

▶ $g \in o(f)$: $\lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} = 0$

▶ $g \in \omega(f)$: $\lim\limits_{n \to \infty} \dfrac{g(n)}{f(n)} = \infty$

Asymptotic Notation

Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

Asymptotic Notation

Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f : \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g : \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.

Asymptotic Notation

Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f : \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g : \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.

3. People write e.g. $h(n) = f(n) + o(g(n))$ when they mean that there exists a function $z : \mathbb{N} \to \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) = f(n) + z(n)$.

Asymptotic Notation

Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f : \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g : \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.

3. People write e.g. $h(n) = f(n) + o(g(n))$ when they mean that there exists a function $z : \mathbb{N} \to \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) = f(n) + z(n)$.

4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here, $\Theta(n)$ stands for an anonymous function in the set $\Theta(n)$ that makes the expression true.

Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here, $\Theta(n)$ stands for an anonymous function in the set $\Theta(n)$ that makes the expression true.

Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

Asymptotic Notation in Equations

How do we interpret an expression like:

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

Asymptotic Notation in Equations

How do we interpret an expression like:

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

Careful!

Asymptotic Notation in Equations

How do we interpret an expression like:

$$\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$$

Careful!

"It is understood" that every occurence of an $\mathcal{O}$-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \cdots + \Theta(n-1) + \Theta(n)$$

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{ f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)$$
$$\text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \Big\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0 \; \forall n \geq n_0 : f(n) > 0$ *(the same for g). Then*

▶ $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0 \, \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0 \, \forall n \geq n_0 : f(n) > 0$ *(the same for g). Then*

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
- $\mathcal{O}(f(n)) \cdot \mathcal{O}(g(n)) = \mathcal{O}(f(n) \cdot g(n))$

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0 \, \forall n \geq n_0 : f(n) > 0$ *(the same for g). Then*

- ▶ $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*
- ▶ $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
- ▶ $\mathcal{O}(f(n)) \cdot \mathcal{O}(g(n)) = \mathcal{O}(f(n) \cdot g(n))$
- ▶ $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(\max\{f(n), g(n)\})$

Asymptotic Notation

Lemma 3

Let f, g be functions with the property
$\exists n_0 > 0 \, \forall n \geq n_0 : f(n) > 0$ *(the same for g). Then*

- ▶ $c \cdot f(n) \in \Theta(f(n))$ *for any constant c*
- ▶ $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
- ▶ $\mathcal{O}(f(n)) \cdot \mathcal{O}(g(n)) = \mathcal{O}(f(n) \cdot g(n))$
- ▶ $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(\max\{f(n), g(n)\})$

The expressions also hold for Ω. Note that this means that
$f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

Asymptotic Notation

Comments

▶ Do not use asymptotic notation within induction proofs.

Asymptotic Notation

Comments

▶ Do not use asymptotic notation within induction proofs.

▶ For any constants a, b we have $\log_a n = \Theta(\log_b n)$.
Therefore, we will usually ignore the base of a logarithm
within asymptotic notation.

Asymptotic Notation

Comments

▶ Do not use asymptotic notation within induction proofs.

▶ For any constants a, b we have $\log_a n = \Theta(\log_b n)$.
  Therefore, we will usually ignore the base of a logarithm
  within asymptotic notation.

▶ In general $\log n = \log_2 n$, i.e., we use 2 as the default base
  for the logarithm.

Funktionen
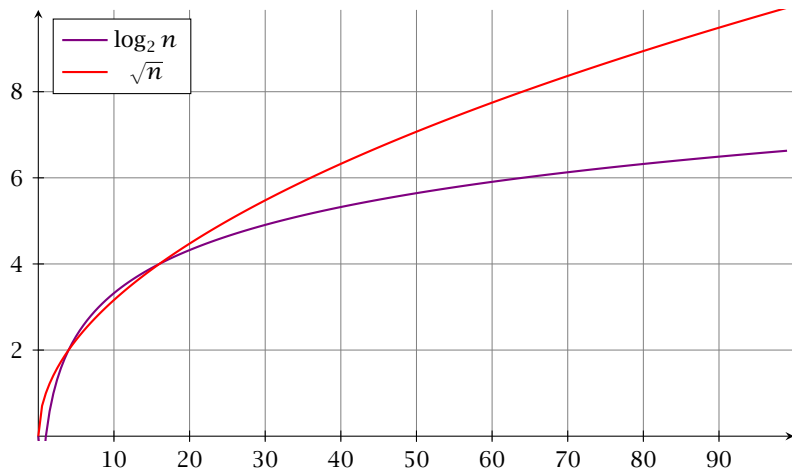
Funktionen
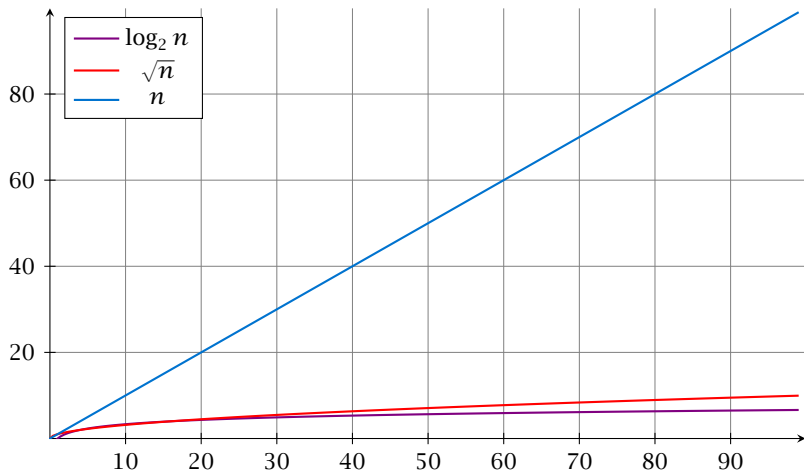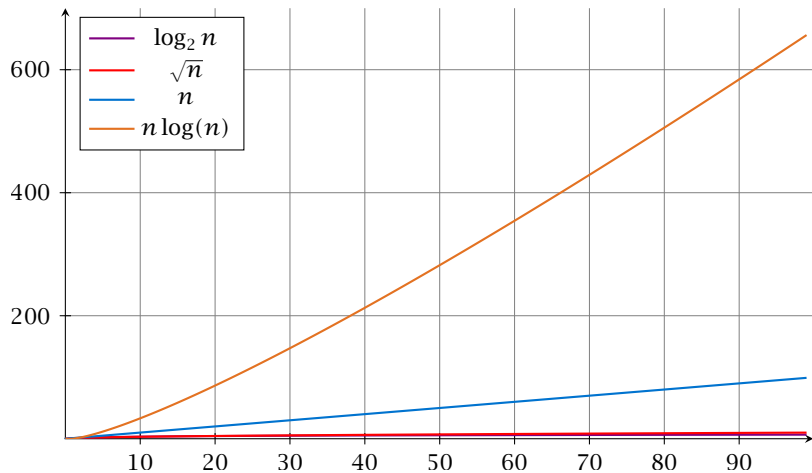
Funktionen
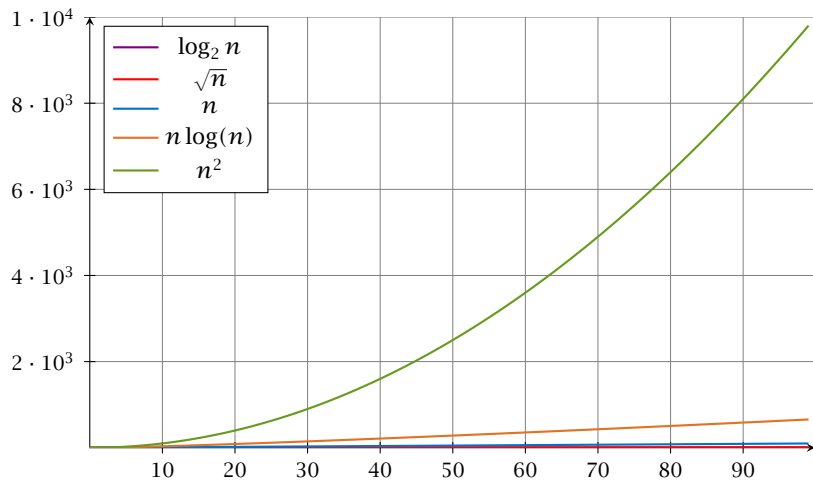
Funktionen
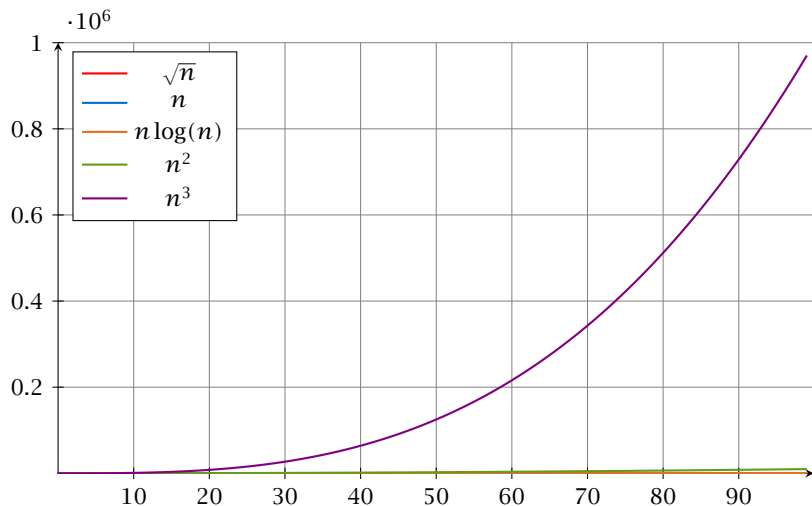
Funktionen
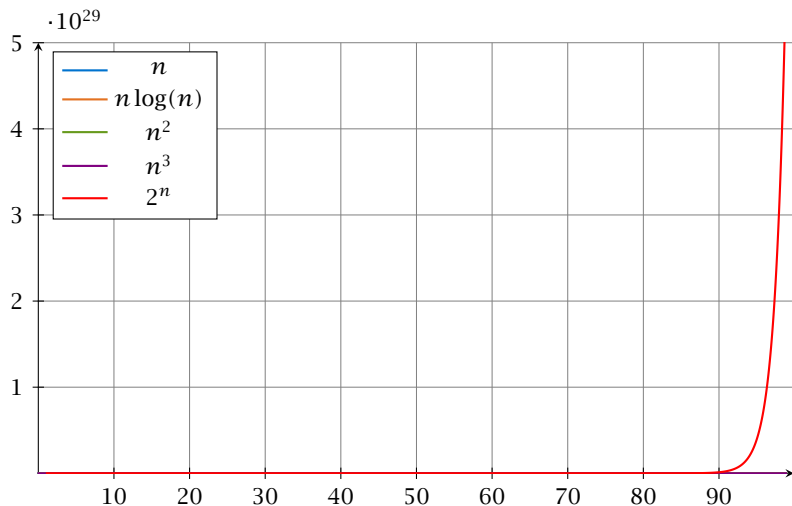
Funktionen

Laufzeiten

Funktion	Eingabelänge n							
$f(n)$	**10**	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\log n$	33ns	66ns	0.1µs	0.1µs	0.2µs	0.2µs	0.2µs	0.3µs
$\sqrt{n}$	32ns	0.1µs	0.3µs	1µs	3.1µs	10µs	31µs	0.1ms
n	100ns	1µs	10µs	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3µs	6.6µs	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3µs	10µs	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1µs	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10µs	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$y					
2^n	10µs	$4 \cdot 10^{14}$y						
$n!$	36ms	$3 \cdot 10^{142}$y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$y

Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

- ▶ However, suppose that I have two algorithms:

Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

- ▶ However, suppose that I have two algorithms:
  - ▶ Algorithm A. Running time $f(n) = 1000 \log n = \mathcal{O}(\log n)$.

Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

- ▶ However, suppose that I have two algorithms:
  - ▶ Algorithm A. Running time $f(n) = 1000 \log n = \mathcal{O}(\log n)$.
  - ▶ Algorithm B. Running time $g(n) = \log^2 n$.

Asymptotic Notation

In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.

- ▶ However, suppose that I have two algorithms:
  - ▶ Algorithm A. Running time $f(n) = 1000 \log n = \mathcal{O}(\log n)$.
  - ▶ Algorithm B. Running time $g(n) = \log^2 n$.

  Clearly $f = o(g)$. However, as long as $\log n \le 1000$ Algorithm B will be more efficient.

Multiple Variables in Asymptotic Notation

Sometimes the input for an algorithm consists of several parameters (e.g., nodes and edges of a graph (n and m)).

Multiple Variables in Asymptotic Notation

Sometimes the input for an algorithm consists of several parameters (e.g., nodes and edges of a graph (n and m)).

If we want to make asympotic statements for $n \to \infty$ and $m \to \infty$ we have to extend the definition to multiple variables.

Multiple Variables in Asymptotic Notation

Sometimes the input for an algorithm consists of several parameters (e.g., nodes and edges of a graph (n and m)).

If we want to make asympotic statements for $n \to \infty$ and $m \to \infty$ we have to extend the definition to multiple variables.

Formal Definition

Let f, g denote functions from $\mathbb{N}^d$ to $\mathbb{R}_0^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0 \; \exists N \in \mathbb{N}_0 \; \forall \vec{n} \text{ with } n_i \geq N \text{ for some } i :$
  $$[g(\vec{n}) \leq c \cdot f(\vec{n})]\}$$
  (set of functions that asymptotically grow not faster than f)

Multiple Variables in Asymptotic Notation

Example 4

▶ $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n - 1$

Multiple Variables in Asymptotic Notation

Example 4

▶ $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n - 1$
   then $f = \mathcal{O}(g)$ does not hold

Multiple Variables in Asymptotic Notation

Example 4

▶ $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n - 1$
  then $f = \mathcal{O}(g)$ does not hold

▶ $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n$

Multiple Variables in Asymptotic Notation

Example 4

- $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n,m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n,m) = n-1$
  then $f = \mathcal{O}(g)$ does not hold

- $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n,m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n,m) = n$
  then: $f = \mathcal{O}(g)$

Multiple Variables in Asymptotic Notation

Example 4

► $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n - 1$
  then $f = \mathcal{O}(g)$ does not hold

► $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n$
  then: $f = \mathcal{O}(g)$

► $f : \mathbb{N}_0 \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N}_0 \to \mathbb{R}_0^+$, $g(n, m) = n$

Multiple Variables in Asymptotic Notation

Example 4

► $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n - 1$
   then $f = \mathcal{O}(g)$ does not hold

► $f : \mathbb{N} \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \to \mathbb{R}_0^+$, $g(n, m) = n$
   then: $f = \mathcal{O}(g)$

► $f : \mathbb{N}_0 \to \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N}_0 \to \mathbb{R}_0^+$, $g(n, m) = n$
   then $f = \mathcal{O}(g)$ does not hold

6 Recurrences

> **Algorithm 2** mergesort(list L)
>
> 1: $n \leftarrow \text{size}(L)$
> 2: **if** $n \leq 1$ **return** L
> 3: $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
> 4: $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
> 5: mergesort(L_1)
> 6: mergesort(L_2)
> 7: $L \leftarrow \text{merge}(L_1, L_2)$
> 8: **return** L

6 Recurrences

Algorithm 2 mergesort(list L)

1: $n \leftarrow \text{size}(L)$
2: **if** $n \leq 1$ **return** L
3: $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
4: $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
5: $\text{mergesort}(L_1)$
6: $\text{mergesort}(L_2)$
7: $L \leftarrow \text{merge}(L_1, L_2)$
8: **return** L

This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons
($\approx$ running time) into a closed form?

Recurrences

How do we bring the expression for the number of comparisons ($\approx$ running time) into a closed form?

For this we need to solve the recurrence.

Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \le \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

Assume that instead we have

$$T(n) \le \begin{cases} 2T(\frac{n}{2}) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \le \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

Assume that instead we have

$$T(n) \le \begin{cases} 2T(\frac{n}{2}) + cn & n \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d.

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d. Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d. Then

$$
\begin{aligned}
T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\
&\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn
\end{aligned}
$$

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d. Then

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$
$$\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn$$
$$= dn(\log n - 1) + cn$$

6.1 Guessing+Induction

Suppose we guess $T(n) \le dn \log n$ for a constant d. Then

$$
\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\
&\le 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn \\
&= dn(\log n - 1) + cn \\
&= dn\log n + (c-d)n
\end{aligned}
$$

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d. Then

$$
\begin{aligned}
T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\
&\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn \\
&= dn(\log n - 1) + cn \\
&= dn \log n + (c - d)n \\
&\leq dn \log n
\end{aligned}
$$

if we choose $d \geq c$.

6.1 Guessing+Induction

Suppose we guess $T(n) \leq dn \log n$ for a constant d. Then

$$
\begin{aligned}
T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\
&\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\
&= dn(\log n - 1) + cn \\
&= dn \log n + (c - d)n \\
&\leq dn \log n
\end{aligned}
$$

if we choose $d \geq c$.

Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

6.1 Guessing+Induction

Guess: $T(n) \leq dn \log n$.

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

▶ **base case** $(2 \leq n < 16)$:

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

▶ **base case** ($2 \leq n < 16$): true if we choose $d \geq b$.

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

- **base case** ($2 \leq n < 16$): true if we choose $d \geq b$.
- **induction step** $n/2 \to n$:

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

▶ **base case** $(2 \leq n < 16)$: true if we choose $d \geq b$.

▶ **induction step** $n/2 \rightarrow n$:

Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

- ▶ **base case** $(2 \leq n < 16)$: true if we choose $d \geq b$.

- ▶ **induction step** $n/2 \to n$:

  Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

  $$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

- ▶ **base case** $(2 \leq n < 16)$: true if we choose $d \geq b$.

- ▶ **induction step** $n/2 \to n$:

  Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

  $$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$
  $$\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn$$

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T\left(\frac{n}{2}\right) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

▶ **base case** $(2 \leq n < 16)$: true if we choose $d \geq b$.

▶ **induction step** $n/2 \to n$:

Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$$\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn$$

$$= dn(\log n - 1) + cn$$

6.1 Guessing+Induction

$$T(n) \le \begin{cases} 2T(\frac{n}{2}) + cn & n \ge 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \le dn \log n$.

Proof. (by induction)

▶ **base case** ($2 \le n < 16$): true if we choose $d \ge b$.

▶ **induction step** $n/2 \to n$:

Let $n = 2^k \ge 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

$$\begin{aligned} T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\ &\le 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn \log n + (c - d)n \end{aligned}$$

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

▶ **base case** $(2 \leq n < 16)$: true if we choose $d \geq b$.

▶ **induction step** $n/2 \to n$:

Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2}\log\frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn\log n + (c - d)n \\ &\leq dn\log n \end{aligned}$$

6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

Guess: $T(n) \leq dn \log n$.

Proof. (by induction)

▶ **base case** $(2 \leq n < 16)$: true if we choose $d \geq b$.

▶ **induction step** $n/2 \to n$:

Let $n = 2^k \geq 16$. Suppose statem. is true for $n' = n/2$. We prove it for n:

$$\begin{aligned} T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\ &\leq 2\left(d\frac{n}{2} \log \frac{n}{2}\right) + cn \\ &= dn(\log n - 1) + cn \\ &= dn \log n + (c - d)n \\ &\leq dn \log n \end{aligned}$$

Hence, statement is true if we choose $d \geq c$.

How do we get a result for all values of n?

6.1 Guessing+Induction

How do we get a result for all values of n?

We consider the following recurrence instead of the original one:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 16 \\ b & \text{otherwise} \end{cases}$$

6.1 Guessing+Induction

How do we get a result for all values of n?

We consider the following recurrence instead of the original one:

$$T(n) \le \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \ge 16 \\ b & \text{otherwise} \end{cases}$$

Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n)$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$
$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1}$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \quad \leq 2(d(n/2 + 1)\log(n/2 + 1)) + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \quad \leq 2\left(d(n/2 + 1) \log(n/2 + 1)\right) + cn$$

$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n}$$

6.1 Guessing+Induction

We also make a guess of $T(n) \le dn \log n$ and get

$$T(n) \le 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\le 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \le \frac{n}{2} + 1} \quad \le 2(d(n/2 + 1)\log(n/2 + 1)) + cn$$

$$\boxed{\frac{n}{2} + 1 \le \frac{9}{16}n} \quad \le dn \log \left(\frac{9}{16}n\right) + 2d \log n + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \quad \leq 2(d(n/2 + 1)\log(n/2 + 1)) + cn$$

$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n} \quad \leq dn \log \left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)}$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \quad \leq 2(d(n/2 + 1) \log(n/2 + 1)) + cn$$

$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n} \quad \leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} \quad = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \quad \leq 2(d(n/2 + 1) \log(n/2 + 1)) + cn$$

$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n} \quad \leq dn \log \left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} \quad = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\boxed{\log n \leq \frac{n}{4}}$$

6.1 Guessing+Induction

We also make a guess of $T(n) \le dn \log n$ and get

$$T(n) \le 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\le 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \le \frac{n}{2} + 1} \quad \le 2(d(n/2 + 1)\log(n/2 + 1)) + cn$$

$$\boxed{\frac{n}{2} + 1 \le \frac{9}{16}n} \quad \le dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} \quad = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\boxed{\log n \le \frac{n}{4}} \quad \le dn \log n + (\log 9 - 3.5)dn + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1}$ $\quad \leq 2(d(n/2 + 1)\log(n/2 + 1)) + cn$

$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n}$ $\quad \leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$

$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)}$ $\quad = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$

$\boxed{\log n \leq \frac{n}{4}}$ $\quad \leq dn \log n + (\log 9 - 3.5)dn + cn$

$$\leq dn \log n - 0.33dn + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \le dn \log n$ and get

$$T(n) \le 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\le 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \le \frac{n}{2} + 1} \quad \le 2(d(n/2 + 1)\log(n/2 + 1)) + cn$$

$$\boxed{\frac{n}{2} + 1 \le \frac{9}{16}n} \quad \le dn \log\left(\frac{9}{16}n\right) + 2d\log n + cn$$

$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} \quad = dn\log n + (\log 9 - 4)dn + 2d\log n + cn$$

$$\boxed{\log n \le \frac{n}{4}} \quad \le dn\log n + (\log 9 - 3.5)dn + cn$$

$$\le dn\log n - 0.33dn + cn$$

$$\le dn\log n$$

for a suitable choice of d.

6.2 Master Theorem

Lemma 5

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.
If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.
If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
$k \geq 0$.

Case 3.
If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
$af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

$$\boxed{n}$$

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

$f(n)$

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \ .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}}$$

Case 1. Now suppose that $f(n) \le cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

Case 1. Now suppose that $f(n) \leq c n^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} \quad = c n^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1} - 1}{q - 1}} \quad = c n^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1)/(b^{\epsilon} - 1)$$

$$= c n^{\log_b a - \epsilon} (n^{\epsilon} - 1)/(b^{\epsilon} - 1)$$

$$= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^{\epsilon} - 1)/(n^{\epsilon})$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} = cn^{\log_b a - \epsilon}(b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b(a)}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i}(b^{\log_b a})^{-i} = b^{\epsilon i}a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^\epsilon)^i$$

$$\boxed{\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1)/(b^\epsilon - 1)$$

$$= cn^{\log_b a - \epsilon}(n^\epsilon - 1)/(b^\epsilon - 1)$$

$$= \frac{c}{b^\epsilon - 1} n^{\log_b a}(n^\epsilon - 1)/(n^\epsilon)$$

Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1\right) n^{\log_b(a)} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}$.

Now suppose that $f(n) \leq c n^{\log_b a}$.

$$T(n) - n^{\log_b a}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

Case 2. Now suppose that $f(n) \le c n^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1
\end{aligned}
$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= c n^{\log_b a} \log_b n
\end{aligned}
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$
T(n) = \mathcal{O}(n^{\log_b a} \log_b n)
$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\
&= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1 \\
&= cn^{\log_b a} \log_b n
\end{aligned}
$$

Hence,

$$
T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).}
$$

6.2 Master Theorem

Case 2. Now suppose that $f(n) \geq c n^{\log_b a}$.

Case 2. Now suppose that $f(n) \geq c n^{\log_b a}$.

$$T(n) - n^{\log_b a}$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Harald Räcke

Case 2. Now suppose that $f(n) \geq c n^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

Case 2. Now suppose that $f(n) \geq c n^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

Case 2. Now suppose that $f(n) \geq c n^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= c n^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= c n^{\log_b a} \log_b n$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

Case 2. Now suppose that $f(n) \geq cn^{\log_b a}$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\geq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}$$

$$= cn^{\log_b a} \sum_{i=0}^{\log_b n - 1} 1$$

$$= cn^{\log_b a} \log_b n$$

Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \qquad \boxed{\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

Case 2. Now suppose that $f(n) \le c n^{\log_b a} (\log_b(n))^k$.

$$T(n) - n^{\log_b a}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a} (\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a} (\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad = c n^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \\
\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\
&= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k
\end{aligned}$$

Case 2. Now suppose that $f(n) \le cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\le c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

Case 2. Now suppose that $f(n) \leq c n^{\log_b a} (\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$ $= c n^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$

$$= c n^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= c n^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \frac{1}{k} \ell^{k+1}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} \quad = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}(\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$\boxed{n = b^\ell \Rightarrow \ell = \log_b n}$ $\quad = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \qquad \boxed{\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 3. Now suppose that $f(n) \geq d n^{\log_b a + \epsilon}$, and that for sufficiently large n: $a f(n/b) \leq c f(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}}$$

Case 3. Now suppose that $f(n) \geq d n^{\log_b a + \epsilon}$, and that for sufficiently large n: $a f(n/b) \leq c f(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}} \quad &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

Case 3. Now suppose that $f(n) \geq d n^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

$$\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a})$$

$$\boxed{q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}} \quad \leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n: $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$
\begin{aligned}
T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\
&\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\
&\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a})
\end{aligned}
$$

$$\boxed{q < 1: \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n)) \qquad \boxed{\Rightarrow T(n) = \Theta(f(n)).}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \quad A$$

$$1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \quad B$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$\begin{array}{ccccccccc|c}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
\end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{\mathbf{1}} & \mathbf{1} \\
\hline
 & & & & & & & & \mathbf{0}
\end{array}
\quad
\begin{array}{c}
A \\
B
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
 & & & & & & & & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
 & & & & & & 0 & 0 & 0 &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \quad A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & \quad B \\
 & & & & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\
\hline
 & & & & & 1 & 0 & 0 & 0 &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
 & & & & 0 & 1 & 1 & 1 & \\
\hline
 & & & & & 1 & 0 & 0 & 0 &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
& & _1 & _1 & _0 & _1 & _1 & _1 & & \\
\hline
& & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
& 0 & 1 & 1 & 0 & 1 & 1 & 1 & & \\
\hline
& 1 & 0 & 0 & 1 & 0 & 0 & 0 & &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
& 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{cccccccccc}
 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &
\end{array}
$$

Example: Multiplying Two Integers

Suppose we want to multiply two n-bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B:

$$
\begin{array}{ccccccccc}
\color{red}{1} & \color{red}{1} & \color{red}{0} & \color{red}{1} & \color{red}{1} & \color{red}{0} & \color{red}{1} & \color{red}{0} & \color{red}{1} \\
\color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
\qquad
\begin{array}{c}
\color{red}{A} \\
\color{blue}{B} \\
\\
\end{array}
$$

This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$\underline{1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\mathbf{1} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{1} \quad \times \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \quad \boxed{\mathbf{1}}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\underline{1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ \boxed{1}}$$
$$1\ 0\ 0\ 0\ 1$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$\underline{1 \ 0 \ 0 \ 0 \ 1 \ \times \ 1 \ 0 \ \boxed{1} \ 1}$$
$$1 \ 0 \ 0 \ 0 \ 1$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$1 \; 0 \; 0 \; 0 \; 1 \; \times \; 1 \; 0 \; \boxed{1} \; 1$$
$$1 \; 0 \; 0 \; 0 \; 1$$
$$0$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & \boxed{1} & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & \boxed{0} & 1 & 1 \\
\hline
 & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$1\ 0\ 0\ 0\ 1 \times 1\ \boxed{0}\ 1\ 1$$

$$
\begin{array}{cccccc}
 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & 0 & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
  &   &   &   &   & 1 & 0 & 0 & 0 & 1 \\
  &   &   &   & 1 & 0 & 0 & 0 & 1 & 0 \\
  &   &   & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
 & 1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{r}
1 \ 0 \ 0 \ 0 \ 1 \ \times \ \boxed{1} \ 0 \ 1 \ 1 \\
\hline
1 \ 0 \ 0 \ 0 \ 1 \\
1 \ 0 \ 0 \ 0 \ 1 \ 0 \\
0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \\
0 \ 0 \ 0
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 1 & \times & \boxed{1} & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

```
      1  0  0  0  1  ×  1  0  1  1
                        1  0  0  0  1
                     1  0  0  0  1  0
                  0  0  0  0  0  0  0
               1  0  0  0  1  0  0  0
               1  0  1  1  1  0  1  1
```

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \leq n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Time requirement:

▶ Computing intermediate results: $\mathcal{O}(nm)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n-bit integer A and an m-bit integer B ($m \le n$).

$$
\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 \\
\hline
 & & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Time requirement:

▶ Computing intermediate results: $\mathcal{O}(nm)$.

▶ Adding m numbers of length $\le 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:
Suppose that integers A and B are of length $n = 2^k$, for some k.

Example: Multiplying Two Integers

A recursive approach:
Suppose that integers A and B are of length $n = 2^k$, for some k.

B	$\times$	A

Example: Multiplying Two Integers

A recursive approach:
Suppose that integers A and B are of length $n = 2^k$, for some k.

| b_{n-1} | $\cdots$ | b_0 | $\times$ | a_{n-1} | $\cdots$ | a_0 |

Example: Multiplying Two Integers

A recursive approach:
Suppose that integers A and B are of length $n = 2^k$, for some k.

| b_{n-1} | $\cdots$ | $b_{\frac{n}{2}}$ $b_{\frac{n}{2}-1}$ | $\cdots$ | b_0 | $\times$ | a_{n-1} | $\cdots$ | $a_{\frac{n}{2}}$ $a_{\frac{n}{2}-1}$ | $\cdots$ | a_0 |

Example: Multiplying Two Integers

A recursive approach:
Suppose that integers A and B are of length $n = 2^k$, for some k.

B_1	B_0	×	A_1	A_0

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

| B_1 | B_0 | $\times$ | A_1 | A_0 |

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k.

B_1	B_0	$\times$	A_1	A_0

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

> **Algorithm 3** mult(A, B)
>
> 1: **if** $|A| = |B| = 1$ **then**
> 2:   **return** $a_0 \cdot b_0$
> 3: split A into A_0 and A_1
> 4: split B into B_0 and B_1
> 5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
> 6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
> 7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
> 8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

> **Algorithm 3** mult(A, B)
> ___
> 1: **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
> 2:   **return** $a_0 \cdot b_0$
> 3: split A into A_0 and A_1
> 4: split B into B_0 and B_1
> 5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
> 6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
> 7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
> 8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

> **Algorithm 3** mult(A, B)
>
> 1: **if** $|A| = |B| = 1$ **then** $\quad\mathcal{O}(1)$
> 2:   **return** $a_0 \cdot b_0$ $\quad\mathcal{O}(1)$
> 3: split A into A_0 and A_1
> 4: split B into B_0 and B_1
> 5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
> 6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
> 7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
> 8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)	
1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	
5: $Z_2 \leftarrow$ mult(A_1, B_1)	
6: $Z_1 \leftarrow$ mult$(A_1, B_0) +$ mult(A_0, B_1)	
7: $Z_0 \leftarrow$ mult(A_0, B_0)	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$	
1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)	
1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)	
1: **if** $\lvert A \rvert = \lvert B \rvert = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow$ mult(A_1, B_1)	$T(\frac{n}{2})$
6: $Z_1 \leftarrow$ mult(A_1, B_0) + mult(A_0, B_1)	$2T(\frac{n}{2}) + \mathcal{O}(n)$
7: $Z_0 \leftarrow$ mult(A_0, B_0)	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)	
1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow$ mult(A_1, B_1)	$T(\frac{n}{2})$
6: $Z_1 \leftarrow$ mult(A_1, B_0) + mult(A_0, B_1)	$2T(\frac{n}{2}) + \mathcal{O}(n)$
7: $Z_0 \leftarrow$ mult(A_0, B_0)	$T(\frac{n}{2})$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$	
1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)	
1: **if** $\|A\| = \|B\| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	$2T(\frac{n}{2}) + \mathcal{O}(n)$
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$

▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- ▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\qquad T(n) = \Theta(n^{\log_b a})$

▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\;\; T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\qquad T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$

▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

$\Longrightarrow$ Not better then the "school method".

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - A_1 B_1 - A_0 B_0$$

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$
\begin{aligned}
Z_1 &= A_1 B_0 + A_0 B_1 \\
    &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1 B_1}_{= Z_2} - \underbrace{A_0 B_0}_{= Z_0}
\end{aligned}
$$

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$

$$= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1 B_1}_{= Z_2} - \underbrace{A_0 B_0}_{= Z_0}$$

Hence,

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}$$

Hence,

Algorithm 4 mult(A, B)

1: **if** $|A| = |B| = 1$ **then**
2:        **return** $a_0 \cdot b_0$
3: split A into A_0 and A_1
4: split B into B_0 and B_1
5: $Z_2 \leftarrow$ mult(A_1, B_1)
6: $Z_0 \leftarrow$ mult(A_0, B_0)
7: $Z_1 \leftarrow$ mult$(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1 \qquad \overbrace{= Z_2}^{} \quad \overbrace{= Z_0}^{}$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1 B_1}_{} - \underbrace{A_0 B_0}_{}$$

Hence,

Algorithm 4 mult(A, B)

1: **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
2:     **return** $a_0 \cdot b_0$
3: split A into A_0 and A_1
4: split B into B_0 and B_1
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1 \qquad \overbrace{}^{= Z_2} \; \overbrace{}^{= Z_0}$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - A_1 B_1 - A_0 B_0$$

Hence,

Algorithm 4 mult(A, B)

1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	
4: split B into B_0 and B_1	
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$

$$= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}$$

Hence,

Algorithm 4 mult(A, B)

1: **if** $\lvert A \rvert = \lvert B \rvert = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	
5: $Z_2 \leftarrow$ mult(A_1, B_1)	
6: $Z_0 \leftarrow$ mult(A_0, B_0)	
7: $Z_1 \leftarrow$ mult$(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1 \qquad \overbrace{= Z_2}^{} \quad \overbrace{= Z_0}^{}$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1 B_1}_{} - \underbrace{A_0 B_0}_{}$$

Hence,

Algorithm 4 mult(A, B)	
1: **if** $\|A\| = \|B\| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}$$

Hence,

Algorithm 4 mult(A, B)	
1: **if** $\lvert A \rvert = \lvert B \rvert = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow$ mult(A_1, B_1)	$T(\frac{n}{2})$
6: $Z_0 \leftarrow$ mult(A_0, B_0)	
7: $Z_1 \leftarrow$ mult$(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}$$

Hence,

Algorithm 4 mult(A, B)	
1: **if** $\|A\| = \|B\| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}$$

Hence,

Algorithm 4 mult(A, B)	
1: **if** $|A| = |B| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

We can use the following identity to compute Z_1:

$$Z_1 = A_1 B_0 + A_0 B_1$$
$$= (A_0 + A_1) \cdot (B_0 + B_1) - \overbrace{A_1 B_1}^{= Z_2} - \overbrace{A_0 B_0}^{= Z_0}$$

Hence,

Algorithm 4 mult(A, B)	
1: **if** $\|A\| = \|B\| = 1$ **then**	$\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	$T(\frac{n}{2})$
6: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	$T(\frac{n}{2})$
7: $Z_1 \leftarrow \text{mult}(A_0 + A_1, B_0 + B_1) - Z_2 - Z_0$	$T(\frac{n}{2}) + \mathcal{O}(n)$
8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\qquad T(n) = \Theta(n^{\log_b a})$

▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\qquad T(n) = \Theta(f(n))$

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$

▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of
$\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$

▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

▶ Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of
$\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

▶ $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

▶ $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.

▶ The recurrence is linear as there are no products of $T[n]$'s.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

▶ $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.

▶ The recurrence is linear as there are no products of $T[n]$'s.

▶ If $f(n) = 0$ then the recurrence relation becomes a linear, homogenous recurrence relation of order k.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

▶ $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.

▶ The recurrence is linear as there are no products of $T[n]$'s.

▶ If $f(n) = 0$ then the recurrence relation becomes a linear, homogenous recurrence relation of order k.

Note that we ignore boundary conditions for the moment.

6.3 The Characteristic Polynomial

Observations:

6.3 The Characteristic Polynomial

Observations:

▶ The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

6.3 The Characteristic Polynomial

Observations:

▶ The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

▶ In fact, any k consecutive values completely determine the solution.

6.3 The Characteristic Polynomial

Observations:

▶ The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

▶ In fact, any k consecutive values completely determine the solution.

▶ k non-conceccutive values might not be an appropriate set of boundary conditions (depends on the problem).

6.3 The Characteristic Polynomial

Observations:

▶ The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

▶ In fact, any k consecutive values completely determine the solution.

▶ k non-conceccutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

6.3 The Characteristic Polynomial

Observations:

- ▶ The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

- ▶ In fact, any k consecutive values completely determine the solution.

- ▶ k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- ▶ First determine all solutions that satisfy recurrence relation.

6.3 The Characteristic Polynomial

Observations:

- ▶ The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

- ▶ In fact, any k consecutive values completely determine the solution.

- ▶ k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- ▶ First determine all solutions that satisfy recurrence relation.

- ▶ Then pick the right one by analyzing boundary conditions.

6.3 The Characteristic Polynomial

Observations:

- ▶ The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.

- ▶ In fact, any k consecutive values completely determine the solution.

- ▶ k non-conceccutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- ▶ First determine all solutions that satisfy recurrence relation.

- ▶ Then pick the right one by analyzing boundary conditions.

- ▶ First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \;\Big|\; \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space.

The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \ \Big| \ \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha \mathcal{T}_1 + \beta \mathcal{T}_2 \in S$, for arbitrary constants α, β.

The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \ldots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha \mathcal{T}_1 + \beta \mathcal{T}_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \ \Big| \ \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha \mathcal{T}_1 + \beta \mathcal{T}_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens.

The Homogenous Case

The solution space

$$S = \Big\{ \mathcal{T} = T[1], T[2], T[3], \ldots \ \Big| \ \mathcal{T} \text{ fulfills recurrence relation} \Big\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha \mathcal{T}_1 + \beta \mathcal{T}_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i.

The Homogenous Case

Lemma 6

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

The Homogenous Case

Lemma 6

Assume that the characteristic polynomial has k *distinct roots* $\lambda_1, \ldots, \lambda_k$. *Then* all *solutions to the recurrence relation are of the form*

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n .$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

The Homogenous Case

Lemma 6

Assume that the characteristic polynomial has k *distinct roots* $\lambda_1, \ldots, \lambda_k$. *Then* all *solutions to the recurrence relation are of the form*

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\alpha_1 \cdot \lambda_1 \;+\; \alpha_2 \cdot \lambda_2 \;+\; \cdots \;+\; \alpha_k \cdot \lambda_k \;=\; T[1]$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$
\begin{array}{ccccccccc}
\alpha_1 \cdot \lambda_1 & + & \alpha_2 \cdot \lambda_2 & + & \cdots & + & \alpha_k \cdot \lambda_k & = & T[1] \\
\alpha_1 \cdot \lambda_1^2 & + & \alpha_2 \cdot \lambda_2^2 & + & \cdots & + & \alpha_k \cdot \lambda_k^2 & = & T[2]
\end{array}
$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$
\begin{array}{ccccccccc}
\alpha_1 \cdot \lambda_1 & + & \alpha_2 \cdot \lambda_2 & + & \cdots & + & \alpha_k \cdot \lambda_k & = & T[1] \\
\alpha_1 \cdot \lambda_1^2 & + & \alpha_2 \cdot \lambda_2^2 & + & \cdots & + & \alpha_k \cdot \lambda_k^2 & = & T[2] \\
& & & & \vdots & & & &
\end{array}
$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$
\begin{array}{ccccccccc}
\alpha_1 \cdot \lambda_1 & + & \alpha_2 \cdot \lambda_2 & + & \cdots & + & \alpha_k \cdot \lambda_k & = & T[1] \\
\alpha_1 \cdot \lambda_1^2 & + & \alpha_2 \cdot \lambda_2^2 & + & \cdots & + & \alpha_k \cdot \lambda_k^2 & = & T[2] \\
& & & & \vdots & & & & \\
\alpha_1 \cdot \lambda_1^k & + & \alpha_2 \cdot \lambda_2^k & + & \cdots & + & \alpha_k \cdot \lambda_k^k & = & T[k]
\end{array}
$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i's$ such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k
\end{vmatrix} =
$$

Computing the Determinant

$$
\begin{vmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\
\vdots & \vdots & & \vdots & \vdots \\
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k
\end{vmatrix}
= \prod_{i=1}^{k} \lambda_i \cdot
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\vdots & \vdots & & \vdots & \vdots \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1}
\end{vmatrix}
$$

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

Computing the Determinant

$$
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\
1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1}
\end{vmatrix} =
$$

Computing the Determinant

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix} =$$

$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix} =$$

Computing the Determinant

$$
\begin{vmatrix}
1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\
1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2}
\end{vmatrix} =
$$

$$
\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix}
$$

Computing the Determinant

$$
\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix} =
$$

Computing the Determinant

$$
\begin{vmatrix}
1 & 0 & \cdots & 0 & 0 \\
1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2}
\end{vmatrix} =
$$

$$
\prod_{i=2}^{k} (\lambda_i - \lambda_1) \cdot
\begin{vmatrix}
1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\
\vdots & \vdots & & \vdots & \vdots \\
1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2}
\end{vmatrix}
$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i > \ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

This means

$$c_0 n \lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$c_0 n \lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

This means

$$c_0 n \lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + \underbrace{c_k(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

The Homogeneous Case

Suppose λ_i has multiplicity j.

The Homogeneous Case

Suppose λ_i has multiplicity j. We know that

$$c_0 n \lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

The Homogeneous Case

Suppose λ_i has multiplicity j. We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \cdots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \cdots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

The Homogeneous Case

Suppose λ_i has multiplicity j. We know that

$$c_0 n \lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

The Homogeneous Case

Suppose λ_i has multiplicity j. We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \cdots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \cdots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \ldots, j-1$.

The Homogeneous Case

Lemma 7

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

$T[0] = 0$ gives $\alpha + \beta = 0$.

Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

$T[0] = 0$ gives $\alpha + \beta = 0$.

$T[1] = 1$ gives

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1$$

Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

$T[0] = 0$ gives $\alpha + \beta = 0$.

$T[1] = 1$ gives

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

The Inhomogeneous Case

Example:
$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

The Inhomogeneous Case

Example:
$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,
$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

The Inhomogeneous Case

Example:
$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,
$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or
$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

The Inhomogeneous Case

Example:
$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,
$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or
$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

$T[0] = 1$ gives $\alpha = 1$.

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \implies \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2$$

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2 = T[n - 2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] + 2n - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
$$= 2T[n-2] - T[n-3] + 2n - 3$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
$$= 2T[n-2] - T[n-3] + 2n - 3$$

Difference:

$$T[n] - T[n-1] = 2T[n-1] - T[n-2] + 2n - 1$$
$$- 2T[n-2] + T[n-3] - 2n + 3$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
$$= 2T[n-2] - T[n-3] + 2n - 3$$

Difference:

$$T[n] - T[n-1] = 2T[n-1] - T[n-2] + 2n - 1$$
$$- 2T[n-2] + T[n-3] - 2n + 3$$

$$T[n] = 3T[n-1] - 3T[n-2] + T[n-3] + 2$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
$$= 2T[n-2] - T[n-3] + 2n - 3$$

Difference:

$$T[n] - T[n-1] = 2T[n-1] - T[n-2] + 2n - 1$$
$$- 2T[n-2] + T[n-3] - 2n + 3$$

$$T[n] = 3T[n-1] - 3T[n-2] + T[n-3] + 2$$

and so on...

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n \geq 0} a_n z^n \; ;$$

6.4 Generating Functions

Definition 8 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

▶ generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n \geq 0} a_n z^n \ ;$$

▶ exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n \ .$$

6.4 Generating Functions

Example 9

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

$$F(z) = 1 \,.$$

6.4 Generating Functions

Example 9

1. The generating function of the sequence $(1, 0, 0, \ldots)$ is

$$F(z) = 1 \,.$$

2. The generating function of the sequence $(1, 1, 1, \ldots)$ is

$$F(z) = \frac{1}{1 - z} \,.$$

6.4 Generating Functions

There are two different views:

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

▶ **Equality:** f and g are equal if $a_n = b_n$ for all n.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

▶ **Equality:** f and g are equal if $a_n = b_n$ for all n.

▶ **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

- ▶ **Equality:** f and g are equal if $a_n = b_n$ for all n.
- ▶ **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.
- ▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

▶ **Equality:** f and g are equal if $a_n = b_n$ for all n.

▶ **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.

▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

There are no convergence issues here.

6.4 Generating Functions

The arithmetic view:

6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

6.4 Generating Functions

What does $\sum_{n \geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

6.4 Generating Functions

What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n\geq 0}^{\infty} z^n\right) = 1 \ .$$

6.4 Generating Functions

What does $\sum_{n \geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left(\sum_{n \geq 0}^{\infty} z^n\right) = 1 \ .$$

This is well-defined.

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z} \ .$$

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z} \ .$$

We can compute the derivative:

$$\sum_{n \geq 1} n z^{n-1} = \frac{1}{(1-z)^2}$$

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1 - z} \ .$$

We can compute the derivative:

$$\underbrace{\sum_{n \geq 1} n z^{n-1}}_{\sum_{n \geq 0}(n+1)z^n} = \frac{1}{(1 - z)^2}$$

6.4 Generating Functions

Suppose we are given the generating function

$$\sum_{n \geq 0} z^n = \frac{1}{1-z} \ .$$

We can compute the derivative:

$$\underbrace{\sum_{n \geq 1} n z^{n-1}}_{\sum_{n \geq 0} (n+1) z^n} = \frac{1}{(1-z)^2}$$

Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

6.4 Generating Functions

We can repeat this

6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n+1)z^n = \frac{1}{(1-z)^2} \quad .$$

6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n+1)z^n = \frac{1}{(1-z)^2} \ .$$

Derivative:

$$\sum_{n \geq 1} n(n+1)z^{n-1} = \frac{2}{(1-z)^3}$$

6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n+1) z^n = \frac{1}{(1-z)^2} \ .$$

Derivative:

$$\underbrace{\sum_{n \geq 1} n(n+1) z^{n-1}}_{\sum_{n \geq 0} (n+1)(n+2) z^n} = \frac{2}{(1-z)^3}$$

6.4 Generating Functions

We can repeat this

$$\sum_{n \geq 0} (n+1)z^n = \frac{1}{(1-z)^2} \ .$$

Derivative:

$$\underbrace{\sum_{n \geq 1} n(n+1)z^{n-1}}_{\sum_{n \geq 0} (n+1)(n+2)z^n} = \frac{2}{(1-z)^3}$$

Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n \geq k} n(n-1) \cdot \ldots \cdot (n-k+1) z^{n-k}$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n \geq k} n(n-1) \cdot \ldots \cdot (n-k+1) z^{n-k} = \sum_{n \geq 0} (n+k) \cdot \ldots \cdot (n+1) z^n$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n \geq k} n(n-1) \cdot \ldots \cdot (n-k+1)z^{n-k} = \sum_{n \geq 0} (n+k) \cdot \ldots \cdot (n+1)z^n$$

$$= \frac{k!}{(1-z)^{k+1}} \ .$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n \geq k} n(n-1) \cdot \ldots \cdot (n-k+1) z^{n-k} = \sum_{n \geq 0} (n+k) \cdot \ldots \cdot (n+1) z^n$$

$$= \frac{k!}{(1-z)^{k+1}} \quad .$$

Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \quad .$$

6.4 Generating Functions

Computing the k-th derivative of $\sum z^n$.

$$\sum_{n \geq k} n(n-1) \cdot \ldots \cdot (n-k+1)z^{n-k} = \sum_{n \geq 0} (n+k) \cdot \ldots \cdot (n+1)z^n$$

$$= \frac{k!}{(1-z)^{k+1}} \ .$$

Hence:

$$\sum_{n \geq 0} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}} \ .$$

The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

6.4 Generating Functions

$$\sum_{n\geq 0} nz^n = \sum_{n\geq 0} (n+1)z^n - \sum_{n\geq 0} z^n$$

6.4 Generating Functions

$$\sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n$$

$$= \frac{1}{(1-z)^2} - \frac{1}{1-z}$$

6.4 Generating Functions

$$\sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n$$

$$= \frac{1}{(1-z)^2} - \frac{1}{1-z}$$

$$= \frac{z}{(1-z)^2}$$

6.4 Generating Functions

$$\sum_{n \geq 0} n z^n = \sum_{n \geq 0} (n+1) z^n - \sum_{n \geq 0} z^n$$

$$= \frac{1}{(1-z)^2} - \frac{1}{1-z}$$

$$= \frac{z}{(1-z)^2}$$

The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

6.4 Generating Functions

We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

6.4 Generating Functions

We know

$$\sum_{n \geq 0} y^n = \frac{1}{1 - y}$$

Hence,

$$\sum_{n \geq 0} a^n z^n = \frac{1}{1 - az}$$

6.4 Generating Functions

We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n \geq 0} a^n z^n = \frac{1}{1-az}$$

The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z)$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$A(z) = \sum_{n \geq 0} a_n z^n$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \\
&= z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} z^n
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \\
&= z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} z^n \\
&= z A(z) + \sum_{n \geq 0} z^n
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$$
\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} (a_{n-1} + 1) z^n \\
&= 1 + z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} z^n \\
&= z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} z^n \\
&= zA(z) + \sum_{n \geq 0} z^n \\
&= zA(z) + \frac{1}{1 - z}
\end{aligned}
$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

$$A(z) = \frac{1}{(1-z)^2}$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2}$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n \geq 0} (n+1) z^n$$

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Solving for $A(z)$ gives

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{(1-z)^2} = \sum_{n \geq 0} (n+1) z^n$$

Hence, $a_n = n + 1$.

Some Generating Functions

n-th sequence element	*generating function*

Some Generating Functions

n-th sequence element	*generating function*
1	$\dfrac{1}{1-z}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\dfrac{1}{1-z}$
$n+1$	$\dfrac{1}{(1-z)^2}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\dfrac{1}{1-z}$
$n+1$	$\dfrac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\dfrac{1}{(1-z)^{k+1}}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\dfrac{1}{1-z}$
$n+1$	$\dfrac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\dfrac{1}{(1-z)^{k+1}}$
n	$\dfrac{z}{(1-z)^2}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\dfrac{1}{1-z}$
$n+1$	$\dfrac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\dfrac{1}{(1-z)^{k+1}}$
n	$\dfrac{z}{(1-z)^2}$
a^n	$\dfrac{1}{1-az}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\dfrac{1}{1-z}$
$n+1$	$\dfrac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\dfrac{1}{(1-z)^{k+1}}$
n	$\dfrac{z}{(1-z)^2}$
a^n	$\dfrac{1}{1-az}$
n^2	$\dfrac{z(1+z)}{(1-z)^3}$

Some Generating Functions

n-th sequence element	*generating function*
1	$\dfrac{1}{1-z}$
$n+1$	$\dfrac{1}{(1-z)^2}$
$\binom{n+k}{k}$	$\dfrac{1}{(1-z)^{k+1}}$
n	$\dfrac{z}{(1-z)^2}$
a^n	$\dfrac{1}{1-az}$
n^2	$\dfrac{z(1+z)}{(1-z)^3}$
$\frac{1}{n!}$	e^z

Some Generating Functions

n-th sequence element	*generating function*

Some Generating Functions

n-th sequence element	*generating function*
$c f_n$	cF

Some Generating Functions

n-th sequence element	*generating function*
$c f_n$	cF
$f_n + g_n$	$F + G$

Some Generating Functions

n-th sequence element	*generating function*
$c f_n$	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$

Some Generating Functions

n-th sequence element	*generating function*
$c f_n$	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
f_{n-k} $(n \geq k)$; 0 otw.	$z^k F$

Some Generating Functions

n-th sequence element	generating function
$c\,f_n$	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k}\ \ (n \geq k);\ \ 0$ otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$

Some Generating Functions

n-th sequence element	*generating function*
$c f_n$	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
f_{n-k} $(n \geq k)$; 0 otw.	$z^k F$
$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1 - z}$
$n f_n$	$z \dfrac{\mathrm{d} F(z)}{\mathrm{d} z}$

Some Generating Functions

n-th sequence element	generating function
$c\,f_n$	cF
$f_n + g_n$	$F + G$
$\sum_{i=0}^{n} f_i g_{n-i}$	$F \cdot G$
$f_{n-k}\ \ (n \geq k);\ \ 0\ \text{otw.}$	$z^k F$
$\sum_{i=0}^{n} f_i$	$\dfrac{F(z)}{1-z}$
$n f_n$	$z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$
$c^n f_n$	$F(cz)$

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.
2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.
3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

5. Write $f(z)$ as a formal power series.
   Techniques:

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

5. Write $f(z)$ as a formal power series.
   Techniques:
   - ▶ partial fraction decomposition (Partialbruchzerlegung)

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

5. Write $f(z)$ as a formal power series.
   Techniques:
   ▶ partial fraction decomposition (Partialbruchzerlegung)
   ▶ lookup in tables

Solving Recursions with Generating Functions

1. Set $A(z) = \sum_{n \geq 0} a_n z^n$.

2. Transform the right hand side so that boundary condition and recurrence relation can be plugged in.

3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.

4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

5. Write $f(z)$ as a formal power series.
   Techniques:
   ▶ partial fraction decomposition (Partialbruchzerlegung)
   ▶ lookup in tables

6. The coefficients of the resulting power series are the a_n.

1. Set up generating function:

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n \geq 1} a_n z^n$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n \geq 1} a_n z^n$$

2. Plug in:

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

2. Transform right hand side so that recurrence can be plugged in:

$$A(z) = a_0 + \sum_{n \geq 1} a_n z^n$$

2. Plug in:

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1}) z^n$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

$$= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1}$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

$$= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1}$$

$$= 1 + 2z \sum_{n \geq 0} a_n z^n$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

$$= 1 + 2z \sum_{n \geq 1} a_{n-1} z^{n-1}$$

$$= 1 + 2z \sum_{n \geq 0} a_n z^n$$

$$= 1 + 2z \cdot A(z)$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

$$= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1}$$

$$= 1 + 2z \sum_{n \geq 0} a_n z^n$$

$$= 1 + 2z \cdot A(z)$$

4. Solve for $A(z)$.

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

$$= 1 + 2z \sum_{n \geq 1} a_{n-1}z^{n-1}$$

$$= 1 + 2z \sum_{n \geq 0} a_n z^n$$

$$= 1 + 2z \cdot A(z)$$

4. Solve for $A(z)$.

$$A(z) = \frac{1}{1 - 2z}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$A(z) = \frac{1}{1 - 2z}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{1 - 2z}$$

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

5. Rewrite $f(z)$ as a power series:

$$\sum_{n \geq 0} a_n z^n = A(z) = \frac{1}{1-2z} = \sum_{n \geq 0} 2^n z^n$$

1. Set up generating function:

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

1. Set up generating function:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$A(z) = \sum_{n \geq 0} a_n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$A(z) = \sum_{n \geq 0} a_n z^n$$
$$= a_0 + \sum_{n \geq 1} a_n z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$A(z) = \sum_{n \geq 0} a_n z^n$$
$$= a_0 + \sum_{n \geq 1} a_n z^n$$
$$= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n$$

6.4 Generating Functions

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} a_n z^n \\
&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n \\
&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n
\end{aligned}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

2./3. Transform right hand side:

$$
\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} a_n z^n \\
&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n \\
&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n \\
&= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n
\end{aligned}
$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

2./3. Transform right hand side:

$$\begin{aligned}
A(z) &= \sum_{n \geq 0} a_n z^n \\
&= a_0 + \sum_{n \geq 1} a_n z^n \\
&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n \\
&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n \\
&= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n \\
&= 1 + 3z A(z) + \frac{z}{(1-z)^2}
\end{aligned}$$

6.4 Generating Functions

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

4. Solve for $A(z)$:

$$A(z) = 1 + 3zA(z) + \frac{z}{(1-z)^2}$$

gives

$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n, \, a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \stackrel{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \overset{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

This gives

$$z^2 - z + 1 = A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z)$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \overset{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

This gives

$$
\begin{aligned}
z^2 - z + 1 &= A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z) \\
&= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z)
\end{aligned}
$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

We use partial fraction decomposition:

$$\frac{z^2 - z + 1}{(1 - 3z)(1 - z)^2} \stackrel{!}{=} \frac{A}{1 - 3z} + \frac{B}{1 - z} + \frac{C}{(1 - z)^2}$$

This gives

$$
\begin{aligned}
z^2 - z + 1 &= A(1 - z)^2 + B(1 - 3z)(1 - z) + C(1 - 3z) \\
&= A(1 - 2z + z^2) + B(1 - 4z + 3z^2) + C(1 - 3z) \\
&= (A + 3B)z^2 + (-2A - 4B - 3C)z + (A + B + C)
\end{aligned}
$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1 - 3z} - \frac{1}{4} \cdot \frac{1}{1 - z} - \frac{1}{2} \cdot \frac{1}{(1 - z)^2}$$

$$= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n + 1) z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1 - 3z} - \frac{1}{4} \cdot \frac{1}{1 - z} - \frac{1}{2} \cdot \frac{1}{(1 - z)^2}$$

$$= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n + 1) z^n$$

$$= \sum_{n \geq 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n + 1) \right) z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1 - 3z} - \frac{1}{4} \cdot \frac{1}{1 - z} - \frac{1}{2} \cdot \frac{1}{(1 - z)^2}$$

$$= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n + 1) z^n$$

$$= \sum_{n \geq 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n + 1) \right) z^n$$

$$= \sum_{n \geq 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4} \right) z^n$$

Example: $a_n = 3a_{n-1} + n$, $a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$A(z) = \frac{7}{4} \cdot \frac{1}{1 - 3z} - \frac{1}{4} \cdot \frac{1}{1 - z} - \frac{1}{2} \cdot \frac{1}{(1 - z)^2}$$

$$= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n + 1) z^n$$

$$= \sum_{n \geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n + 1) \Big) z^n$$

$$= \sum_{n \geq 0} \Big(\frac{7}{4} \cdot 3^n - \frac{1}{2}n - \frac{3}{4} \Big) z^n$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2.$$

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \ .$$

Define

$$g_n := \log f_n \ .$$

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \, .$$

Define

$$g_n := \log f_n \, .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$
$$g_1 = \log 2 = 1 (\text{for } \log = \log_2), \ g_0 = 0$$

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$
$$g_1 = \log 2 = 1 (\text{for } \log = \log_2), \ g_0 = 0$$
$$g_n = F_n \ (n\text{-th Fibonacci number})$$

6.5 Transformation of the Recurrence

Example 10

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \geq 2$$
$$g_1 = \log 2 = 1 \text{(for log = } \log_2\text{)}, \ g_0 = 0$$
$$g_n = F_n \text{ (}n\text{-th Fibonacci number)}$$
$$f_n = 2^{F_n}$$

6.5 Transformation of the Recurrence

Example 11

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 \; ;$$

6.5 Transformation of the Recurrence

Example 11

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 \; ;$$

Define

$$g_k := f_{2^k} \; .$$

6.5 Transformation of the Recurrence

Example 11

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 ;$$

Define

$$g_k := f_{2^k} .$$

Then:

$$g_0 = 1$$

6.5 Transformation of the Recurrence

Example 11

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, k \geq 1 \text{ ;}$$

Define

$$g_k := f_{2^k} \; .$$

Then:

$$g_0 = 1$$
$$g_k = 3g_{k-1} + 2^k, \; k \geq 1$$

6 Recurrences

We get

$$g_k = 3\left[g_{k-1}\right] + 2^k$$

6 Recurrences

We get

$$g_k = 3 \left[g_{k-1} \right] + 2^k$$
$$= 3 \left[3 g_{k-2} + 2^{k-1} \right] + 2^k$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 3 \cdot 2^{k-1} + 2^k
\end{aligned}
$$

6 Recurrences

We get

$$\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 3 2^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 3 2^{k-1} + 2^k
\end{aligned}$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 3 2^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 3 2^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 2^{k-1} + 2^k
\end{aligned}
$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k}\left(\frac{3}{2}\right)^i
\end{aligned}
$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 3 2^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 3 2^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 3 2^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2}
\end{aligned}
$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k}\left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$
$$= 3(2^{\log 3})^k - 2 \cdot 2^k$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$
$$= 3(2^{\log 3})^k - 2 \cdot 2^k$$
$$= 3(2^k)^{\log 3} - 2 \cdot 2^k$$

6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$
$$= 3(2^{\log 3})^k - 2 \cdot 2^k$$
$$= 3(2^k)^{\log 3} - 2 \cdot 2^k$$
$$= 3n^{\log 3} - 2n \ .$$

Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- ▶ The key comes from a totally ordered set, and we assume that there is an efficient comparison function.
- ▶ The value can be anything; it usually carries satellite information important for the application that uses the ADT.

Dynamic Set Operations

▶ $S.\,\mathbf{search}(k)$: Returns pointer to object x from S with $\mathrm{key}[x] = k$ or null.

Dynamic Set Operations

- ▶ $S.\,\text{search}(k)$: Returns pointer to object x from S with $\text{key}[x] = k$ or null.

- ▶ $S.\,\text{insert}(x)$: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.

Dynamic Set Operations

- ▶ $S.\,\text{search}(k)$: Returns pointer to object x from S with $\text{key}[x] = k$ or null.

- ▶ $S.\,\text{insert}(x)$: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.

- ▶ $S.\,\text{delete}(x)$: Given pointer to object x from S, delete x from the set.

Dynamic Set Operations

▶ $S.\,\mathbf{search}(k)$: Returns pointer to object x from S with $\text{key}[x] = k$ or null.

▶ $S.\,\mathbf{insert}(x)$: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.

▶ $S.\,\mathbf{delete}(x)$: Given pointer to object x from S, delete x from the set.

▶ $S.\,\mathbf{minimum}()$: Return pointer to object with smallest key-value in S.

Dynamic Set Operations

▶ $S.\,\text{search}(k)$: Returns pointer to object x from S with $\text{key}[x] = k$ or null.

▶ $S.\,\text{insert}(x)$: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.

▶ $S.\,\text{delete}(x)$: Given pointer to object x from S, delete x from the set.

▶ $S.\,\text{minimum}()$: Return pointer to object with smallest key-value in S.

▶ $S.\,\text{maximum}()$: Return pointer to object with largest key-value in S.

Dynamic Set Operations

▶ $S.\,\text{search}(k)$: Returns pointer to object x from S with $\text{key}[x] = k$ or null.

▶ $S.\,\text{insert}(x)$: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.

▶ $S.\,\text{delete}(x)$: Given pointer to object x from S, delete x from the set.

▶ $S.\,\text{minimum}()$: Return pointer to object with smallest key-value in S.

▶ $S.\,\text{maximum}()$: Return pointer to object with largest key-value in S.

▶ $S.\,\text{successor}(x)$: Return pointer to the next larger element in S or null if x is maximum.

Dynamic Set Operations

▶ $S.\,\text{search}(k)$: Returns pointer to object x from S with $\text{key}[x] = k$ or null.

▶ $S.\,\text{insert}(x)$: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.

▶ $S.\,\text{delete}(x)$: Given pointer to object x from S, delete x from the set.

▶ $S.\,\text{minimum}()$: Return pointer to object with smallest key-value in S.

▶ $S.\,\text{maximum}()$: Return pointer to object with largest key-value in S.

▶ $S.\,\text{successor}(x)$: Return pointer to the next larger element in S or null if x is maximum.

▶ $S.\,\text{predecessor}(x)$: Return pointer to the next smaller element in S or null if x is minimum.

Dynamic Set Operations

▶ $S.\,\mathbf{union}(S')$: Sets $S := S \cup S'$. The set S' is destroyed.

Dynamic Set Operations

▶ $S.\operatorname{union}(S')$: Sets $S := S \cup S'$. The set S' is destroyed.
▶ $S.\operatorname{merge}(S')$: Sets $S := S \cup S'$. Requires $S \cap S' = \varnothing$.

Dynamic Set Operations

▶ $S.\text{union}(S')$: Sets $S := S \cup S'$. The set S' is destroyed.

▶ $S.\text{merge}(S')$: Sets $S := S \cup S'$. Requires $S \cap S' = \varnothing$.

▶ $S.\text{split}(k, S')$:
  $S := \{x \in S \mid \text{key}[x] \le k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.

Dynamic Set Operations

▶ **$S.\,\text{union}(S')$:** Sets $S := S \cup S'$. The set S' is destroyed.

▶ **$S.\,\text{merge}(S')$:** Sets $S := S \cup S'$. Requires $S \cap S' = \varnothing$.

▶ **$S.\,\text{split}(k, S')$:**
   $S := \{x \in S \mid \text{key}[x] \le k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.

▶ **$S.\,\text{concatenate}(S')$:** $S := S \cup S'$.
   Requires $\text{key}[S.\,\text{maximum}()] \le \text{key}[S'.\,\text{minimum}()]$.

Dynamic Set Operations

- ▶ **$S.\,\text{union}(S')$**: Sets $S := S \cup S'$. The set S' is destroyed.
- ▶ **$S.\,\text{merge}(S')$**: Sets $S := S \cup S'$. Requires $S \cap S' = \varnothing$.
- ▶ **$S.\,\text{split}(k, S')$**:
  $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.
- ▶ **$S.\,\text{concatenate}(S')$**: $S := S \cup S'$.
  Requires $\text{key}[S.\,\text{maximum}()] \leq \text{key}[S'.\,\text{minimum}()]$.

- ▶ **$S.\,\text{decrease-key}(x, k)$**: Replace $\text{key}[x]$ by $k \leq \text{key}[x]$.

Examples of ADTs

Stack:

▶ $S.\,\textbf{push}(x)$: Insert an element.

▶ $S.\,\textbf{pop}()$: Return the element from S that was inserted most recently; delete it from S.

▶ $S.\,\textbf{empty}()$: Tell if S contains any object.

Examples of ADTs

Stack:

- $S.\,\text{push}(x)$: Insert an element.
- $S.\,\text{pop}()$: Return the element from S that was inserted most recently; delete it from S.
- $S.\,\text{empty}()$: Tell if S contains any object.

Queue:

- $S.\,\text{enqueue}(x)$: Insert an element.
- $S.\,\text{dequeue}()$: Return the element that is longest in the structure; delete it from S.
- $S.\,\text{empty}()$: Tell if S contains any object.

Examples of ADTs

Stack:

- ▶ $S.\,\mathrm{push}(x)$: Insert an element.
- ▶ $S.\,\mathrm{pop}()$: Return the element from S that was inserted most recently; delete it from S.
- ▶ $S.\,\mathrm{empty}()$: Tell if S contains any object.

Queue:

- ▶ $S.\,\mathrm{enqueue}(x)$: Insert an element.
- ▶ $S.\,\mathrm{dequeue}()$: Return the element that is longest in the structure; delete it from S.
- ▶ $S.\,\mathrm{empty}()$: Tell if S contains any object.

Priority-Queue:

- ▶ $S.\,\mathrm{insert}(x)$: Insert an element.
- ▶ $S.\,\mathrm{delete\text{-}min}()$: Return the element with lowest key-value; delete it from S.

7 Dictionary

Dictionary:

- ▶ $S.\mathbf{insert}(x)$: Insert an element x.
- ▶ $S.\mathbf{delete}(x)$: Delete the element pointed to by x.
- ▶ $S.\mathbf{search}(k)$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:
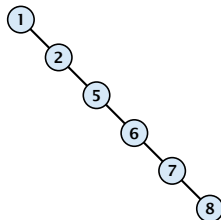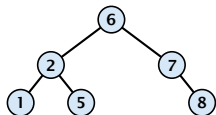
7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.\text{insert}(x)$
- ▶ $T.\text{delete}(x)$
- ▶ $T.\text{search}(k)$
- ▶ $T.\text{successor}(x)$
- ▶ $T.\text{predecessor}(x)$
- ▶ $T.\text{minimum}()$
- ▶ $T.\text{maximum}()$

Binary Search Trees: Searching

Algorithm 1 TreeSearch(x, k)

1: **if** x = null **or** k = key[x] **return** x
2: **if** k < key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 17)

Algorithm 1 TreeSearch(x, k)

1: **if** $x = $ null **or** $k = $ key$[x]$ **return** x
2: **if** $k < $ key$[x]$ **return** TreeSearch(left$[x], k$)
3: **else return** TreeSearch(right$[x], k$)

Binary Search Trees: Searching

TreeSearch(root, 17)

Algorithm 1 TreeSearch(x, k)
1: **if** x = null **or** k = key[x] **return** x
2: **if** k < key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

Algorithm 1 TreeSearch(x, k)
1: **if** x = null **or** k = key[x] **return** x
2: **if** $k <$ key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 17)

Algorithm 1 TreeSearch(x, k)

1: **if** x = null **or** k = key[x] **return** x
2: **if** k < key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 17)

Algorithm 1 TreeSearch(x, k)
1: **if** $x =$ null **or** $k =$ key$[x]$ **return** x
2: **if** $k <$ key$[x]$ **return** TreeSearch(left$[x], k$)
3: **else return** TreeSearch(right$[x], k$)

Binary Search Trees: Searching

TreeSearch(root, 17)

Algorithm 1 TreeSearch(x, k)
1: **if** x = null **or** k = key[x] **return** x
2: **if** k < key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

Algorithm 1 TreeSearch(x, k)

1: **if** $x = $ null **or** $k = $ key$[x]$ **return** x
2: **if** $k < $ key$[x]$ **return** TreeSearch(left$[x], k$)
3: **else return** TreeSearch(right$[x], k$)

Binary Search Trees: Searching

TreeSearch(root, 8)

Algorithm 1 TreeSearch(x, k)

1: **if** x = null **or** k = key[x] **return** x
2: **if** k < key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 8)

Algorithm 1 TreeSearch(x, k)
1: **if** x = null **or** k = key[x] **return** x
2: **if** k < key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 8)

Algorithm 1 TreeSearch(x, k)
1: **if** x = null **or** k = key$[x]$ **return** x
2: **if** k < key$[x]$ **return** TreeSearch(left$[x], k)$
3: **else return** TreeSearch(right$[x], k)$

Binary Search Trees: Searching

TreeSearch(root, 8)

Algorithm 1 TreeSearch(x, k)

1: **if** x = null **or** k = key[x] **return** x
2: **if** $k <$ key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 8)

Algorithm 1 TreeSearch(x, k)
1: **if** $x = $ null **or** $k = $ key$[x]$ **return** x
2: **if** $k < $ key$[x]$ **return** TreeSearch(left$[x], k$)
3: **else return** TreeSearch(right$[x], k$)

Binary Search Trees: Searching

TreeSearch(root, 8)

Algorithm 1 TreeSearch(x, k)

1: **if** $x = $ null **or** $k = $ key[x] **return** x
2: **if** $k < $ key[x] **return** TreeSearch(left[x], k)
3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x
2: **return** TreeMin(left[x])

Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x

2: **return** TreeMin(left[x])

Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

1: **if** $x = $ null **or** left$[x] = $ null **return** x
2: **return** TreeMin(left$[x]$)

Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)
1: **if** x = null **or** left[x] = null **return** x
2: **return** TreeMin(left[x])

Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

1: **if** $x =$ null **or** left[x] $=$ null **return** x
2: **return** TreeMin(left[x])

Binary Search Trees: Minimum

Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x
2: **return** TreeMin(left[x])

Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)
1: **if** right[x] ≠ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:      $x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)

1: **if** right$[x] \neq$ null **return** TreeMin(right$[x]$)
2: $y \leftarrow$ parent$[x]$
3: **while** $y \neq$ null **and** $x =$ right$[y]$ **do**
4: $\quad x \leftarrow y; y \leftarrow$ parent$[x]$
5: **return** y;

Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)

1: **if** right[x] ≠ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:     $x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)

1: **if** right[x] ≠ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:      $x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)

1: **if** right[x] ≠ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:     $x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)

1: **if** right[x] $\neq$ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:     $x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor

Algorithm 3 TreeSucc(x)

1: **if** right[x] $\neq$ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4:     $x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Insert

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
3:     **return**;
4: **if** key$[x]$ > key$[z]$ **then**
5:     **if** left$[x]$ = null **then**
6:         left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
7:     **else** TreeInsert(left$[x], z$);
8: **else**
9:     **if** right$[x]$ = null **then**
10:        right$[x] \leftarrow z$; parent$[z] \leftarrow x$;
11:     **else** TreeInsert(right$[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     root[T] ← z; parent[z] ← null;
3:     **return**;
4: **if** key[x] > key[z] **then**
5:     **if** left[x] = null **then**
6:         left[x] ← z; parent[z] ← x;
7:     **else** TreeInsert(left[x], z);
8: **else**
9:     **if** right[x] = null **then**
10:        right[x] ← z; parent[z] ← x;
11:     **else** TreeInsert(right[x], z);

Binary Search Trees: Insert

Insert element **not** in the tree.

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     root[T] ← z; parent[z] ← null;
3:     **return**;
4: **if** key[x] > key[z] **then**
5:     **if** left[x] = null **then**
6:         left[x] ← z; parent[z] ← x;
7:     **else** TreeInsert(left[x], z);
8: **else**
9:     **if** right[x] = null **then**
10:        right[x] ← z; parent[z] ← x;
11:     **else** TreeInsert(right[x], z);

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:       root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
3:       **return**;
4: **if** key$[x] >$ key$[z]$ **then**
5:       **if** left$[x]$ = null **then**
6:           left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
7:       **else** TreeInsert(left$[x], z$);
8: **else**
9:       **if** right$[x]$ = null **then**
10:          right$[x] \leftarrow z$; parent$[z] \leftarrow x$;
11:       **else** TreeInsert(right$[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2: $\quad$ root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
3: $\quad$ **return**;
4: **if** key$[x] >$ key$[z]$ **then**
5: $\quad$ **if** left$[x]$ = null **then**
6: $\quad\quad$ left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
7: $\quad$ **else** TreeInsert(left$[x], z$);
8: **else**
9: $\quad$ **if** right$[x]$ = null **then**
10: $\quad\quad$ right$[x] \leftarrow z$; parent$[z] \leftarrow x$;
11: $\quad$ **else** TreeInsert(right$[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     root[T] ← z; parent[z] ← null;
3:     **return**;
4: **if** key[x] > key[z] **then**
5:     **if** left[x] = null **then**
6:         left[x] ← z; parent[z] ← x;
7:     **else** TreeInsert(left[x], z);
8: **else**
9:     **if** right[x] = null **then**
10:        right[x] ← z; parent[z] ← x;
11:    **else** TreeInsert(right[x], z);

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

```
1: if x = null then
2:     root[T] ← z; parent[z] ← null;
3:     return;
4: if key[x] > key[z] then
5:     if left[x] = null then
6:         left[x] ← z; parent[z] ← x;
7:     else TreeInsert(left[x], z);
8: else
9:     if right[x] = null then
10:        right[x] ← z; parent[z] ← x;
11:    else TreeInsert(right[x], z);
```

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow$ null;
3:     **return**;
4: **if** $\text{key}[x] > \text{key}[z]$ **then**
5:     **if** $\text{left}[x]$ = null **then**
6:         $\text{left}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
7:     **else** TreeInsert($\text{left}[x], z$);
8: **else**
9:     **if** $\text{right}[x]$ = null **then**
10:         $\text{right}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
11:     **else** TreeInsert($\text{right}[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

1: **if** x = null **then**
2:     root[T] ← z; parent[z] ← null;
3:     **return**;
4: **if** key[x] > key[z] **then**
5:     **if** left[x] = null **then**
6:         left[x] ← z; parent[z] ← x;
7:     **else** TreeInsert(left[x], z);
8: **else**
9:     **if** right[x] = null **then**
10:         right[x] ← z; parent[z] ← x;
11:     **else** TreeInsert(right[x], z);

Binary Search Trees: Delete

Binary Search Trees: Delete

Case 1:

Element does not have any children

▶ Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete

Case 1:

Element does not have any children

▶ Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete

Case 1:

Element does not have any children

▶ Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete

Case 2:

Element has exactly one child

▶ Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete

Case 2:

Element has exactly one child

- ▶ Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete

Case 2:

Element has exactly one child

▶ Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete

Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

 1: **if** left[z] = null **or** right[z] = null
 2:     **then** $y \leftarrow z$ **else** $y \leftarrow$ TreeSucc(z);   select y to splice out
 3: **if** left[y] ≠ null
 4:     **then** $x \leftarrow$ left[y] **else** $x \leftarrow$ right[y]; x is child of y (or null)
 5: **if** x ≠ null **then** parent[x] $\leftarrow$ parent[y];   parent[x] is correct
 6: **if** parent[y] = null **then**
 7:     root[T] $\leftarrow x$
 8: **else**
 9:     **if** y = left[parent[y]] **then**
10:         left[parent[y]] $\leftarrow x$ fix pointer to x
11:     **else**
12:         right[parent[y]] $\leftarrow x$
13: **if** y ≠ z **then** copy y-data to z

Balanced Binary Search Trees

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees
With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees
With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

7.2 Red Black Trees

Definition 12

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

7.2 Red Black Trees

Definition 12

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.

7.2 Red Black Trees

Definition 12

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.

2. All leaf nodes are black.

7.2 Red Black Trees

Definition 12

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.

2. All leaf nodes are black.

3. For each node, all paths to descendant leaves contain the same number of black nodes.

7.2 Red Black Trees

Definition 12

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

7.2 Red Black Trees

Definition 12

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

Red Black Trees: Example
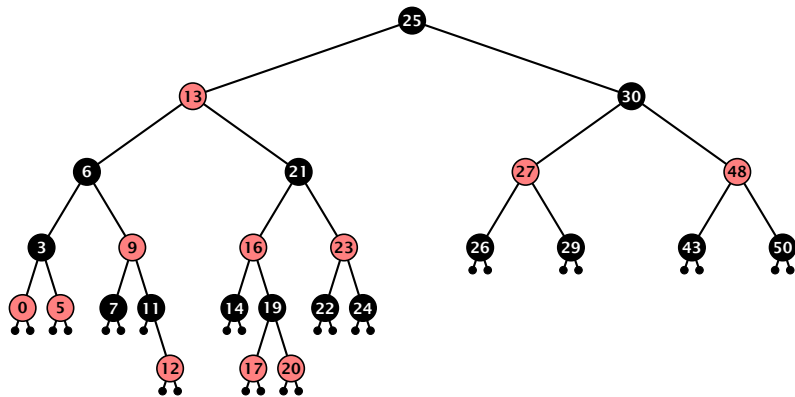
7.2 Red Black Trees

Lemma 13

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

7.2 Red Black Trees

Lemma 13

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

Definition 14

The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

7.2 Red Black Trees

Lemma 13
A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

Definition 14
The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 15
A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

Proof of Lemma 15.

7.2 Red Black Trees

Proof of Lemma 15.

Induction on the height of v.

7.2 Red Black Trees

Proof of Lemma 15.

Induction on the height of v.

base case ($\text{height}(v) = 0$)

▶ If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

7.2 Red Black Trees

Proof of Lemma 15.

Induction on the height of v.

base case ($\text{height}(v) = 0$)

▶ If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

▶ The black height of v is 0.

7.2 Red Black Trees

Proof of Lemma 15.

Induction on the height of v.

base case ($\text{height}(v) = 0$)

▶ If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

▶ The black height of v is 0.

▶ The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

7.2 Red Black Trees

Proof (cont.)

7.2 Red Black Trees

Proof (cont.)

induction step

▶ Supose ν is a node with $\text{height}(\nu) > 0$.

7.2 Red Black Trees

Proof (cont.)

induction step

▶ Supose v is a node with $\text{height}(v) > 0$.

▶ v has two children with strictly smaller height.

7.2 Red Black Trees

Proof (cont.)

induction step

- ▶ Supose v is a node with $\text{height}(v) > 0$.
- ▶ v has two children with strictly smaller height.
- ▶ These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.

7.2 Red Black Trees

Proof (cont.)

induction step

- ▶ Supose v is a node with $\text{height}(v) > 0$.
- ▶ v has two children with strictly smaller height.
- ▶ These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- ▶ By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.

7.2 Red Black Trees

Proof (cont.)

induction step

- ▶ Supose v is a node with $\text{height}(v) > 0$.

- ▶ v has two children with strictly smaller height.

- ▶ These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.

- ▶ By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.

- ▶ Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

□

7.2 Red Black Trees

Proof of Lemma 13.

7.2 Red Black Trees

Proof of Lemma 13.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

7.2 Red Black Trees

Proof of Lemma 13.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

7.2 Red Black Trees

Proof of Lemma 13.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

7.2 Red Black Trees

Proof of Lemma 13.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

7.2 Red Black Trees

Proof of Lemma 13.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. $\qquad\square$

7.2 Red Black Trees

Definition 1
A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

7.2 Red Black Trees

We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:

LeftRotate(x)

RightRotate(z)

Red Black Trees: Insert

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

Insert:

▶ first make a normal insert into a binary search tree

▶ then fix red-black properties

Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node

Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node
- ▶ the black-height property is fulfilled at every node

Red Black Trees: Insert

Invariant of the fix-up algorithm:

▶ z is a red node

▶ the black-height property is fulfilled at every node

▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$

Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$
  - ▶ either both of them are red
    (most important case)

Red Black Trees: Insert

Invariant of the fix-up algorithm:

▶ z is a red node

▶ the black-height property is fulfilled at every node

▶ the only violation of red-black properties occurs at z and parent$[z]$

   ▶ either both of them are red
     (most important case)
   ▶ or the parent does not exist
     (violation since root must be black)

Red Black Trees: Insert

Invariant of the fix-up algorithm:

▶ z is a red node

▶ the black-height property is fulfilled at every node

▶ the only violation of red-black properties occurs at z and
  parent$[z]$
  - ▶ either both of them are red
    (most important case)
  - ▶ or the parent does not exist
    (violation since root must be black)

If z has a parent but no grand-parent we could simply color the
parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)
1: **while** parent[z] $\neq$ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then**
3:         *uncle* $\leftarrow$ right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**
5:             col[p[z]] $\leftarrow$ black; col[u] $\leftarrow$ black;
6:             col[gp[z]] $\leftarrow$ red; $z \leftarrow$ grandparent[z];
7:         **else**
8:             **if** z = right[parent[z]] **then**
9:                 $z \leftarrow$ p[z]; LeftRotate(z);
10:             col[p[z]] $\leftarrow$ black; col[gp[z]] $\leftarrow$ red;
11:             RightRotate(gp[z]);
12:     **else** same as then-clause but right and left exchanged
13: col(root[T]) $\leftarrow$ black;

Red Black Trees: Insert

Algorithm 10 InsertFix(z)
1: **while** parent[z] ≠ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then** *z in left subtree of grandparent*
3:         *uncle* ← right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**
5:             col[p[z]] ← black; col[u] ← black;
6:             col[gp[z]] ← red; z ← grandparent[z];
7:         **else**
8:             **if** z = right[parent[z]] **then**
9:                 z ← p[z]; LeftRotate(z);
10:            col[p[z]] ← black; col[gp[z]] ← red;
11:            RightRotate(gp[z]);
12:    **else** same as then-clause but right and left exchanged
13: col(root[T]) ← black;

Red Black Trees: Insert

Algorithm 10 InsertFix(z)
1: **while** parent[z] $\neq$ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then**
3:         *uncle* ← right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**   Case 1: uncle red
5:             col[p[z]] ← black; col[u] ← black;
6:             col[gp[z]] ← red; z ← grandparent[z];
7:         **else**
8:             **if** z = right[parent[z]] **then**
9:                 z ← p[z]; LeftRotate(z);
10:             col[p[z]] ← black; col[gp[z]] ← red;
11:             RightRotate(gp[z]);
12:     **else** same as then-clause but right and left exchanged
13: col(root[T]) ← black;

Red Black Trees: Insert

Algorithm 10 InsertFix(z)
1: **while** parent[z] ≠ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then**
3:         *uncle* ← right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**
5:             col[p[z]] ← black; col[u] ← black;
6:             col[gp[z]] ← red; z ← grandparent[z];
7:         **else**   Case 2: uncle black
8:             **if** z = right[parent[z]] **then**
9:                 z ← p[z]; LeftRotate(z);
10:            col[p[z]] ← black; col[gp[z]] ← red;
11:            RightRotate(gp[z]);
12:     **else** same as then-clause but right and left exchanged
13: col(root[T]) ← black;

Red Black Trees: Insert

Algorithm 10 InsertFix(z)
1: **while** parent[z] ≠ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then**
3:         *uncle* ← right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**
5:             col[p[z]] ← black; col[u] ← black;
6:             col[gp[z]] ← red; z ← grandparent[z];
7:         **else**
8:             **if** z = right[parent[z]] **then**   2a: z right child
9:                 z ← p[z]; LeftRotate(z);
10:            col[p[z]] ← black; col[gp[z]] ← red;
11:            RightRotate(gp[z]);
12:     **else** same as then-clause but right and left exchanged
13: col(root[T]) ← black;

Red Black Trees: Insert

Algorithm 10 InsertFix(z)
1: **while** parent[z] ≠ null **and** col[parent[z]] = red **do**
2:     **if** parent[z] = left[gp[z]] **then**
3:         *uncle* ← right[grandparent[z]]
4:         **if** col[*uncle*] = red **then**
5:             col[p[z]] ← black; col[u] ← black;
6:             col[gp[z]] ← red; z ← grandparent[z];
7:         **else**
8:             **if** z = right[parent[z]] **then**
9:                 z ← p[z]; LeftRotate(z);
10:             col[p[z]] ← black; col[gp[z]] ← red; 2b: *z* left child
11:             RightRotate(gp[z]);
12:     **else** same as then-clause but right and left exchanged
13: col(root[T]) ← black;
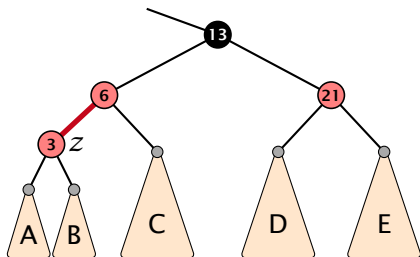
Case 1: Red Uncle

Case 1: Red Uncle
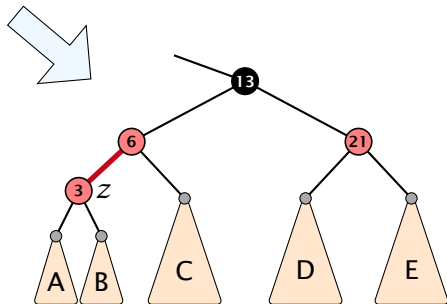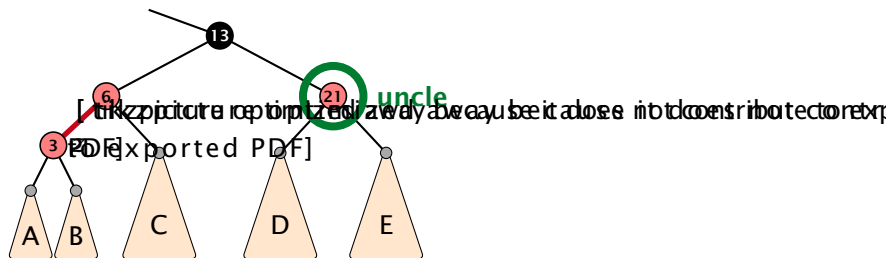
[This picture is optimized away because it does not contribute to exported PDF]

z. uncle

Case 1: Red Uncle

Uncle

[this picture re optimized away because it does not contribute context PDF exported PDF]

1. recolour

Case 1: Red Uncle

uncle

1. recolour

Case 1: Red Uncle

[tikz picture optimized away because it does not contribute to exported PDF]
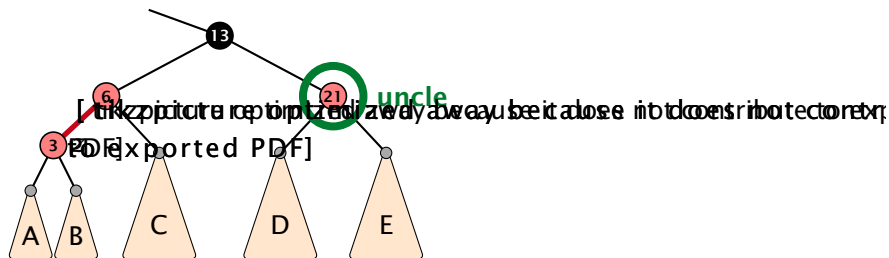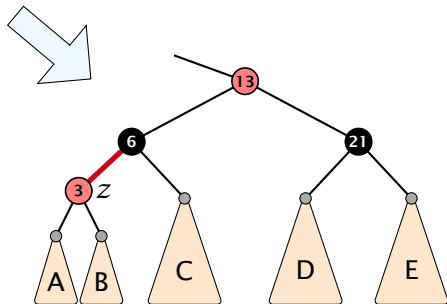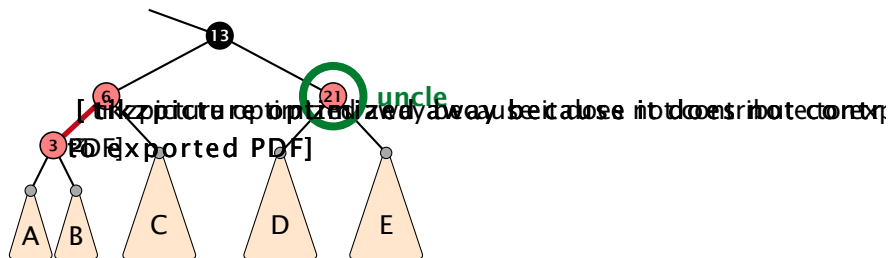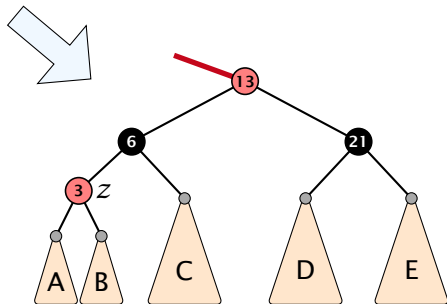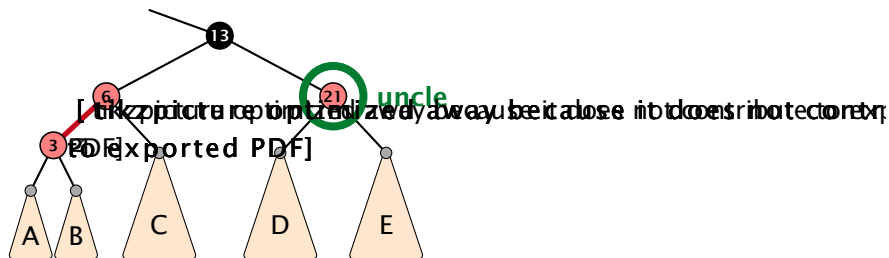
1. recolour
2. move z to grand-parent
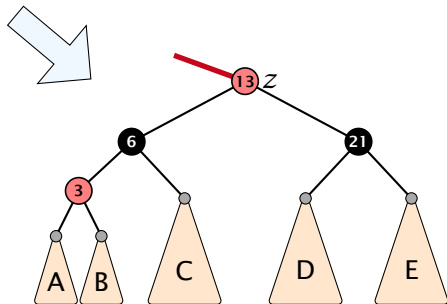
Case 1: Red Uncle
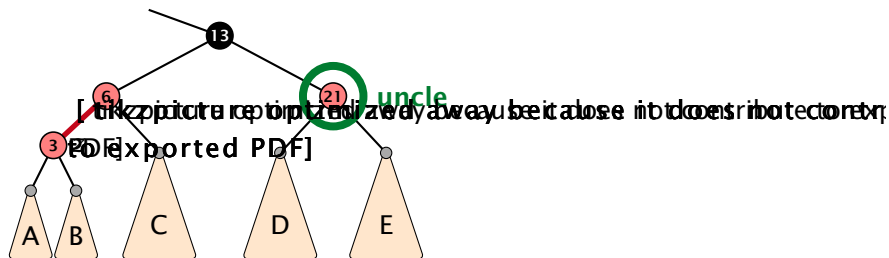
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
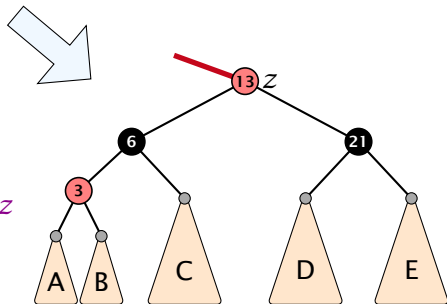
Case 1: Red Uncle
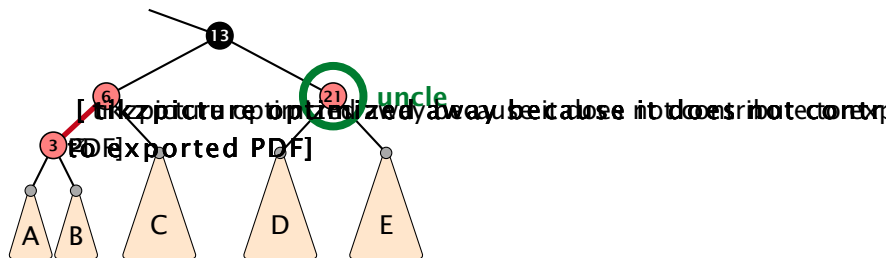
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress
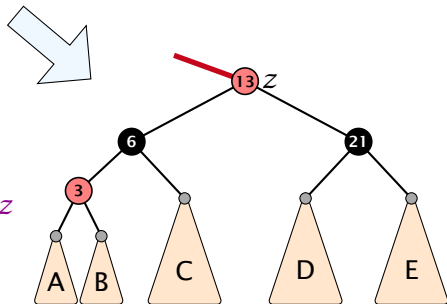
Case 2b: Black uncle and z is left child
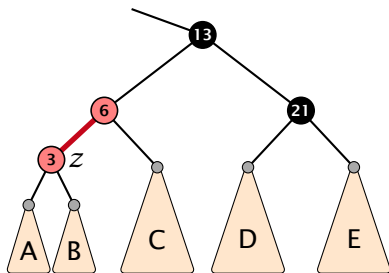
Case 2b: Black uncle and *z* is left child

[tikzpicture optimized away because it does not contribute to ex PDF]
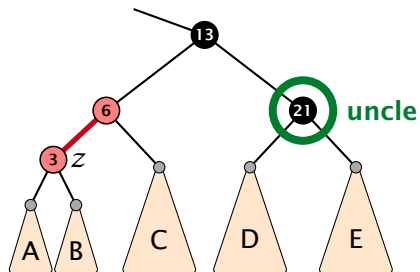
Case 2b: Black uncle and z is left child

1. rotate around grandparent

[tikz picture reportimized away because it does not contribute to exported PDF]

Case 2b: Black uncle and z is left child
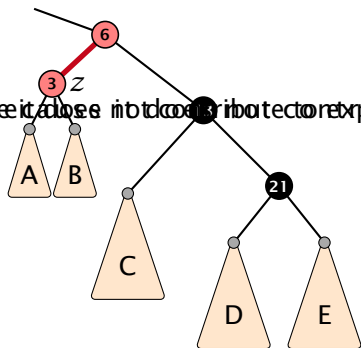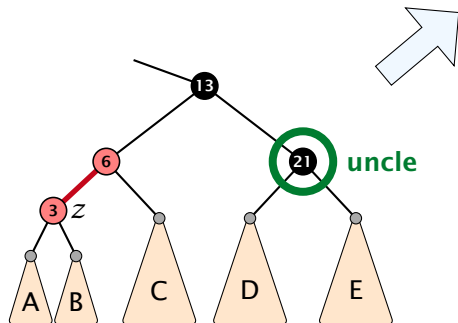
1. rotate around grandparent

2. [recolor to ensure that red-black property holds] exchange colors of 6 and 21, black-height property holds
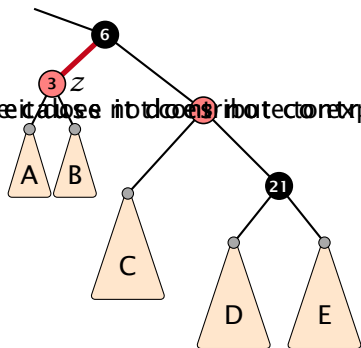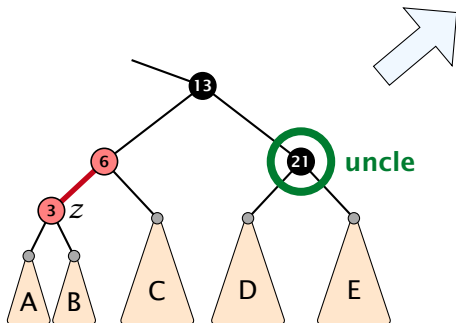
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. [recolour to repair red black property] recolour so that the red black property holds
3. you have a red black tree
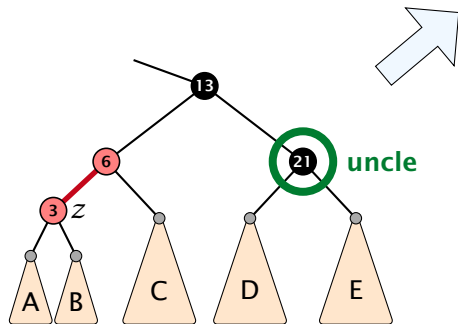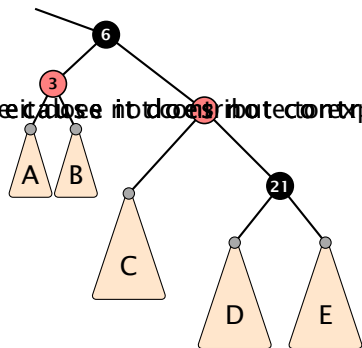
Case 2a: Black uncle and *z* is right child
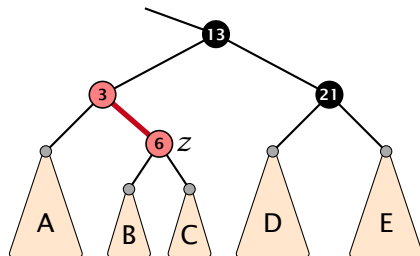
Case 2a: Black uncle and *z* is right child

[tikzpicture optimized away because it does not contribute to ex PDF]
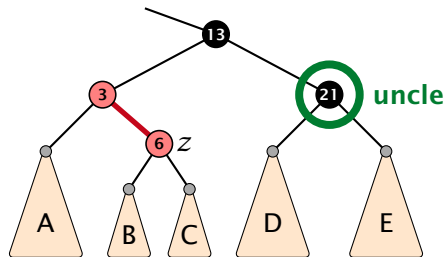
Case 2a: Black uncle and *z* is right child

1. rotate around parent

   [tikz picture reoptimized away because it does not contribute to the exported PDF]
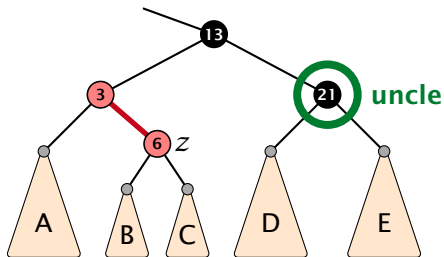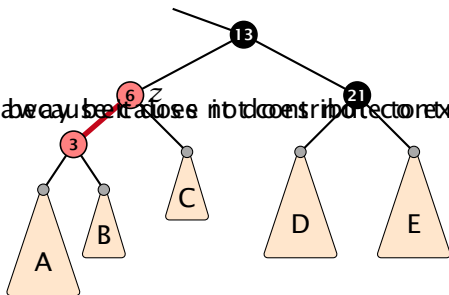
Case 2a: Black uncle and *z* is right child

1. rotate around parent

2. [nike picture upsidedized away because not constructor in PDF exported PDF]
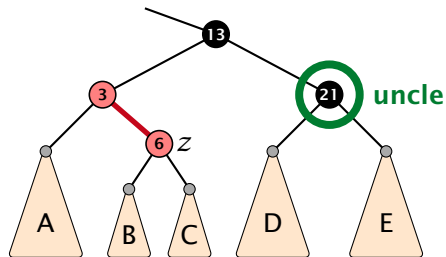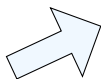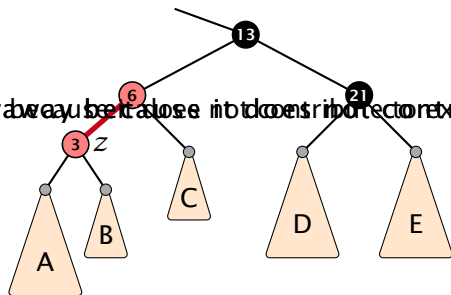
Case 2a: Black uncle and *z* is right child

1. rotate around parent
2. [keep it down optimized way away because it does not correct]
3. you have Case 2b
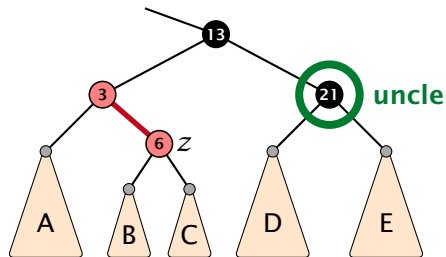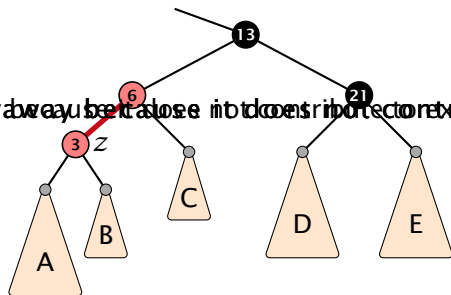
Red Black Trees: Insert

Running time:

▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete

Harald Räcke

Red Black Trees: Delete

First do a standard delete.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of x were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete

Red Black Trees: Delete

Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

Delete:

▶ deleting black node messes up black-height property

Red Black Trees: Delete
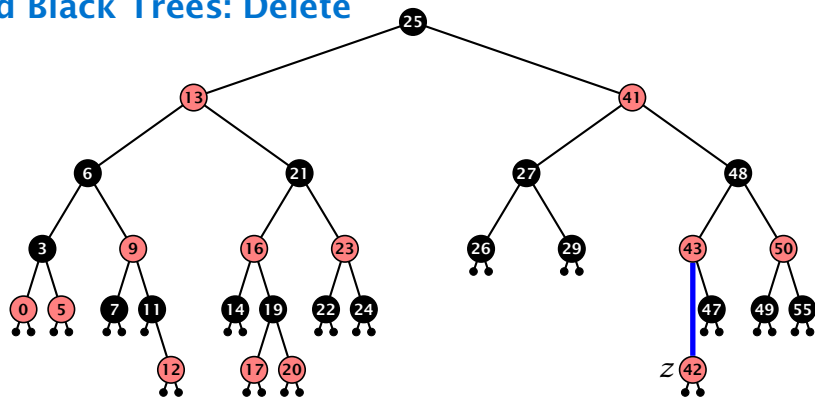
Delete:
- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete
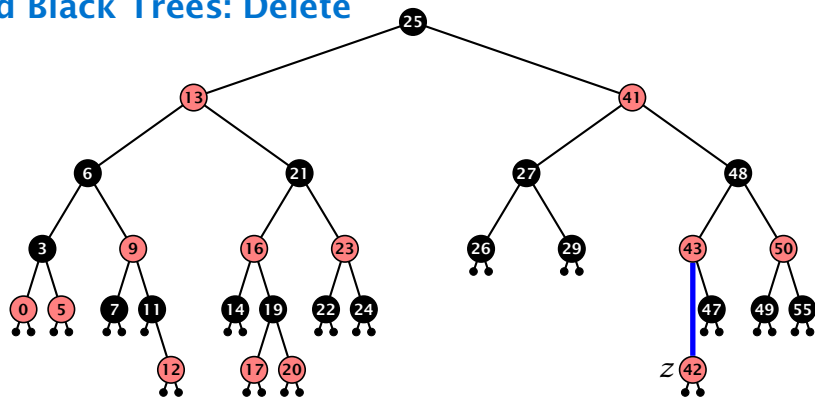
Delete:

▶ deleting black node messes up black-height property

▶ if z is red, we can simply color it black and everything is fine

▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm
- ▶ the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

▶ the node z is black

▶ if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Red Black Trees: Delete

Invariant of the fix-up algorithm

▶ the node z is black

▶ if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.
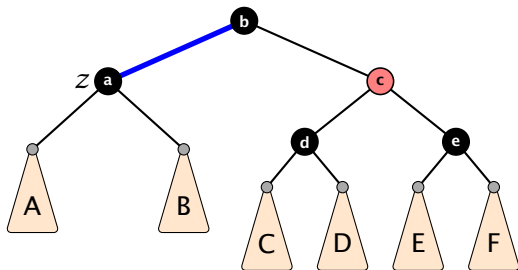
Case 1: Sibling of z is red

Case 1: Sibling of z is red

[tikzpicture optimized away because it does not contribute to exp...
PDF]

Case 1: Sibling of *z* is red

sibling

[tikz picture reoptimized away, because it does not contribute to exported PDF]
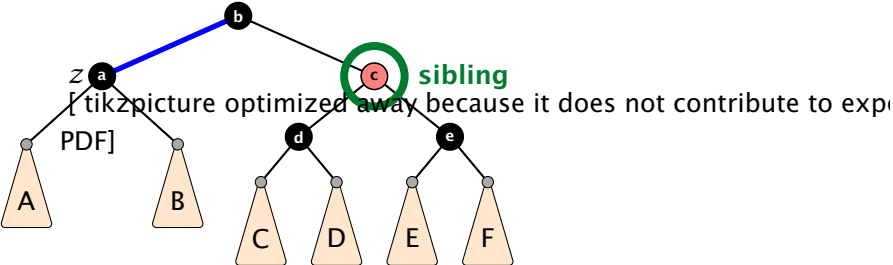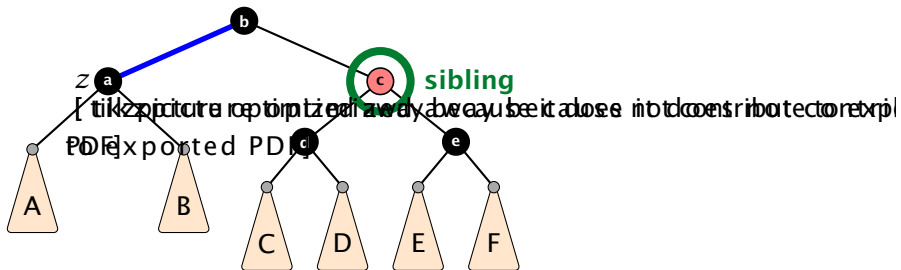
1. left-rotate around parent of *z*
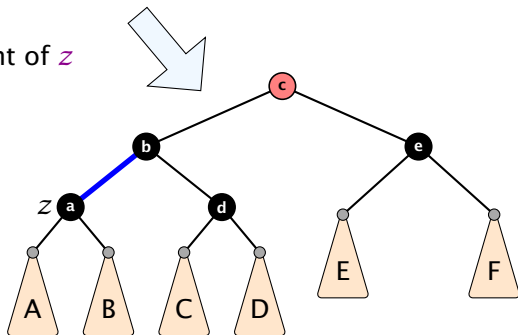
Case 1: Sibling of z is red
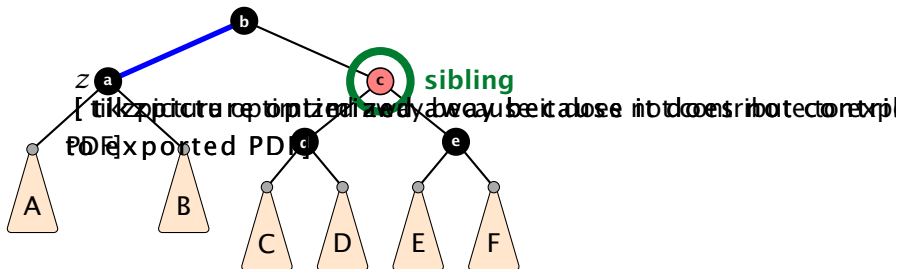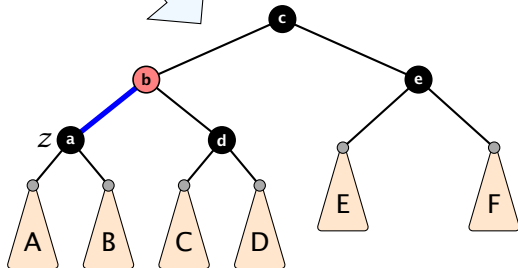
1. left-rotate around parent of z
2. recolor nodes b and c

Case 1: Sibling of z is red
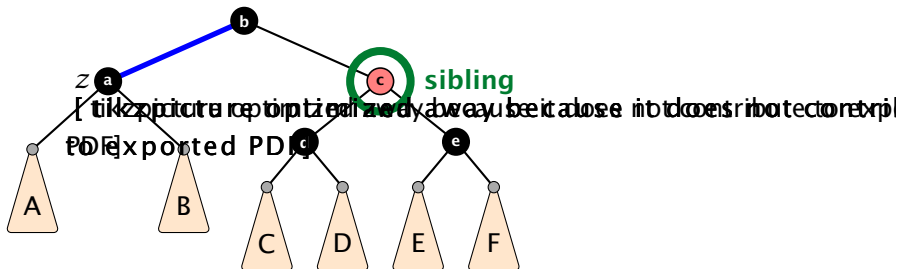
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
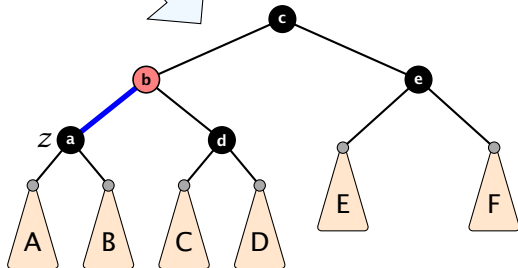
Case 1: Sibling of z is red
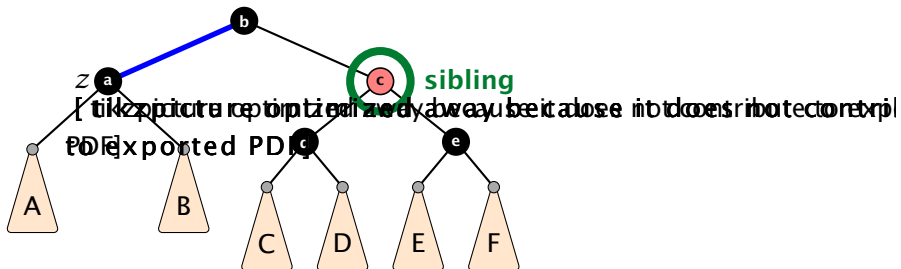
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4
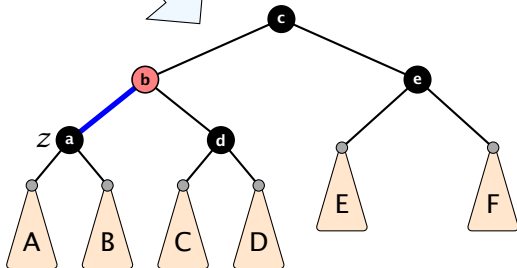
Case 2: Sibling is black with two black children
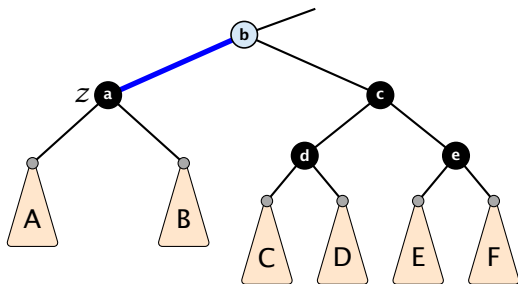
Case 2: Sibling is black with two black children
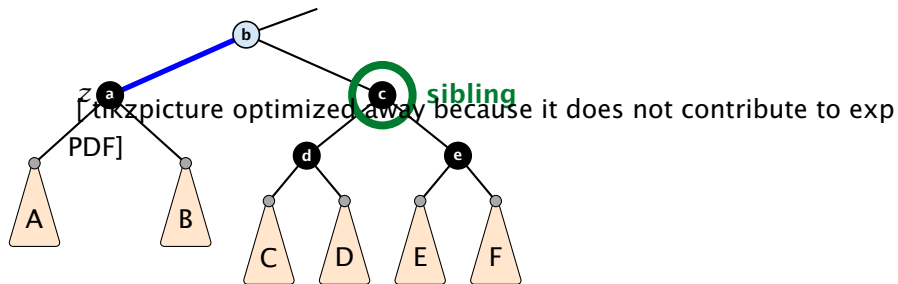
Case 2: Sibling is black with two black children
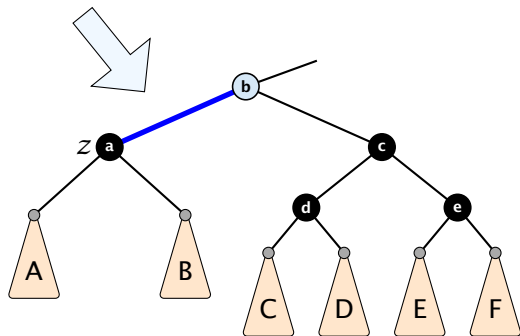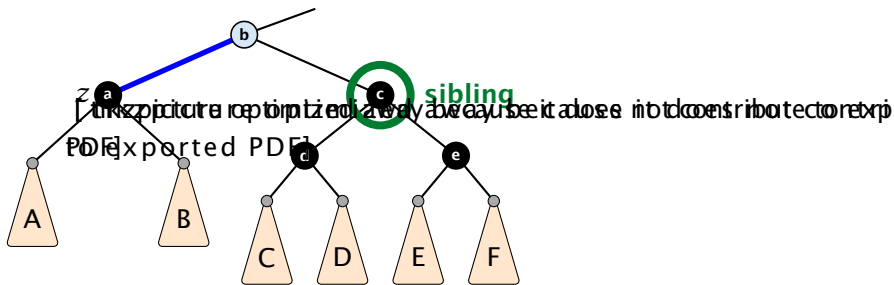
Case 2: Sibling is black with two black children
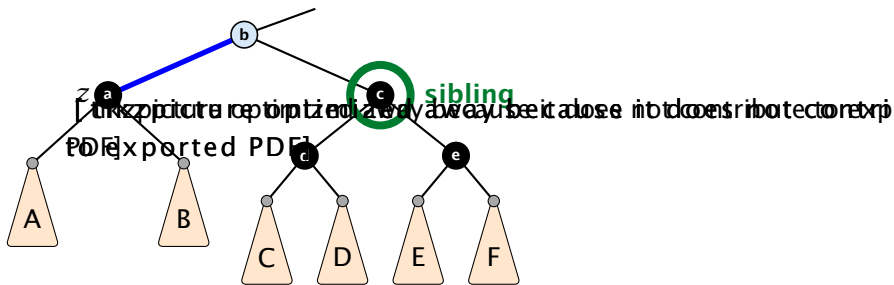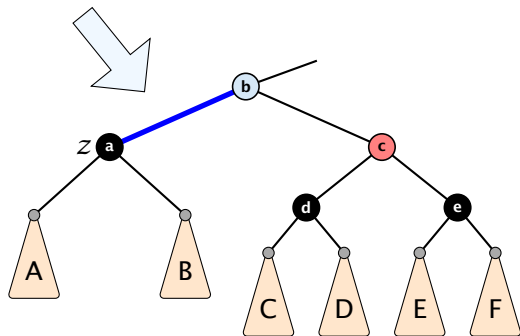
1. re-color node c

Case 2: Sibling is black with two black children
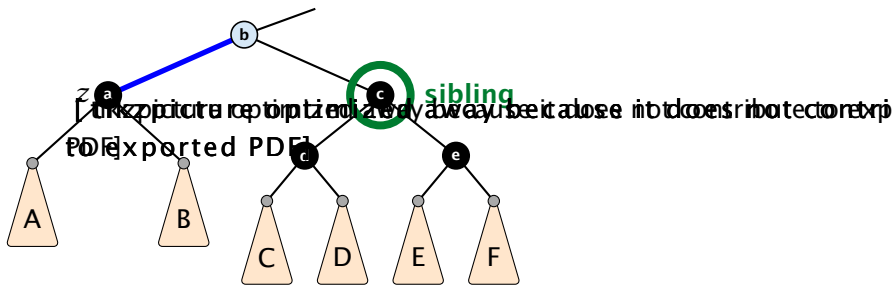
1. re-color node c
2. move fake black unit upwards
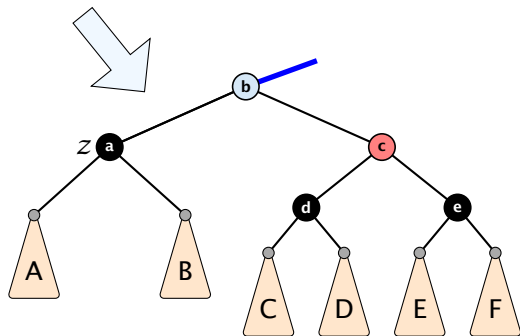
Case 2: Sibling is black with two black children

sibling

[this picture optimized away because it does not contribute to exported PDF]
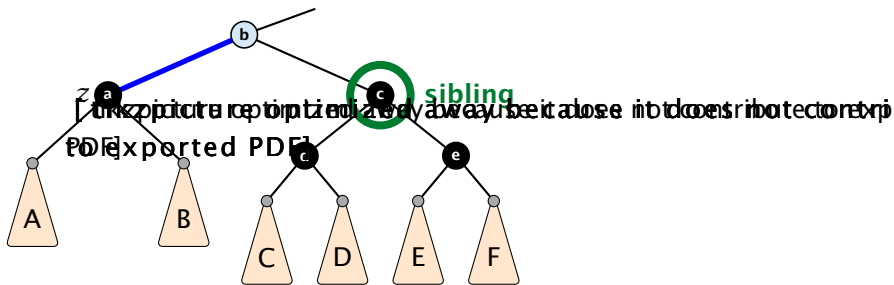
1. re-color node c
2. move fake black unit upwards
3. move z upwards

Case 2: Sibling is black with two black children

[this picture optimized away because it does not contribute to exported PDF]
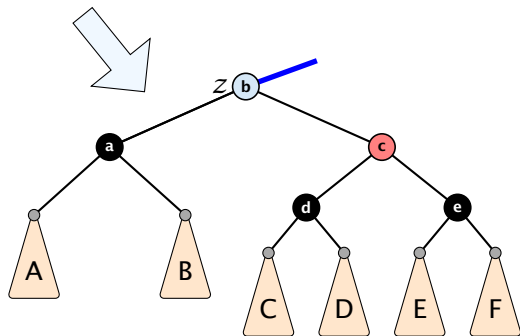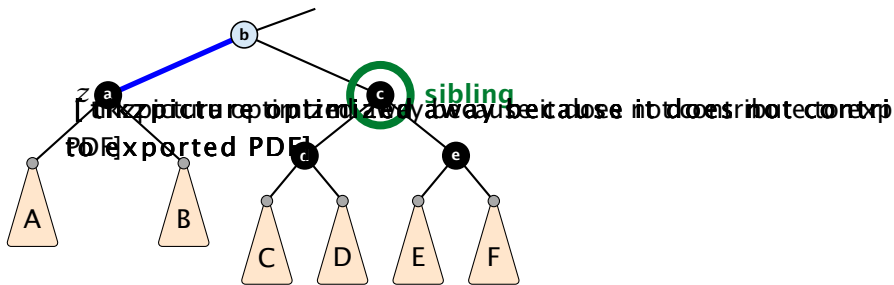
1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress

Case 2: Sibling is black with two black children

sibling

[this picture is optimized away because it does not contribute to the exported PDF]
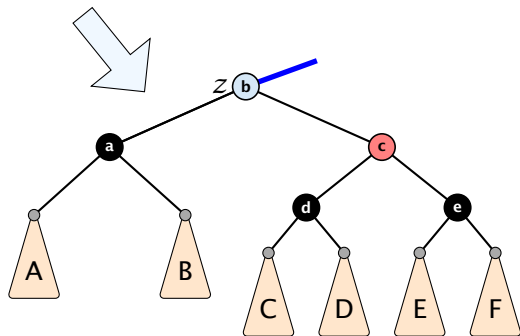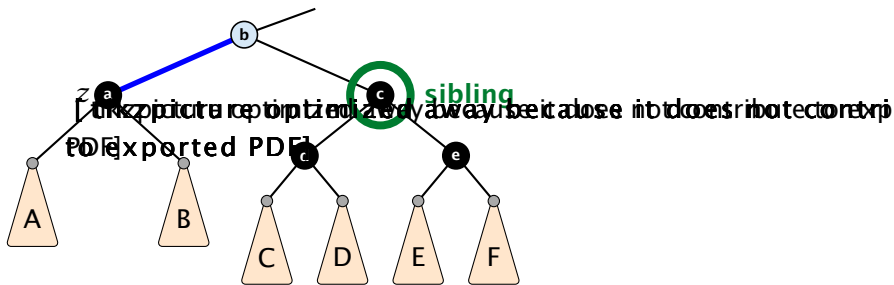
1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done
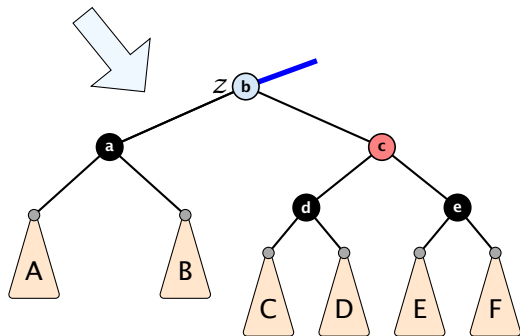
Case 3: Sibling black with one black child to the right
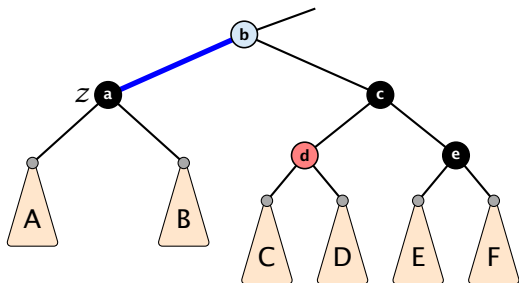
Case 3: Sibling black with one black child to the right

[tikzpicture optimized away because it does not contribute to ex
PDF]
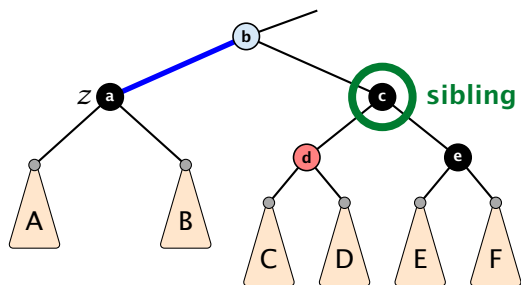
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling

[tikz picture reproduced away because it does not contribute to exported PDF]
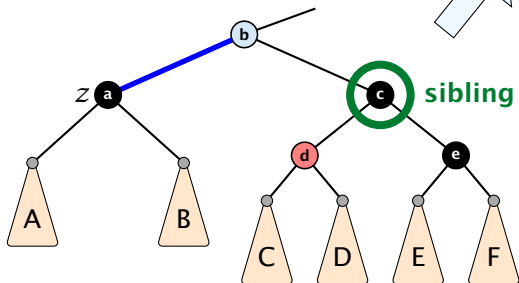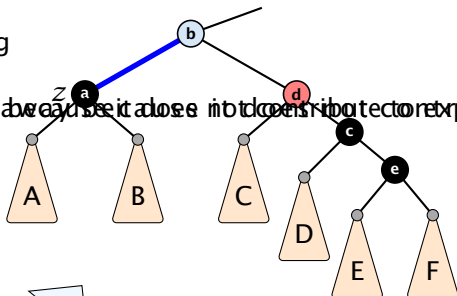
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d

[tikz picture replaced by a usetikzlibrary external context replaced by tikz picture optimized away because it does not contribute to exported PDF]

Case 3: Sibling black with one black child to the right
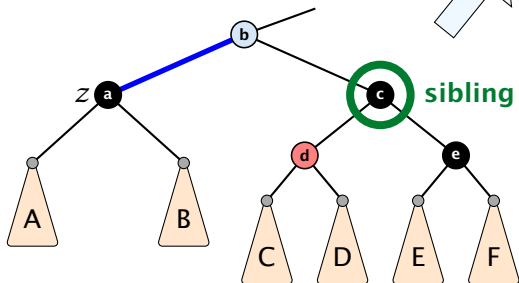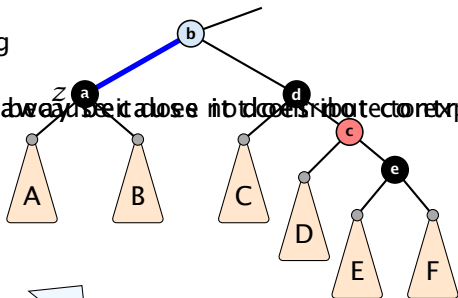
1. do a right-rotation at sibling
2. recolor c and d
   [tikz picture optimized away, because it does not contribute to the exported PDF]
3. new sibling is black with red right child (Case 4)

Case 4: Sibling is black with red right child
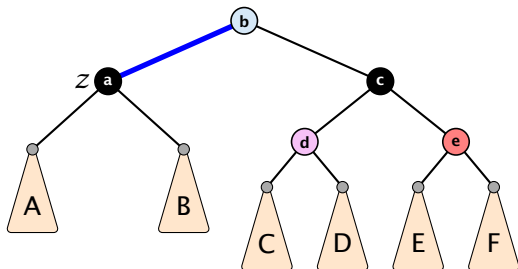
Case 4: Sibling is black with red right child
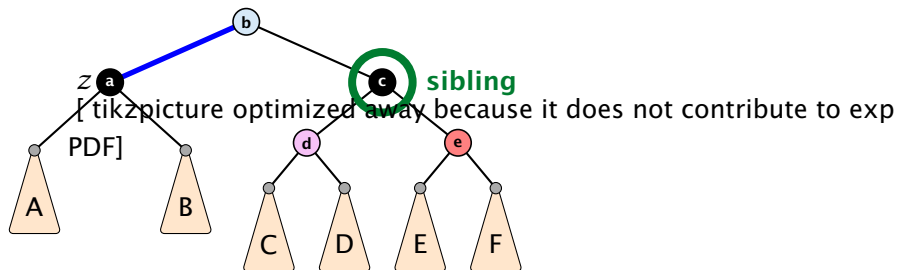
Case 4: Sibling is black with red right child
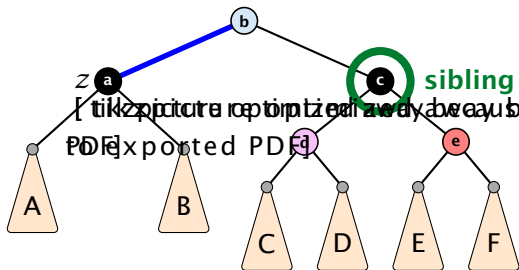
sibling

1. left-rotate around b
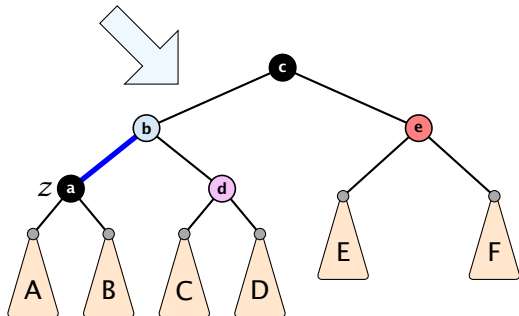
Case 4: Sibling is black with red right child
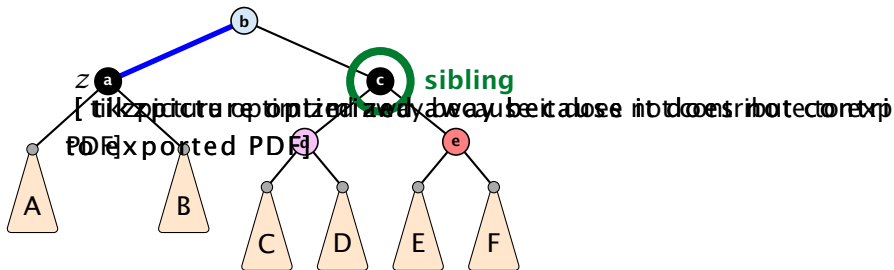
1. left-rotate around b
2. remove the fake black unit

Case 4: Sibling is black with red right child

sibling

[tikz picture reoptimized away, because it does not contribute to exported PDF]
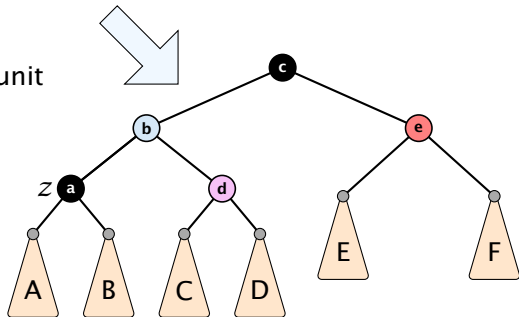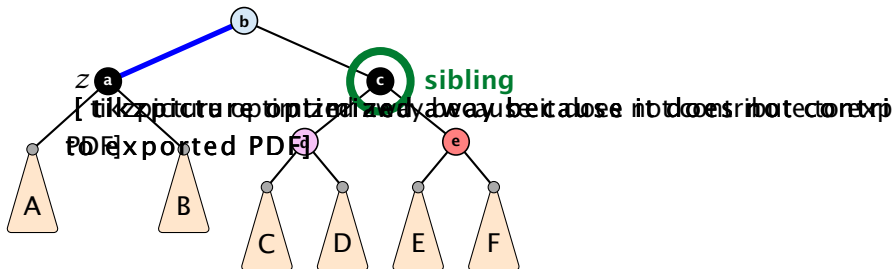
1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
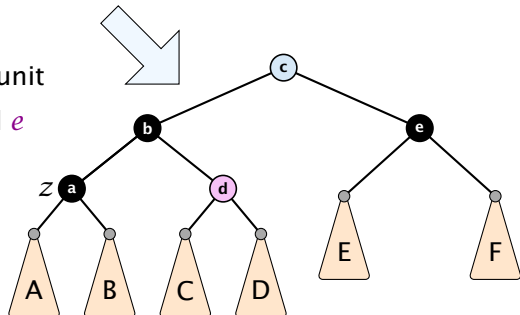
Case 4: Sibling is black with red right child
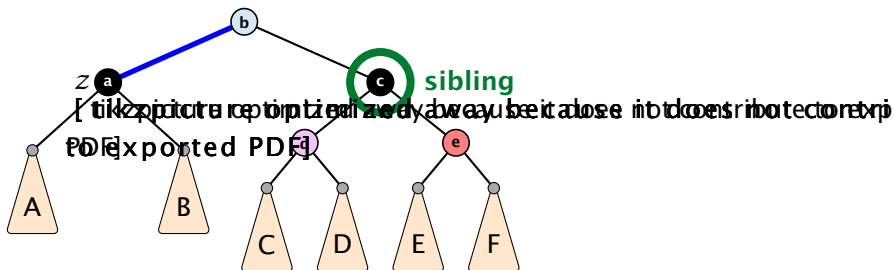
1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree
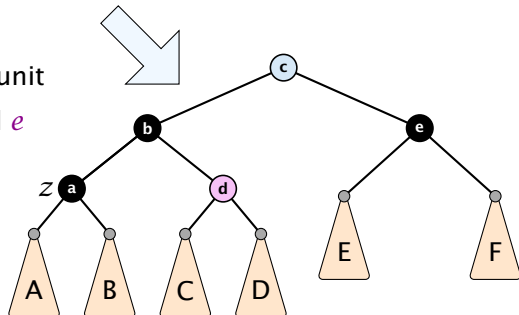
Running time:

▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree

Running time:

▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

▶ Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree

▶ Case 3 → Case 4 → red black tree

Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

- ▶ Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree

- ▶ Case 3 → Case 4 → red black tree

- ▶ Case 4 → red black tree

Running time:

▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree

▶ Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree

▶ Case 3 → Case 4 → red black tree

▶ Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

Splay Trees

Disadvantage of balanced search trees:

Splay Trees

Disadvantage of balanced search trees:

- – worst case; no advantage for easy inputs

Splay Trees

Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required

Splay Trees

Disadvantage of balanced search trees:

- – worst case; no advantage for easy inputs
- – additional memory required
- – complicated implementation

Splay Trees

Disadvantage of balanced search trees:

- – worst case; no advantage for easy inputs
- – additional memory required
- – complicated implementation

Splay Trees:

Splay Trees

Disadvantage of balanced search trees:

- – worst case; no advantage for easy inputs

- – additional memory required

- – complicated implementation

Splay Trees:

- + after access, an element is moved to the root; splay(x) repeated accesses are faster

Splay Trees

Disadvantage of balanced search trees:

- − worst case; no advantage for easy inputs
- − additional memory required
- − complicated implementation

Splay Trees:

- + after access, an element is moved to the root; $\text{splay}(x)$ repeated accesses are faster
- − only amortized guarantee

Splay Trees

Disadvantage of balanced search trees:

- − worst case; no advantage for easy inputs
- − additional memory required
- − complicated implementation

Splay Trees:

- + after access, an element is moved to the root; splay(x) repeated accesses are faster
- − only amortized guarantee
- − read-operations change the tree

Splay Trees

find(x)

- ▶ search for x according to a search tree
- ▶ let $\bar{x}$ be last element on search-path
- ▶ splay($\bar{x}$)

Splay Trees

insert(x)

- ▶ search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)

- ▶ splay($\bar{x}$) moves $\bar{x}$ to the root

- ▶ insert x as new root
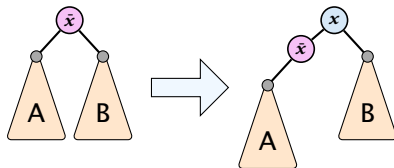
Splay Trees

delete(x)

- ▶ search for x; splay(x); remove x
- ▶ search largest element $\bar{x}$ in A
- ▶ splay($\bar{x}$) (on subtree A)
- ▶ connect root of B as right child of $\bar{x}$
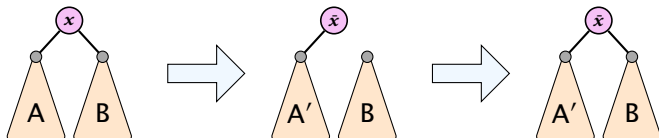
Move to Root
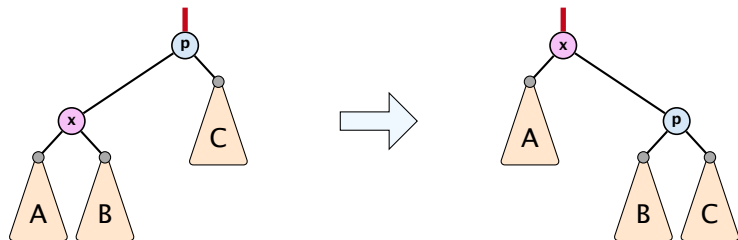
How to bring element to root?

▶ one (bad) option: moveToRoot(x)

▶ iteratively do rotation around parent of x until x is root

▶ if x is left child do right rotation otw. left rotation

Splay: Zig Case
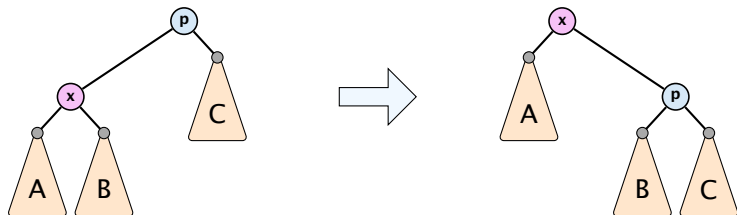
better option splay(x):

▶ zig case: if x is child of root do left rotation or right rotation around parent

Splay: Zigzag Case
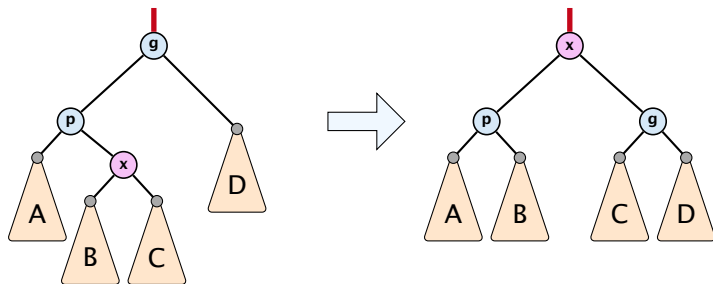
better option splay(x):

▶ zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)

▶ do double right rotation around grand-parent (resp. double left rotation)

Double Rotations

LeftRotate(*y*)

RightRotate(*x*)

DoubleRightRotate(*x*)
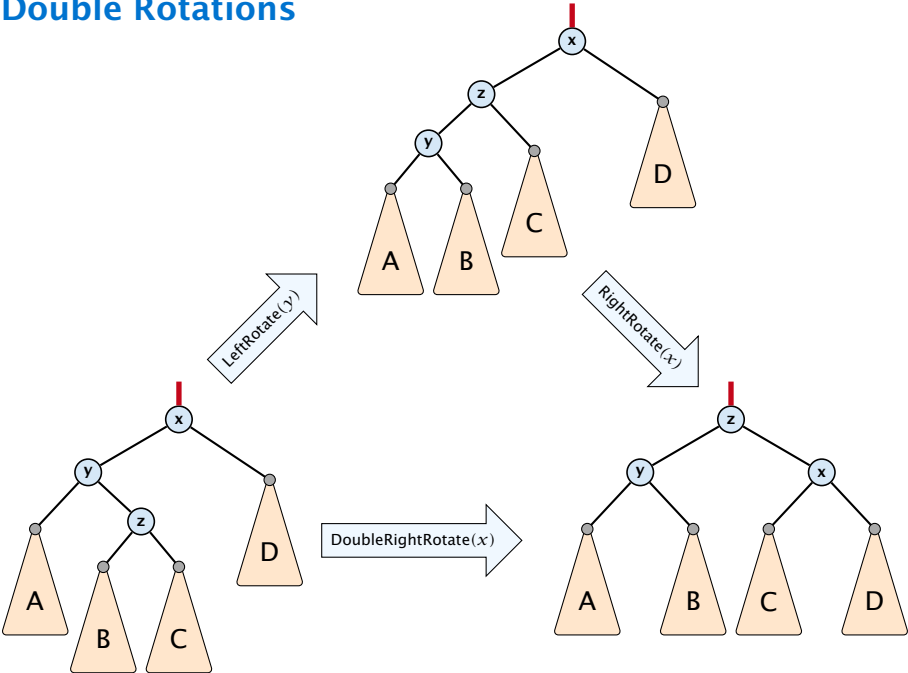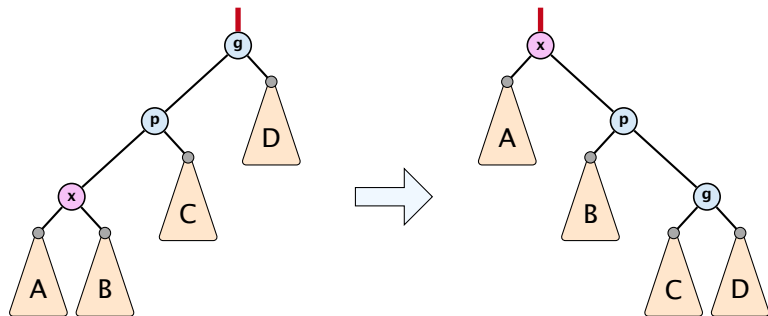
Splay: Zigzig Case

better option splay(x):

▶ zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)

▶ do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root
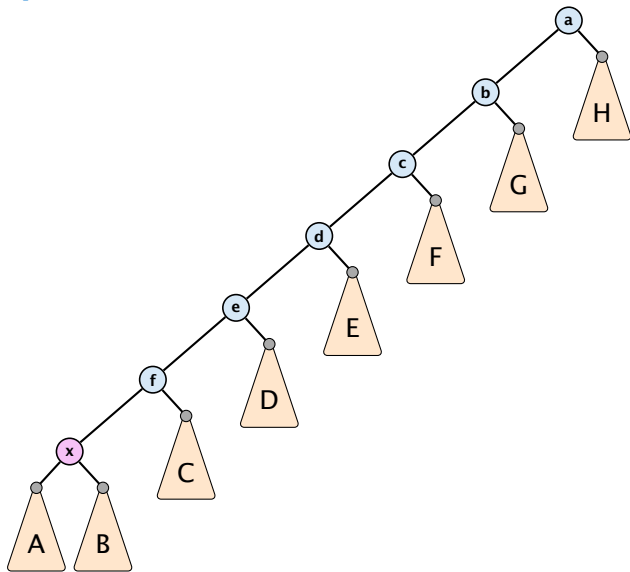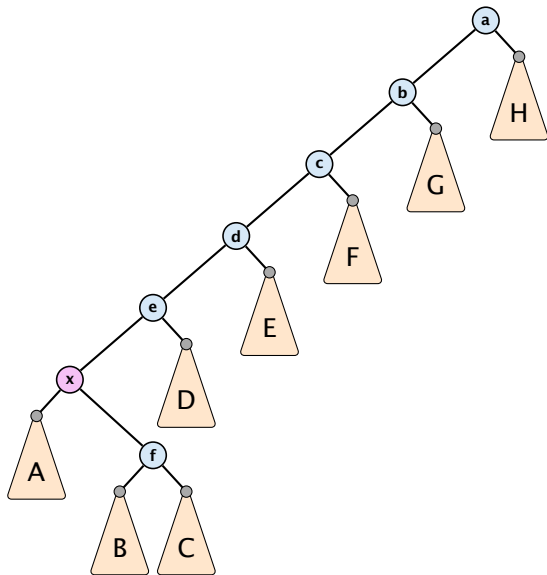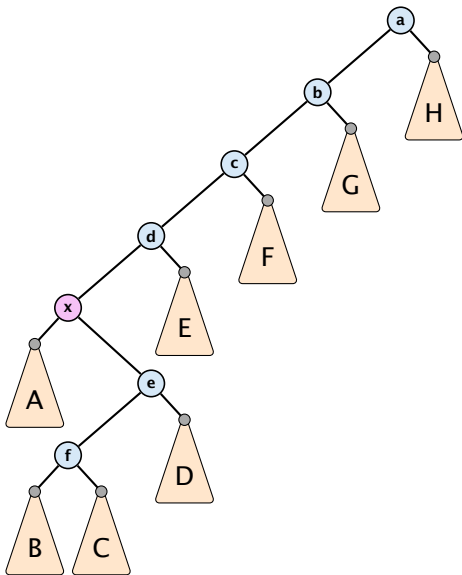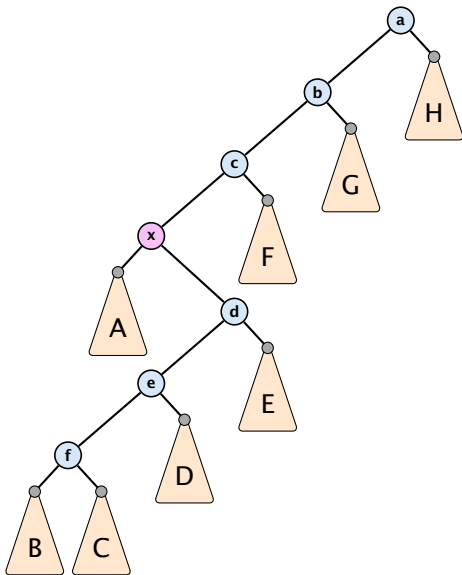
Splay vs. Move to Root
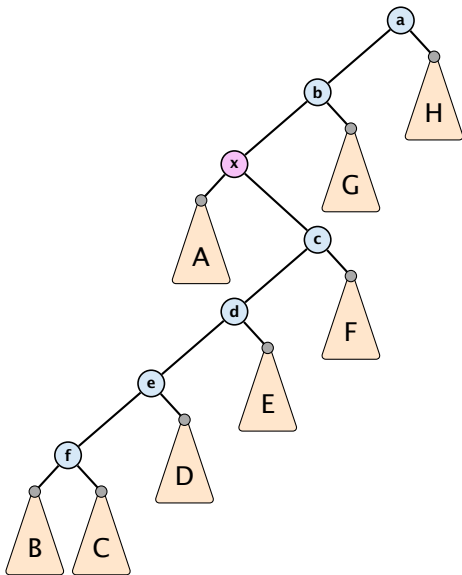
Splay vs. Move to Root
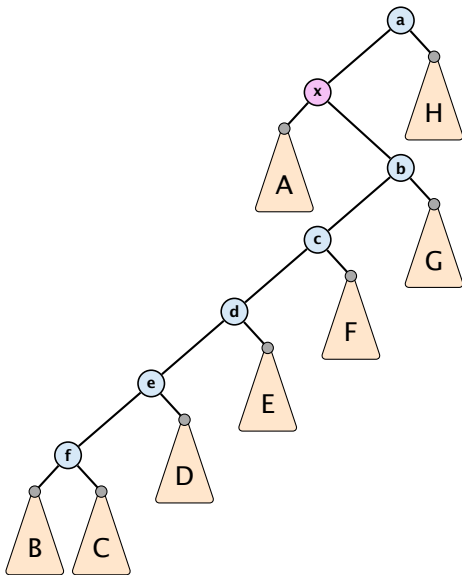
Splay vs. Move to Root
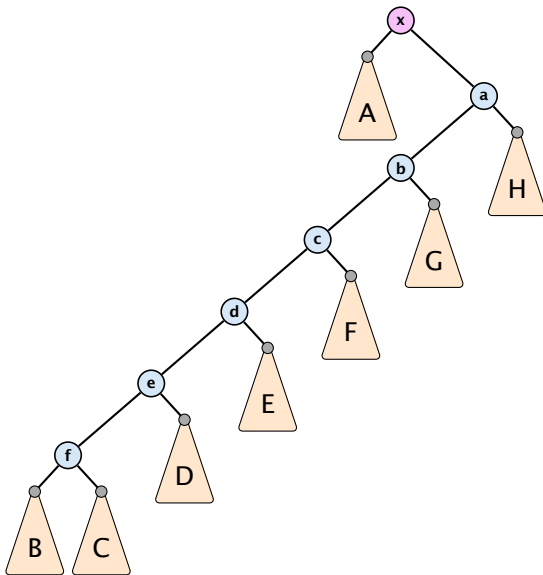
Splay vs. Move to Root
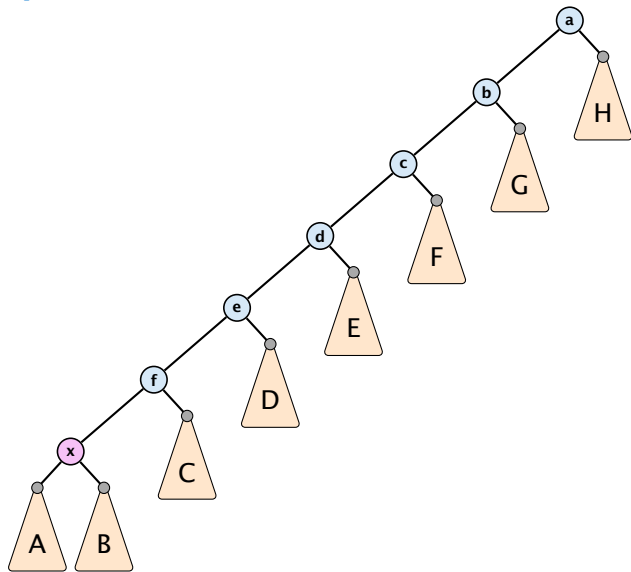
Splay vs. Move to Root
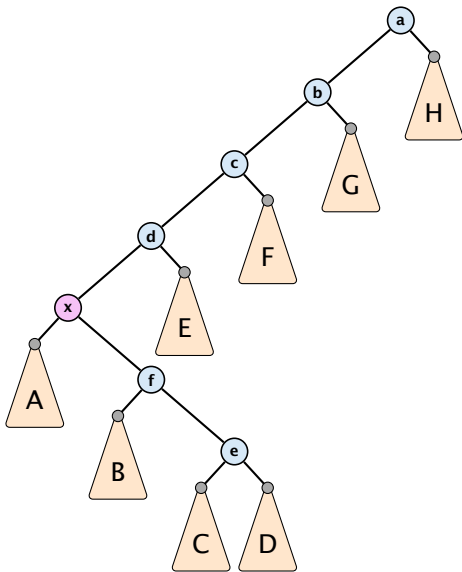
Splay vs. Move to Root
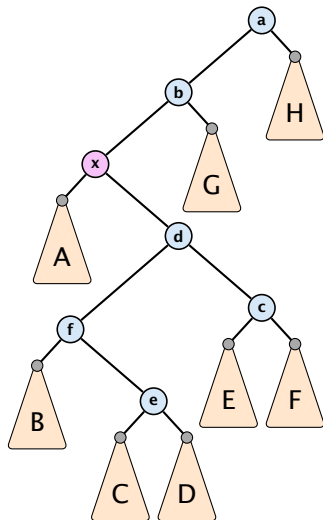
Splay vs. Move to Root
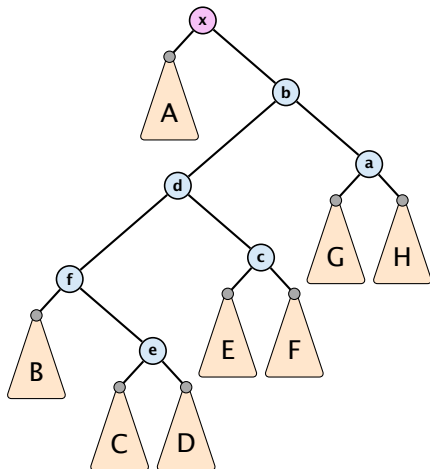
Static Optimality

Suppose we have a sequence of m find-operations. $\text{find}(x)$ appears h_x times in this sequence.

The cost of a static search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \, \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\min}))$, where $T_{\min}$ is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- ▶ the cost for accessing element x is $1 + \operatorname{depth}(x)$;
- ▶ after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\operatorname{cost}(A, S))$, for processing S.

Lemma 16

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ for all operations.

Amortized Analysis

Definition 17

A data structure with operations $\text{op}_1(), \ldots, \text{op}_k()$ has amortized running times $t_1, \ldots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\text{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

Potential Method

Introduce a potential for the data structure.

▶ $\Phi(D_i)$ is the potential after the i-th operation.

Potential Method

Introduce a potential for the data structure.

- ▶ $\Phi(D_i)$ is the potential after the i-th operation.
- ▶ Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

Potential Method

Introduce a potential for the data structure.

- $\Phi(D_i)$ is the potential after the i-th operation.
- Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Potential Method

Introduce a potential for the data structure.

- $\Phi(D_i)$ is the potential after the i-th operation.
- Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i$$

Potential Method

Introduce a potential for the data structure.

- ▶ $\Phi(D_i)$ is the potential after the i-th operation.
- ▶ Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0)$$

Potential Method

Introduce a potential for the data structure.

- ▶ $\Phi(D_i)$ is the potential after the i-th operation.
- ▶ Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- ▶ $S.\mathbf{push()}$
- ▶ $S.\mathbf{pop()}$
- ▶ $S.\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

Example: Stack

Stack

- ▶ $S.\,\mathbf{push}()$
- ▶ $S.\,\mathbf{pop}()$
- ▶ $S.\,\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- ▶ $S.\,\mathbf{push}()$: cost 1.
- ▶ $S.\,\mathbf{pop}()$: cost 1.
- ▶ $S.\,\mathbf{multipop}(k)$: cost $\min\{\text{size}, k\} = k$.

Example: Stack

Use potential function $\Phi(S) =$ number of elements on the stack.

Example: Stack

Use potential function $\Phi(S) =$ number of elements on the stack.

Amortized cost:

- ▶ $S.\,\mathbf{push()}$: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \leq 2 \ .$$

Example: Stack

Use potential function $\Phi(S) =$ number of elements on the stack.

Amortized cost:

▶ $S.\,\mathbf{push()}$: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \le 2 \ .$$

▶ $S.\,\mathbf{pop()}$: cost

$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \le 0 \ .$$

Example: Stack

Use potential function $\Phi(S) =$ number of elements on the stack.

Amortized cost:

▶ $S.\,\mathbf{push()}$: cost

$$\hat{C}_{\mathrm{push}} = C_{\mathrm{push}} + \Delta\Phi = 1 + 1 \le 2 \ .$$

▶ $S.\,\mathbf{pop()}$: cost

$$\hat{C}_{\mathrm{pop}} = C_{\mathrm{pop}} + \Delta\Phi = 1 - 1 \le 0 \ .$$

▶ $S.\,\mathbf{multipop(k)}$: cost

$$\hat{C}_{\mathrm{mp}} = C_{\mathrm{mp}} + \Delta\Phi = \min\{\mathrm{size}, k\} - \min\{\mathrm{size}, k\} \le 0 \ .$$

Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs
one time-unit.

Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n-bit binary counter may require to examine n-bits, and maybe change them.

Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n-bit binary counter may require to examine n-bits, and maybe change them.

Actual cost:

▶ Changing bit from 0 to 1: cost 1.

▶ Changing bit from 1 to 0: cost 1.

▶ Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

Example: Binary Counter

Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

► Changing bit from 0 to 1:

$$\hat{C}_{0 \to 1} = C_{0 \to 1} + \Delta\Phi = 1 + 1 \leq 2 \ .$$

Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

▶ Changing bit from 0 to 1:

$$\hat{C}_{0\to 1} = C_{0\to 1} + \Delta\Phi = 1 + 1 \leq 2 \ .$$

▶ Changing bit from 1 to 0:

$$\hat{C}_{1\to 0} = C_{1\to 0} + \Delta\Phi = 1 - 1 \leq 0 \ .$$

Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

▶ Changing bit from 0 to 1:

$$\hat{C}_{0 \to 1} = C_{0 \to 1} + \Delta\Phi = 1 + 1 \leq 2 \ .$$

▶ Changing bit from 1 to 0:

$$\hat{C}_{1 \to 0} = C_{1 \to 0} + \Delta\Phi = 1 - 1 \leq 0 \ .$$

▶ Increment: Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $(1 \to 0)$-operations, and one $(0 \to 1)$-operation.

Hence, the amortized cost is $k\hat{C}_{1 \to 0} + \hat{C}_{0 \to 1} \leq 2$.

Splay Trees

potential function for splay trees:

- ▶ size $s(x) = |T_x|$
- ▶ rank $r(x) = \log_2(s(x))$
- ▶ $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case

$$\Delta \Phi =$$
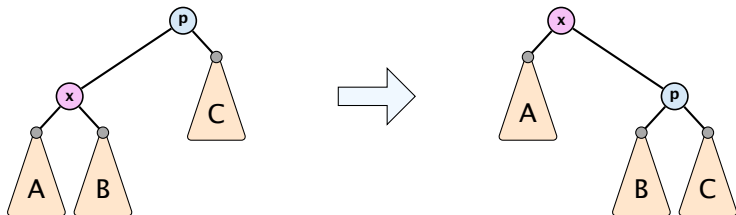
Splay: Zig Case
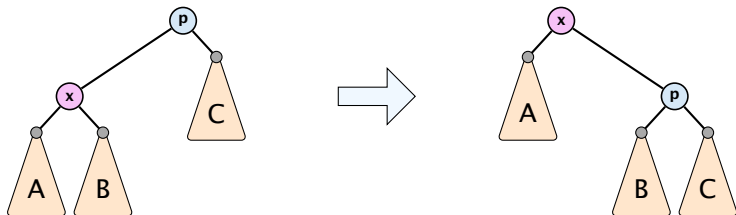
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

Splay: Zig Case
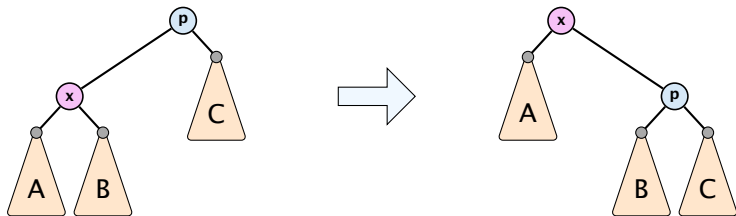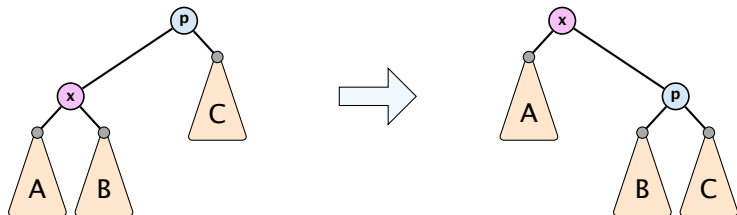
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$

Splay: Zig Case
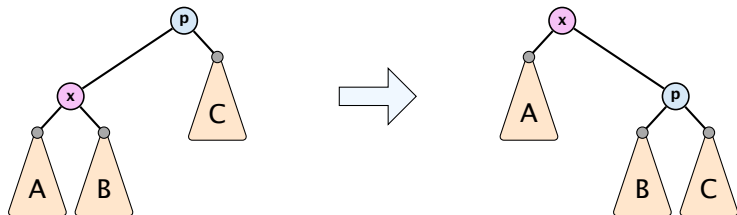
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$
$$\leq r'(x) - r(x)$$

Splay: Zig Case

$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$
$$\leq r'(x) - r(x)$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

Splay: Zigzig Case

$\Delta\Phi =$
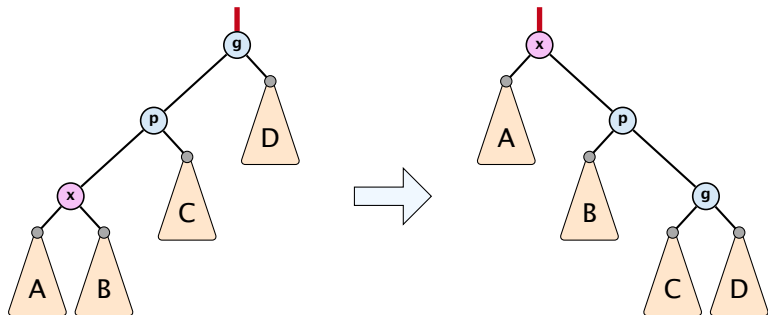
Splay: Zigzig Case
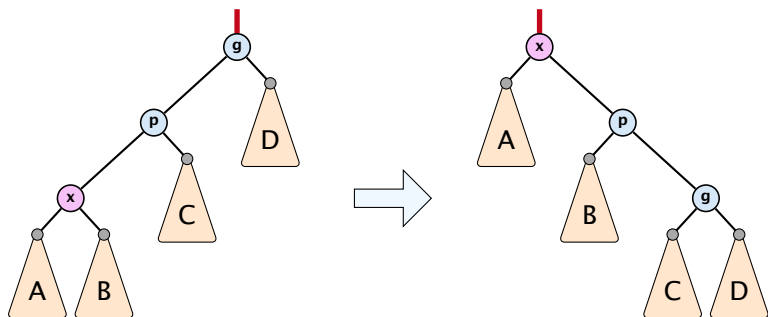
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzig Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
$$= r'(p) + r'(g) - r(x) - r(p)$$

Splay: Zigzig Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
$$= r'(p) + r'(g) - r(x) - r(p)$$
$$\leq r'(x) + r'(g) - r(x) - r(x)$$

Splay: Zigzig Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

$$= r'(p) + r'(g) - r(x) - r(p)$$

$$\leq r'(x) + r'(g) - r(x) - r(x)$$

$$= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)$$

Splay: Zigzig Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

$$= r'(p) + r'(g) - r(x) - r(p)$$

$$\leq r'(x) + r'(g) - r(x) - r(x)$$

$$= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)$$

$$= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))$$

Splay: Zigzig Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
$$= r'(p) + r'(g) - r(x) - r(p)$$
$$\leq r'(x) + r'(g) - r(x) - r(x)$$
$$= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)$$
$$= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))$$
$$\leq -2 + 3(r'(x) - r(x))$$

Splay: Zigzig Case

$$\Delta \Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

$$= r'(p) + r'(g) - r(x) - r(p)$$

$$\leq r'(x) + r'(g) - r(x) - r(x)$$

$$= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)$$

$$= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))$$

$$\leq -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzig}} \leq 3(r'(x) - r(x))$$
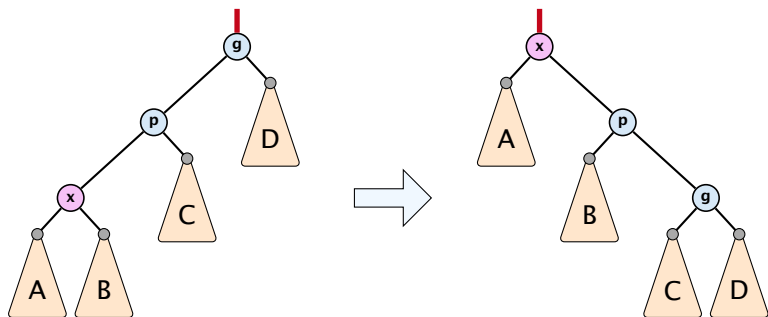
Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\le \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \le \log\Big(\frac{1}{2}\Big)$$

Splay: Zigzig Case

$$\frac{1}{2}\Big(r(x) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s(x)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$= \frac{1}{2}\log\Big(\frac{s(x)}{s'(x)}\Big) + \frac{1}{2}\log\Big(\frac{s'(g)}{s'(x)}\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s(x)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big) = -1$$

Splay: Zigzag Case

$\Delta\Phi =$

Splay: Zigzag Case
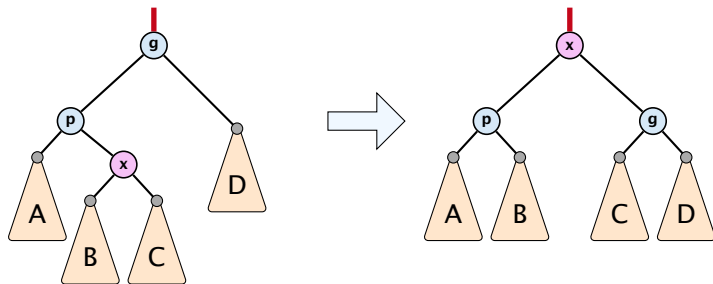
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

Splay: Zigzag Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
$$= r'(p) + r'(g) - r(x) - r(p)$$

Splay: Zigzag Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
$$= r'(p) + r'(g) - r(x) - r(p)$$
$$\leq r'(p) + r'(g) - r(x) - r(x)$$

Splay: Zigzag Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

$$= r'(p) + r'(g) - r(x) - r(p)$$
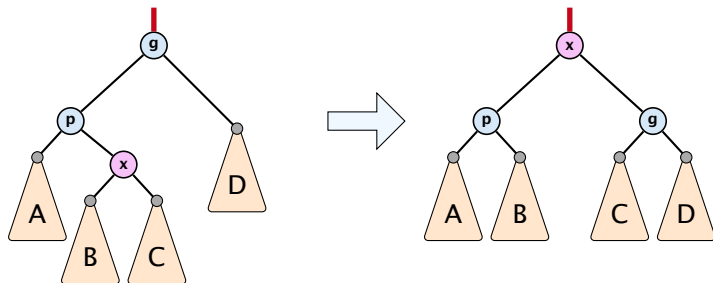
$$\leq r'(p) + r'(g) - r(x) - r(x)$$

$$= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)$$

Splay: Zigzag Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$
$$= r'(p) + r'(g) - r(x) - r(p)$$
$$\leq r'(p) + r'(g) - r(x) - r(x)$$
$$= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)$$
$$\leq -2 + 2(r'(x) - r(x))$$

Splay: Zigzag Case

$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

$$= r'(p) + r'(g) - r(x) - r(p)$$

$$\leq r'(p) + r'(g) - r(x) - r(x)$$

$$= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)$$

$$\leq -2 + 2(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))$$
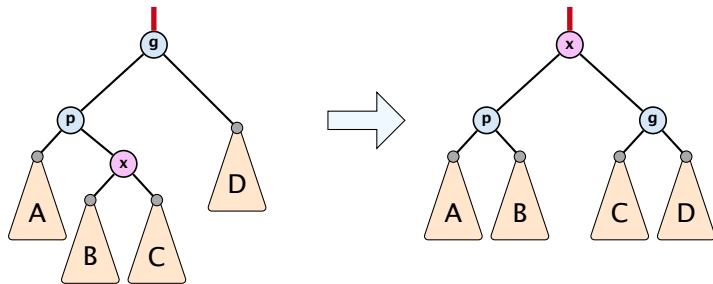
Splay: Zigzag Case

$$\frac{1}{2}\big(r'(p) + r'(g) - 2r'(x)\big)$$

Splay: Zigzag Case

$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

Splay: Zigzag Case

$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$
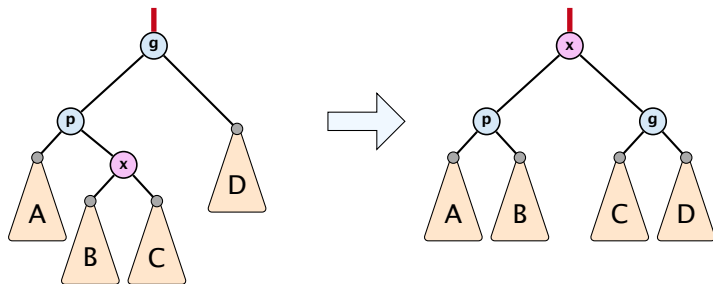
$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big)$$

Splay: Zigzag Case

$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big)$$

Splay: Zigzag Case

$$\frac{1}{2}\Big(r'(p) + r'(g) - 2r'(x)\Big)$$

$$= \frac{1}{2}\Big(\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))\Big)$$

$$\leq \log\Big(\frac{1}{2}\frac{s'(p)}{s'(x)} + \frac{1}{2}\frac{s'(g)}{s'(x)}\Big) \leq \log\Big(\frac{1}{2}\Big) = -1$$

Amortized cost of the whole splay operation:

$$\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x))$$

$$= 2 + 3(r(\text{root}) - r_0(x))$$

$$\leq \mathcal{O}(\log n)$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x):** insert element x.
- ▶ **Search(k):** search for element with key k.
- ▶ **Delete(x):** delete element referenced by pointer x.
- ▶ **find-by-rank(ℓ):** return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x):** insert element x.
- ▶ **Search(k):** search for element with key k.
- ▶ **Delete(x):** delete element referenced by pointer x.
- ▶ **find-by-rank(ℓ):** return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

2. determine additional information to be stored in the underlying structure

3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

7.4 Augmenting Data Structures

How to augment a data-structure

1. choose an underlying data-structure

2. determine additional information to be stored in the underlying structure

3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.

4. develop the new operations

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v.

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.

2. We store in each node v the size of the sub-tree rooted at v.

3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

4. How does find-by-rank work?
Find-by-rank(k) := Select(root, k) with

Algorithm 1 Select(x, i)

1: **if** $x =$ null **then return** error
2: **if** left$[x] \neq$ null **then** $r \leftarrow$ left$[x]$. size $+1$ **else** $r \leftarrow 1$
3: **if** $i = r$ **then return** x
4: **if** $i < r$ **then**
5:      **return** Select(left$[x], i$)
6: **else**
7:      **return** Select(right$[x], i - r$)
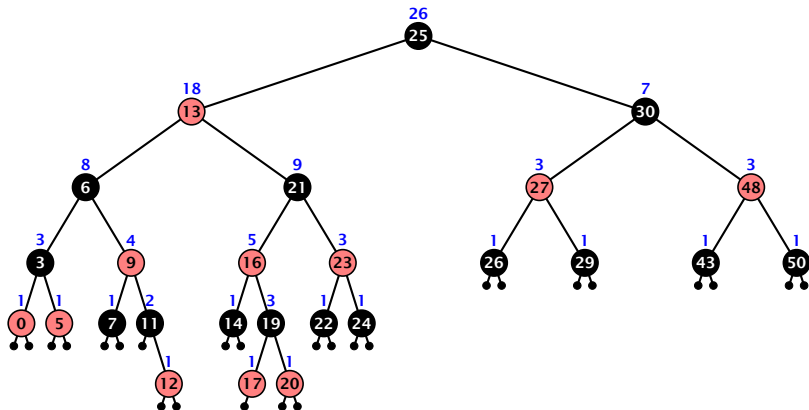
Select(x, i)

Find-by-rank:

▶ decide whether you have to proceed into the left or right sub-tree

▶ adjust the rank that you are searching for if you go right

Select(x, i)

Select(25, 14)

Find-by-rank:

▶ decide whether you have to proceed into the left or right sub-tree

▶ adjust the rank that you are searching for if you go right

Select(x, i)

Select(⑬, 14)

Find-by-rank:

▶ decide whether you have to proceed into the left or right
sub-tree

▶ adjust the rank that you are searching for if you go right

Select(x, i)

Select(21, 5)

Find-by-rank:

▶ decide whether you have to proceed into the left or right sub-tree

▶ adjust the rank that you are searching for if you go right

Select(x, i)

Select(⑯, 5)

Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

Select(x, i)

Select(19 , 3)

Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

Select(x, i)

Select(⑳, 1)
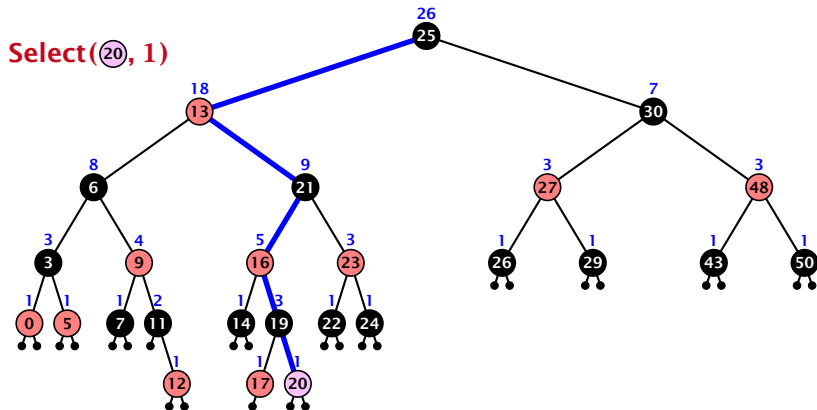
Find-by-rank:
- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

   3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:
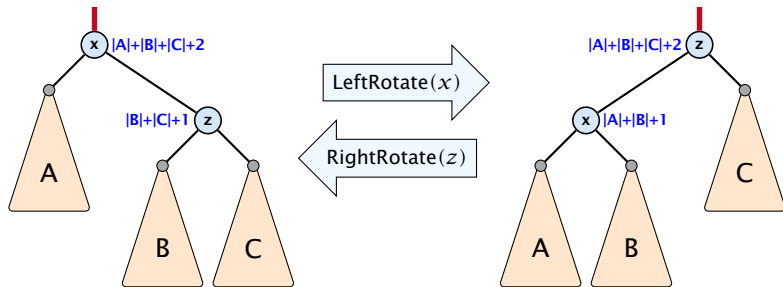
The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- time for search $\Theta(n)$
- time for insert $\Theta(n)$ (dominated by searching the item)
- time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search $\Theta(n)$
- ▶ time for insert $\Theta(n)$ (dominated by searching the item)
- ▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

▶ time for search $\Theta(n)$

▶ time for insert $\Theta(n)$ (dominated by searching the item)

▶ time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.5 Skip Lists

How can we improve the search-operation?

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

7.5 Skip Lists

How can we improve the search-operation?

Add an express lane:

Let $|L_1|$ denote the number of elements in the "express lane", and $|L_0| = n$ the number of all elements (ignoring dummy elements).

7.5 Skip Lists

How can we improve the search-operation?
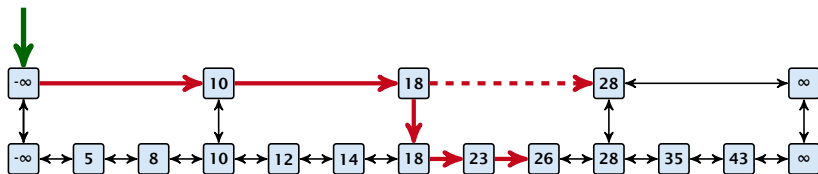
Add an express lane:

Let $|L_1|$ denote the number of elements in the "express lane", and $|L_0| = n$ the number of all elements (ignoring dummy elements).

Worst case search time: $|L_1| + \frac{|L_0|}{|L_1|}$ (ignoring additive constants)

7.5 Skip Lists

How can we improve the search-operation?
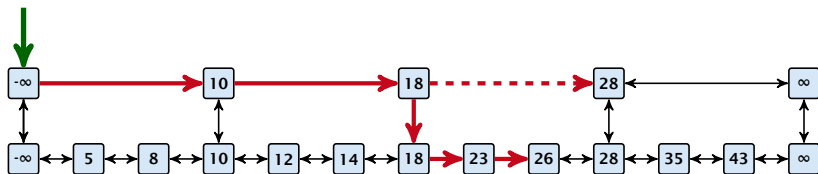
Add an express lane:

Let $|L_1|$ denote the number of elements in the "express lane", and $|L_0| = n$ the number of all elements (ignoring dummy elements).

Worst case search time: $|L_1| + \frac{|L_0|}{|L_1|}$ (ignoring additive constants)

Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)

▶ Find the largest item in list L_k that is smaller than x. At most $|L_k| + 2$ steps.

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)

- ▶ Find the largest item in list L_k that is smaller than x. At most $|L_k| + 2$ steps.
- ▶ Find the largest item in list L_{k-1} that is smaller than x. At most $\left\lceil \frac{|L_{k-1}|}{|L_k|+1} \right\rceil + 2$ steps.

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)

▶ Find the largest item in list L_k that is smaller than x. At most $|L_k| + 2$ steps.

▶ Find the largest item in list L_{k-1} that is smaller than x. At most $\lceil \frac{|L_{k-1}|}{|L_k|+1} \rceil + 2$ steps.

▶ Find the largest item in list L_{k-2} that is smaller than x. At most $\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \rceil + 2$ steps.

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)

▶ Find the largest item in list L_k that is smaller than x. At most $|L_k| + 2$ steps.

▶ Find the largest item in list L_{k-1} that is smaller than x. At most $\left\lceil \frac{|L_{k-1}|}{|L_k|+1} \right\rceil + 2$ steps.

▶ Find the largest item in list L_{k-2} that is smaller than x. At most $\left\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \right\rceil + 2$ steps.

▶ ...

7.5 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

Search(x) ($k + 1$ lists $L_0, \ldots, L_k$)

▶ Find the largest item in list L_k that is smaller than x. At most $|L_k| + 2$ steps.

▶ Find the largest item in list L_{k-1} that is smaller than x. At most $\lceil \frac{|L_{k-1}|}{|L_k|+1} \rceil + 2$ steps.

▶ Find the largest item in list L_{k-2} that is smaller than x. At most $\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \rceil + 2$ steps.

▶ ...

▶ At most $|L_k| + \sum_{i=1}^{k} \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$.

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$.
Choose $r = n^{\frac{1}{k+1}}$. Then

$$r^{-k}n + kr$$

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$.
Choose $r = n^{\frac{1}{k+1}}$. Then

$$r^{-k}n + kr = \left(n^{\frac{1}{k+1}}\right)^{-k} n + kn^{\frac{1}{k+1}}$$

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$.
Choose $r = n^{\frac{1}{k+1}}$. Then

$$r^{-k}n + kr = \left(n^{\frac{1}{k+1}}\right)^{-k}n + kn^{\frac{1}{k+1}}$$
$$= n^{1 - \frac{k}{k+1}} + kn^{\frac{1}{k+1}}$$

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$.
Choose $r = n^{\frac{1}{k+1}}$. Then

$$\begin{aligned}
r^{-k}n + kr &= \left(n^{\frac{1}{k+1}}\right)^{-k}n + kn^{\frac{1}{k+1}} \\
&= n^{1-\frac{k}{k+1}} + kn^{\frac{1}{k+1}} \\
&= (k+1)n^{\frac{1}{k+1}} \ .
\end{aligned}$$

7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$.
Choose $r = n^{\frac{1}{k+1}}$. Then

$$\begin{aligned}
r^{-k}n + kr &= \left(n^{\frac{1}{k+1}}\right)^{-k} n + kn^{\frac{1}{k+1}} \\
&= n^{1 - \frac{k}{k+1}} + kn^{\frac{1}{k+1}} \\
&= (k+1)n^{\frac{1}{k+1}} \ .
\end{aligned}$$

Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

7.5 Skip Lists

How to do insert and delete?

7.5 Skip Lists

How to do insert and delete?

▶ If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

7.5 Skip Lists

How to do insert and delete?

▶ If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

Use randomization instead!

Insert:

7.5 Skip Lists

Insert:

▶ A search operation gives you the insert position for element x in every list.

7.5 Skip Lists

Insert:

▶ A search operation gives you the insert position for element x in every list.

▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.

7.5 Skip Lists

Insert:

▶ A search operation gives you the insert position for element x in every list.

▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.

▶ Insert x into lists $L_0, \dots, L_{t-1}$.

7.5 Skip Lists

Insert:

- ▶ A search operation gives you the insert position for element x in every list.
- ▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- ▶ Insert x into lists $L_0, \dots, L_{t-1}$.

Delete:

7.5 Skip Lists

Insert:

▶ A search operation gives you the insert position for element x in every list.

▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.

▶ Insert x into lists $L_0, \dots, L_{t-1}$.

Delete:

▶ You get all predecessors via backward pointers.

7.5 Skip Lists

Insert:

▶ A search operation gives you the insert position for element x in every list.

▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.

▶ Insert x into lists $L_0, \dots, L_{t-1}$.

Delete:

▶ You get all predecessors via backward pointers.

▶ Delete x in all lists it actually appears in.

7.5 Skip Lists

Insert:

- ▶ A search operation gives you the insert position for element x in every list.
- ▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- ▶ Insert x into lists $L_0, \dots, L_{t-1}$.

Delete:

- ▶ You get all predecessors via backward pointers.
- ▶ Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

7.5 Skip Lists

Insert (35):
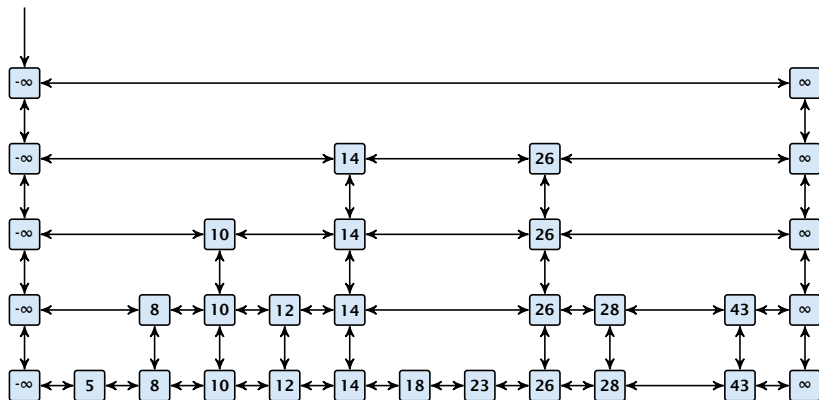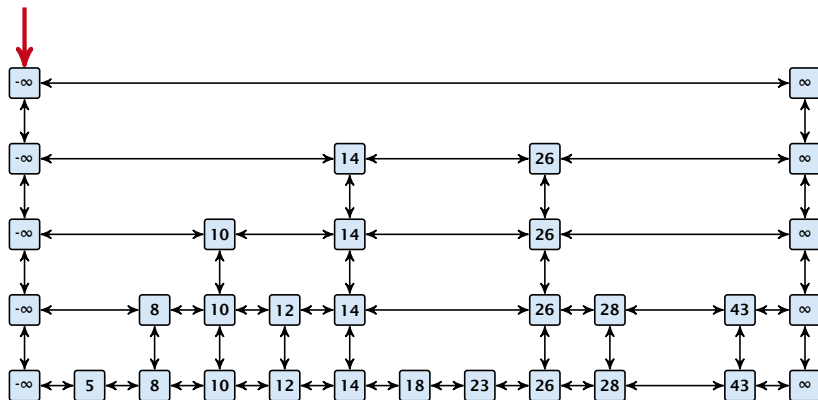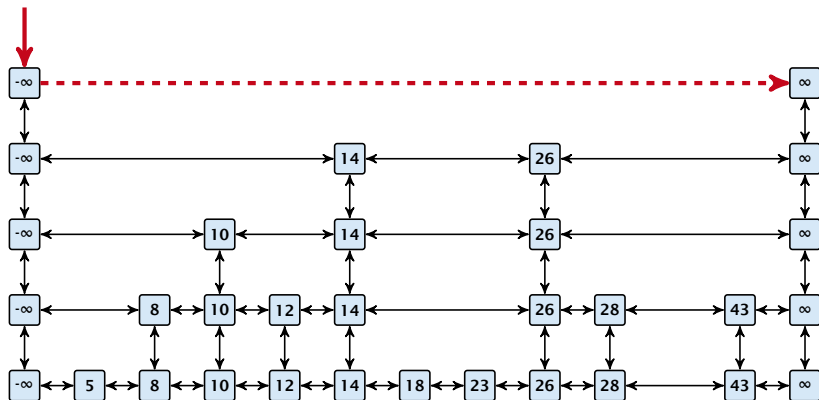
7.5 Skip Lists

Insert (35):

7.5 Skip Lists

Insert (35):

7.5 Skip Lists
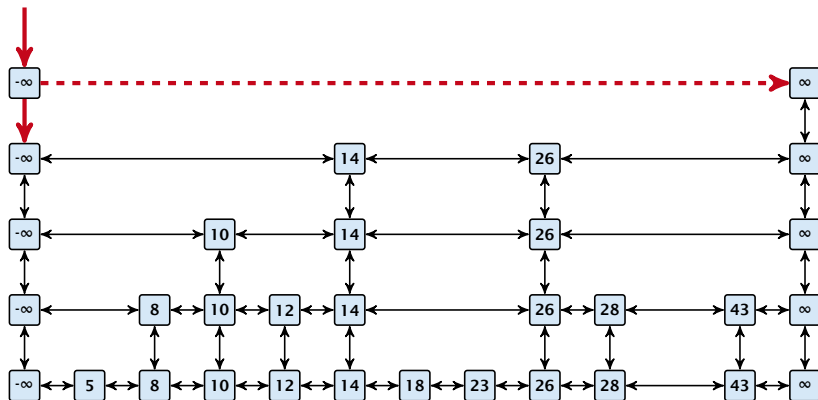
Insert (35):

7.5 Skip Lists
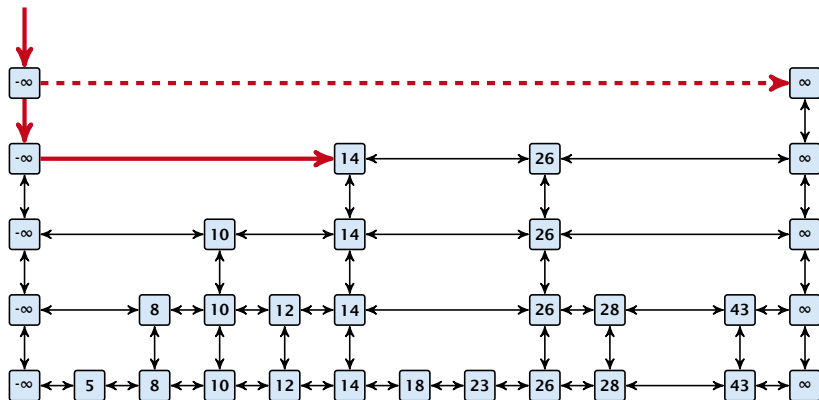
Insert (35):

7.5 Skip Lists
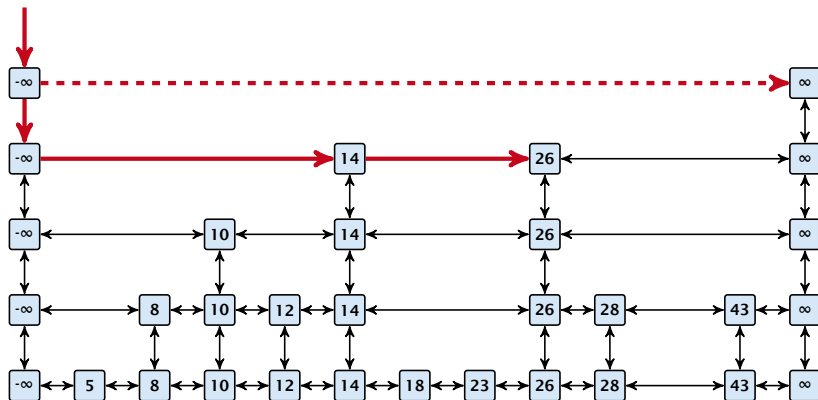
Insert (35):

7.5 Skip Lists
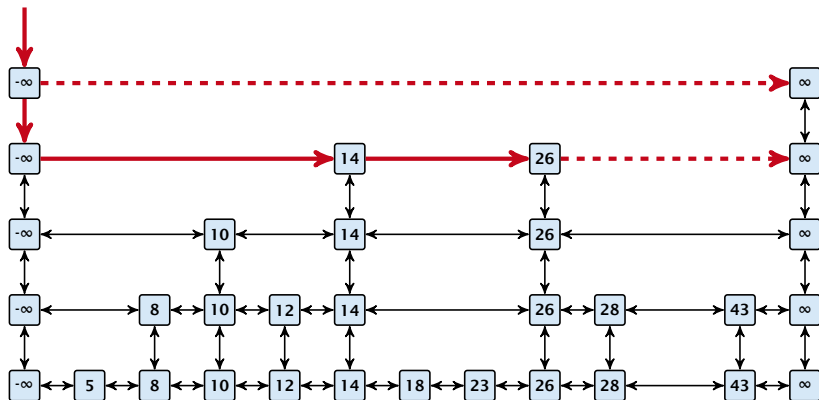
Insert (35):

7.5 Skip Lists
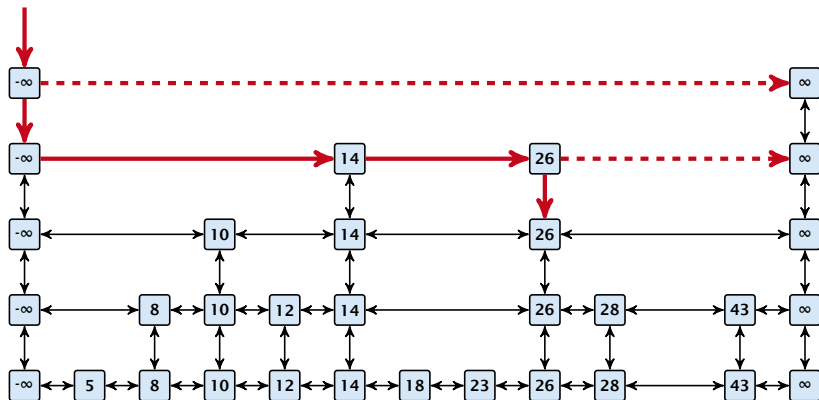
Insert (35):

7.5 Skip Lists
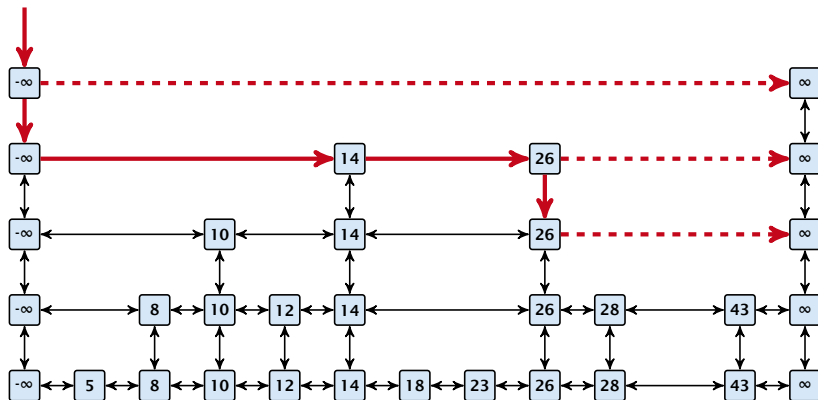
Insert (35):

7.5 Skip Lists
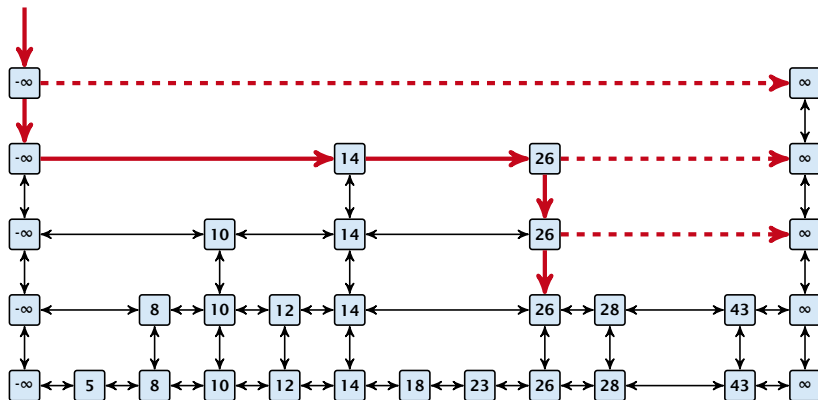
Insert (35):

7.5 Skip Lists
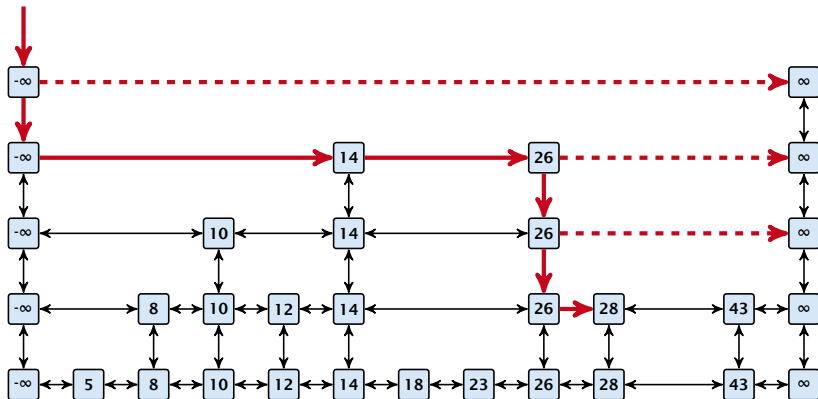
Insert (35):

7.5 Skip Lists
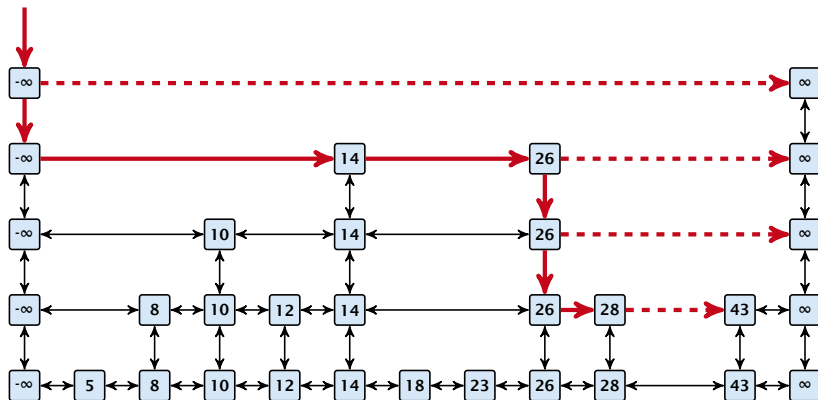
Insert (35):

7.5 Skip Lists
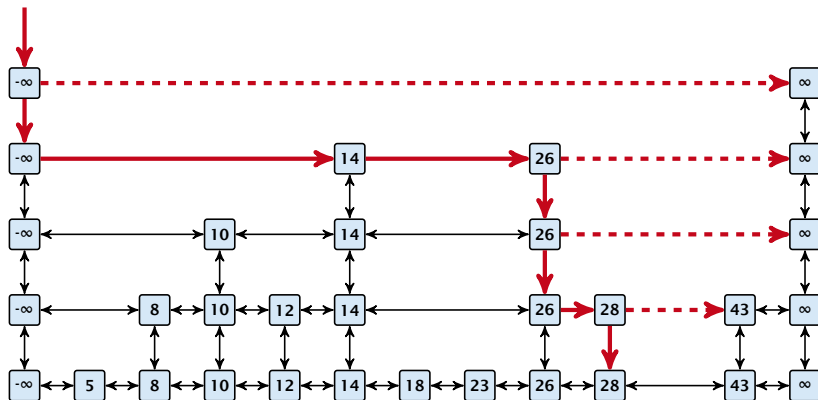
Insert (35):

7.5 Skip Lists
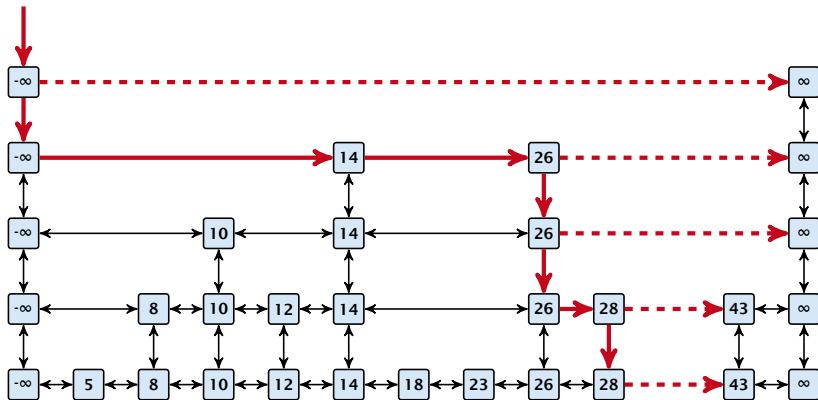
Insert (35):

7.5 Skip Lists
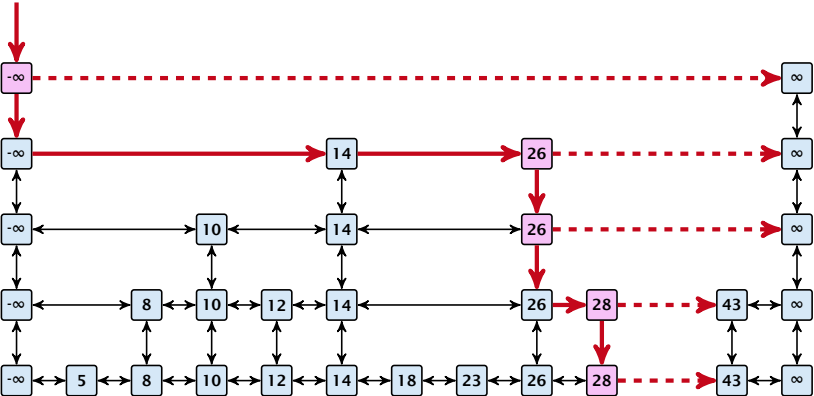
Insert (35):

7.5 Skip Lists

Insert (35):
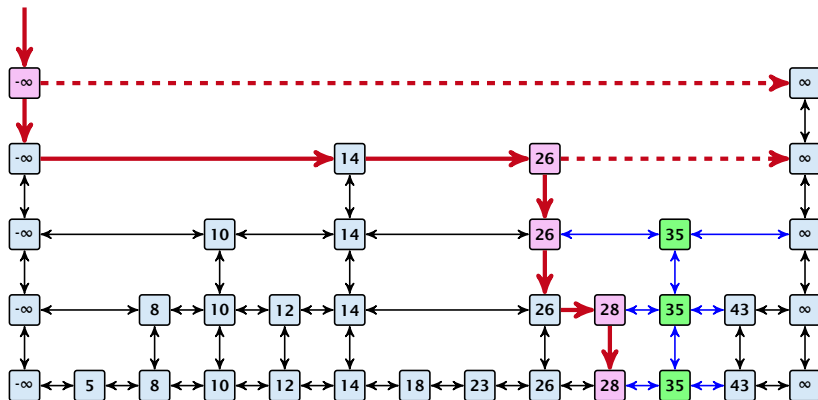
High Probability

Definition 18 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

High Probability

Definition 18 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Here the $\mathcal{O}$-notation hides a constant that may depend on α.

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell]$$

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell] = 1 - \Pr[\bar{E}_1 \vee \cdots \vee \bar{E}_\ell]$$

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell] = 1 - \Pr[\bar{E}_1 \vee \cdots \vee \bar{E}_\ell]$$
$$\geq 1 - n^c \cdot n^{-\alpha}$$

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell] = 1 - \Pr[\bar{E}_1 \vee \cdots \vee \bar{E}_\ell]$$
$$\geq 1 - n^c \cdot n^{-\alpha}$$
$$= 1 - n^{c-\alpha} \ .$$

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_\ell] = 1 - \Pr[\bar{E}_1 \vee \cdots \vee \bar{E}_\ell]$$
$$\geq 1 - n^c \cdot n^{-\alpha}$$
$$= 1 - n^{c-\alpha} .$$

This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

7.5 Skip Lists

Lemma 19

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

7.5 Skip Lists

Backward analysis:
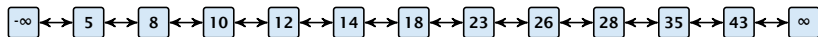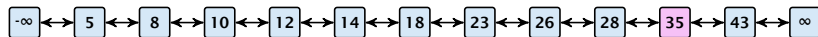
7.5 Skip Lists

Backward analysis:

7.5 Skip Lists
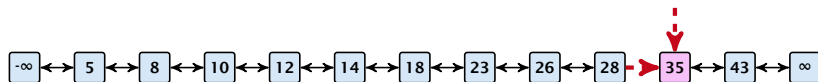
Backward analysis:

7.5 Skip Lists

Backward analysis:

7.5 Skip Lists
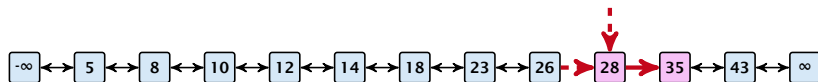
Backward analysis:

7.5 Skip Lists
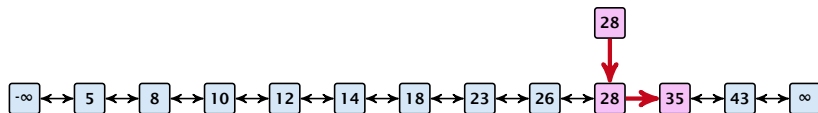
Backward analysis:

7.5 Skip Lists
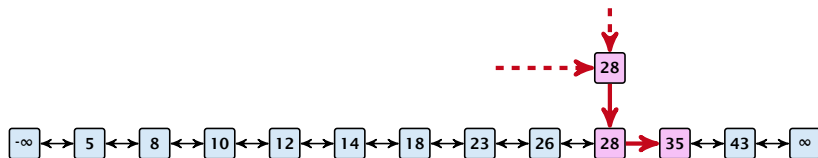
Backward analysis:

7.5 Skip Lists
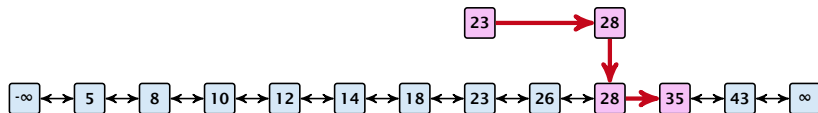
Backward analysis:

7.5 Skip Lists
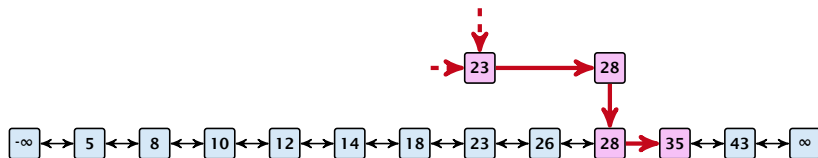
Backward analysis:

7.5 Skip Lists
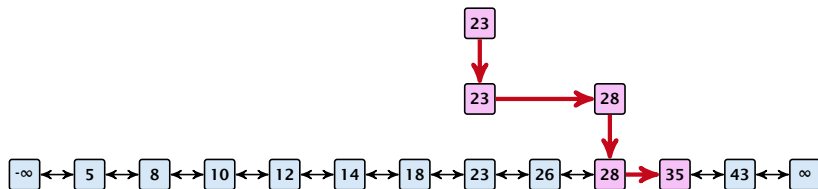
Backward analysis:

7.5 Skip Lists
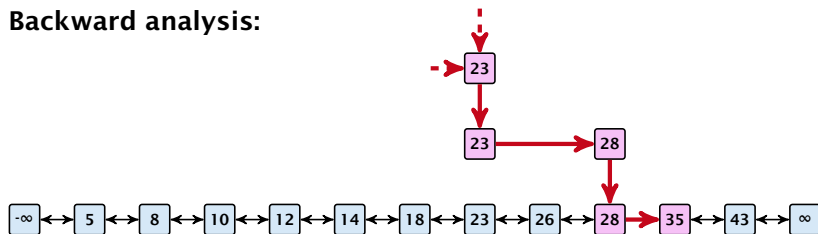
Backward analysis:

7.5 Skip Lists
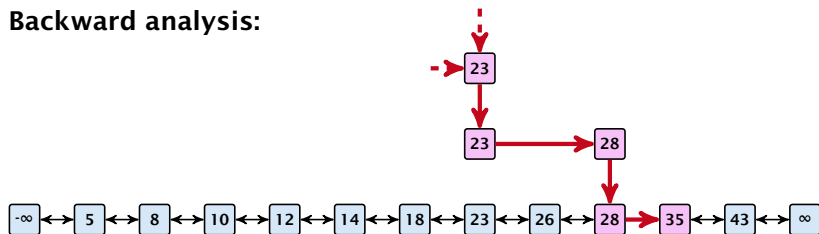
Backward analysis:

At each point the path goes up with probability $1/2$ and left with probability $1/2$.

7.5 Skip Lists

Backward analysis:
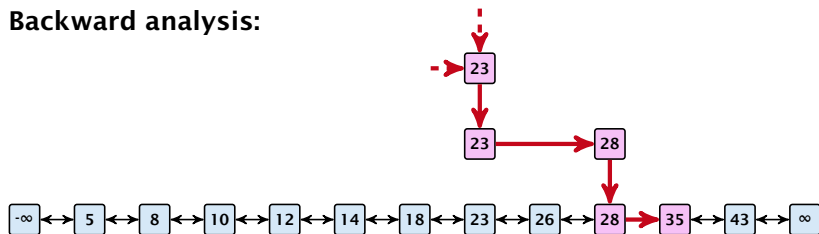
At each point the path goes up with probability $1/2$ and left with probability $1/2$.

We show that w.h.p:

▶ A "long" search path must also go very high.

7.5 Skip Lists

Backward analysis:
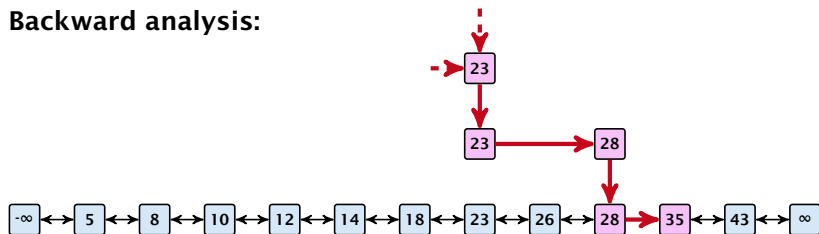
At each point the path goes up with probability $1/2$ and left with probability $1/2$.

We show that w.h.p:

- ▶ A "long" search path must also go very high.
- ▶ There are no elements in high lists.

7.5 Skip Lists

Backward analysis:
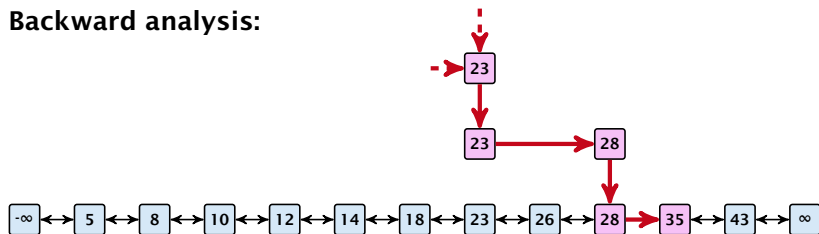
At each point the path goes up with probability $1/2$ and left with probability $1/2$.

We show that w.h.p:

- ▶ A "long" search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \ge \left(\frac{n}{k}\right)^k$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \ge \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}$$

$$= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \ge \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \le \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}$$

$$= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \le \left(\frac{n}{k}\right)^k \cdot \sum_{i \ge 0} \frac{k^i}{i!}$$

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}$$

$$= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{n}{k}\right)^k \cdot \sum_{i \geq 0} \frac{k^i}{i!} = \left(\frac{en}{k}\right)^k$$

7.5 Skip Lists

7.5 Skip Lists

Let $E_{z,k}$ denote the event that a search path is of length z (number of edges) but does not visit a list above L_k.

7.5 Skip Lists

Let $E_{z,k}$ denote the event that a search path is of length z (number of edges) but does not visit a list above L_k.

In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

7.5 Skip Lists

$\Pr[E_{z,k}]$

7.5 Skip Lists

$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

$$\leq \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^k n^{-\alpha}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \le \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\le \binom{z}{k} 2^{-(z-k)} \le \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \le \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \ge 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\le \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma \alpha} \le \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

$$\le \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^k n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

$$\leq \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^k n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

$$\leq \left(\frac{42\alpha}{64^\alpha}\right)^k n^{-\alpha}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \le \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\le \binom{z}{k} 2^{-(z-k)} \le \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \le \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \ge 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\le \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \le \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

$$\le \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^k n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

$$\le \left(\frac{42\alpha}{64^\alpha}\right)^k n^{-\alpha} \le n^{-\alpha}$$

7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

$$\leq \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^k n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

$$\leq \left(\frac{42\alpha}{64^\alpha}\right)^k n^{-\alpha} \leq n^{-\alpha}$$

for $\alpha \geq 1$.

7.5 Skip Lists

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\Pr[\text{search requires } z \text{ steps}]$$

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$
$$\leq n^{-\alpha} + n^{-(\gamma-1)}$$

7.5 Skip Lists

So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} \ .$$

For the search to take at least $z = 7\alpha\gamma \log n$ steps either the event $E_{z,k}$ or the event A_{k+1} must hold.
Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$
$$\leq n^{-\alpha} + n^{-(\gamma-1)}$$

This means, the search requires at most z steps, w. h. p.

7.6 van Emde Boas Trees

Dynamic Set Data Structure S:

- ▶ $S.\mathrm{insert}(x)$
- ▶ $S.\mathrm{delete}(x)$
- ▶ $S.\mathrm{search}(x)$
- ▶ $S.\mathrm{min}()$
- ▶ $S.\mathrm{max}()$
- ▶ $S.\mathrm{succ}(x)$
- ▶ $S.\mathrm{pred}(x)$

7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

- ▶ $S.\,\mathbf{insert}(x)$: Inserts x into S.

- ▶ $S.\,\mathbf{delete}(x)$: Deletes x from S. Usually assumes that $x \in S$.

- ▶ $S.\,\mathbf{member}(x)$: Returns 1 if $x \in S$ and 0 otw.

- ▶ $S.\,\mathbf{min}()$: Returns the value of the minimum element in S.

- ▶ $S.\,\mathbf{max}()$: Returns the value of the maximum element in S.

- ▶ $S.\,\mathbf{succ}(x)$: Returns successor of x in S. Returns null if x is maximum or larger than any element in S. Note that x needs not to be in S.

- ▶ $S.\,\mathbf{pred}(x)$: Returns the predecessor of x in S. Returns null if x is minimum or smaller than any element in S. Note that x needs not to be in S.

7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \ldots, u-1\}$, where u denotes the size of the universe.

Implementation 1: Array
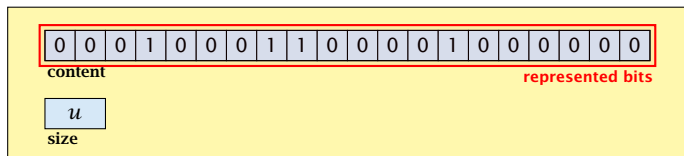
one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 1 array.insert(x)

1: content$[x] \leftarrow 1$;

Algorithm 2 array.delete(x)

1: content$[x] \leftarrow 0$;

Algorithm 3 array.member(x)

1: **return** content$[x]$;

▶ Note that we assume that x is valid, i.e., it falls within the array boundaries.

▶ Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 array.max()
1: **for** (i = size -1; $i \geq 0$; i--) **do**
2:     **if** content[i] = 1 **then return** i;
3: **return** null;

Implementation 1: Array

Algorithm 4 array.max()
1: **for** ($i = $ size -1; $i \geq 0$; $i--$) **do**
2:     **if** content[i] $= 1$ **then return** i;
3: **return** null;

Algorithm 5 array.min()
1: **for** ($i = 0$; $i < $ size; $i++$) **do**
2:     **if** content[i] $= 1$ **then return** i;
3: **return** null;

Implementation 1: Array

Algorithm 4 array.max()

1: **for** $(i = \text{size} - 1;\ i \geq 0;\ i--)$ **do**
2:      **if** content$[i] = 1$ **then return** i;
3: **return** null;

Algorithm 5 array.min()

1: **for** $(i = 0;\ i < \text{size};\ i++)$ **do**
2:      **if** content$[i] = 1$ **then return** i;
3: **return** null;

▶ Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 1: Array

Algorithm 6 array.succ(x)

1: **for** ($i = x + 1$; $i < $ size; i++) **do**
2:     **if** content[i] = 1 **then return** i;
3: **return** null;

Algorithm 7 array.pred(x)

1: **for** ($i = x - 1$; $i \geq 0$; i--) **do**
2:     **if** content[i] = 1 **then return** i;
3: **return** null;

▶ Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array
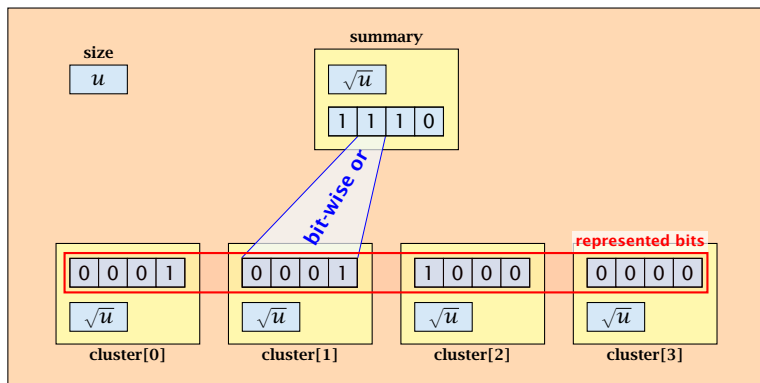
- ▶ $\sqrt{u}$ cluster-arrays of $\sqrt{u}$ bits.
- ▶ One summary-array of $\sqrt{u}$ bits. The i-th bit in the summary array stores the bit-wise or of the bits in the i-th cluster.

Implementation 2: Summary Array

Implementation 2: Summary Array

The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

Implementation 2: Summary Array

The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

Implementation 2: Summary Array

The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\text{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\text{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

> **Algorithm 8** member(x)
> 1: **return** cluster[high(x)]. member(low(x));

Implementation 2: Summary Array

Algorithm 8 member(x)

1: **return** cluster[high(x)].member(low(x));

Algorithm 9 insert(x)

1: cluster[high(x)].insert(low(x));
2: summary.insert(high(x));

Implementation 2: Summary Array

> **Algorithm 8** member(x)
> 1: **return** cluster[high(x)].member(low(x));

> **Algorithm 9** insert(x)
> 1: cluster[high(x)].insert(low(x));
> 2: summary.insert(high(x));

▶ The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

> **Algorithm 10** delete(x)
> 1: cluster[high(x)]. delete(low(x));
> 2: **if** cluster[high(x)]. min() = null **then**
> 3:   summary . delete(high(x));

Implementation 2: Summary Array

Algorithm 10 delete(x)

1: cluster[high(x)]. delete(low(x));
2: **if** cluster[high(x)]. min() = null **then**
3:     summary . delete(high(x));

▶ The running time is dominated by the cost of a minimum
  computation on an array of size $\sqrt{u}$. Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

> **Algorithm 11** max()
> 1: *maxcluster* ← summary . max();
> 2: **if** *maxcluster* = null **return** null;
> 3: *offs* ← cluster[*maxcluster*]. max()
> 4: **return** *maxcluster* ∘ *offs*;

Implementation 2: Summary Array

Algorithm 11 max()

1: *maxcluster* ← summary.max();
2: **if** *maxcluster* = null **return** null;
3: *offs* ← cluster[*maxcluster*].max()
4: **return** *maxcluster* ∘ *offs*;

Algorithm 12 min()

1: *mincluster* ← summary.min();
2: **if** *mincluster* = null **return** null;
3: *offs* ← cluster[*mincluster*].min();
4: **return** *mincluster* ∘ *offs*;

Implementation 2: Summary Array

Algorithm 11 max()

1: *maxcluster* ← summary . max();
2: **if** *maxcluster* = null **return** null;
3: *offs* ← cluster[*maxcluster*]. max()
4: **return** *maxcluster* ∘ *offs*;

> The operator ∘ stands for the concatenation of two bitstrings. This means if $x = 0111_2$ and $y = 0001_2$ then $x \circ y = 01110001_2$.

Algorithm 12 min()

1: *mincluster* ← summary . min();
2: **if** *mincluster* = null **return** null;
3: *offs* ← cluster[*mincluster*]. min();
4: **return** *mincluster* ∘ *offs*;

▶ Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 $\text{succ}(x)$

1: $m \leftarrow \text{cluster}[\text{high}(x)].\,\text{succ}(\text{low}(x))$

2: **if** $m \neq$ null **then return** $\text{high}(x) \circ m$;

3: $succcluster \leftarrow \text{summary}.\,\text{succ}(\text{high}(x))$;

4: **if** $succcluster \neq$ null **then**

5: $\qquad offs \leftarrow \text{cluster}[succcluster].\,\text{min}()$;

6: $\qquad$ **return** $succcluster \circ offs$;

7: **return** null;

Implementation 2: Summary Array

Algorithm 13 succ(x)

1: $m \leftarrow$ cluster[high(x)].succ(low(x))
2: **if** $m \neq$ null **then return** high(x) $\circ$ m;
3: *succcluster* $\leftarrow$ summary.succ(high(x));
4: **if** *succcluster* $\neq$ null **then**
5:     *offs* $\leftarrow$ cluster[*succcluster*].min();
6:     **return** *succcluster* $\circ$ *offs*;
7: **return** null;

▶ Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 pred(x)

1: $m \leftarrow$ cluster[high(x)]. pred(low(x))
2: **if** $m \neq$ null **then return** high(x) $\circ$ m;
3: $predcluster \leftarrow$ summary . pred(high(x));
4: **if** $predcluster \neq$ null **then**
5: $\quad$ $offs \leftarrow$ cluster[$predcluster$]. max();
6: $\quad$ **return** $predcluster \circ offs$;
7: **return** null;

▶ Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:
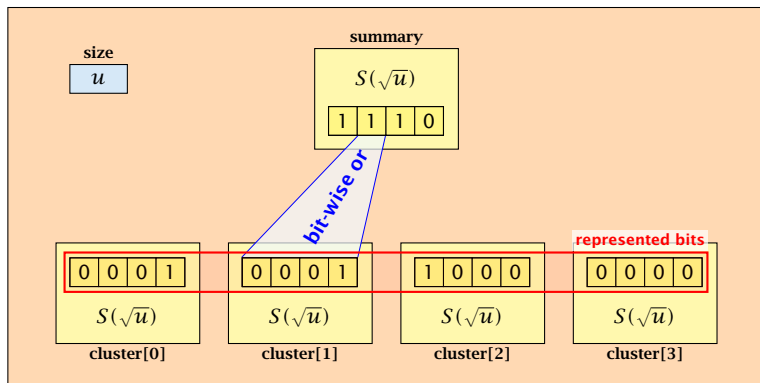
Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k.

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like cluster[1].min() from within the data-structure $S(4)$ is not a recursive call as it will call the function array.min().

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\mathrm{cluster}[1].\min()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\mathrm{array}.\min()$.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

> **Algorithm 15** member(x)
> 1: **return** cluster[high(x)]. member(low(x));

- $T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 16 insert(x)

1: cluster[high(x)].insert(low(x));
2: summary.insert(high(x));

▶ $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 17 delete(x)

1: cluster[high(x)]. delete(low(x));
2: **if** cluster[high(x)]. min() = null **then**
3:     summary. delete(high(x));

▶ $T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 18 min()

1: *mincluster* ← summary . min();
2: **if** *mincluster* = null **return** null;
3: *offs* ← cluster[*mincluster*]. min();
4: **return** *mincluster* ∘ *offs*;

▶ $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 19 succ(x)

1: $m \leftarrow$ cluster[high(x)].succ(low(x))
2: **if** $m \neq$ null **then return** high(x) $\circ$ m;
3: *succcluster* $\leftarrow$ summary.succ(high(x));
4: **if** *succcluster* $\neq$ null **then**
5:     *offs* $\leftarrow$ cluster[*succcluster*].min();
6:     **return** *succcluster* $\circ$ *offs*;
7: **return** null;

▶ $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1.$

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Implementation 3: Recursion

$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^{\ell})$.

Implementation 3: Recursion

$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^{\ell})$. Then

Implementation 3: Recursion

$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell)$$

Implementation 3: Recursion

$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\text{mem}}(2^\ell)$$

Implementation 3: Recursion

$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u)$$

Implementation 3: Recursion

$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$$

Implementation 3: Recursion

$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$$
$$= T_{\mathrm{mem}}(2^{\frac{\ell}{2}}) + 1$$

Implementation 3: Recursion

$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$$
$$= T_{\mathrm{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \ .$$

Implementation 3: Recursion

$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$. Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$$
$$= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \ .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^{\ell})$. Then

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell)$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\text{ins}}(2^\ell)$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u)$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$$
$$= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$$
$$= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \ .$$

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$$
$$= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \ .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

Implementation 3: Recursion

$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^{\ell})$. Then

$$X(\ell) = T_{\text{ins}}(2^{\ell}) = T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$$
$$= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \ .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

The same holds for $T_{\max}(u)$ and $T_{\min}(u)$.

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^{\ell})$.

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c \log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell)$$

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell)$$

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c \log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u)$$

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c \log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u$$

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c\log u$$
$$= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell$$

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c\log u$$
$$= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X(\tfrac{\ell}{2}) + c\ell \ .$$

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \leq 2T_{\text{del}}(\sqrt{u}) + c \log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^{\ell})$. Then

$$X(\ell) = T_{\text{del}}(2^{\ell}) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u$$
$$= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X(\tfrac{\ell}{2}) + c\ell \ .$$

Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$.

Implementation 3: Recursion

$$T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1 \le 2T_{\text{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{del}}(2^\ell)$. Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c\log u$$
$$= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X(\tfrac{\ell}{2}) + c\ell .$$

Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$.

The same holds for $T_{\text{pred}}(u)$ and $T_{\text{succ}}(u)$.

Implementation 4: van Emde Boas Trees
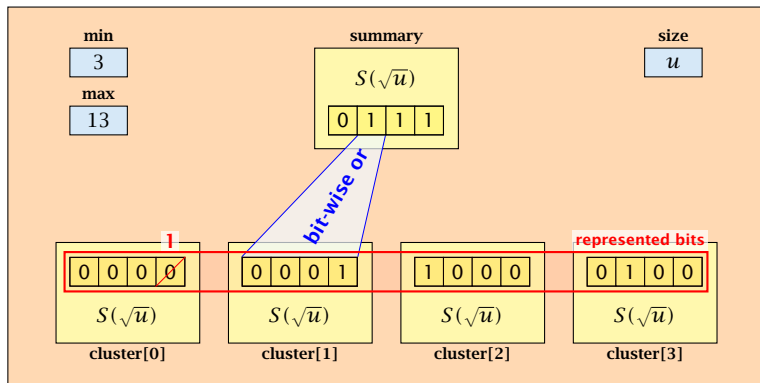
▶ The bit referenced by min is not set within sub-datastructures.

▶ The bit referenced by max is set within sub-datastructures (if max ≠ min).

Advantages of having max/min pointers:

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

▶ Recursive calls for min and max are constant time.

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

- ▶ Recursive calls for min and max are constant time.

- ▶ min = null means that the data-structure is empty.

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

▶ Recursive calls for min and max are constant time.

▶ min = null means that the data-structure is empty.

▶ min = max ≠ null means that the data-structure contains exactly one element.

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

- ▶ Recursive calls for min and max are constant time.
- ▶ min = null means that the data-structure is empty.
- ▶ min = max ≠ null means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting min = max = x.

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

- ▶ Recursive calls for min and max are constant time.
- ▶ min = null means that the data-structure is empty.
- ▶ min = max ≠ null means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting min = max = x.
- ▶ We can delete from a data-structure that just contains one element in constant time by setting min = max = null.

Algorithm 20 max()

1: **return** max;

Algorithm 21 min()

1: **return** min;

▶ Constant time.

Implementation 4: van Emde Boas Trees

> **Algorithm 22** member(x)
> ___
> 1: **if** $x = \min$ **then return** 1; // TRUE
> 2: **return** cluster[high(x)]. member(low(x));

▶ $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Longrightarrow T(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 $\text{succ}(x)$

1: **if** $\min \neq$ null $\land$ $x < \min$ **then return** $\min$;
2: *maxincluster* $\leftarrow$ cluster[high(x)]. max();
3: **if** *maxincluster* $\neq$ null $\land$ low(x) < *maxincluster* **then**
4:     *offs* $\leftarrow$ cluster[high(x)]. succ(low(x));
5:     **return** high(x) $\circ$ *offs*;
6: **else**
7:     *succcluster* $\leftarrow$ summary . succ(high(x));
8:     **if** *succcluster* = null **then return** null;
9:     *offs* $\leftarrow$ cluster[*succcluster*]. min();
10:     **return** *succcluster* $\circ$ *offs*;

▶ $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 35 insert(x)

 1: **if** min = null **then**
 2:     min = x; max = x;
 3: **else**
 4:     **if** x < min **then** exchange x and min;
 5:     **if** x > max **then** max = x;
 6:     **if** cluster[high(x)]. min = null; **then**
 7:         summary . insert(high(x));
 8:         cluster[high(x)]. insert(low(x));
 9:     **else**
10:         cluster[high(x)]. insert(low(x));

▶ $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \implies T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$.

Implementation 4: van Emde Boas Trees

Note that the recusive call in Line 8 takes constant time as the if-condition in Line 6 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 7 and in Line 10. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

▶ **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

1: **if** min = max **then**
2:     min = max = null;
3: **else**
4:     **if** x = min **then**
5:         *firstcluster* ← summary. min();
6:         *offs* ← cluster[*firstcluster*]. min();
7:         x ← *firstcluster* ∘ *offs*;
8:         min ← x;
9:     cluster[high(x)]. delete(low(x));

<div align="center">continued...</div>

Implementation 4: van Emde Boas Trees

▶ **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

1: **if** min = max **then**
2:      min = max = null;
3: **else**
4:      **if** x = min **then**   find new minimum
5:          *firstcluster* ← summary . min();
6:          *offs* ← cluster[*firstcluster*] . min();
7:          x ← *firstcluster* ∘ *offs*;
8:          min ← x;
9:      cluster[high(x)] . delete(low(x));
                    continued...

Implementation 4: van Emde Boas Trees

▶ **Assumes that x is contained in the structure.**

Algorithm 36 delete(x)

1: **if** min = max **then**
2:     min = max = null;
3: **else**
4:     **if** x = min **then**
5:         *firstcluster* ← summary.min();
6:         *offs* ← cluster[*firstcluster*].min();
7:         x ← *firstcluster* ∘ *offs*;
8:         min ← x;
9:     cluster[high(x)].delete(low(x));   delete
                    continued...

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

<div align="center">...continued</div>

10:     **if** cluster[high(x)].min() = null **then**
11:         summary.delete(high(x));
12:         **if** x = max **then**
13:             *summax* ← summary.max();
14:             **if** *summax* = null **then** max ← min;
15:             **else**
16:                 *offs* ← cluster[*summax*].max();
17:                 max ← *summax* ∘ *offs*
18:     **else**
19:         **if** x = max **then**
20:             *offs* ← cluster[high(x)].max();
21:             max ← high(x) ∘ *offs*;

Implementation 4: van Emde Boas Trees

Algorithm 36 delete(x)

	...continued	fix maximum

10:     **if** cluster[high(x)].min() = null **then**
11:         summary.delete(high(x));
12:         **if** x = max **then**
13:             *summax* ← summary.max();
14:             **if** *summax* = null **then** max ← min;
15:             **else**
16:                 *offs* ← cluster[*summax*].max();
17:                 max ← *summax* ∘ *offs*
18:     **else**
19:         **if** x = max **then**
20:             *offs* ← cluster[high(x)].max();
21:             max ← high(x) ∘ *offs*;

Implementation 4: van Emde Boas Trees

Note that only one of the possible recusive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\text{cluster}[\text{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c \ .$$

This gives $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$.

7.6 van Emde Boas Trees

Space requirements:

- ▶ The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) \ .$$

- ▶ Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- ▶ One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- Let the "real" recurrence relation be

$$S(k^2) = (k + 1)S(k) + c_1 \cdot k; \; S(4) = c_2$$

- Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k + 1)R(k) + ck; \; R(4) = 1$$

  where $c = c_1/c_2 < 1$.

- Now, we show $R(k) \leq k - 2$ for squares $k \geq 4$.
  - Obviously, this holds for $k = 4$.
  - For $k = \ell^2 > 4$ with ℓ integral we have

$$R(k) = (1 + \ell)R(\ell) + c\ell$$
$$\leq (1 + \ell)(\ell - 2) + \ell \leq k - 2$$

  - This shows that $R(k)$ and, hence, $S(k)$ grows linearly.

7.7 Hashing

Dictionary:

- ▶ $S.\,\textbf{insert}(x)$: Insert an element x.

- ▶ $S.\,\textbf{delete}(x)$: Delete the element pointed to by x.

- ▶ $S.\,\textbf{search}(k)$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

7.7 Hashing

Dictionary:

- ▶ $S.\,\mathbf{insert}(x)$: Insert an element x.
- ▶ $S.\,\mathbf{delete}(x)$: Delete the element pointed to by x.
- ▶ $S.\,\mathbf{search}(k)$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

7.7 Hashing

Dictionary:

- ▶ $S.\,\text{insert}(x)$: Insert an element x.
- ▶ $S.\,\text{delete}(x)$: Delete the element pointed to by x.
- ▶ $S.\,\text{search}(k)$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

7.7 Hashing

Dictionary:

- ▶ $S.\,\mathbf{insert}(x)$: Insert an element x.
- ▶ $S.\,\mathbf{delete}(x)$: Delete the element pointed to by x.
- ▶ $S.\,\mathbf{search}(k)$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

7.7 Hashing

Definitions:

▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.

7.7 Hashing

Definitions:

- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.

7.7 Hashing

Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- ▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- ▶ Array $T[0, \ldots, n-1]$ hash-table.

7.7 Hashing

Definitions:

▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.

▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.

▶ Array $T[0, \ldots, n-1]$ hash-table.

▶ Hash function $h : U \to [0, \ldots, n-1]$.

7.7 Hashing

Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- ▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- ▶ Array $T[0, \ldots, n-1]$ hash-table.
- ▶ Hash function $h : U \to [0, \ldots, n-1]$.

The hash-function h should fulfill:

7.7 Hashing

Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- ▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- ▶ Array $T[0, \ldots, n-1]$ hash-table.
- ▶ Hash function $h : U \to [0, \ldots, n-1]$.

The hash-function h should fulfill:

- ▶ Fast to evaluate.

7.7 Hashing

Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- ▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- ▶ Array $T[0, \ldots, n-1]$ hash-table.
- ▶ Hash function $h : U \to [0, \ldots, n-1]$.

The hash-function h should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.

7.7 Hashing

Definitions:

- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- Array $T[0, \ldots, n-1]$ hash-table.
- Hash function $h : U \to [0, \ldots, n-1]$.

The hash-function h should fulfill:

- Fast to evaluate.
- Small storage requirement.
- Good distribution of elements over the whole table.

Direct Addressing

Ideally the hash function maps all keys to different memory locations.
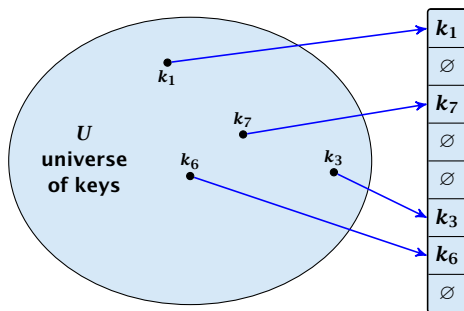
This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.
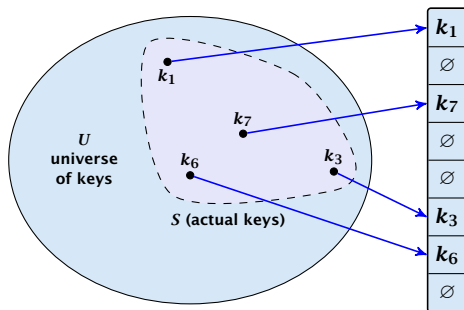
Such a hash function h is called a perfect hash function for set S.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions
Usually the universe U is much larger than the table-size n.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions

Usually the universe U is much larger than the table-size n.

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 20

The probability of having a collision when hashing m elements into a table of size n under uniform hashing *is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} \ .$$

Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 20

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} \ .$$

Uniform hashing:
Choose a hash function uniformly at random from all functions
$f : U \to [0, \ldots, n-1]$.

Collisions

Proof.
Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

Collisions

Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

$$\Pr[A_{m,n}]$$

Collisions

Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^{m} \frac{n - \ell + 1}{n}$$

Collisions

Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^{m} \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right)$$

Collisions

Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^{m} \frac{n-\ell+1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right)$$

$$\leq \prod_{j=0}^{m-1} e^{-j/n}$$

Collisions

Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^{m} \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right)$$

$$\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}}$$

Collisions

Proof.

Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^{m} \frac{n-\ell+1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right)$$

$$\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}} \ .$$

Collisions

Proof.
Let $A_{m,n}$ denote the event that inserting m keys into a table of size n does not generate a collision. Then

$$\Pr[A_{m,n}] = \prod_{\ell=1}^{m} \frac{n-\ell+1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right)$$

$$\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}} \ .$$

Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions
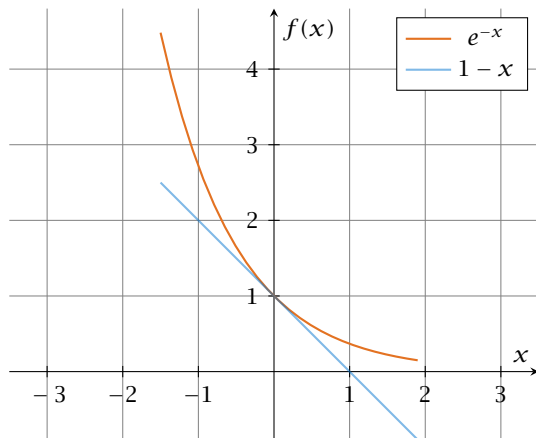
The inequality $1 - x \le e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

Hashing with Chaining

Arrange elements that map to the same position in a linear list.

▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.

▶ Insert: insert at the front of the list.
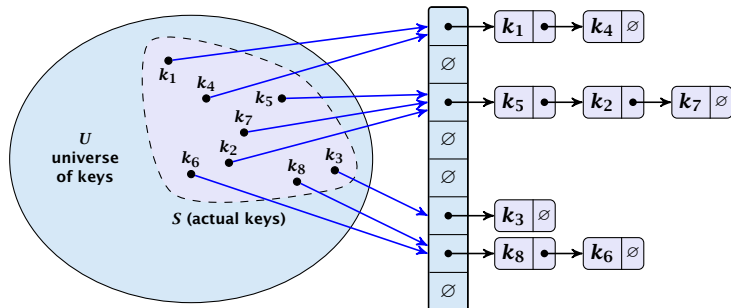
Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- ▶ A^+ denotes the average time for a **successful** search when using A;

Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- ▶ A^+ denotes the average time for a **successful** search when using A;

- ▶ A^- denotes the average time for an **unsuccessful** search when using A;

Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- ▶ A^+ denotes the average time for a **successful** search when using A;

- ▶ A^- denotes the average time for an **unsuccessful** search when using A;

- ▶ We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called fill factor of the hash-table.

Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

▶ A^+ denotes the average time for a **successful** search when using A;

▶ A^- denotes the average time for an **unsuccessful** search when using A;

▶ We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called fill factor of the hash-table.

We assume uniform hashing for the following analysis.

Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined.

Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is $\alpha = \frac{m}{n}$.

Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is $\alpha = \frac{m}{n}$. Hence, if A is the collision resolving strategy "Hashing with Chaining" we have

$$A^- = 1 + \alpha \ .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1 + \sum_{j=i+1}^{m} X_{ij}\right)\right]$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1 + \overbrace{\sum_{j=i+1}^{m} X_{ij}}^{\text{keys before } k_i}\right)\right]$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\left[\frac{1}{m} \sum_{i=1}^{m} \left(1 + \sum_{j=i+1}^{m} X_{ij} \right) \right]$$

$$\underbrace{\phantom{\left(1 + \sum_{j=i+1}^{m} X_{ij} \right)}}_{\text{cost for key } k_i}$$

Hashing with Chaining

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right]$$

Hashing with Chaining

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}X_{ij}\right)\right] = \frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\right)$$

Hashing with Chaining

$$\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}X_{ij}\Big)\Big] = \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\mathrm{E}\,[X_{ij}]\Big)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\frac{1}{n}\Big)$$

Hashing with Chaining

$$\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}X_{ij}\Big)\Big] = \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\Big)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\frac{1}{n}\Big)$$

$$= 1 + \frac{1}{mn}\sum_{i=1}^{m}(m-i)$$

Hashing with Chaining

$$\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}X_{ij}\Big)\Big] = \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\Big)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\frac{1}{n}\Big)$$

$$= 1+\frac{1}{mn}\sum_{i=1}^{m}(m-i)$$

$$= 1+\frac{1}{mn}\Big(m^2-\frac{m(m+1)}{2}\Big)$$

Hashing with Chaining

$$
\begin{aligned}
\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}X_{ij}\Big)\Big] &= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\Big)\\
&= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\frac{1}{n}\Big)\\
&= 1+\frac{1}{mn}\sum_{i=1}^{m}(m-i)\\
&= 1+\frac{1}{mn}\Big(m^2-\frac{m(m+1)}{2}\Big)\\
&= 1+\frac{m-1}{2n}
\end{aligned}
$$

Harald Räcke

Hashing with Chaining

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1 + \sum_{j=i+1}^{m} X_{ij}\right)\right] = \frac{1}{m}\sum_{i=1}^{m}\left(1 + \sum_{j=i+1}^{m} \mathrm{E}\left[X_{ij}\right]\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left(1 + \sum_{j=i+1}^{m} \frac{1}{n}\right)$$

$$= 1 + \frac{1}{mn}\sum_{i=1}^{m}(m - i)$$

$$= 1 + \frac{1}{mn}\left(m^2 - \frac{m(m+1)}{2}\right)$$

$$= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} \ .$$

Hashing with Chaining

$$
\begin{aligned}
\mathrm{E}\Big[\frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}X_{ij}\Big)\Big] &= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\mathrm{E}\left[X_{ij}\right]\Big) \\
&= \frac{1}{m}\sum_{i=1}^{m}\Big(1+\sum_{j=i+1}^{m}\frac{1}{n}\Big) \\
&= 1+\frac{1}{mn}\sum_{i=1}^{m}(m-i) \\
&= 1+\frac{1}{mn}\Big(m^2-\frac{m(m+1)}{2}\Big) \\
&= 1+\frac{m-1}{2n} = 1+\frac{\alpha}{2}-\frac{\alpha}{2m} \ .
\end{aligned}
$$

Hence, the expected cost for a successful search is $A^+ \le 1+\frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

Open Addressing

All objects are stored in the table itself.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0), \ldots, h(k, n - 1)$ must form a permutation of $0, \ldots, n - 1$.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0), \ldots, h(k, n-1)$ must form a permutation of $0, \ldots, n-1$.

Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1)$, $h(k, 2)$, $\ldots$.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k,0),\ldots,h(k, n-1)$ must form a permutation of $0, \ldots, n-1$.

Search(k): Try position $h(k,0)$; if it is empty your search fails; otw. continue with $h(k,1), h(k,2), \ldots$.

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n-1)$, and this slot is non-empty then your table is full.

Open Addressing

Choices for $h(k, j)$:

▶ Linear probing:
$h(k, i) = h(k) + i \mod n$
(sometimes: $h(k, i) = h(k) + ci \mod n$).

Open Addressing

Choices for $h(k, j)$:

- ▶ Linear probing:
  $h(k, i) = h(k) + i \mod n$
  (sometimes: $h(k, i) = h(k) + ci \mod n$).

- ▶ Quadratic probing:
  $h(k, i) = h(k) + c_1 i + c_2 i^2 \mod n$.

Open Addressing

Choices for $h(k, j)$:

▶ Linear probing:
  $h(k, i) = h(k) + i \mod n$
  (sometimes: $h(k, i) = h(k) + ci \mod n$).

▶ Quadratic probing:
  $h(k, i) = h(k) + c_1 i + c_2 i^2 \mod n$.

▶ Double hashing:
  $h(k, i) = h_1(k) + i h_2(k) \mod n$.

Open Addressing

Choices for $h(k, j)$:

- ▶ Linear probing:
  $h(k, i) = h(k) + i \mod n$
  (sometimes: $h(k, i) = h(k) + ci \mod n$).

- ▶ Quadratic probing:
  $h(k, i) = h(k) + c_1 i + c_2 i^2 \mod n$.

- ▶ Double hashing:
  $h(k, i) = h_1(k) + i h_2(k) \mod n$.

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

Linear Probing

▶ Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

Linear Probing

- ▶ Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Linear Probing

- ▶ Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 21

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1 - \alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1 - \alpha)^2}\right)$$

Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

Quadratic Probing

▶ Not as cache-efficient as Linear Probing.

▶ Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

Lemma 22

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

Double Hashing

▶ Any probe into the hash-table usually creates a cache-miss.

Double Hashing

▶ Any probe into the hash-table usually creates a cache-miss.

Lemma 23

Let D be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right)$$

$$D^- \approx \frac{1}{1-\alpha}$$

Open Addressing

Some values:

α	Linear Probing		Quadratic Probing		Double Hashing	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing
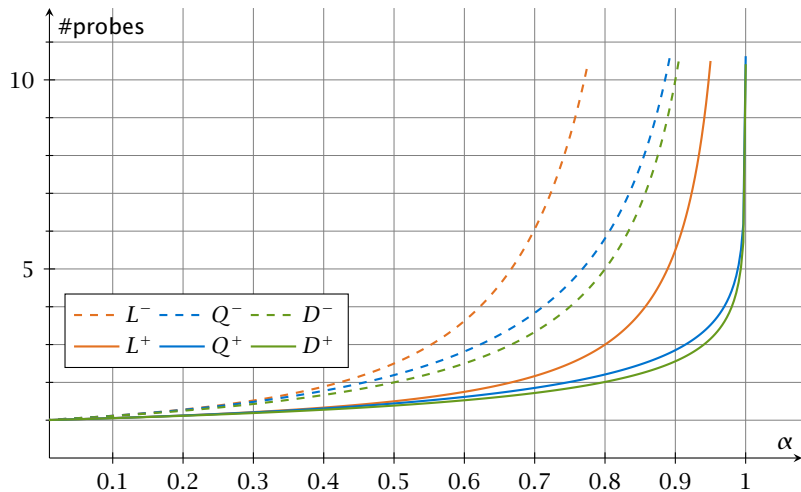
Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

Analysis of Idealized Open Address Hashing

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Let A_i denote the event that the i-th probe occurs and is to a non-empty slot.

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Let A_i denote the event that the i-th probe occurs and is to a non-empty slot.

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$
$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot$$
$$\ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]$$

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Let A_i denote the event that the i-th probe occurs and is to a non-empty slot.

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$
$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot$$
$$\ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]$$

$$\Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Let A_i denote the event that the i-th probe occurs and is to a non-empty slot.

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$
$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot$$
$$\ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]$$

$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Let A_i denote the event that the i-th probe occurs and is to a non-empty slot.

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$
$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot$$
$$\ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]$$

$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$
$$\leq \left(\frac{m}{n}\right)^{i-1}$$

Analysis of Idealized Open Address Hashing

Let X denote a random variable describing the number of probes in an unsuccessful search.

Let A_i denote the event that the i-th probe occurs and is to a non-empty slot.

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$
$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot$$
$$\ldots \cdot \Pr[A_{i-1} \mid A_1 \cap \cdots \cap A_{i-2}]$$

$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

$$\leq \left(\frac{m}{n}\right)^{i-1} = \alpha^{i-1} .$$

Analysis of Idealized Open Address Hashing

$\mathrm{E}[X]$

Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1}$$

Harald Räcke

Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i$$

Analysis of Idealized Open Address Hashing

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} \mathrm{Pr}[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} \ .$$

Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} \ .$$

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing
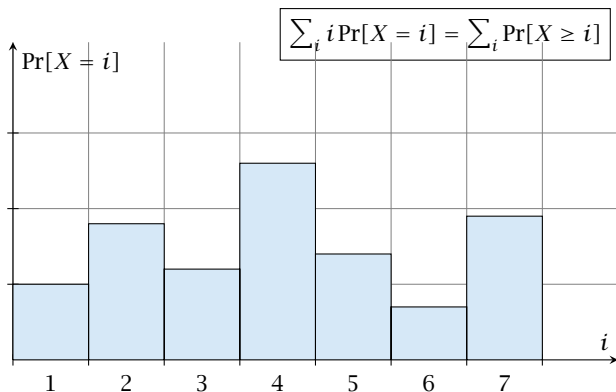
$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

$i = 1$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

$i = 2$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$
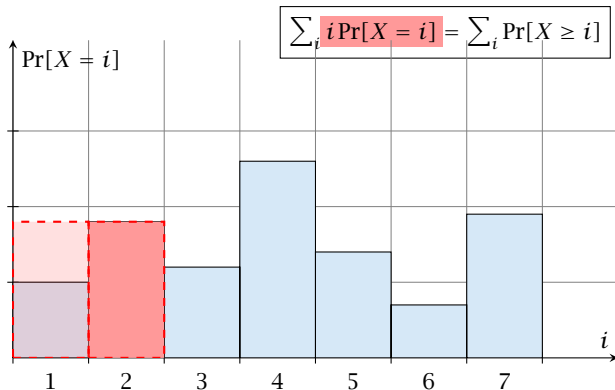
Analysis of Idealized Open Address Hashing

$i = 3$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

$i = 4$

Analysis of Idealized Open Address Hashing

$i = 1$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$
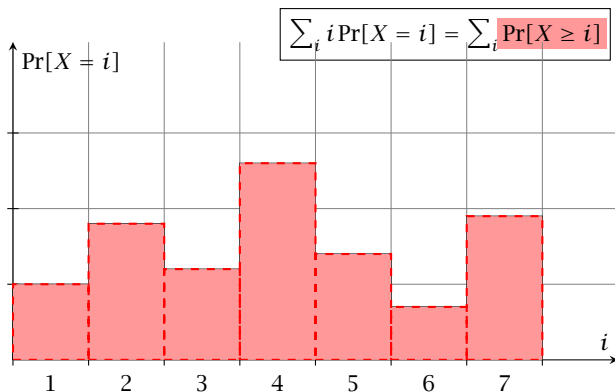
Analysis of Idealized Open Address Hashing

$i = 2$

Analysis of Idealized Open Address Hashing

Analysis of Idealized Open Address Hashing

$i = 4$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

$$\sum_i i\Pr[X=i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$
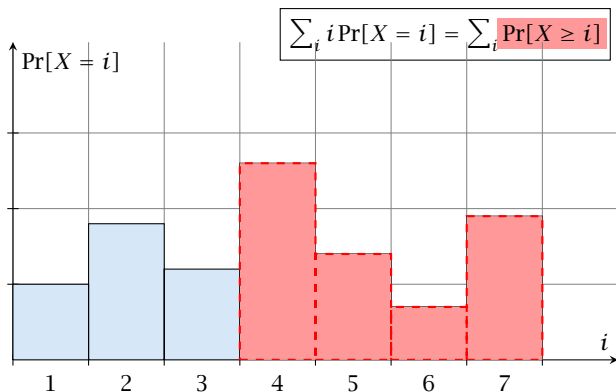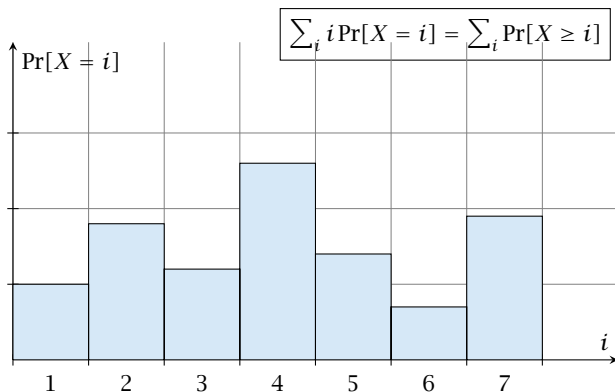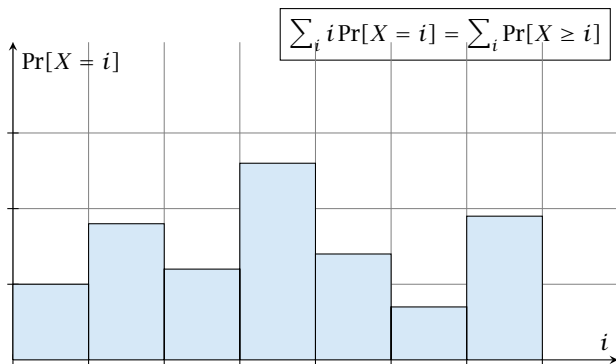
The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

Analysis of Idealized Open Address Hashing

Analysis of Idealized Open Address Hashing

The number of probes in a successful search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Let k be the $i + 1$-st element. The expected time for a search for k is at most $\frac{1}{1 - i/n} = \frac{n}{n-i}$.

Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Let k be the $i + 1$-st element. The expected time for a search for k is at most $\frac{1}{1 - i/n} = \frac{n}{n - i}$.

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n - i}$$

Analysis of Idealized Open Address Hashing

The number of probes in a successful search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Let k be the $i + 1$-st element. The expected time for a search for k is at most $\frac{1}{1 - i/n} = \frac{n}{n - i}$.

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n - i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n - i}$$

Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Let k be the $i+1$-st element. The expected time for a search for k is at most $\frac{1}{1-i/n} = \frac{n}{n-i}$.

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^{n} \frac{1}{k}$$

Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Let k be the $i + 1$-st element. The expected time for a search for k is at most $\frac{1}{1 - i/n} = \frac{n}{n - i}$.

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^{n} \frac{1}{k}$$

$$\leq \frac{1}{\alpha} \int_{n-m}^{n} \frac{1}{x} \mathrm{d}x$$

Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Let k be the $i+1$-st element. The expected time for a search for k is at most $\frac{1}{1-i/n} = \frac{n}{n-i}$.

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^{n} \frac{1}{k}$$

$$\leq \frac{1}{\alpha} \int_{n-m}^{n} \frac{1}{x} \mathrm{d}x = \frac{1}{\alpha} \ln \frac{n}{n-m}$$

Analysis of Idealized Open Address Hashing

The number of probes in a successful search for k is equal to the number of probes made in an unsuccessful search for k at the time that k is inserted.

Let k be the $i + 1$-st element. The expected time for a search for k is at most $\frac{1}{1 - i/n} = \frac{n}{n - i}$.

$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^{n} \frac{1}{k}$$

$$\leq \frac{1}{\alpha} \int_{n-m}^{n} \frac{1}{x} dx = \frac{1}{\alpha} \ln \frac{n}{n-m} = \frac{1}{\alpha} \ln \frac{1}{1-\alpha} \ .$$

Analysis of Idealized Open Address Hashing

$$\sum_{k=m-n+1}^{n} \frac{1}{k} \leq \int_{m-n}^{n} \frac{1}{x} \, \mathrm{d}x$$

Deletions in Hashtables

How do we delete in a hash-table?

▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

Deletions in Hashtables

How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

- ▶ For open addressing this is difficult.

Deletions in Hashtables

▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

Deletions in Hashtables

▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

▶ One can delete an element by replacing it with a deleted-marker.

Deletions in Hashtables

- Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

- One can delete an element by replacing it with a *deleted*-marker.
  - During an insertion if a *deleted*-marker is encountered an element can be inserted there.

Deletions in Hashtables

▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

▶ One can delete an element by replacing it with a deleted-marker.

  ▶ During an insertion if a deleted-marker is encountered an element can be inserted there.

  ▶ During a search a deleted-marker must not be used to terminate the probe sequence.

Deletions in Hashtables

▶ Simply removing a key might interrupt the probe sequence of
  other keys which then cannot be found anymore.

▶ One can delete an element by replacing it with a
  deleted-marker.

  ▶ During an insertion if a deleted-marker is encountered an
    element can be inserted there.
  ▶ During a search a deleted-marker must not be used to
    terminate the probe sequence.

▶ The table could fill up with deleted-markers leading to bad
  performance.

Deletions in Hashtables

▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

▶ One can delete an element by replacing it with a deleted-marker.

   ▶ During an insertion if a deleted-marker is encountered an element can be inserted there.
   ▶ During a search a deleted-marker must not be used to terminate the probe sequence.

▶ The table could fill up with deleted-markers leading to bad performance.

▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

► For Linear Probing one can delete elements without using deletion-markers.

Deletions for Linear Probing

- ▶ For Linear Probing one can delete elements without using deletion-markers.
- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

Deletions for Linear Probing

Algorithm 37 delete(p)
1: $T[p] \leftarrow$ null
2: $p \leftarrow$ succ(p)
3: **while** $T[p] \neq$ null **do**
4:      $y \leftarrow T[p]$
5:      $T[p] \leftarrow$ null
6:      $p \leftarrow$ succ(p)
7:      insert(y)

p is the index into the table-cell that contains the object to be deleted.

Deletions for Linear Probing

Algorithm 37 delete(p)
1: $T[p] \leftarrow$ null
2: $p \leftarrow \text{succ}(p)$
3: **while** $T[p] \neq$ null **do**
4: $\quad\quad y \leftarrow T[p]$
5: $\quad\quad T[p] \leftarrow$ null
6: $\quad\quad p \leftarrow \text{succ}(p)$
7: $\quad\quad \text{insert}(y)$

p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

Universal Hashing

Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

Universal Hashing

Definition 24

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n - 1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 24

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w.r.t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

Universal Hashing

Definition 25

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

▶ For any key $u \in U$, and $t \in \{0, \ldots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.

▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

Universal Hashing

Definition 25

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

▶ For any key $u \in U$, and $t \in \{0, \ldots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.

▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

This requirement clearly implies a universal hash-function.

Universal Hashing

Definition 26

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called k-independent if for any choice of $\ell \le k$ distinct keys $u_1, \ldots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \ldots, t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{1}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 27

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called (μ, k)-independent if for any choice of $\ell \leq k$ distinct keys $u_1, \ldots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \ldots, t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Universal Hashing

Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Universal Hashing

Let $U := \{0, \ldots, p - 1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p - 1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p - 1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Define
$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Universal Hashing

Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Define
$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 28

The class
$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \ldots, n-1\}$.

Universal Hashing

Universal Hashing

Proof.
Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

Universal Hashing

Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

- $ax + b \not\equiv ay + b \pmod{p}$

Universal Hashing

Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

- ▶ $ax + b \not\equiv ay + b \pmod{p}$

  If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Universal Hashing

Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

▶ $ax + b \not\equiv ay + b \pmod{p}$

If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

Universal Hashing

Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

▶ $ax + b \not\equiv ay + b \pmod{p}$

If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

Universal Hashing

▶ The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

Universal Hashing

▶ The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

This holds because we can compute a and b when given t_x and t_y:

Universal Hashing

▶ The hash-function does not generate collisions before the $(\mod n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

This holds because we can compute a and b when given t_x and t_y:

$$t_x \equiv ax + b \qquad\qquad (\mod p)$$
$$t_y \equiv ay + b \qquad\qquad (\mod p)$$

Universal Hashing

▶ The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

This holds because we can compute a and b when given t_x and t_y:

$$t_x \equiv ax + b \qquad\qquad (\bmod\ p)$$
$$t_y \equiv ay + b \qquad\qquad (\bmod\ p)$$

$$t_x - t_y \equiv a(x - y) \qquad\qquad (\bmod\ p)$$
$$t_y \equiv ay + b \qquad\qquad (\bmod\ p)$$

Universal Hashing

▶ The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

This holds because we can compute a and b when given t_x and t_y:

$$t_x \equiv ax + b \qquad (\bmod\ p)$$
$$t_y \equiv ay + b \qquad (\bmod\ p)$$

$$t_x - t_y \equiv a(x - y) \qquad (\bmod\ p)$$
$$t_y \equiv ay + b \qquad (\bmod\ p)$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \qquad (\bmod\ p)$$
$$b \equiv t_y - ay \qquad (\bmod\ p)$$

Universal Hashing

Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\bmod n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\mathrm{mod}\ n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the $\mathrm{mod}\ n$ operation?

Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\bmod n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the mod n-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the mod n operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \ldots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

Universal Hashing

Universal Hashing

As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1$$

Universal Hashing

As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1$$

Universal Hashing

As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

Universal Hashing

As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

possibilities for choosing t_y such that the final hash-value creates a collision.

Universal Hashing

As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

possibilities for choosing t_y such that the final hash-value creates a collision.

This happens with probability at most $\frac{1}{n}$.

Universal Hashing

Universal Hashing

It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right]$$

Universal Hashing

It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \le \Pr_{t_x \ne t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ t_y \overset{\wedge}{\bmod} n = h_2 \end{array} \right] \le \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

Universal Hashing

It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \le \Pr_{t_x \ne t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right] \le \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 29

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$.
Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

Universal Hashing

For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i \Big) \bmod q$$

Universal Hashing

For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i \Big) \bmod q$$

The polynomial is defined by $d + 1$ distinct points.

Universal Hashing

Universal Hashing

Fix $\ell \leq d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q - 1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Universal Hashing

Fix $\ell \leq d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\vec{a}} \in \mathcal{H} \mid h_{\vec{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$

Universal Hashing

Fix $\ell \leq d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\vec{a}} \in \mathcal{H} \mid h_{\vec{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$

Then

$\quad h_{\vec{a}} \in A^\ell \Leftrightarrow h_{\vec{a}} = f_{\vec{a}} \mod n$ and

$$f_{\vec{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

Universal Hashing

Fix $\ell \le d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q - 1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\vec{a}} \in \mathcal{H} \mid h_{\vec{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then

$$h_{\vec{a}} \in A^\ell \Leftrightarrow h_{\vec{a}} = f_{\vec{a}} \bmod n \text{ and}$$

$$f_{\vec{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

Universal Hashing

Fix $\ell \leq d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q - 1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.

Universal Hashing

Fix $\ell \le d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q - 1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Therefore we have

$$|B_1| \cdot \ldots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose $\bar{a}$ such that $h_{\bar{a}} \in A_\ell$.

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}}$$

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{(\frac{q+n}{n})^\ell}{q^\ell}$$

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell}$$

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \Big(\frac{q+n}{q}\Big)^\ell \cdot \frac{1}{n^\ell}$$

$$\leq \Big(1 + \frac{1}{\ell}\Big)^\ell \cdot \frac{1}{n^\ell}$$

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n}\rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \Big(\frac{q+n}{q}\Big)^\ell \cdot \frac{1}{n^\ell}$$

$$\leq \Big(1+\frac{1}{\ell}\Big)^\ell \cdot \frac{1}{n^\ell} \leq \frac{e}{n^\ell} \ .$$

Universal Hashing

Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} \le \frac{(\frac{q+n}{n})^\ell}{q^\ell} \le \Big(\frac{q+n}{q}\Big)^\ell \cdot \frac{1}{n^\ell}$$

$$\le \Big(1 + \frac{1}{\ell}\Big)^\ell \cdot \frac{1}{n^\ell} \le \frac{e}{n^\ell} \ .$$

This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.
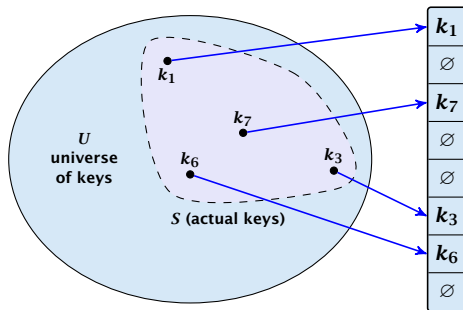
Perfect Hashing

Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is
$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$\mathrm{E}[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \;.$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

Perfect Hashing

Perfect Hashing

We can find such a hash-function by a few trials.

Perfect Hashing

We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

Perfect Hashing

We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Perfect Hashing

We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

Perfect Hashing

$$\sum_i m_i = m$$
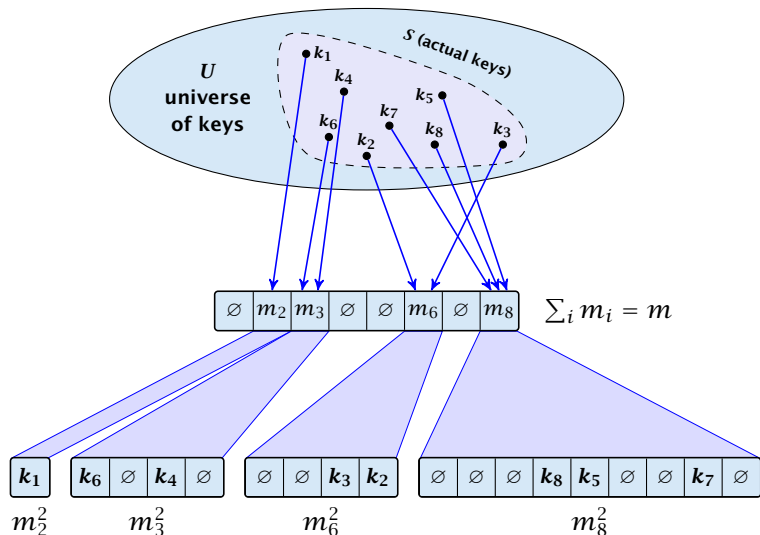
Perfect Hashing

Perfect Hashing

The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$.
Note that m_j is a random variable.

$$\mathrm{E}\Big[\sum_j m_j^2\Big]$$

Perfect Hashing

The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$. Note that m_j is a random variable.

$$\mathrm{E}\left[\sum_j m_j^2\right] = \mathrm{E}\left[2 \sum_j \binom{m_j}{2} + \sum_j m_j\right]$$

Perfect Hashing

The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$.
Note that m_j is a random variable.

$$\mathrm{E}\left[\sum_j m_j^2\right] = \mathrm{E}\left[2\sum_j \binom{m_j}{2} + \sum_j m_j\right]$$

$$= 2\,\mathrm{E}\left[\sum_j \binom{m_j}{2}\right] + \mathrm{E}\left[\sum_j m_j\right]$$

Perfect Hashing

The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$.
Note that m_j is a random variable.

$$\mathrm{E}\Big[\sum_j m_j^2\Big] = \mathrm{E}\Big[2\sum_j \binom{m_j}{2} + \sum_j m_j\Big]$$

$$= 2\,\mathrm{E}\Big[\sum_j \binom{m_j}{2}\Big] + \mathrm{E}\Big[\sum_j m_j\Big]$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

Perfect Hashing

The total memory that is required by all hash-tables is $\mathcal{O}(\sum_j m_j^2)$. Note that m_j is a random variable.

$$\mathrm{E}\left[\sum_j m_j^2\right] = \mathrm{E}\left[2\sum_j \binom{m_j}{2} + \sum_j m_j\right]$$

$$= 2\,\mathrm{E}\left[\sum_j \binom{m_j}{2}\right] + \mathrm{E}\left[\sum_j m_j\right]$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

▶ Two hash-tables $T_1[0, \ldots, n-1]$ and $T_2[0, \ldots, n-1]$, with hash-functions h_1, and h_2.

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables $T_1[0, \ldots, n-1]$ and $T_2[0, \ldots, n-1]$, with hash-functions h_1, and h_2.
- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables $T_1[0, \ldots, n-1]$ and $T_2[0, \ldots, n-1]$, with hash-functions h_1, and h_2.
- ▶ An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
- ▶ A search clearly takes constant time if the above constraint is met.

Cuckoo Hashing

Insert:

Cuckoo Hashing

Insert:

Cuckoo Hashing

Insert:

Cuckoo Hashing

Insert:

Cuckoo Hashing

Insert:

Cuckoo Hashing

Algorithm 38 Cuckoo-Insert(x)

 1: **if** $T_1[h_1(x)] = x \lor T_2[h_2(x)] = x$ **then return**
 2: steps $\leftarrow 1$
 3: **while** steps $\leq$ maxsteps **do**
 4:      exchange x and $T_1[h_1(x)]$
 5:      **if** $x =$ null **then return**
 6:      exchange x and $T_2[h_2(x)]$
 7:      **if** $x =$ null **then return**
 8:      steps $\leftarrow$ steps $+1$
 9: rehash() // change hash-functions; rehash everything
10: Cuckoo-Insert(x)

Cuckoo Hashing

▶ We call one iteration through the while-loop a step of the algorithm.

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a step of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.

Cuckoo Hashing

▶ We call one iteration through the while-loop a step of the algorithm.

▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.

▶ We say a phase is successful if it is not terminated by the maxstep-condition, but the while loop is left because $x = $ null.

Cuckoo Hashing

Cuckoo Hashing

What is the expected time for an insert-operation?

Cuckoo Hashing

What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

Cuckoo Hashing

What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

Cuckoo Hashing: Insert

T_1 T_2

Cuckoo Hashing: Insert

Cuckoo Hashing: Insert

Cuckoo Hashing: Insert

Cuckoo Hashing: Insert

Cuckoo Hashing: Insert

Cuckoo Hashing: Insert

Cuckoo Hashing: Insert
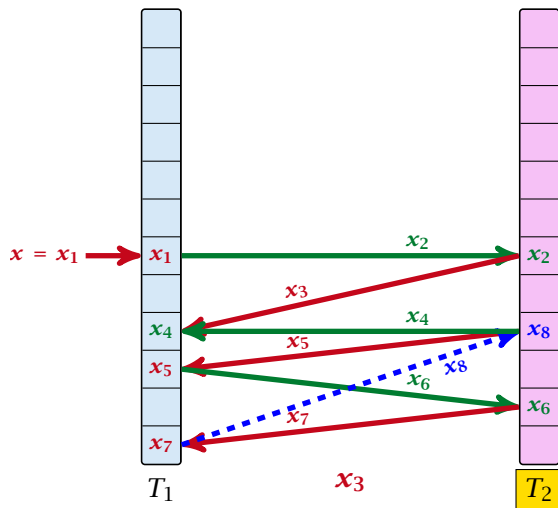
Cuckoo Hashing: Insert
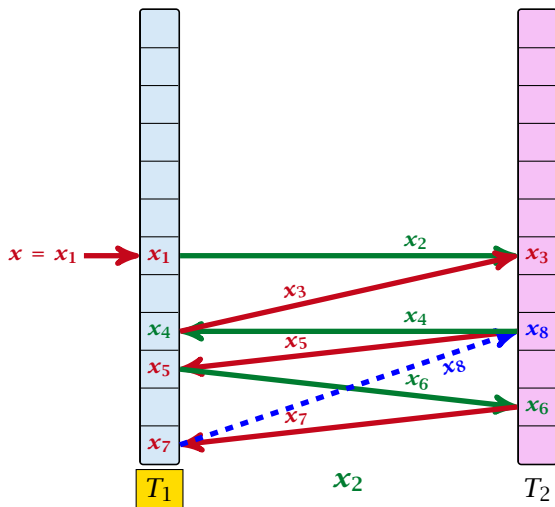
Cuckoo Hashing: Insert
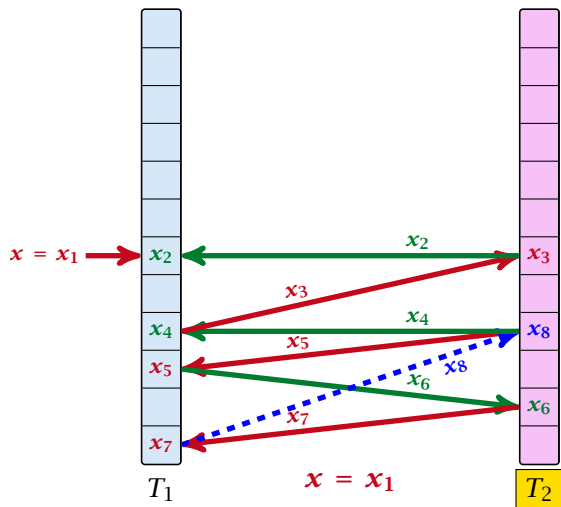
Cuckoo Hashing: Insert
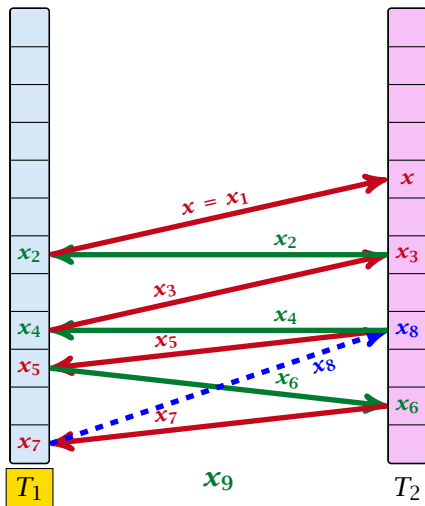
Cuckoo Hashing: Insert
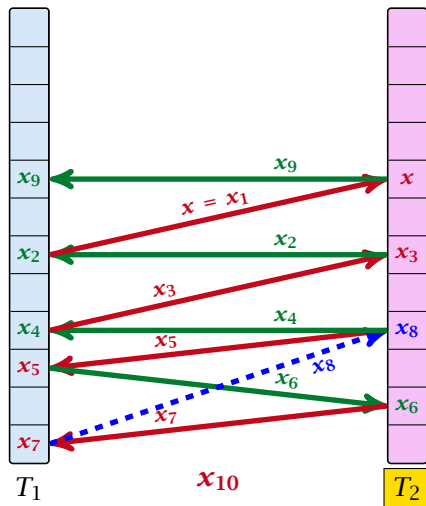
Cuckoo Hashing: Insert
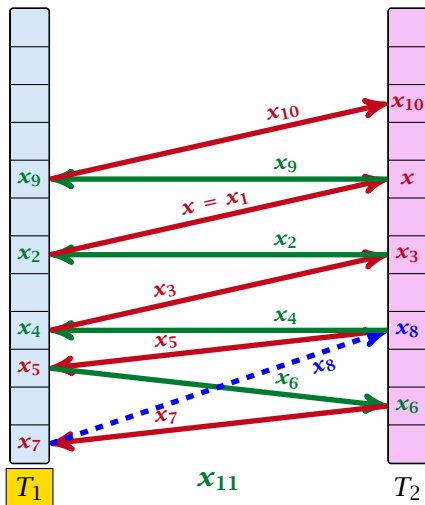
Cuckoo Hashing: Insert
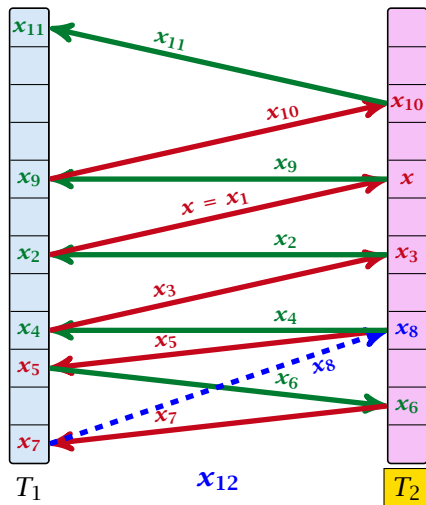
Cuckoo Hashing: Insert

Cuckoo Hashing: Insert
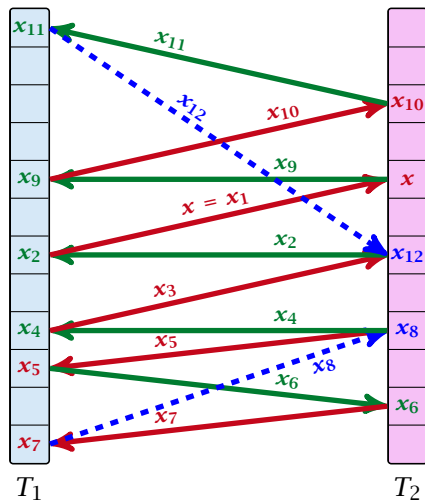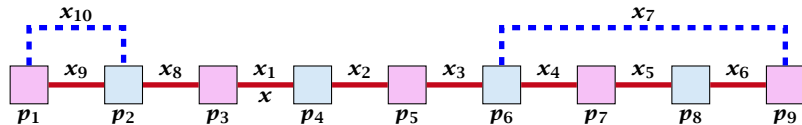
Cuckoo Hashing

A cycle-structure of size s is defined by

Cuckoo Hashing

A cycle-structure of size s is defined by

► $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

Cuckoo Hashing

A cycle-structure of size s is defined by

▶ $s-1$ different cells (alternating btw. cells from T_1 and T_2).

▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.

Cuckoo Hashing
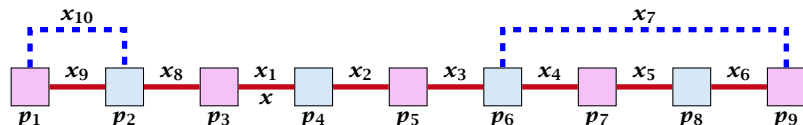
A cycle-structure of size s is defined by

- ▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).
- ▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.
- ▶ The leftmost cell is "linked forward" to some cell on the right.

Cuckoo Hashing
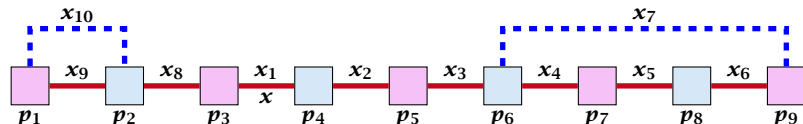
A cycle-structure of size s is defined by

▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.

▶ The leftmost cell is "linked forward" to some cell on the right.

▶ The rightmost cell is "linked backward" to a cell on the left.

Cuckoo Hashing
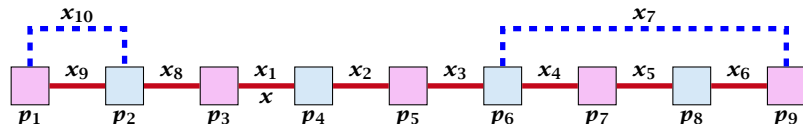
A cycle-structure of size s is defined by

▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.

▶ The leftmost cell is "linked forward" to some cell on the right.

▶ The rightmost cell is "linked backward" to a cell on the left.

▶ One link represents key x; this is where the counting starts.

Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Cuckoo Hashing

A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \ .$$

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \; .$$

- ▶ There are at most s^2 possibilities where to attach the forward and backward links.

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most s^2 possibilities where to attach the forward and backward links.
- ▶ There are at most s possibilities to choose where to place key x.

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \, .$$

- ▶ There are at most s^2 possibilities where to attach the forward and backward links.
- ▶ There are at most s possibilities to choose where to place key x.
- ▶ There are m^{s-1} possibilities to choose the keys apart from x.

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

▶ There are at most s^2 possibilities where to attach the forward and backward links.

▶ There are at most s possibilities to choose where to place key x.

▶ There are m^{s-1} possibilities to choose the keys apart from x.

▶ There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right) \quad .$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right) \quad .$$

Here we used the fact that $(1 + \epsilon)m \leq n$.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

$$\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right) \ .$$

Here we used the fact that $(1 + \epsilon)m \leq n$.

Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \ .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.
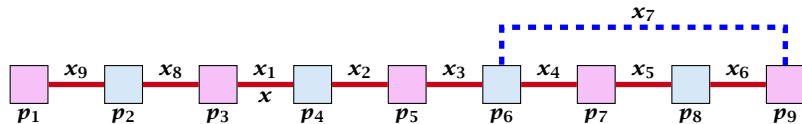
Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 30

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 \ .$$

Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. $\qquad\square$
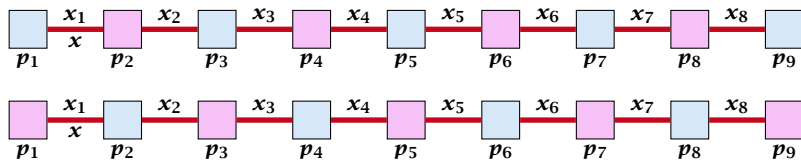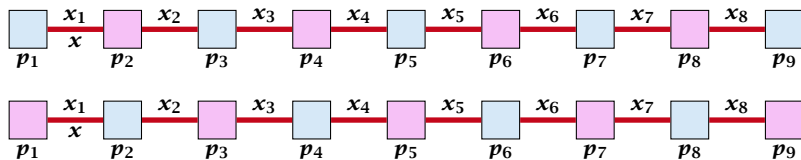
A path-structure of size s is defined by

Cuckoo Hashing

A path-structure of size s is defined by

▶ $s + 1$ different cells (alternating btw. cells from T_1 and T_2).
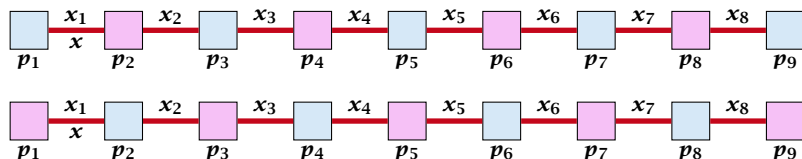
Cuckoo Hashing

A path-structure of size s is defined by

▶ $s + 1$ different cells (alternating btw. cells from T_1 and T_2).

▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.
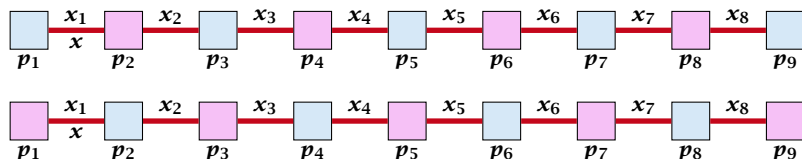
A path-structure of size s is defined by

▶ $s + 1$ different cells (alternating btw. cells from T_1 and T_2).

▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.

▶ The leftmost cell is either from T_1 or T_2.

Cuckoo Hashing

A path-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$
$$\leq 2\mu^2 \left(\frac{m}{n} \right)^{s-1}$$

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$
$$\leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

$$\leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Plugging in $s = (2t+2)/3$ gives

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$
$$\leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Plugging in $s = (2t+2)/3$ gives

$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1}$$

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$
$$\leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1}$$

Plugging in $s = (2t+2)/3$ gives

$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1} = 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} \quad .$$

Cuckoo Hashing

We choose maxsteps $\geq 3\ell/2 + 1/2$.

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\Pr[\text{unsuccessful} \mid \text{no cycle}]$$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\Pr[\text{unsuccessful} \mid \text{no cycle}]$$
$$\leq \Pr[\exists \text{ active path-structure of size at least } \tfrac{2\text{maxsteps}+2}{3}]$$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\Pr[\text{unsuccessful} \mid \text{no cycle}]$$
$$\leq \Pr[\exists \text{ active path-structure of size at least } \tfrac{2\text{maxsteps}+2}{3}]$$
$$\leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1]$$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$\Pr[\text{unsuccessful} \mid \text{no cycle}]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1]$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$\Pr[\text{unsuccessful} \mid \text{no cycle}]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1]$

$\qquad \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{\ell}$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$\Pr[\text{unsuccessful} \mid \text{no cycle}]$

$\leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}]$

$\leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1]$

$\leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1]$

$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2}$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\Pr[\text{unsuccessful} \mid \text{no cycle}]$$

$$\leq \Pr[\exists \text{ active path-structure of size at least } \tfrac{2\text{maxsteps}+2}{3}]$$

$$\leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1]$$

$$\leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1]$$

$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2}$$

by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right)/\log\left(\frac{1}{1+\epsilon}\right) = \log\left(2\mu^2 m^2\right)/\log\left(1+\epsilon\right)$

Cuckoo Hashing

We choose $\text{maxsteps} \geq 3\ell/2 + 1/2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$\Pr[\text{unsuccessful} \mid \text{no cycle}]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}+2}{3}]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1]$

$\qquad \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1]$

$\qquad \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2}$

by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right)/\log\left(\frac{1}{1+\epsilon}\right) = \log\left(2\mu^2 m^2\right)/\log\left(1+\epsilon\right)$

This gives $\text{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\textsf{cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

$$\Pr[\textsf{unsuccessful} \mid \textsf{no cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

Observe that

$$\Pr[\text{successful}] = \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}]$$

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

Observe that

$$\Pr[\text{successful}] = \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}]$$
$$\ge c \cdot \Pr[\text{no cycle}]$$

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

Observe that

$$\Pr[\text{successful}] = \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}]$$
$$\ge c \cdot \Pr[\text{no cycle}]$$

for a suitable constant $c > 0$.

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

$$E[\text{number of steps} \mid \text{phase successful}]$$

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

$$
\mathrm{E}[\text{number of steps} \mid \text{phase successful}]
$$
$$
= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}]
$$

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

$\mathrm{E}[\text{number of steps} \mid \text{phase successful}]$

$\qquad = \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}]$

We have

$\Pr[\text{search at least } t \text{ steps} \mid \text{successful}]$

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

$$E[\text{number of steps} \mid \text{phase successful}]$$
$$= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}]$$

We have

$$\Pr[\text{search at least } t \text{ steps} \mid \text{successful}]$$
$$= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}]$$

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

$$E[\text{number of steps} \mid \text{phase successful}]$$
$$= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}]$$

We have

$$\Pr[\text{search at least } t \text{ steps} \mid \text{successful}]$$
$$= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}]$$
$$\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}]$$

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

$E[\text{number of steps} \mid \text{phase successful}]$

$$= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}]$$

We have

$\Pr[\text{search at least } t \text{ steps} \mid \text{successful}]$

$= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}]$

$\leq \dfrac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}]$

$\leq \dfrac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{no cycle}] / \Pr[\text{no cycle}]$

Cuckoo Hashing

The expected number of complete steps in the successful phase of an insert operation is:

$$E[\text{number of steps} \mid \text{phase successful}]$$
$$= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}]$$

We have

$$\Pr[\text{search at least } t \text{ steps} \mid \text{successful}]$$
$$= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}]$$
$$\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}]$$
$$\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{no cycle}] / \Pr[\text{no cycle}]$$
$$= \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] .$$

Cuckoo Hashing

Hence,

$\mathrm{E}[\text{number of steps} \mid \text{phase successful}]$

Cuckoo Hashing

Hence,

$\mathrm{E}[$number of steps | phase successful$]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps | no cycle}]$$

Cuckoo Hashing

Hence,

$$E[\text{number of steps} \mid \text{phase successful}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \Big(\frac{1}{1+\epsilon}\Big)^{(2t-1)/3}$$

Cuckoo Hashing

Hence,

$\mathrm{E}[\text{number of steps} \mid \text{phase successful}]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \Big(\frac{1}{1+\epsilon}\Big)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \Big(\frac{1}{1+\epsilon}\Big)^{(2(t+1)-1)/3}$$

Cuckoo Hashing

Hence,

$\mathrm{E}[\text{number of steps} \mid \text{phase successful}]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \Big(\frac{1}{1+\epsilon}\Big)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \Big(\frac{1}{1+\epsilon}\Big)^{(2(t+1)-1)/3}$$

$$= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \Big(\frac{1}{(1+\epsilon)^{2/3}}\Big)^t$$

Cuckoo Hashing

Hence,

$\mathrm{E}[\text{number of steps} \mid \text{phase successful}]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \Big(\frac{1}{1+\epsilon}\Big)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \Big(\frac{1}{1+\epsilon}\Big)^{(2(t+1)-1)/3}$$

$$= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \Big(\frac{1}{(1+\epsilon)^{2/3}}\Big)^t = \mathcal{O}(1) \ .$$

Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}]$$

$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2(t+1)-1)/3}$$

$$= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \left(\frac{1}{(1+\epsilon)^{2/3}}\right)^t = \mathcal{O}(1) \ .$$

This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is
$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is
$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

Therefore the expected cost for re-hashes is $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m + 1$ insertions fails):

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m + 1$ insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m + 1$ insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p \ .$$

Let Z_i denote the event that the i-th rehash occurs:

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m + 1$ insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p \ .$$

Let Z_i denote the event that the i-th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m + 1$ insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p \ .$$

Let Z_i denote the event that the i-th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

Let X_i^s, $s \in \{1, \ldots, m + 1\}$ denote the cost for inserting the s-th element during the i-th rehash (assuming i-th rehash occurs):

$$\mathrm{E}[X_i^s]$$

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m + 1$ insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p \ .$$

Let Z_i denote the event that the i-th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

Let X_i^s, $s \in \{1, \ldots, m + 1\}$ denote the cost for inserting the s-th element during the i-th rehash (assuming i-th rehash occurs):

$$E[X_i^s] = E[\text{steps} \mid \text{phase successful}] \cdot \Pr[\text{phase sucessful}]$$
$$+ \text{maxsteps} \cdot \Pr[\text{not sucessful}]$$

Formal Proof

Let Y_i denote the event that the i-th rehash occurs and does not lead to a valid configuration (i.e., one of the $m + 1$ insertions fails):

$$\Pr[Y_i | Z_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p \ .$$

Let Z_i denote the event that the i-th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=0}^{i-1} Y_j] \leq p^i$$

Let X_i^s, $s \in \{1, \ldots, m + 1\}$ denote the cost for inserting the s-th element during the i-th rehash (assuming i-th rehash occurs):

$$\begin{aligned} \mathrm{E}[X_i^s] = {} & \mathrm{E}[\text{steps} \mid \text{phase successful}] \cdot \Pr[\text{phase sucessful}] \\ & + \text{maxsteps} \cdot \Pr[\text{not sucessful}] = \mathcal{O}(1) \ . \end{aligned}$$

The expected cost for all rehashes is

$$\mathrm{E}\Big[\sum_i \sum_s Z_i X_i^s\Big]$$

The expected cost for all rehashes is

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_i^s\right]$$

Note that Z_i is independent of X_j^s, $j \geq i$ (however, it is not independent of X_j^s, $j < i$). Hence,

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_s^i\right] = \sum_i \sum_s \mathrm{E}[Z_i] \cdot \mathrm{E}[X_s^i]$$

The expected cost for all rehashes is

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_i^s\right]$$

Note that Z_i is independent of X_j^s, $j \geq i$ (however, it is not independent of X_j^s, $j < i$). Hence,

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_s^i\right] = \sum_i \sum_s \mathrm{E}[Z_i] \cdot \mathrm{E}[X_s^i]$$
$$\leq \mathcal{O}(m) \cdot \sum_i p^i$$

The expected cost for all rehashes is

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_i^s\right]$$

Note that Z_i is independent of X_j^s, $j \geq i$ (however, it is not independent of X_j^s, $j < i$). Hence,

$$\begin{aligned}
\mathrm{E}\left[\sum_i \sum_s Z_i X_s^i\right] &= \sum_i \sum_s \mathrm{E}[Z_i] \cdot \mathrm{E}[X_s^i] \\
&\leq \mathcal{O}(m) \cdot \sum_i p^i \\
&\leq \mathcal{O}(m) \cdot \frac{p}{1-p}
\end{aligned}$$

The expected cost for all rehashes is

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_i^s\right]$$

Note that Z_i is independent of X_j^s, $j \geq i$ (however, it is not independent of X_j^s, $j < i$). Hence,

$$\begin{aligned}
\mathrm{E}\left[\sum_i \sum_s Z_i X_s^i\right] &= \sum_i \sum_s \mathrm{E}[Z_i] \cdot \mathrm{E}[X_s^i] \\
&\leq \mathcal{O}(m) \cdot \sum_i p^i \\
&\leq \mathcal{O}(m) \cdot \frac{p}{1-p} \\
&= \mathcal{O}(1) .
\end{aligned}$$

Cuckoo Hashing

What kind of hash-functions do we need?

Cuckoo Hashing

What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Cuckoo Hashing

What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

Cuckoo Hashing

How do we make sure that $n \geq (1 + \epsilon)m$?

▶ Let $\alpha := 1/(1 + \epsilon)$.

Cuckoo Hashing

How do we make sure that $n \geq (1 + \epsilon)m$?

▶ Let $\alpha := 1/(1 + \epsilon)$.

▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).

Cuckoo Hashing

How do we make sure that $n \geq (1 + \epsilon)m$?

▶ Let $\alpha := 1/(1 + \epsilon)$.

▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).

▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).

Cuckoo Hashing

How do we make sure that $n \geq (1 + \epsilon)m$?

- ▶ Let $\alpha := 1/(1 + \epsilon)$.

- ▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).

- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).

- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.

Cuckoo Hashing

How do we make sure that $n \geq (1 + \epsilon)m$?

- Let $\alpha := 1/(1 + \epsilon)$.

- Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).

- Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).

- Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.

- Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

Cuckoo Hashing

Lemma 31

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Cuckoo Hashing

Lemma 31

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

▶ $S.\,\mathbf{build}(x_1, \ldots, x_n)$: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

▶ $S.\operatorname{build}(x_1, \ldots, x_n)$: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.

▶ $S.\operatorname{insert}(x)$: Adds element x to the data-structure.

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

- ▶ $S.\,\text{build}(x_1, \ldots, x_n)$: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.

- ▶ $S.\,\text{insert}(x)$: Adds element x to the data-structure.

- ▶ element $S.\,\text{minimum}()$: Returns an element $x \in S$ with minimum key-value $\text{key}[x]$.

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

- ▶ **$S.\,\text{build}(x_1, \ldots, x_n)$:** Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.

- ▶ **$S.\,\text{insert}(x)$:** Adds element x to the data-structure.

- ▶ **element $S.\,\text{minimum}()$:** Returns an element $x \in S$ with minimum key-value $\text{key}[x]$.

- ▶ **element $S.\,\text{delete-min}()$:** Deletes the element with minimum key-value from S and returns it.

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

▶ $S.\,\mathbf{build}(x_1, \ldots, x_n)$: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.

▶ $S.\,\mathbf{insert}(x)$: Adds element x to the data-structure.

▶ element $S.\,\mathbf{minimum}()$: Returns an element $x \in S$ with minimum key-value $\text{key}[x]$.

▶ element $S.\,\mathbf{delete\text{-}min}()$: Deletes the element with minimum key-value from S and returns it.

▶ boolean $S.\,\mathbf{is\text{-}empty}()$: Returns true if the data-structure is empty and false otherwise.

8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

- ▶ $S.\,\mathbf{build}(x_1, \dots, x_n)$: Creates a data-structure that contains just the elements $x_1, \dots, x_n$.

- ▶ $S.\,\mathbf{insert}(x)$: Adds element x to the data-structure.

- ▶ element $S.\,\mathbf{minimum}()$: Returns an element $x \in S$ with minimum key-value $\mathrm{key}[x]$.

- ▶ element $S.\,\mathbf{delete\text{-}min}()$: Deletes the element with minimum key-value from S and returns it.

- ▶ boolean $S.\,\mathbf{is\text{-}empty}()$: Returns true if the data-structure is empty and false otherwise.

Sometimes we also have

- ▶ $S.\,\mathbf{merge}(S')$: $S := S \cup S'$; $S' := \varnothing$.

8 Priority Queues

An addressable Priority Queue also supports:

8 Priority Queues

An addressable Priority Queue also supports:

▶ **handle $S.$ insert(x):** Adds element x to the data-structure, and returns a handle to the object for future reference.

8 Priority Queues

An addressable Priority Queue also supports:

▶ **handle $S.$ insert(x):** Adds element x to the data-structure, and returns a handle to the object for future reference.

▶ **$S.$ delete(h):** Deletes element specified through handle h.

8 Priority Queues

An addressable Priority Queue also supports:

▶ **handle $S.$ insert(x):** Adds element x to the data-structure, and returns a handle to the object for future reference.

▶ **$S.$ delete(h):** Deletes element specified through handle h.

▶ **$S.$ decrease-key(h, k):** Decreases the key of the element specified by handle h to k. Assumes that the key is at least k before the operation.

Dijkstra's Shortest Path Algorithm

Algorithm 1 Shortest-Path($G = (V, E, d), s \in V$)

1: **Input:** weighted graph $G = (V, E, d)$; start vertex s;
2: **Output:** key-field of every node contains distance from s;
3: S.build(); // build empty priority queue
4: **for all** $v \in V \setminus \{s\}$ **do**
5:      $v.\text{key} \leftarrow \infty$;
6:      $h_v \leftarrow S.\text{insert}(v)$;
7: $s.\text{key} \leftarrow 0$; $S.\text{insert}(s)$;
8: **while** $S.\text{is-empty}() = \text{false}$ **do**
9:      $v \leftarrow S.\text{delete-min}()$;
10:      **for all** $x \in V$ s.t. $(v, x) \in E$ **do**
11:          **if** $x.\text{key} > v.\text{key} + d(v, x)$ **then**
12:             $S.\text{decrease-key}(h_x, v.\text{key} + d(v, x))$;
13:             $x.\text{key} \leftarrow v.\text{key} + d(v, x)$;

Prim's Minimum Spanning Tree Algorithm

Algorithm 2 Prim-MST($G = (V, E, d), s \in V$)

1: **Input:** weighted graph $G = (V, E, d)$; start vertex s;
2: **Output:** pred-fields encode MST;
3: S.build(); // build empty priority queue
4: **for all** $v \in V \setminus \{s\}$ **do**
5:     $v.\text{key} \leftarrow \infty$;
6:     $h_v \leftarrow S.\text{insert}(v)$;
7: $s.\text{key} \leftarrow 0$; $S.\text{insert}(s)$;
8: **while** $S.\text{is-empty}() = \text{false}$ **do**
9:     $v \leftarrow S.\text{delete-min}()$;
10:     **for all** $x \in V$ s.t. $\{v, x\} \in E$ **do**
11:         **if** $x.\text{key} > d(v, x)$ **then**
12:             $S.\text{decrease-key}(h_x, d(v, x))$;
13:             $x.\text{key} \leftarrow d(v, x)$;
14:             $x.\text{pred} \leftarrow v$;

Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
- ▶ $|V|$ delete-min() operations
- ▶ $|V|$ is-empty() operations
- ▶ $|E|$ decrease-key() operations

Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
- ▶ $|V|$ delete-min() operations
- ▶ $|V|$ is-empty() operations
- ▶ $|E|$ decrease-key() operations

How good a running time can we obtain?

8 Priority Queues

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

8 Priority Queues

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap*
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$**	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Note that most applications use **build()** only to create an empty heap which then costs time 1.

8 Priority Queues

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Note that most applications use **build()** only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

8 Priority Queues

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Note that most applications use **build()** only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an amortized guarantee.

8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time
$\mathcal{O}((|V| + |E|) \log |V|)$.

Using Fibonacci Heaps, Prim and Dijkstra run in time
$\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps
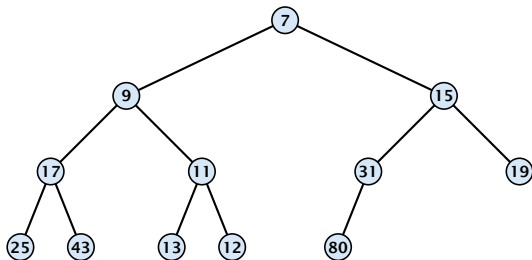
8.1 Binary Heaps

▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
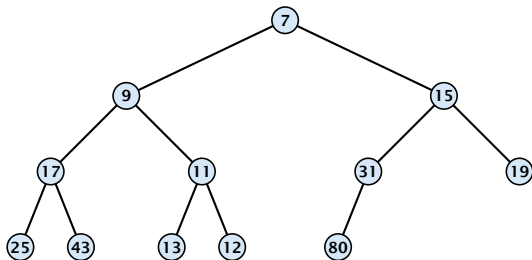
8.1 Binary Heaps

▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.

▶ Heap property: A node's key is not larger than the key of one of its children.
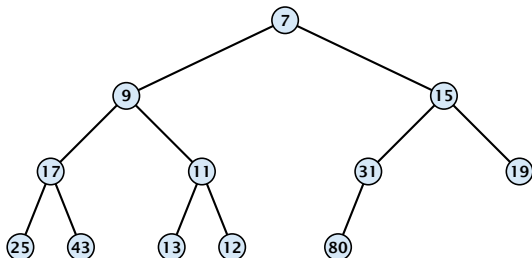
Binary Heaps

Operations:

Binary Heaps

Operations:

- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.

Binary Heaps

Operations:

- ▶ **minimum():** return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty():** check whether root-pointer is null. Time $\mathcal{O}(1)$.

8.1 Binary Heaps

Maintain a pointer to the last element x.
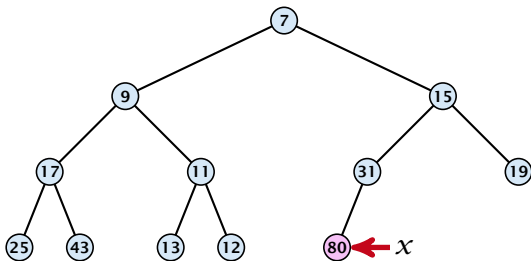
8.1 Binary Heaps

Maintain a pointer to the last element x.

▶ We can compute the predecessor of x
  (last element when x is deleted) in time $\mathcal{O}(\log n)$.
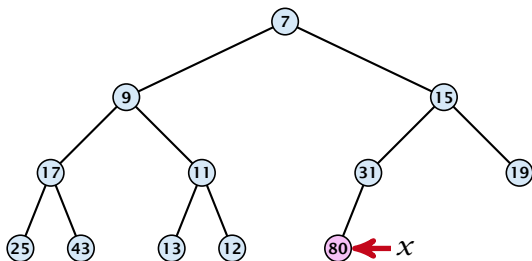
8.1 Binary Heaps

Maintain a pointer to the last element x.

▶ We can compute the predecessor of x
  (last element when x is deleted) in time $\mathcal{O}(\log n)$.

  go up until the last edge used was a right edge.
  go left; go right until you reach a leaf
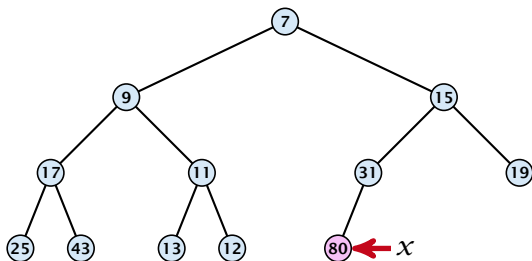
8.1 Binary Heaps

Maintain a pointer to the last element x.

- ▶ We can compute the predecessor of x
  (last element when x is deleted) in time $\mathcal{O}(\log n)$.

  go up until the last edge used was a right edge.
  go left; go right until you reach a leaf

  if you hit the root on the way up, go to the rightmost element
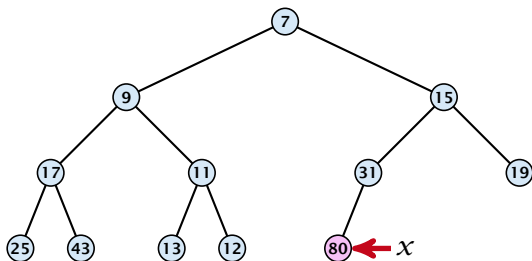
8.1 Binary Heaps

Maintain a pointer to the last element x.
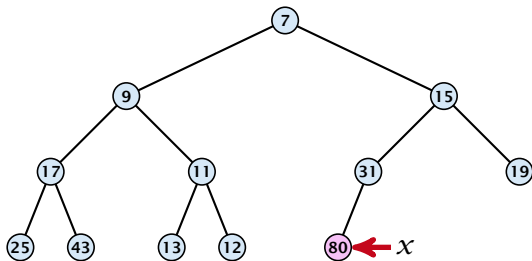
8.1 Binary Heaps

Maintain a pointer to the last element x.

- ▶ We can compute the successor of x
  (last element when an element is inserted) in time $\mathcal{O}(\log n)$.
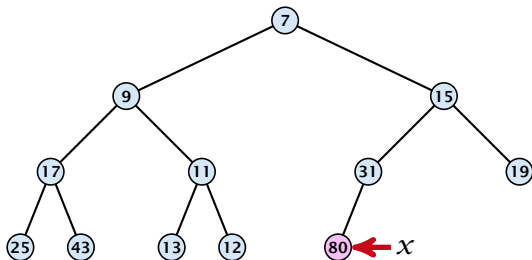
8.1 Binary Heaps

Maintain a pointer to the last element x.

▶ We can compute the successor of x
  (last element when an element is inserted) in time $\mathcal{O}(\log n)$.

  go up until the last edge used was a left edge.
  go right; go left until you reach a null-pointer.
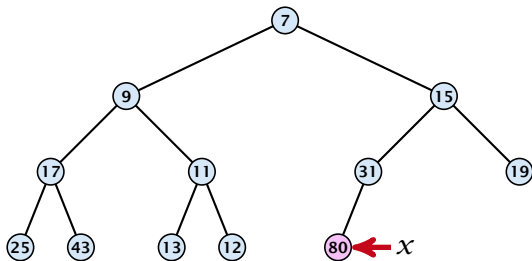
8.1 Binary Heaps

Maintain a pointer to the last element x.

▶ We can compute the successor of x
  (last element when an element is inserted) in time $\mathcal{O}(\log n)$.

  go up until the last edge used was a left edge.
  go right; go left until you reach a null-pointer.

  if you hit the root on the way up, go to the leftmost element;
  insert a new element as a left child;
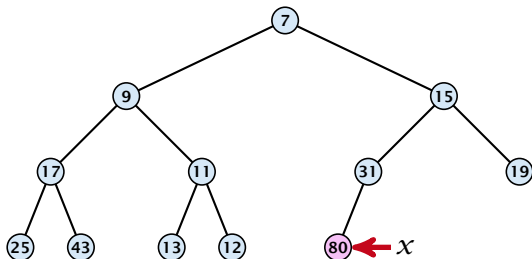
Insert

1. Insert element at successor of x.
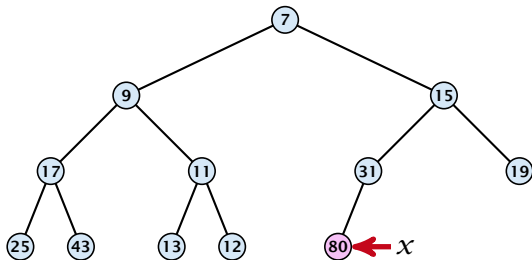
Insert

1. Insert element at successor of x.

2. Exchange with parent until heap property is fulfilled.
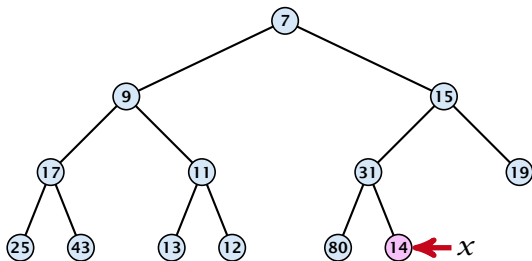
Insert

1. Insert element at successor of x.
2. Exchange with parent until heap property is fulfilled.
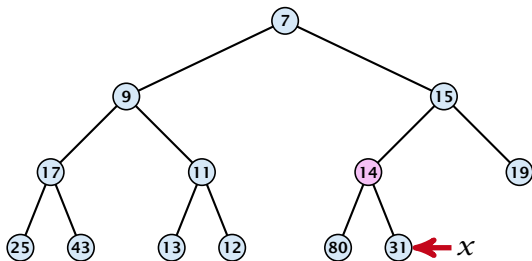
Insert

1. Insert element at successor of x.
2. Exchange with parent until heap property is fulfilled.
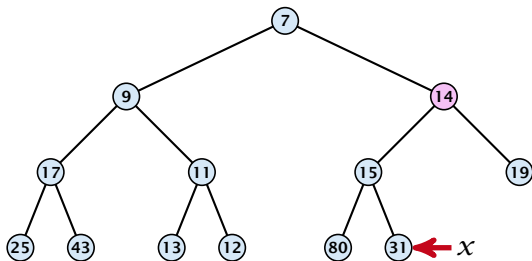
Insert

1. Insert element at successor of x.
2. Exchange with parent until heap property is fulfilled.
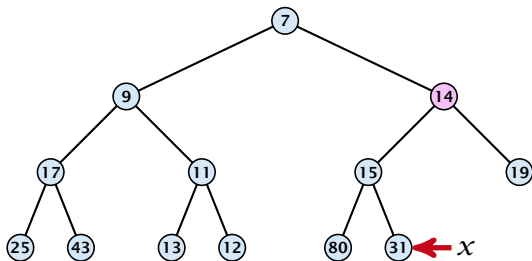
Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

Delete

1. Exchange the element to be deleted with the element e pointed to by x.
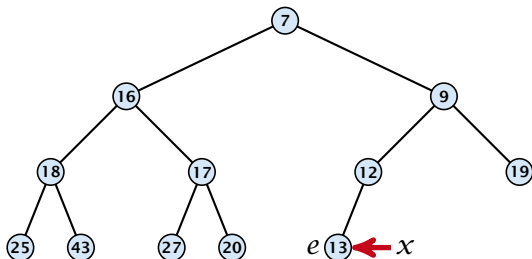
Delete

1. Exchange the element to be deleted with the element e pointed to by x.

2. Restore the heap-property for the element e.
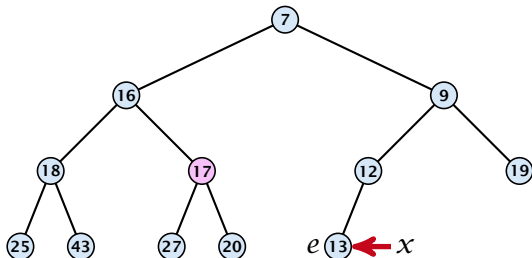
Delete

1. Exchange the element to be deleted with the element e pointed to by x.

2. Restore the heap-property for the element e.
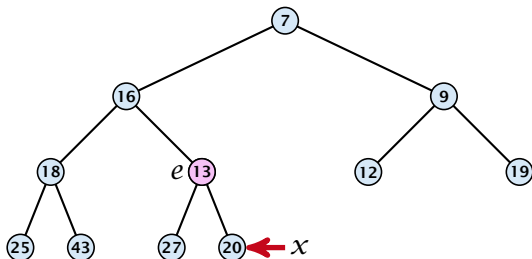
Delete

1. Exchange the element to be deleted with the element e pointed to by x.

2. Restore the heap-property for the element e.
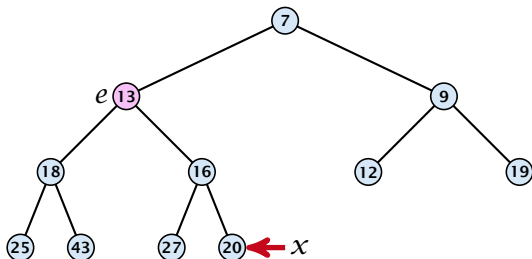
Delete

1. Exchange the element to be deleted with the element e pointed to by x.

2. Restore the heap-property for the element e.
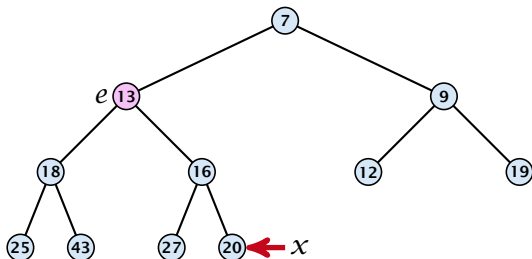
At its new position e may either travel up or down in the tree (but not both directions).

Binary Heaps

Operations:

- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.

- ▶ **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.

- ▶ **insert(k)**: insert at successor of x and bubble up. Time $\mathcal{O}(\log n)$.

- ▶ **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

Build Heap

We can build a heap in linear time:

$$\sum_{\text{levels } \ell} 2^{\ell} \cdot (h - \ell) = \sum_{i} i 2^{h-i} = \mathcal{O}(2^h) = \mathcal{O}(n)$$

Binary Heaps

Operations:

- ▶ **minimum():** Return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty():** Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- ▶ **insert(k):** Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h):** Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- ▶ **build($x_1, \ldots, x_n$):** Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \ldots, n-1]$ be an array

- ▶ The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- ▶ The left child of i-th element is at position $2i + 1$.
- ▶ The right child of i-th element is at position $2i + 2$.

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \ldots, n-1]$ be an array

- The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- The left child of i-th element is at position $2i + 1$.
- The right child of i-th element is at position $2i + 2$.

Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \ldots, n-1]$ be an array

- ▶ The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- ▶ The left child of i-th element is at position $2i + 1$.
- ▶ The right child of i-th element is at position $2i + 2$.

Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

8.2 Binomial Heaps

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	**$\log n$**	1

Binomial Trees
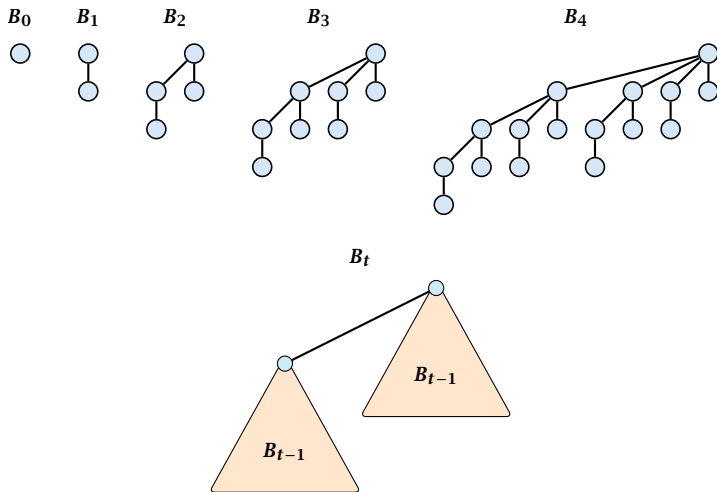
Binomial Trees

Properties of Binomial Trees

▶ B_k has 2^k nodes.

Binomial Trees

Properties of Binomial Trees

- ▶ B_k has 2^k nodes.
- ▶ B_k has height k.

Binomial Trees

Properties of Binomial Trees

▶ B_k has 2^k nodes.

▶ B_k has height k.

▶ The root of B_k has degree k.

Binomial Trees

Properties of Binomial Trees

- ▶ B_k has 2^k nodes.

- ▶ B_k has height k.

- ▶ The root of B_k has degree k.

- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ.

Binomial Trees

Properties of Binomial Trees

▶ B_k has 2^k nodes.

▶ B_k has height k.

▶ The root of B_k has degree k.

▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ.

▶ Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

Binomial Trees
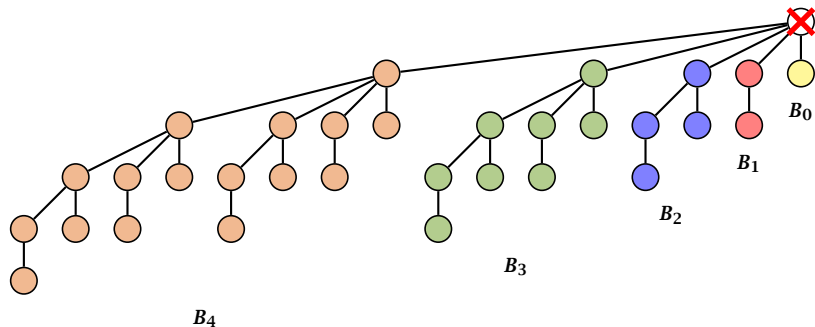
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.
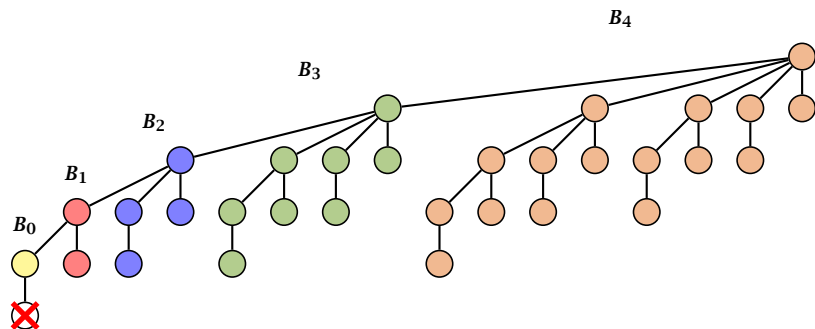
Binomial Trees

Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.
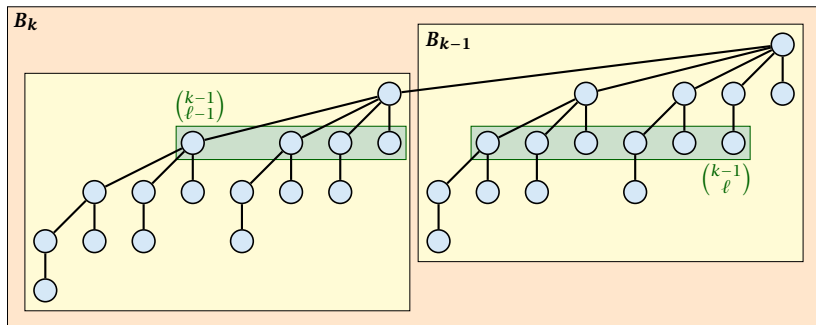
Binomial Trees

The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees
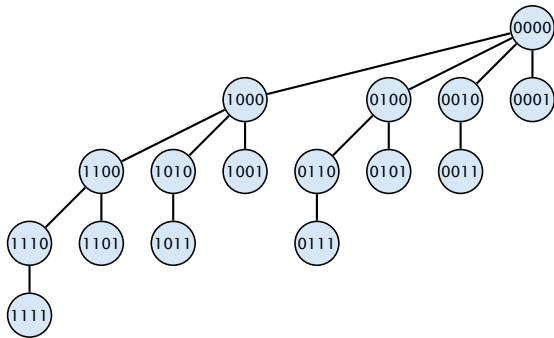
Binomial Trees
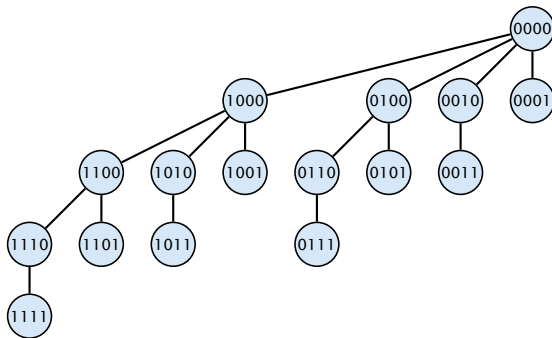
The binomial tree B_k is a sub-graph of the hypercube H_k.

Binomial Trees
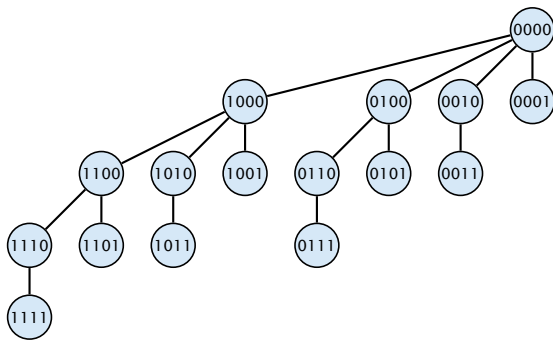
The binomial tree B_k is a sub-graph of the hypercube H_k.

The parent of a node with label $b_k, \ldots, b_1$ is obtained by setting the least significant 1-bit to 0.

Binomial Trees
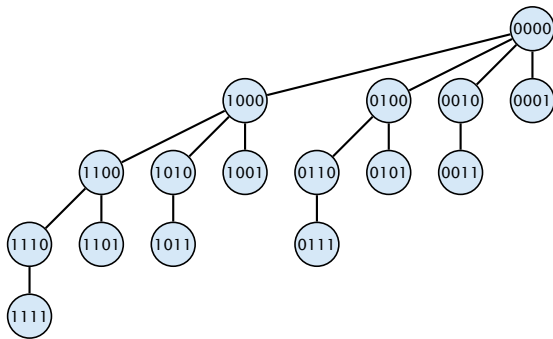
The binomial tree B_k is a sub-graph of the hypercube H_k.

The parent of a node with label $b_k, \ldots, b_1$ is obtained by setting the least significant 1-bit to 0.

The ℓ-th level contains nodes that have ℓ 1's in their label.

8.2 Binomial Heaps

How do we implement trees with non-constant degree?

▶ The children of a node are arranged in a circular linked list.
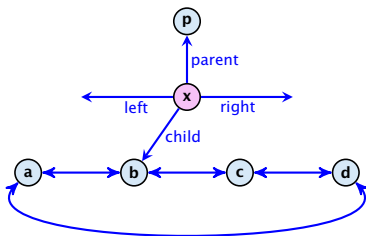
8.2 Binomial Heaps

How do we implement trees with non-constant degree?

▶ The children of a node are arranged in a circular linked list.

▶ A child-pointer points to an arbitrary node within the list.
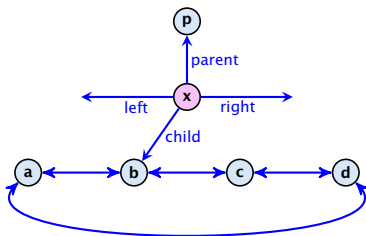
8.2 Binomial Heaps

How do we implement trees with non-constant degree?

▶ The children of a node are arranged in a circular linked list.

▶ A child-pointer points to an arbitrary node within the list.

▶ A parent-pointer points to the parent node.
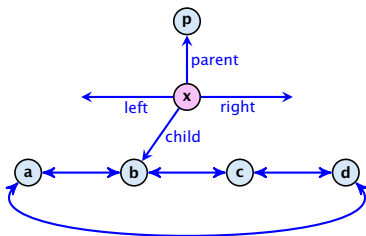
8.2 Binomial Heaps

How do we implement trees with non-constant degree?

▶ The children of a node are arranged in a circular linked list.

▶ A child-pointer points to an arbitrary node within the list.

▶ A parent-pointer points to the parent node.

▶ Pointers $x.\mathrm{left}$ and $x.\mathrm{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\mathrm{left} = x.\mathrm{right} = x$).
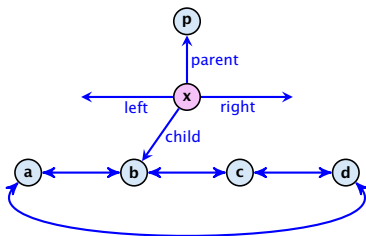
8.2 Binomial Heaps

▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.

▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap
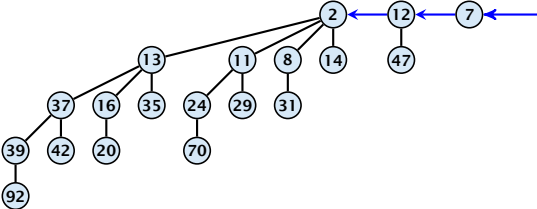
Binomial Heap
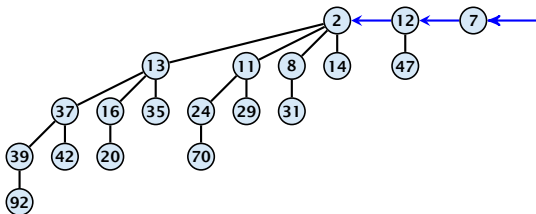
In a binomial heap the keys are arranged in a collection of
binomial trees.

Binomial Heap
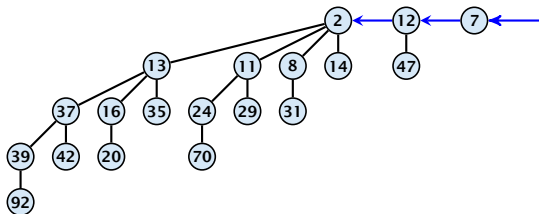
In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

Binomial Heap
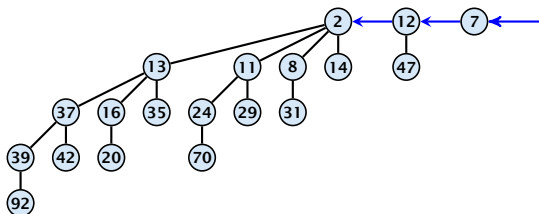
In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

Binomial Heap: Merge

Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let B_{k_1}, B_{k_2}, B_{k_3}, $k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let B_{k_1}, B_{k_2}, B_{k_3}, $k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.
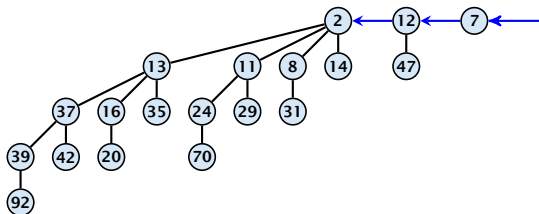
Binomial Heap

Properties of a heap with n keys:

Binomial Heap

Properties of a heap with n keys:

▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.
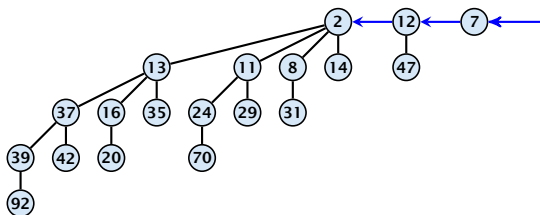
Binomial Heap

Properties of a heap with n keys:

▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.

▶ The heap contains tree B_i iff $b_i = 1$.
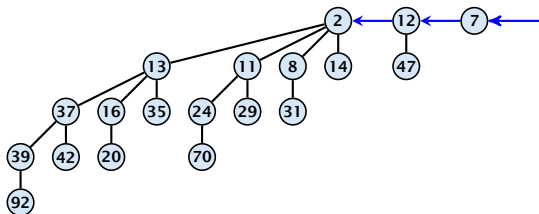
Binomial Heap

Properties of a heap with n keys:

- ▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.
- ▶ The heap contains tree B_i iff $b_i = 1$.
- ▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.
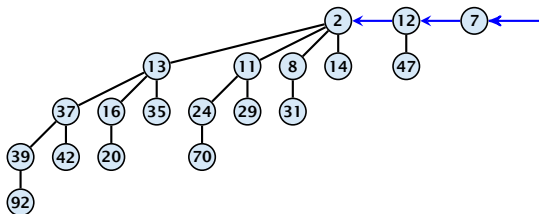
Binomial Heap

Properties of a heap with n keys:

▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.

▶ The heap contains tree B_i iff $b_i = 1$.

▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.

▶ The minimum must be contained in one of the roots.
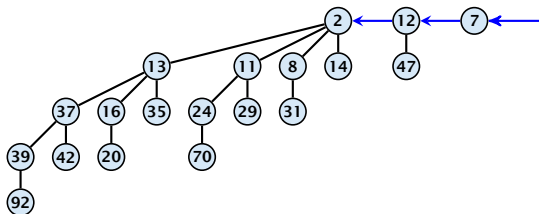
Binomial Heap

Properties of a heap with n keys:

▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.

▶ The heap contains tree B_i iff $b_i = 1$.

▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.

▶ The minimum must be contained in one of the roots.

▶ The height of the largest tree is at most $\lfloor \log n \rfloor$.
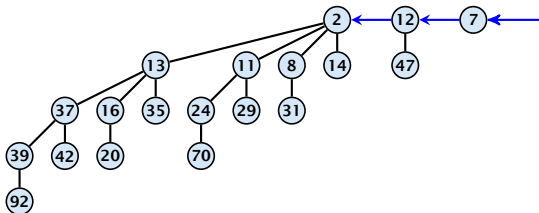
Binomial Heap

Properties of a heap with n keys:

▶ Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.

▶ The heap contains tree B_i iff $b_i = 1$.

▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.

▶ The minimum must be contained in one of the roots.

▶ The height of the largest tree is at most $\lfloor \log n \rfloor$.

▶ The trees are stored in a single-linked list; ordered by dimension/size.
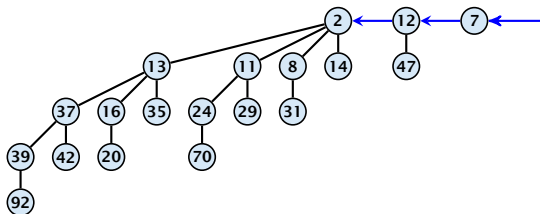
Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.
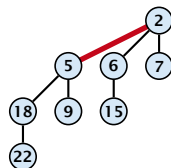
Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.

tikzpicture optimized away because it does not contribute to exported PDF]

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tik
tu

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

[tikzpicture optimized away because it does not contribute to exported PDF]

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tikzpic-
ture opti-
mized
away be-
cause
it does
not con-
tribute
to ex-
ported
PDF]

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tik
tu

14 4 19
16 22 39
29

2 14 40
3 19 90 5 17
9 18 21 27 30 94
46 13 26 42
54

4 19
14 39 40
17

19
40

tikzpic-
ture opti-
mized
away be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tikzpic-
ture
opti-
mized
away
be-
cause
it
does
not
con-
tribute
to ex-
ported
PDF]

tik
tu

[tikzpicture optimized away because it does not contribute to exported PDF]

tikzpicture optimized away because it does not contribute to exported PDF]

8.2 Binomial Heaps

S_1. merge(S_2):

- ▶ Analogous to binary addition.

8.2 Binomial Heaps

S_1. merge(S_2):

- ▶ Analogous to binary addition.
- ▶ Time is proportional to the number of trees in both heaps.

8.2 Binomial Heaps

$S_1.$ **merge**(S_2):

- ▶ Analogous to binary addition.
- ▶ Time is proportional to the number of trees in both heaps.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

All other operations can be reduced to merge().

$S.\ \text{insert}(x)$:

▶ Create a new heap S' that contains just the element x.

8.2 Binomial Heaps

All other operations can be reduced to merge().

$S.\,\mathrm{insert}(x)$:

- ▶ Create a new heap S' that contains just the element x.
- ▶ Execute $S.\,\mathrm{merge}(S')$.

8.2 Binomial Heaps

All other operations can be reduced to merge().

$S.$ **insert(x):**

▶ Create a new heap S' that contains just the element x.

▶ Execute $S.\,\text{merge}(S')$.

▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. minimum():

▶ Find the minimum key-value among all roots.

▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. **delete-min():**

8.2 Binomial Heaps

S. **delete-min():**

▶ Find the minimum key-value among all roots.

8.2 Binomial Heaps

S. **delete-min():**

- ▶ Find the minimum key-value among all roots.
- ▶ Remove the corresponding tree $T_{\min}$ from the heap.

8.2 Binomial Heaps

S. **delete-min()**:

- ▶ Find the minimum key-value among all roots.
- ▶ Remove the corresponding tree $T_{\min}$ from the heap.
- ▶ Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).

8.2 Binomial Heaps

S. delete-min():

- ▶ Find the minimum key-value among all roots.
- ▶ Remove the corresponding tree $T_{\min}$ from the heap.
- ▶ Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- ▶ Compute $S.\text{merge}(S')$.

8.2 Binomial Heaps

S. delete-min():

- ▶ Find the minimum key-value among all roots.
- ▶ Remove the corresponding tree $T_{\min}$ from the heap.
- ▶ Create a new heap S' that contains the trees obtained from $T_{\min}$ after deleting the root (note that these are just $\mathcal{O}(\log n)$ trees).
- ▶ Compute $S.\,\mathrm{merge}(S')$.
- ▶ Time: $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

$S.$ **decrease-key(handle h):**

8.2 Binomial Heaps

$S.$ **decrease-key(handle h):**

▶ Decrease the key of the element pointed to by h.

8.2 Binomial Heaps

$S.$ **decrease-key(handle h):**

- ▶ Decrease the key of the element pointed to by h.
- ▶ Bubble the element up in the tree until the heap property is fulfilled.

8.2 Binomial Heaps

S. **decrease-key(handle h):**

- ▶ Decrease the key of the element pointed to by h.
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time: $\mathcal{O}(\log n)$ since the trees have height $\mathcal{O}(\log n)$.

8.2 Binomial Heaps

S. **delete(handle h):**

8.2 Binomial Heaps

S. delete(handle h):

▶ Execute S. decrease-key$(h, -\infty)$.

8.2 Binomial Heaps

S. delete(handle h):

▶ Execute S. decrease-key$(h, -\infty)$.

▶ Execute S. delete-min$()$.

8.2 Binomial Heaps

S. delete(handle h):

- ▶ Execute S. decrease-key$(h, -\infty)$.
- ▶ Execute S. delete-min().
- ▶ Time: $\mathcal{O}(\log n)$.

8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.
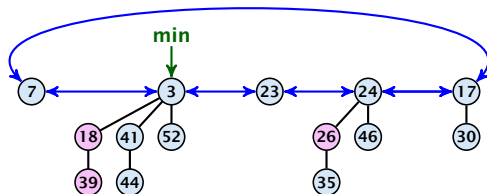
8.3 Fibonacci Heaps

Additional implementation details:

▶ Every node x stores its degree in a field $x.\text{degree}$. Note that this can be updated in constant time when adding a child to x.

▶ Every node stores a boolean value $x.\text{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

▶ $t(S)$ denotes the number of trees in the heap.

▶ $m(S)$ denotes the number of marked nodes.

▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.
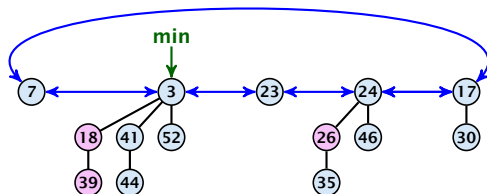
The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen "big enough" (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

$S.$ **minimum()**

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps
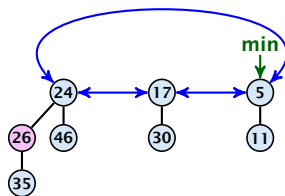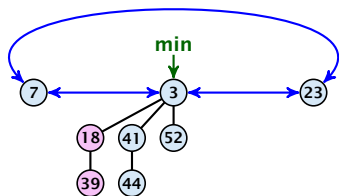
$S.\,\mathbf{merge}(S')$

▶ Merge the root lists.

▶ Adjust the min-pointer

8.3 Fibonacci Heaps

S. merge(S')

▶ Merge the root lists.

▶ Adjust the min-pointer
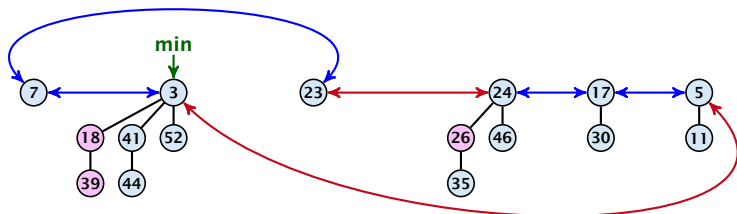
Running time:

▶ Actual cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\,\mathbf{merge}(S')$

▶ Merge the root lists.

▶ Adjust the min-pointer
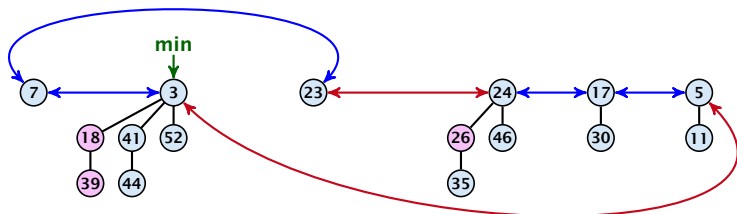
Running time:

▶ Actual cost $\mathcal{O}(1)$.

▶ No change in potential.

8.3 Fibonacci Heaps

$S.\,\mathbf{merge}(S')$

▶ Merge the root lists.

▶ Adjust the min-pointer
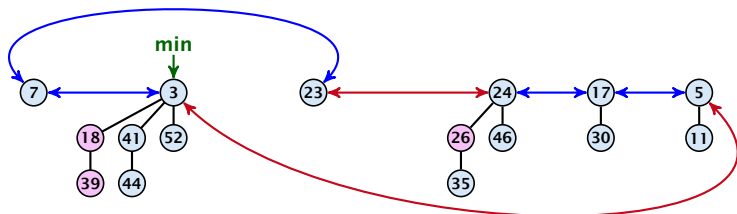
Running time:

▶ Actual cost $\mathcal{O}(1)$.

▶ No change in potential.

▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\text{insert}(x)$

- ▶ Create a new tree containing x.
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.
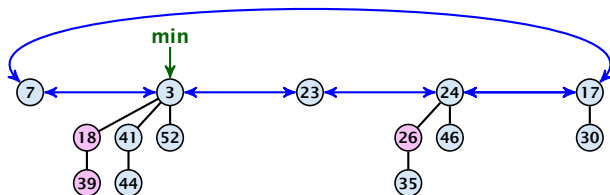
8.3 Fibonacci Heaps

$S.\,\mathbf{insert}(x)$

- ▶ Create a new tree containing x.
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.
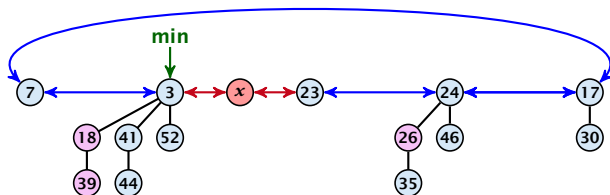
8.3 Fibonacci Heaps

$S.\, \text{insert}(x)$

▶ Create a new tree containing x.
▶ Insert x into the root-list.
▶ Update min-pointer, if necessary.
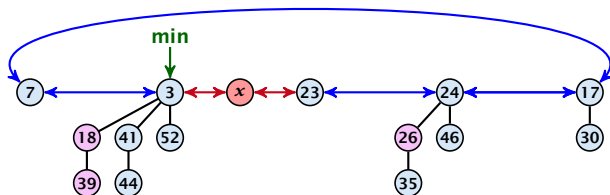
Running time:

▶ Actual cost $\mathcal{O}(1)$.
▶ Change in potential is $+1$.
▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps
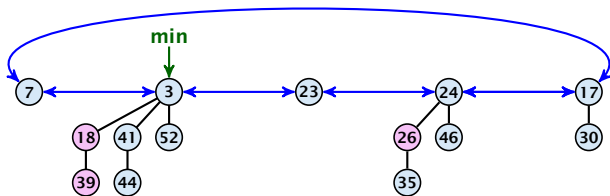
$S.$ **delete-min(x)**

8.3 Fibonacci Heaps
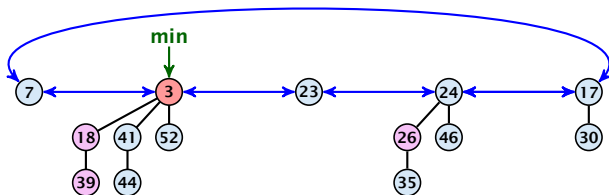
$S.$ delete-min(x)

▶ Delete minimum; add child-trees to heap;
time: $D(\min) \cdot \mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. **delete-min(*x*)**

▶ Delete minimum; add child-trees to heap;
   time: $D(\min) \cdot \mathcal{O}(1)$.

▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.
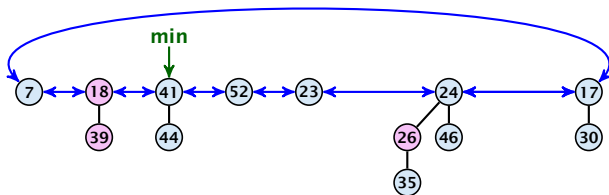
8.3 Fibonacci Heaps

$S.$ **delete-min**(x)

▶ Delete minimum; add child-trees to heap;
  time: $D(\min) \cdot \mathcal{O}(1)$.

▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.
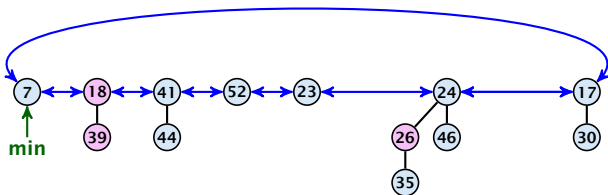
8.3 Fibonacci Heaps

$S.$ delete-min(x)

▶ Delete minimum; add child-trees to heap;
time: $D(\min) \cdot \mathcal{O}(1)$.

▶ Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.
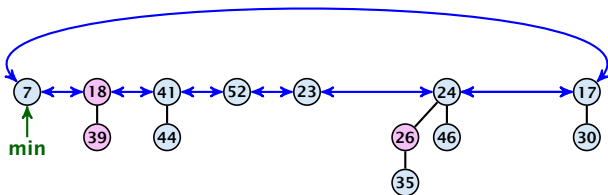
▶ Consolidate root-list so that no roots have the same degree.
Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Consolidate:

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \le D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \le D_n + 1 - t$;

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
   Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

▶ The amortized cost is

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

▶ The amortized cost is

  $c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
   Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

▶ The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
$$\leq (c_1 + c)D_n + (c_1 - c)t + c$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \le D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \le D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

▶ The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
$$\le (c_1 + c)D_n + (c_1 - c)t + c \le 2c(D_n + 1)$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \leq D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

▶ The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
$$\leq (c_1 + c)D_n + (c_1 - c)t + c \leq 2c(D_n + 1) \leq \mathcal{O}(D_n)$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

▶ At most $D_n + t$ elements in root-list before consolidate.

▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
  Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

▶ $t' \le D_n + 1$ as degrees are different after consolidating.

▶ Therefore $\Delta\Phi \le D_n + 1 - t$;

▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.

▶ The amortized cost is

$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$
$$\le (c_1 + c)D_n + (c_1 - c)t + c \le 2c(D_n + 1) \le \mathcal{O}(D_n)$$

for $c \ge c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$.

Case 1: decrease-key does not violate heap-property

▶ Just decrease the key-value of element referenced by h. Nothing else to do.

Case 1: decrease-key does not violate heap-property

▶ Just decrease the key-value of element referenced by h.
  Nothing else to do.

Fibonacci Heaps: decrease-key(handle h, v)

Case 1: decrease-key does not violate heap-property

▶ Just decrease the key-value of element referenced by h.
  Nothing else to do.

Fibonacci Heaps: decrease-key(handle h, v)

Case 1: decrease-key does not violate heap-property

▶ Just decrease the key-value of element referenced by h. Nothing else to do.

Fibonacci Heaps: decrease-key(handle h, v)

Case 2: heap-property is violated, but parent is not marked

▶ Decrease key-value of element x reference by h.

▶ If the heap-property is violated, cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Mark the (previous) parent of x (unless it's a root).

Fibonacci Heaps: decrease-key(handle h, v)

Case 2: heap-property is violated, but parent is not marked

▶ Decrease key-value of element x reference by h.

▶ If the heap-property is violated, cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Mark the (previous) parent of x (unless it's a root).

Fibonacci Heaps: decrease-key(handle h, v)

Case 2: heap-property is violated, but parent is not marked

▶ Decrease key-value of element x reference by h.
▶ If the heap-property is violated, cut the parent edge of x, and make x into a root.
▶ Adjust min-pointers, if necessary.
▶ Mark the (previous) parent of x (unless it's a root).

Fibonacci Heaps: decrease-key(handle h, v)

Case 2: heap-property is violated, but parent is not marked

▶ Decrease key-value of element x reference by h.

▶ If the heap-property is violated, cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Mark the (previous) parent of x (unless it's a root).

Fibonacci Heaps: decrease-key(handle h, v)

Case 2: heap-property is violated, but parent is not marked

▶ Decrease key-value of element x reference by h.

▶ If the heap-property is violated, cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Mark the (previous) parent of x (unless it's a root).

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h.
- ▶ Cut the parent edge of x, and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Continue cutting the parent until you arrive at an unmarked node.

Fibonacci Heaps: decrease-key(handle h, v)

Case 3: heap-property is violated, and parent is marked

▶ Decrease key-value of element x reference by h.

▶ Cut the parent edge of x, and make x into a root.

▶ Adjust min-pointers, if necessary.

▶ Execute the following:

$p \leftarrow \mathrm{parent}[x]$;
while (p is marked)
$\quad pp \leftarrow \mathrm{parent}[p]$;
$\quad$ cut of p; make it into a root; unmark it;
$\quad p \leftarrow pp$;
if p is unmarked and not a root mark it;

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

▶ Constant cost for decreasing the value.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

▶ Constant cost for decreasing the value.

▶ Constant cost for each of ℓ cuts.

▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \le m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \le m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \le \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

▶ Constant cost for decreasing the value.

▶ Constant cost for each of ℓ cuts.

▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

▶ $t' = t + \ell$, as every cut creates one new root.

▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.

▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$

▶ Amortized cost is at most

$$c_2(\ell + 1) + c(4 - \ell)$$

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

  $$c_2(\ell+1) + c(4-\ell) \leq (c_2-c)\ell + 4c + c_2$$

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

$$c_2(\ell+1) + c(4-\ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$

  if $c \geq c_2$.

Delete node

$H. \textbf{delete}(x)$:

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 32

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \textit{if } i = 1 \\ i - 2 & \textit{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

▶ When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.

8.3 Fibonacci Heaps

Proof

▶ When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.

▶ Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.

8.3 Fibonacci Heaps

Proof

▶ When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.

▶ Hence, at this time $\mathrm{degree}(x) \geq i - 1$, and therefore also $\mathrm{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.

▶ Since, then y_i has lost at most one child.

8.3 Fibonacci Heaps

Proof

▶ When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.

▶ Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.

▶ Since, then y_i has lost at most one child.

▶ Therefore, $\text{degree}(y_i) \geq i - 2$.

8.3 Fibonacci Heaps

▶ Let s_k be the minimum possible size of a sub-tree rooted at a
node of degree k that can occur in a Fibonacci heap.

8.3 Fibonacci Heaps

▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.

▶ s_k monotonically increases with k

8.3 Fibonacci Heaps

▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.

▶ s_k monotonically increases with k

▶ $s_0 = 1$ and $s_1 = 2$.

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let $y_1, \ldots, y_k$ be its children.

$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$

8.3 Fibonacci Heaps

▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.

▶ s_k monotonically increases with k

▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let $y_1, \ldots, y_k$ be its children.

$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$

$$\geq 2 + \sum_{i=2}^{k} s_{i-2}$$

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let $y_1, \ldots, y_k$ be its children.

$$
\begin{aligned}
s_k &= 2 + \sum_{i=2}^{k} \text{size}(y_i) \\
&\geq 2 + \sum_{i=2}^{k} s_{i-2} \\
&= 2 + \sum_{i=0}^{k-2} s_i
\end{aligned}
$$

8.3 Fibonacci Heaps

Definition 33

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0:	$1 = F_0 \geq \Phi^0 = 1$
k=1:	$2 = F_1 \geq \Phi^1 \approx 1.61$
k-2,k-1→ k:	$F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi+1)}^{\Phi^2} = \Phi^k$

k=2:	$3 = F_2 = 2 + 1 = 2 + F_0$
k-1→ k:	$F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$

9 Union Find

Union Find Data Structure $\mathcal{P}$**:** Maintains a partition of disjoint sets over elements.

9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

▶ $\mathcal{P}.\operatorname{makeset}(x)$: Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.

9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- ▶ $\mathcal{P}.\,\textbf{makeset}(x)$: Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.

- ▶ $\mathcal{P}.\,\textbf{find}(x)$: Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.

9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- ▶ $\mathcal{P}.\operatorname{makeset}(x)$: Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.

- ▶ $\mathcal{P}.\operatorname{find}(x)$: Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.

- ▶ $\mathcal{P}.\operatorname{union}(x, y)$: Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

9 Union Find

Applications:

- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- ▶ Kruskals Minimum Spanning Tree Algorithm

9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

1: $A \leftarrow \emptyset$;
2: **for all** $v \in V$ **do**
3:      $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$
4: sort edges in non-decreasing order of weight w
5: **for all** $(u, v) \in E$ in non-decreasing order **do**
6:      **if** $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$ **then**
7:          $A \leftarrow A \cup \{(u, v)\}$
8:          $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$

List Implementation

▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

List Implementation

▶ The elements of a set are stored in a list; each node has a
  backward pointer to the head.
▶ The head of the list contains the identifier for the set and a
  field that stores the size of the set.

List Implementation

▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

▶ The head of the list contains the identifier for the set and a field that stores the size of the set.
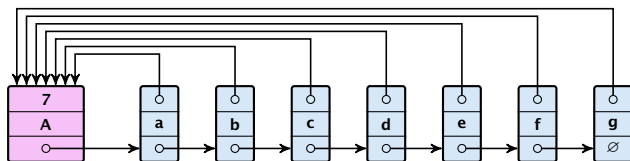
▶ makeset(x) can be performed in constant time.

List Implementation

▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

▶ The head of the list contains the identifier for the set and a field that stores the size of the set.
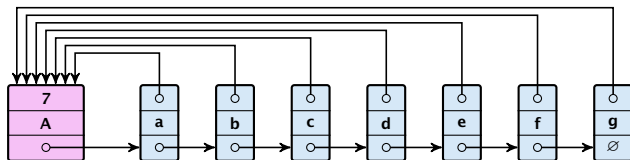
▶ makeset(x) can be performed in constant time.

▶ find(x) can be performed in constant time.

List Implementation

union(*x*, *y*)
- ▶ Determine sets S_x and S_y.

List Implementation

union(*x*, *y*)

▶ Determine sets S_x and S_y.

▶ Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.

List Implementation

union(x, y)

- ▶ Determine sets S_x and S_y.
- ▶ Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
- ▶ Insert list S_y at the head of S_x.

List Implementation

union(x, y)

- ▶ Determine sets S_x and S_y.
- ▶ Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
- ▶ Insert list S_y at the head of S_x.
- ▶ Adjust the size-field of list S_x.

List Implementation

union(x, y)

- ▶ Determine sets S_x and S_y.
- ▶ Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
- ▶ Insert list S_y at the head of S_x.
- ▶ Adjust the size-field of list S_x.
- ▶ Time: $\min\{|S_x|, |S_y|\}$.

List Implementation
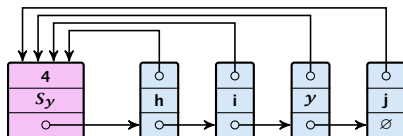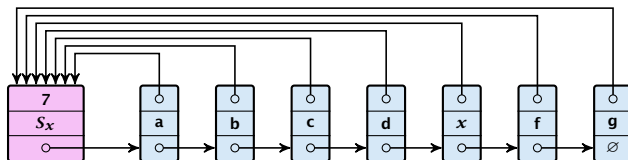
List Implementation
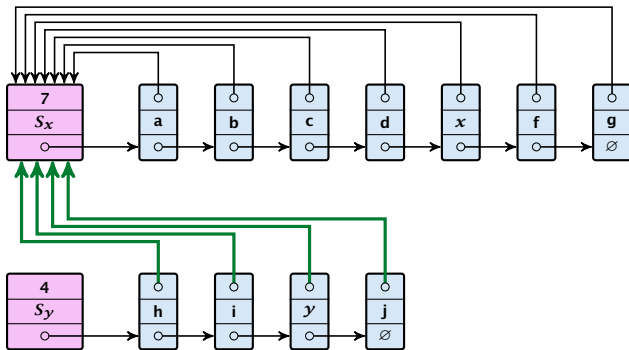
List Implementation
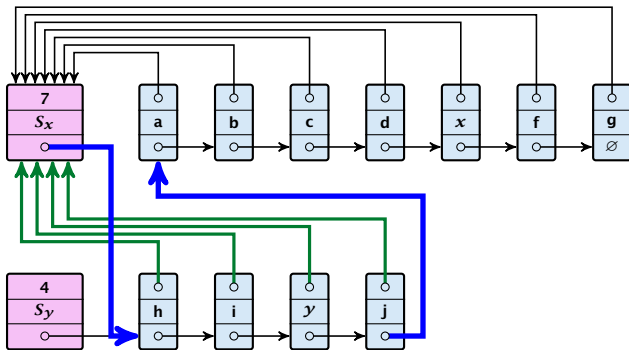
List Implementation
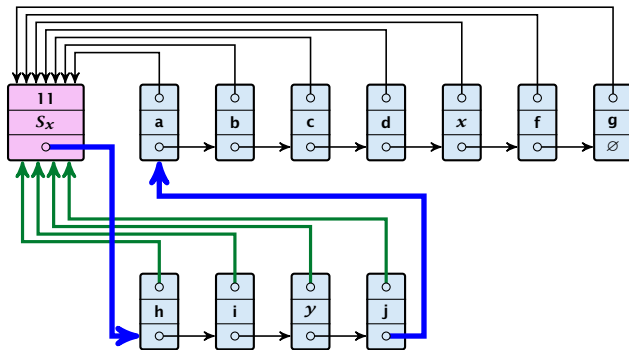
List Implementation

Running times:

▶ find(x): constant

▶ makeset(x): constant

▶ union(x, y): $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 34

The list implementation for the ADT union find fulfills the following amortized time bounds:

- find(x): $\mathcal{O}(1)$.
- makeset(x): $\mathcal{O}(\log n)$.
- union(x, y): $\mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

▶ There is a bank account for every element in the data structure.

The Accounting Method for Amortized Time Bounds

▶ There is a bank account for every element in the data structure.

▶ Initially the balance on all accounts is zero.

The Accounting Method for Amortized Time Bounds

- ▶ There is a bank account for every element in the data structure.

- ▶ Initially the balance on all accounts is zero.

- ▶ Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.

The Accounting Method for Amortized Time Bounds

▶ There is a bank account for every element in the data structure.

▶ Initially the balance on all accounts is zero.

▶ Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.

▶ Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.

The Accounting Method for Amortized Time Bounds

▶ There is a bank account for every element in the data structure.

▶ Initially the balance on all accounts is zero.

▶ Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.

▶ Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.

▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.

List Implementation

▶ For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.

▶ In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).

List Implementation

- ▶ For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- ▶ In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- ▶ For each element a makeset operation occurs as the first operation involving this element.

List Implementation

▶ For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.

▶ In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).

▶ For each element a makeset operation occurs as the first operation involving this element.

▶ We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.

List Implementation

▶ For an operation whose actual cost exceeds the amortized cost we charge the **excess** to the elements involved.

▶ In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).

▶ For each element a makeset operation occurs as the first operation involving this element.

▶ We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.

▶ Later operations charge the account but the balance never drops below zero.

List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

▶ If $S_x = S_y$ the cost is constant; no bank accounts change.

List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

▶ If $S_x = S_y$ the cost is constant; no bank accounts change.

▶ Otw. the actual cost is $\mathcal{O}(\min\{|S_x|, |S_y|\})$.

List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

▶ If $S_x = S_y$ the cost is constant; no bank accounts change.

▶ Otw. the actual cost is $\mathcal{O}(\min\{|S_x|, |S_y|\})$.

▶ Assume wlog. that S_x is the smaller set; let c denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.

List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

▶ If $S_x = S_y$ the cost is constant; no bank accounts change.

▶ Otw. the actual cost is $\mathcal{O}(\min\{|S_x|, |S_y|\})$.

▶ Assume wlog. that S_x is the smaller set; let c denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.

▶ Charge c to every element in set S_x.

List Implementation

Lemma 35

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

Harald Räcke

List Implementation

Lemma 35

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

Implementation via Trees

- ▶ Maintain nodes of a set in a tree.
- ▶ The root of the tree is the label of the set.
- ▶ Only pointer to parent exists; we cannot list all elements of a given set.

Implementation via Trees

▶ Maintain nodes of a set in a tree.

▶ The root of the tree is the label of the set.

▶ Only pointer to parent exists; we cannot list all elements of a given set.

▶ Example:
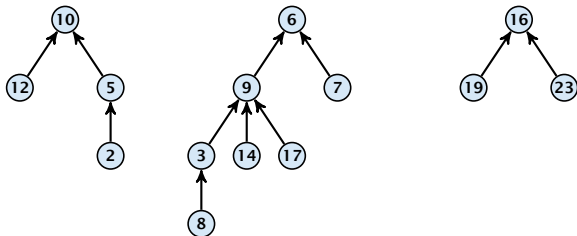
Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

Implementation via Trees

makeset(x)

- ▶ Create a singleton tree. Return pointer to the root.

Implementation via Trees

makeset(x)

- ▶ Create a singleton tree. Return pointer to the root.
- ▶ Time: $\mathcal{O}(1)$.

Implementation via Trees

makeset(x)

- ▶ Create a singleton tree. Return pointer to the root.
- ▶ Time: $\mathcal{O}(1)$.

find(x)

- ▶ Start at element x in the tree. Go upwards until you reach the root.

Implementation via Trees

makeset(*x*)

- ▶ Create a singleton tree. Return pointer to the root.
- ▶ Time: $\mathcal{O}(1)$.

find(*x*)

- ▶ Start at element x in the tree. Go upwards until you reach the root.
- ▶ Time: $\mathcal{O}(\text{level}(x))$, where $\text{level}(x)$ is the distance of element x to the root in its tree. Not constant.

Implementation via Trees

To support union we store the size of a tree in its root.

Implementation via Trees

To support union we store the size of a tree in its root.

union(x, y)

▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.

Implementation via Trees

To support union we store the size of a tree in its root.

union(x, y)

▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.

▶ $\text{link}(a, b)$ attaches the smaller tree as the child of the larger.

Implementation via Trees

To support union we store the size of a tree in its root.

union(x, y)

- ▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.
- ▶ $\text{link}(a, b)$ attaches the smaller tree as the child of the larger.
- ▶ In addition it updates the size-field of the new root.

Implementation via Trees

To support union we store the size of a tree in its root.

union(x, y)

▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.

▶ $\text{link}(a, b)$ attaches the smaller tree as the child of the larger.

▶ In addition it updates the size-field of the new root.

Implementation via Trees

To support union we store the size of a tree in its root.

union(x, y)

▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.

▶ $\text{link}(a, b)$ attaches the smaller tree as the child of the larger.

▶ In addition it updates the size-field of the new root.

Implementation via Trees

To support union we store the size of a tree in its root.

union(x, y)

- ▶ Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.
- ▶ $\text{link}(a, b)$ attaches the smaller tree as the child of the larger.
- ▶ In addition it updates the size-field of the new root.

- ▶ Time: constant for $\text{link}(a, b)$ plus two find-operations.

Implementation via Trees

Lemma 36

The running time (non-amortized!!!) for $\mathrm{find}(x)$ is $\mathcal{O}(\log n)$.

Implementation via Trees

Lemma 36
The running time (non-amortized!!!) for $\mathrm{find}(x)$ is $\mathcal{O}(\log n)$.

Proof.

▶ When we attach a tree with root c to become a child of a tree with root p, then $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, where size denotes the value of the size-field right after the operation.

Implementation via Trees

Lemma 36

The running time (non-amortized!!!) for $\mathrm{find}(x)$ *is* $\mathcal{O}(\log n)$.

Proof.

▶ When we attach a tree with root c to become a child of a tree with root p, then $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, where size denotes the value of the size-field right after the operation.

▶ After that the value of $\mathrm{size}(c)$ stays fixed, while the value of $\mathrm{size}(p)$ may still increase.

Implementation via Trees

Lemma 36

The running time (non-amortized!!!) for $\mathrm{find}(x)$ is $\mathcal{O}(\log n)$.

Proof.

- ▶ When we attach a tree with root c to become a child of a tree with root p, then $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, where size denotes the value of the size-field right after the operation.

- ▶ After that the value of $\mathrm{size}(c)$ stays fixed, while the value of $\mathrm{size}(p)$ may still increase.

- ▶ Hence, at any point in time a tree fulfills $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, for any pair of nodes (p, c), where p is a parent of c.

Implementation via Trees

Lemma 36
The running time (non-amortized!!!) for $\mathrm{find}(x)$ is $\mathcal{O}(\log n)$.

Proof.

▶ When we attach a tree with root c to become a child of a tree with root p, then $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, where size denotes the value of the size-field right after the operation.

▶ After that the value of $\mathrm{size}(c)$ stays fixed, while the value of $\mathrm{size}(p)$ may still increase.

▶ Hence, at any point in time a tree fulfills $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, for any pair of nodes (p, c), where p is a parent of c.

$\square$

Path Compression

find(x):

▶ Go upward until you find the root.

Path Compression

find(x):

- ▶ Go upward until you find the root.
- ▶ Re-attach all visited nodes as children of the root.

Path Compression

find(x):

- ▶ Go upward until you find the root.
- ▶ Re-attach all visited nodes as children of the root.
- ▶ Speeds up successive find-operations.

Path Compression

find(x):

▶ Go upward until you find the root.

▶ Re-attach all visited nodes as children of the root.

▶ Speeds up successive find-operations.

Path Compression

find(*x*):

- ▶ Go upward until you find the root.
- ▶ Re-attach all visited nodes as children of the root.
- ▶ Speeds up successive find-operations.

Path Compression

find(x):

▶ Go upward until you find the root.

▶ Re-attach all visited nodes as children of the root.

▶ Speeds up successive find-operations.

Path Compression

find(x):

▶ Go upward until you find the root.

▶ Re-attach all visited nodes as children of the root.

▶ Speeds up successive find-operations.

Path Compression

find(x):

- ▶ Go upward until you find the root.
- ▶ Re-attach all visited nodes as children of the root.
- ▶ Speeds up successive find-operations.

Path Compression

find(x):

▶ Go upward until you find the root.

▶ Re-attach all visited nodes as children of the root.

▶ Speeds up successive find-operations.

Path Compression

find(x):

▶ Go upward until you find the root.

▶ Re-attach all visited nodes as children of the root.

▶ Speeds up successive find-operations.

▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

Amortized Analysis

Definitions:

Amortized Analysis

Definitions:

▶ $\text{size}(v) \coloneqq$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

Note that this is the same as the size of v's subtree in the case that there are no find-operations.

Amortized Analysis

Definitions:

▶ $\mathrm{size}(v) \coloneqq$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find-operations.

▶ $\mathrm{rank}(v) \coloneqq \lfloor \log(\mathrm{size}(v)) \rfloor$.

Amortized Analysis

Definitions:

▶ $\text{size}(v) \coloneqq$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

Note that this is the same as the size of v's subtree in the case that there are no find-operations.

▶ $\text{rank}(v) \coloneqq \lfloor \log(\text{size}(v)) \rfloor$.

▶ $\implies \text{size}(v) \geq 2^{\text{rank}(v)}$.

Amortized Analysis

Definitions:

- ▶ $\text{size}(v) :=$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find-operations.

- ▶ $\text{rank}(v) := \lfloor \log(\text{size}(v)) \rfloor$.

- ▶ $\implies \text{size}(v) \geq 2^{\text{rank}(v)}$.

Lemma 37

The rank of a parent must be strictly larger than the rank of a child.

Amortized Analysis

Lemma 38

There are at most $n/2^s$ nodes of rank s.

Amortized Analysis

Lemma 38

There are at most $n/2^s$ nodes of rank s.

Proof.

▶ Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.

Amortized Analysis

Lemma 38
There are at most $n/2^s$ nodes of rank s.

Proof.

- Let's say a node v **sees** node x if v is in x's sub-tree at the time that x becomes a child.

- A node v sees at most one node of rank s during the running time of the algorithm.

□

Amortized Analysis

Lemma 38

There are at most $n/2^s$ nodes of rank s.

Proof.

► Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.

► A node v sees at most one node of rank s during the running time of the algorithm.

► This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

□

Amortized Analysis

Lemma 38

There are at most $n/2^s$ nodes of rank s.

Proof.

▶ Let's say a node v *sees* node x if v is in x's sub-tree at the time that x becomes a child.

▶ A node v sees at most one node of rank s during the running time of the algorithm.

▶ This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.

▶ Hence, every node *sees* at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

Amortized Analysis

We define

$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases}$$

Amortized Analysis

We define

$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases} \qquad \text{tow}(i) = 2^{2^{2^{2^{2^2}}}} \Big\} i \text{ times}$$

Amortized Analysis

We define

$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases} \qquad \text{tow}(i) = 2^{2^{2^{2^2}}} \Big\} i \text{ times}$$

and

$$\log^*(n) := \min\{i \mid \text{tow}(i) \geq n\} \ .$$

Amortized Analysis

We define

$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases} \qquad \text{tow}(i) = \left. 2^{2^{2^{2^{2}}}} \right\} i \text{ times}$$

and

$$\log^*(n) := \min\{i \mid \text{tow}(i) \geq n\} \ .$$

Theorem 39
Union find with path compression fulfills the following amortized running times:

▶ $\text{makeset}(x) : \mathcal{O}(\log^*(n))$

▶ $\text{find}(x) : \mathcal{O}(\log^*(n))$

▶ $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

Amortized Analysis

In the following we assume $n \geq 2$.

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

▶ A node with rank $\mathrm{rank}(v)$ is in rank group $\log^*(\mathrm{rank}(v))$.

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

▶ A node with rank $\mathrm{rank}(v)$ is in rank group $\log^*(\mathrm{rank}(v))$.

▶ The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

▶ A node with rank $\mathrm{rank}(v)$ is in rank group $\log^*(\mathrm{rank}(v))$.

▶ The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.

▶ A rank group $g \geq 1$ contains ranks $\mathrm{tow}(g-1) + 1, \ldots, \mathrm{tow}(g)$.

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

▶ A node with rank $\text{rank}(v)$ is in rank group $\log^*(\text{rank}(v))$.

▶ The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.

▶ A rank group $g \geq 1$ contains ranks $\text{tow}(g-1) + 1, \ldots, \text{tow}(g)$.

▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

▶ A node with rank $\mathrm{rank}(v)$ is in rank group $\log^*(\mathrm{rank}(v))$.

▶ The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.

▶ A rank group $g \geq 1$ contains ranks $\mathrm{tow}(g-1) + 1, \ldots, \mathrm{tow}(g)$.

▶ The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).

▶ Hence, the total number of rank-groups is at most $\log^* n$.

Amortized Analysis

Amortized Analysis

Accounting Scheme:

▶ create an account for every find-operation

Amortized Analysis

Accounting Scheme:

- ▶ create an account for every find-operation
- ▶ create an account for every node v

Amortized Analysis

Accounting Scheme:

- ▶ create an account for every find-operation
- ▶ create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\text{parent}[v]$ as follows:

Amortized Analysis

Accounting Scheme:

- ▶ create an account for every find-operation
- ▶ create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\text{parent}[v]$ as follows:

- ▶ If $\text{parent}[v]$ is the root we charge the cost to the find-account.

Amortized Analysis

Accounting Scheme:

- ▶ create an account for every find-operation
- ▶ create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to parent$[v]$ as follows:

- ▶ If parent$[v]$ is the root we charge the cost to the find-account.

- ▶ If the group-number of rank(v) is the same as that of rank(parent$[v]$) (before starting path compression) we charge the cost to the node-account of v.

Amortized Analysis

Accounting Scheme:

▶ create an account for every find-operation

▶ create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\mathrm{parent}[v]$ as follows:

▶ If $\mathrm{parent}[v]$ is the root we charge the cost to the find-account.

▶ If the group-number of $\mathrm{rank}(v)$ is the same as that of $\mathrm{rank}(\mathrm{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.

▶ Otherwise we charge the cost to the find-account.

Amortized Analysis

Observations:

Amortized Analysis

Observations:

▶ A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).

Amortized Analysis

Observations:

▶ A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).

▶ After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.

Amortized Analysis

Observations:

▶ A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).

▶ After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.

▶ After some charges to v the parent will be in a larger rank-group. $\implies v$ will never be charged again.

Amortized Analysis

Observations:

▶ A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).

▶ After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.

▶ After some charges to v the parent will be in a larger rank-group. $\implies v$ will never be charged again.

▶ The total charge made to a node in rank-group g is at most $\text{tow}(g) - \text{tow}(g-1) - 1 \le \text{tow}(g)$.

Amortized Analysis

What is the total charge made to nodes?

Amortized Analysis

What is the total charge made to nodes?

▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) \ ,$$

where $n(g)$ is the number of nodes in group g.

Amortized Analysis

For $g \geq 1$ we have

$$n(g)$$

Amortized Analysis

For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s}$$

Amortized Analysis

For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s}$$

Amortized Analysis

For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s}$$

$$= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s}$$

Amortized Analysis

For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s}$$

$$= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2$$

Amortized Analysis

For $g \geq 1$ we have

$$
\begin{aligned}
n(g) &\leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s} \\
&= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2 \\
&= \frac{n}{2^{\text{tow}(g-1)}}
\end{aligned}
$$

Amortized Analysis

For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s}$$

$$= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2$$

$$= \frac{n}{2^{\text{tow}(g-1)}} = \frac{n}{\text{tow}(g)} \ .$$

Amortized Analysis

For $g \geq 1$ we have

$$
\begin{aligned}
n(g) &\leq \sum_{s=\mathrm{tow}(g-1)+1}^{\mathrm{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\mathrm{tow}(g-1)+1}^{\infty} \frac{n}{2^s} \\
&= \frac{n}{2^{\mathrm{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\mathrm{tow}(g-1)+1}} \cdot 2 \\
&= \frac{n}{2^{\mathrm{tow}(g-1)}} = \frac{n}{\mathrm{tow}(g)} \ .
\end{aligned}
$$

Hence,

$$
\sum_g n(g) \, \mathrm{tow}(g)
$$

Amortized Analysis

For $g \geq 1$ we have

$$n(g) \leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s}$$

$$= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2$$

$$= \frac{n}{2^{\text{tow}(g-1)}} = \frac{n}{\text{tow}(g)} \ .$$

Hence,

$$\sum_{g} n(g) \, \text{tow}(g) \leq n(0) \, \text{tow}(0) + \sum_{g \geq 1} n(g) \, \text{tow}(g)$$

Amortized Analysis

For $g \geq 1$ we have

$$
\begin{aligned}
n(g) &\leq \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \leq \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s} \\
&= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2 \\
&= \frac{n}{2^{\text{tow}(g-1)}} = \frac{n}{\text{tow}(g)} \quad .
\end{aligned}
$$

Hence,

$$
\sum_g n(g)\,\text{tow}(g) \leq n(0)\,\text{tow}(0) + \sum_{g \geq 1} n(g)\,\text{tow}(g) \leq n \log^*(n)
$$

Amortized Analysis

Without loss of generality we can assume that all makeset-operations occur at the start.

Amortized Analysis

Without loss of generality we can assume that all makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

Amortized Analysis

Amortized Analysis

The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

Amortized Analysis

The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m, n))$, where $\alpha(m, n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

There is also a lower bound of $\Omega(\alpha(m, n))$.

Amortized Analysis

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

Amortized Analysis

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

▶ $A(0, y) = y + 1$
▶ $A(1, y) = y + 2$
▶ $A(2, y) = 2y + 3$
▶ $A(3, y) = 2^{y+3} - 3$
▶ $A(4, y) = \underbrace{2^{2^{2^{2}}}}_{y+3 \text{ times}} - 3$

Part IV

Flows and Cuts

The following slides are partially based on slides by Kevin Wayne.

10 Introduction
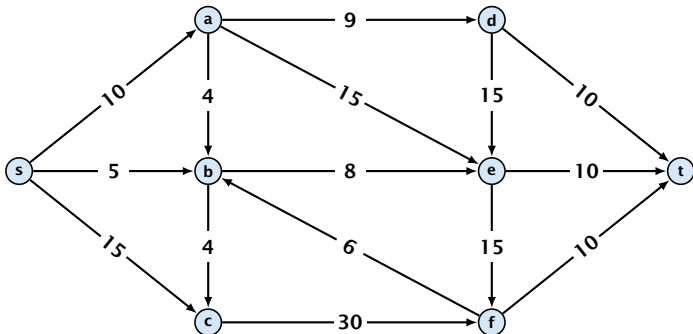
Flow Network

▶ directed graph $G = (V, E)$; edge capacities $c(e)$

10 Introduction

Flow Network

- ▶ directed graph $G = (V, E)$; edge capacities $c(e)$
- ▶ two special nodes: source s; target t;
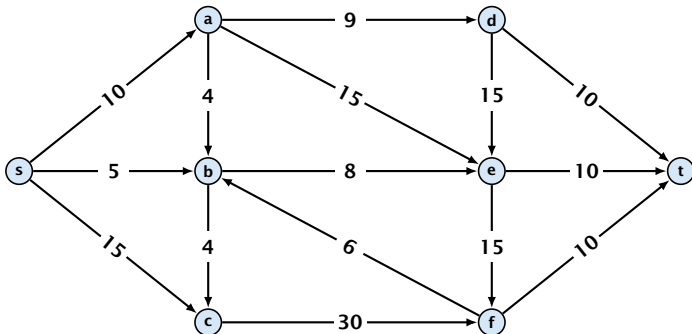
10 Introduction

Flow Network

- ▶ directed graph $G = (V, E)$; edge capacities $c(e)$
- ▶ two special nodes: source s; target t;
- ▶ no edges entering s or leaving t;
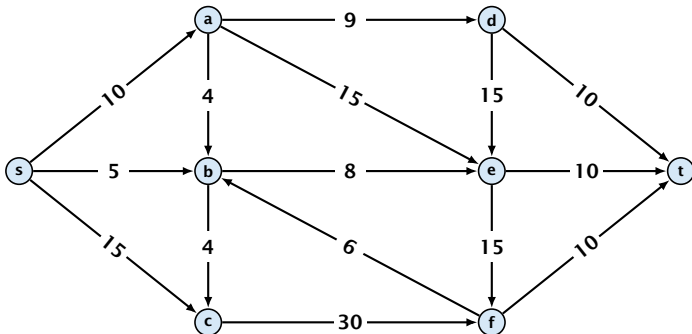
10 Introduction

Flow Network

▶ directed graph $G = (V, E)$; edge capacities $c(e)$

▶ two special nodes: source s; target t;

▶ no edges entering s or leaving t;

▶ at least for now: no parallel edges;
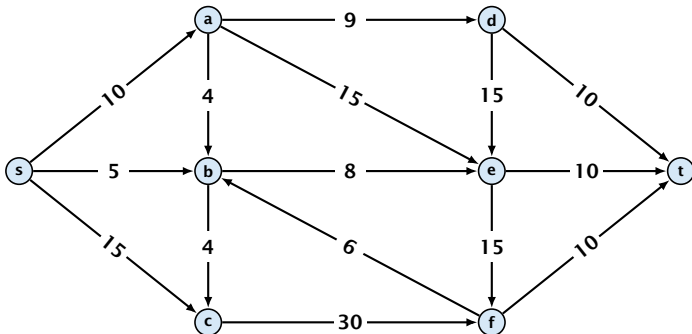
Cuts

Definition 40

An (s, t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

Cuts

Definition 40

An (s, t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

Definition 41

The capacity of a cut A is defined as

$$\text{cap}(A, V \setminus A) := \sum_{e \, \in \, \text{out}(A)} c(e) \ ,$$

where $\text{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

Cuts

Definition 40

An (s, t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

Definition 41

The capacity of a cut A is defined as

$$\mathrm{cap}(A, V \setminus A) := \sum_{e \,\in\, \mathrm{out}(A)} c(e) \ ,$$

where $\mathrm{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

Minimum Cut Problem: Find an (s, t)-cut with minimum capacity.

Cuts

Example 42
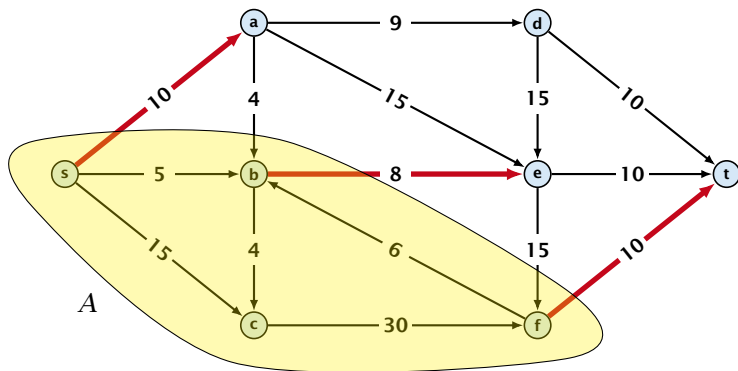
The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

Flows

Definition 43
An (s, t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
$$0 \leq f(e) \leq c(e) \ .$$

    (capacity constraints)

Flows

Definition 43

An (s, t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \ .$$

   (flow conservation constraints)

Flows

Definition 44

The value of an (s, t)-flow f is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) \; .$$

Flows

Definition 44
The value of an (s, t)-flow f is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) \ .$$

Maximum Flow Problem: Find an (s, t)-flow with maximum value.

Flows

Example 45
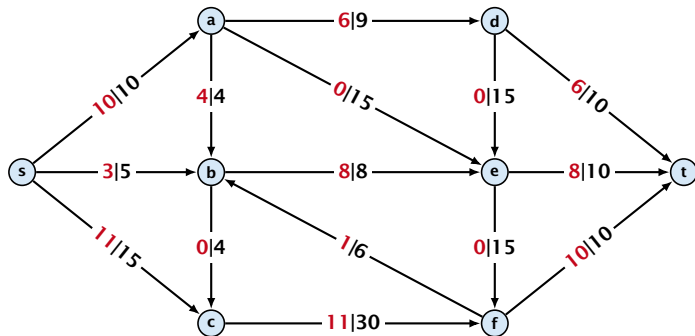
The value of the flow is $\mathrm{val}(f) = 24$.

Flows

Lemma 46 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s, t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \ .$$

Proof.

$\mathrm{val}(f)$

Proof.

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$

Proof.

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$

$$= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)$$

Proof.

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$

$$= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \overbrace{\left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)}^{= 0}$$

Proof.

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$

$$= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)$$

$$= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

Proof.

$$
\begin{aligned}
\mathrm{val}(f) &= \sum_{e \in \mathrm{out}(s)} f(e) \\
&= \sum_{e \in \mathrm{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \mathrm{out}(v)} f(e) - \sum_{e \in \mathrm{in}(v)} f(e) \right) \\
&= \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e)
\end{aligned}
$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. □

Example 47
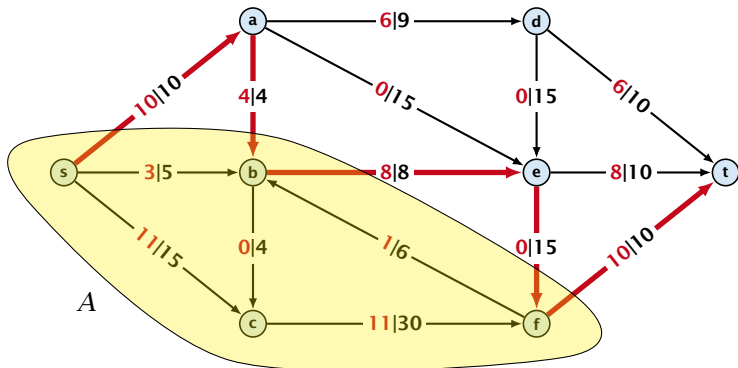
The net-flow across the cut is $\text{val}(f) = 24$.

Corollary 48

Let f be an (s, t)-flow and let A be an (s, t)-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Corollary 48

Let f be an (s, t)-flow and let A be an (s, t)-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Corollary 48

Let f be an (s, t)-flow and let A be an (s, t)-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

Corollary 48

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

$$\text{cap}(A, V \setminus A) < \text{val}(f')$$

□

Corollary 48

Let f be an (s, t)-flow and let A be an (s, t)-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

$$\text{cap}(A, V \setminus A) < \text{val}(f')$$
$$= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e)$$

$\square$

Corollary 48

Let f be an (s, t)-flow and let A be an (s, t)-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

$$\text{cap}(A, V \setminus A) < \text{val}(f')$$
$$= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e)$$
$$\leq \sum_{e \in \text{out}(A)} f'(e)$$

$\square$

Corollary 48

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

$$
\begin{aligned}
\text{cap}(A, V \setminus A) &< \text{val}(f') \\
&= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \\
&\leq \sum_{e \in \text{out}(A)} f'(e) \\
&\leq \text{cap}(A, V \setminus A)
\end{aligned}
$$

$\square$

11 Augmenting Path Algorithms

Greedy-algorithm:

- ▶ start with $f(e) = 0$ everywhere
- ▶ find an s-t path with $f(e) < c(e)$ on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible

flow value: 0

11 Augmenting Path Algorithms

Greedy-algorithm:

▶ start with $f(e) = 0$ everywhere

▶ find an s-t path with $f(e) < c(e)$ on every edge

▶ augment flow along the path

▶ repeat as long as possible
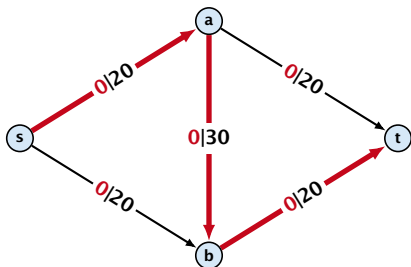
flow value: 0

11 Augmenting Path Algorithms

Greedy-algorithm:

▶ start with $f(e) = 0$ everywhere

▶ find an s-t path with $f(e) < c(e)$ on every edge

▶ augment flow along the path

▶ repeat as long as possible

flow value: 0

11 Augmenting Path Algorithms

Greedy-algorithm:

▶ start with $f(e) = 0$ everywhere

▶ find an s-t path with $f(e) < c(e)$ on every edge

▶ augment flow along the path

▶ repeat as long as possible

flow value: 20

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

▶ Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v.

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

▶ Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v.

▶ G_f has edge e'_1 with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e'_2 with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

▶ Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v.

▶ G_f has edge e_1' with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e_2' with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

G (u) ⟵ **5|20** ⟶ (v)
          6|10

G_f (u) ⟵ **21** ⟶ (v)
            9
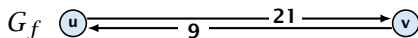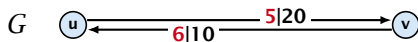
Augmenting Path Algorithm

Definition 49

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Augmenting Path Algorithm

Definition 49

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 1 FordFulkerson($G = (V, E, c)$)

1: Initialize $f(e) \leftarrow 0$ for all edges.

2: **while** $\exists$ augmenting path p in G_f **do**

3:     augment as much flow along p as possible.

Augmenting Paths
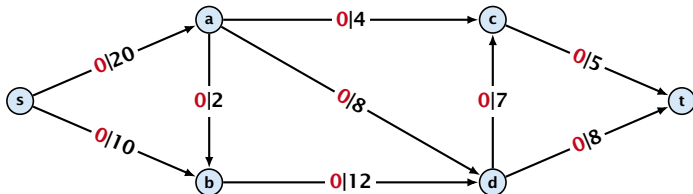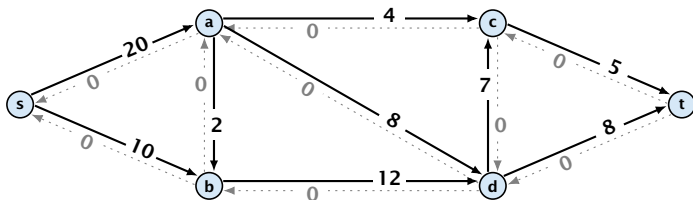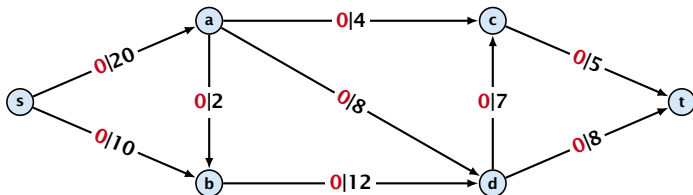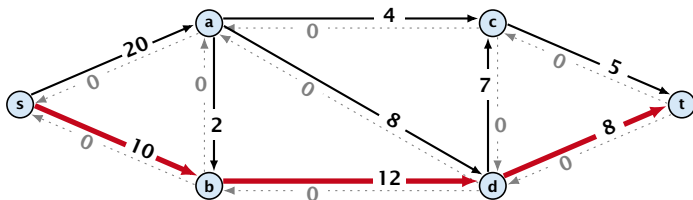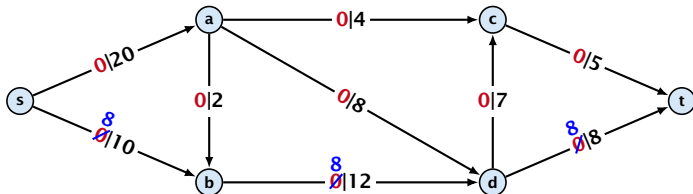
flow value: 0

Augmenting Paths

flow value: 0

Augmenting Paths
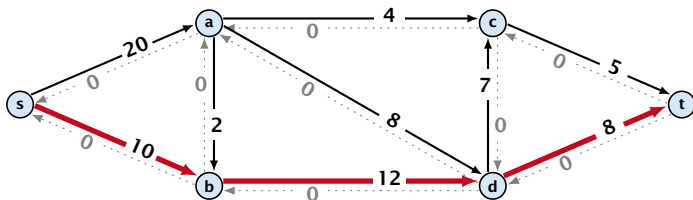
flow value: 0
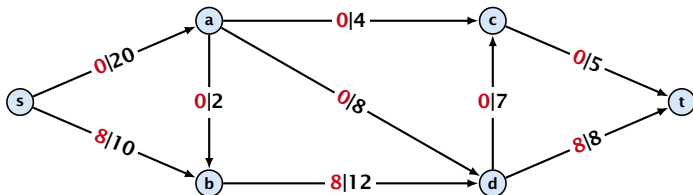
Augmenting Paths
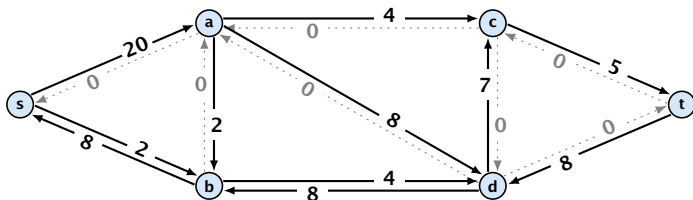
flow value: 8
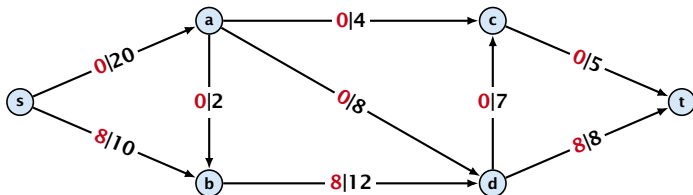
Augmenting Paths
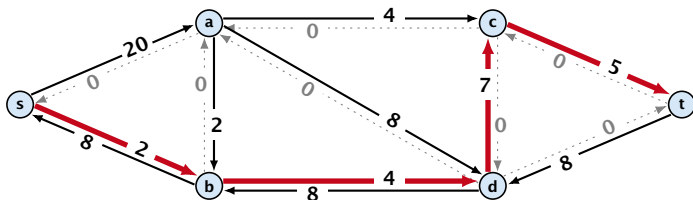
flow value: 8
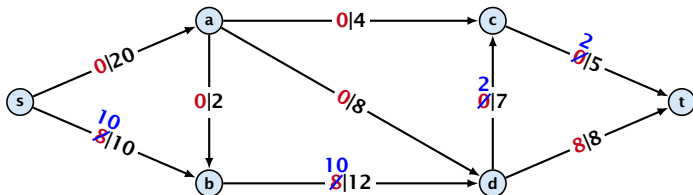
Augmenting Paths
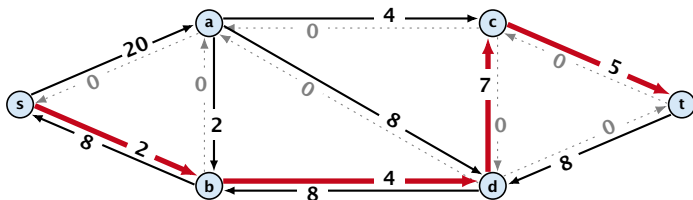
flow value: 8
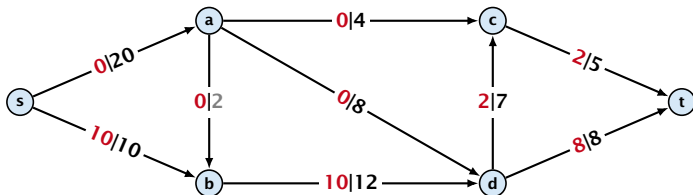
Augmenting Paths
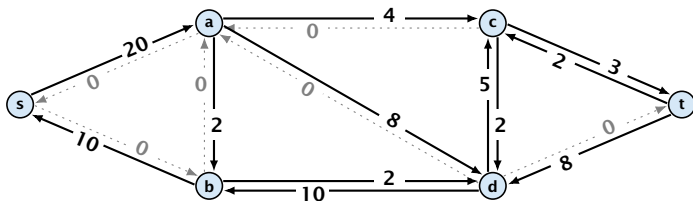
flow value: 10
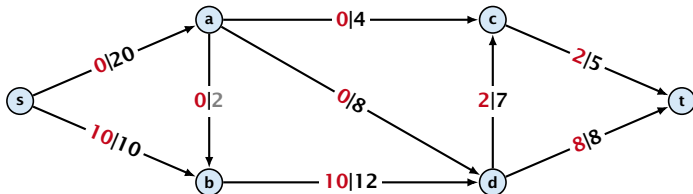
Augmenting Paths
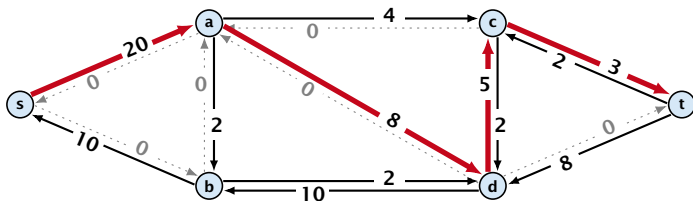
flow value: 10
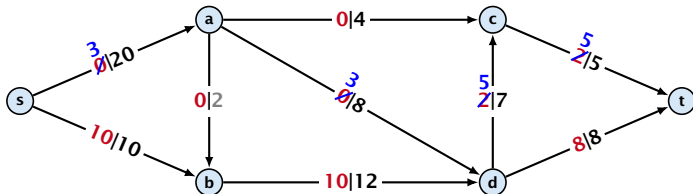
Augmenting Paths
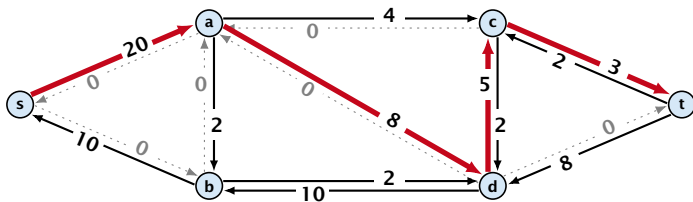
flow value: 10
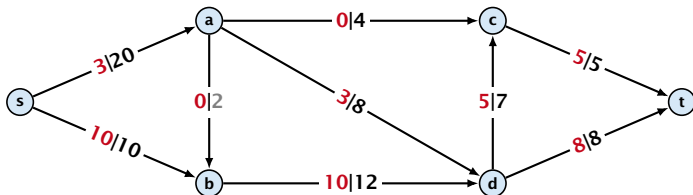
Augmenting Paths

flow value: 13
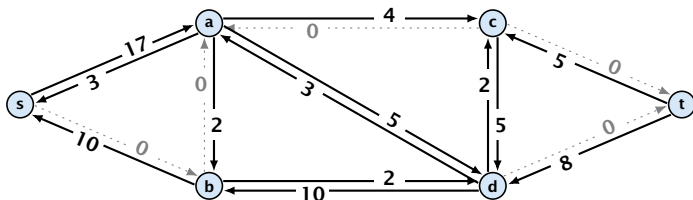
Augmenting Path Algorithm

Augmenting Path Algorithm

Theorem 50

*A flow f is a maximum flow **iff** there are no augmenting paths.*

Augmenting Path Algorithm

Theorem 50

*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 51

The value of a maximum flow is equal to the value of a minimum cut.

Augmenting Path Algorithm

Theorem 50
*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 51
The value of a maximum flow is equal to the value of a minimum cut.

Proof.
Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.

☐

Augmenting Path Algorithm

Theorem 50

*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 51

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\text{val}(f) = \text{cap}(A, V \setminus A)$.

2. Flow f is a maximum flow.

☐

Augmenting Path Algorithm

Theorem 50

*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 51

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f.

□

Augmenting Path Algorithm

Augmenting Path Algorithm

1. $\implies$ 2.
This we already showed.

Augmenting Path Algorithm

1. $\Longrightarrow$ 2.
This we already showed.

2. $\Longrightarrow$ 3.
If there were an augmenting path, we could improve the flow.
Contradiction.

Augmenting Path Algorithm

1. $\implies$ 2.
This we already showed.

2. $\implies$ 3.
If there were an augmenting path, we could improve the flow.
Contradiction.

3. $\implies$ 1.

- ▶ Let f be a flow with no augmenting paths.

Augmenting Path Algorithm

1. $\Longrightarrow$ 2.
This we already showed.

2. $\Longrightarrow$ 3.
If there were an augmenting path, we could improve the flow.
Contradiction.

3. $\Longrightarrow$ 1.

- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.

Augmenting Path Algorithm

1. $\implies$ 2.
This we already showed.

2. $\implies$ 3.
If there were an augmenting path, we could improve the flow.
Contradiction.

3. $\implies$ 1.

- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$$\text{val}(f)$$

Augmenting Path Algorithm

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e)$$

Augmenting Path Algorithm

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e)$$

$$= \sum_{e \in \mathrm{out}(A)} c(e)$$

Augmenting Path Algorithm

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

$$= \sum_{e \in \text{out}(A)} c(e)$$

$$= \text{cap}(A, V \setminus A)$$

Augmenting Path Algorithm

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

$$= \sum_{e \in \text{out}(A)} c(e)$$

$$= \text{cap}(A, V \setminus A)$$

This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

Analysis

Assumption:
All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

Lemma 52

The algorithm terminates in at most $\mathrm{val}(f^*) \leq nC$ iterations, where f^* denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.

Lemma 52

The algorithm terminates in at most $\mathrm{val}(f^*) \leq nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$. *This gives a total running time of* $\mathcal{O}(nmC)$.

Theorem 53

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ *is integral.*

A Bad Input

Problem: The running time may not be polynomial

flow value: 0

A Bad Input

Problem: The running time may not be polynomial

flow value: 0

A Bad Input

Problem: The running time may not be polynomial

flow value: 0

A Bad Input

Problem: The running time may not be polynomial

flow value: 1

A Bad Input

Problem: The running time may not be polynomial

flow value: 1

A Bad Input

Problem: The running time may not be polynomial

flow value: 1

A Bad Input

Problem: The running time may not be polynomial

flow value: 2

A Bad Input

Problem: The running time may not be polynomial

flow value: 2

A Bad Input

Problem: The running time may not be polynomial

flow value: 2

A Bad Input

Problem: The running time may not be polynomial

flow value: 3

A Bad Input

Problem: The running time may not be polynomial

flow value: 3

A Bad Input

Problem: The running time may not be polynomial

flow value: 3

A Bad Input

Problem: The running time may not be polynomial

flow value: 4

A Bad Input

Problem: The running time may not be polynomial

flow value: 4

A Bad Input

Problem: The running time may not be polynomial

flow value: 4

A Bad Input

Problem: The running time may not be polynomial

flow value: 5

A Bad Input

Problem: The running time may not be polynomial

flow value: 5

A Bad Input

Problem: The running time may not be polynomial

flow value: 5

A Bad Input

Problem: The running time may not be polynomial

flow value: 6

A Bad Input

Problem: The running time may not be polynomial
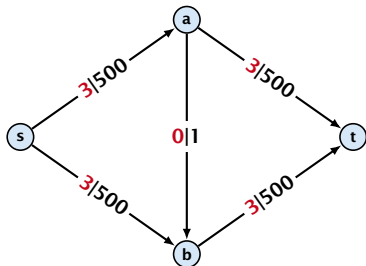
flow value: 6

Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: 0

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: 0

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: 0

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: r^2

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.
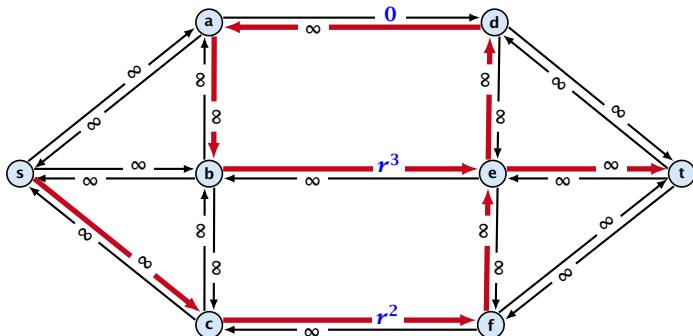
flow value: r^2

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.
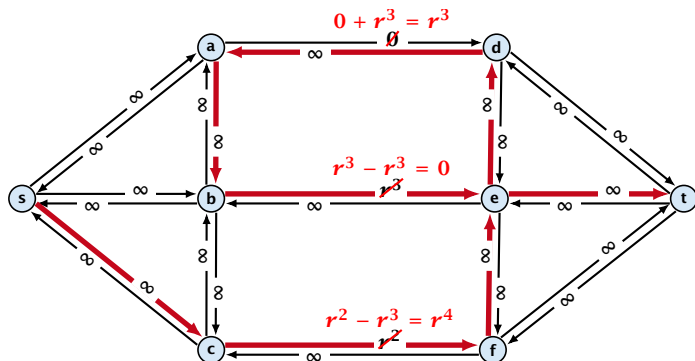
flow value: r^2

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: $r^2 + r^3$

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: $r^2 + r^3$

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: $r^2 + r^3$

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.

flow value: $r^2 + r^3 + r^4$

Running time may be infinite!!!

How to choose augmenting paths?

How to choose augmenting paths?

▶ We need to find paths efficiently.

How to choose augmenting paths?

▶ We need to find paths efficiently.

▶ We want to guarantee a small number of iterations.

How to choose augmenting paths?

▶ We need to find paths efficiently.

▶ We want to guarantee a small number of iterations.

Several possibilities:

How to choose augmenting paths?

▶ We need to find paths efficiently.

▶ We want to guarantee a small number of iterations.

Several possibilities:

▶ Choose path with maximum bottleneck capacity.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

Overview: Shortest Augmenting Paths

Overview: Shortest Augmenting Paths

Lemma 54

The length of the shortest augmenting path never decreases.

Overview: Shortest Augmenting Paths

Lemma 54

The length of the shortest augmenting path never decreases.

Lemma 55

After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 56

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2 n)$.

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 56
The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2 n)$.

Proof.
▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.

□

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 56

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2 n)$.

Proof.

▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.

▶ $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.

□

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

edge of G_f edge of L_G

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u,v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

▶ Bottleneck edges on the chosen path are deleted.

▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

edge of G_f edge of L_G

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

▶ Bottleneck edges on the chosen path are deleted.

▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

edge of G_f edge of L_G

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

edge of G_f edge of L_G

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

▶ Bottleneck edges on the chosen path are deleted.

▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

▶ Bottleneck edges on the chosen path are deleted.

▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

edge of G_f edge of L_G

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

edge of G_f edge of L_G

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

edge of G_f edge of L_G

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

An s-t path in G_f that uses edges not in M has length larger than k, even when using edges added to G_f during the round.

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

An s-t path in G_f that uses edges not in M has length larger than k, even when using edges added to G_f during the round.

In each augmentation an edge is deleted from M.

edge of G_f edge in M

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

An s-t path in G_f that uses edges not in M has length larger than k, even when using edges added to G_f during the round.

In each augmentation an edge is deleted from M.

edge of G_f edge in M

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

An s-t path in G_f that uses edges not in M has length larger than k, even when using edges added to G_f during the round.

In each augmentation an edge is deleted from M.

edge of G_f edge in M

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

An s-t path in G_f that uses edges not in M has length larger than k, even when using edges added to G_f during the round.

In each augmentation an edge is deleted from M.

edge of G_f edge in M

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

An s-t path in G_f that uses edges not in M has length larger than k, even when using edges added to G_f during the round.

In each augmentation an edge is deleted from M.

edge of G_f edge in M

Shortest Augmenting Paths

Shortest Augmenting Paths

Theorem 57
The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Shortest Augmenting Paths

Theorem 57

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 58 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\Omega(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Shortest Augmenting Paths

Theorem 57
The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 58 (without proof)
There exist networks with $m = \Theta(n^2)$ that require $\Omega(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:
There always exists a set of m augmentations that gives a maximum flow (why?).

Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

Shortest Augmenting Paths

We maintain a subset M of the edges of G_f with the guarantee that a shortest s-t path using only edges from M is a shortest augmenting path.

Shortest Augmenting Paths

We maintain a subset M of the edges of G_f with the guarantee that a shortest s-t path using only edges from M is a shortest augmenting path.

With each augmentation some edges are deleted from M.

Shortest Augmenting Paths

We maintain a subset M of the edges of G_f with the guarantee that a shortest s-t path using only edges from M is a shortest augmenting path.

With each augmentation some edges are deleted from M.

When M does not contain an s-t path anymore the distance between s and t strictly increases.

Shortest Augmenting Paths

We maintain a subset M of the edges of G_f with the guarantee that a shortest s-t path using only edges from M is a shortest augmenting path.

With each augmentation some edges are deleted from M.

When M does not contain an s-t path anymore the distance between s and t strictly increases.

Note that M is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between s and t in G_f is k.

Suppose that the initial distance between s and t in G_f is k.

M is initialized as the level graph L_G.

Suppose that the initial distance between s and t in G_f is k.

M is initialized as the level graph L_G.

Perform a DFS search to find a path from s to t using edges from M.

Suppose that the initial distance between s and t in G_f is k.

M is initialized as the level graph L_G.

Perform a DFS search to find a path from s to t using edges from M.

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

Suppose that the initial distance between s and t in G_f is k.

M is initialized as the level graph L_G.

Perform a DFS search to find a path from s to t using edges from M.

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from M.

Analysis

Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing M for the phase takes time $\mathcal{O}(m)$.

Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing M for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in M and takes time $\mathcal{O}(n)$.

Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing M for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in M and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in M for the next search.

Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing M for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in M and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in M for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

How to choose augmenting paths?

▶ We need to find paths efficiently.

How to choose augmenting paths?

▶ We need to find paths efficiently.

▶ We want to guarantee a small number of iterations.

How to choose augmenting paths?

▶ We need to find paths efficiently.

▶ We want to guarantee a small number of iterations.

Several possibilities:

How to choose augmenting paths?

▶ We need to find paths efficiently.

▶ We want to guarantee a small number of iterations.

Several possibilities:

▶ Choose path with maximum bottleneck capacity.

▶ Choose path with sufficiently large bottleneck capacity.

▶ Choose the shortest augmenting path.

Capacity Scaling

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.

Capacity Scaling

Intuition:

▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.

▶ Don't worry about finding the exact bottleneck.

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ.

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ.
- ▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ.
- ▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.

Capacity Scaling

Algorithm 1 maxflow(G, s, t, c)

1: **foreach** $e \in E$ **do** $f_e \leftarrow 0$;
2: $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
3: **while** $\Delta \geq 1$ **do**
4:     $G_f(\Delta) \leftarrow \Delta$-residual graph
5:         **while** there is augmenting path P in $G_f(\Delta)$ **do**
6:             $f \leftarrow$ augment(f, c, P)
7:             update($G_f(\Delta)$)
8:     $\Delta \leftarrow \Delta/2$
9: **return** f

Capacity Scaling

Capacity Scaling

Assumption:

All capacities are integers between 1 and C.

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

▶ because of integrality we have $G_f(1) = G_f$

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

▶ because of integrality we have $G_f(1) = G_f$

▶ therefore after the last phase there are no augmenting paths anymore

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

▶ because of integrality we have $G_f(1) = G_f$

▶ therefore after the last phase there are no augmenting paths anymore

▶ this means we have a maximum flow.

Capacity Scaling

Harald Räcke

Capacity Scaling

Lemma 59

There are $\lceil \log C \rceil + 1$ iterations over Δ.

Proof: obvious.

Capacity Scaling

Lemma 59

There are $\lceil \log C \rceil + 1$ iterations over Δ.

Proof: obvious.

Lemma 60

Let f be the flow at the end of a Δ-phase. Then the maximum flow is smaller than $\mathrm{val}(f) + m\Delta$.

Proof: less obvious, but simple:

Capacity Scaling

Lemma 59

There are $\lceil \log C \rceil + 1$ iterations over Δ.

Proof: obvious.

Lemma 60

Let f be the flow at the end of a Δ-phase. Then the maximum flow is smaller than $\mathrm{val}(f) + m\Delta$.

Proof: less obvious, but simple:

▶ There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.

Capacity Scaling

Lemma 59

There are $\lceil \log C \rceil + 1$ iterations over Δ.

Proof: obvious.

Lemma 60

Let f be the flow at the end of a Δ-phase. Then the maximum flow is smaller than $\mathrm{val}(f) + m\Delta$.

Proof: less obvious, but simple:

▶ There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.

▶ In G_f this cut can have capacity at most $m\Delta$.

Capacity Scaling

Lemma 59

There are $\lceil \log C \rceil + 1$ iterations over Δ.

Proof: obvious.

Lemma 60

Let f be the flow at the end of a Δ-phase. Then the maximum flow is smaller than $\mathrm{val}(f) + m\Delta$.

Proof: less obvious, but simple:

▶ There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.

▶ In G_f this cut can have capacity at most $m\Delta$.

▶ This gives me an upper bound on the flow that I can still add.

Capacity Scaling

Capacity Scaling

Lemma 61

There are at most $2m$ augmentations per scaling-phase.

Capacity Scaling

Lemma 61

There are at most $2m$ augmentations per scaling-phase.

Proof:

▶ Let f be the flow at the end of the previous phase.

Capacity Scaling

Lemma 61

There are at most $2m$ augmentations per scaling-phase.

Proof:

▶ Let f be the flow at the end of the previous phase.

▶ $\text{val}(f^*) \le \text{val}(f) + 2m\Delta$

Capacity Scaling

Lemma 61

There are at most $2m$ augmentations per scaling-phase.

Proof:

- ▶ Let f be the flow at the end of the previous phase.
- ▶ $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- ▶ Each augmentation increases flow by Δ.

Capacity Scaling

Lemma 61

There are at most $2m$ augmentations per scaling-phase.

Proof:

▶ Let f be the flow at the end of the previous phase.

▶ $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$

▶ Each augmentation increases flow by Δ.

Theorem 62

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Matching

▶ Input: undirected graph $G = (V, E)$.

▶ $M \subseteq E$ is a matching if each node appears in at most one edge in M.

▶ Maximum Matching: find a matching of maximum cardinality
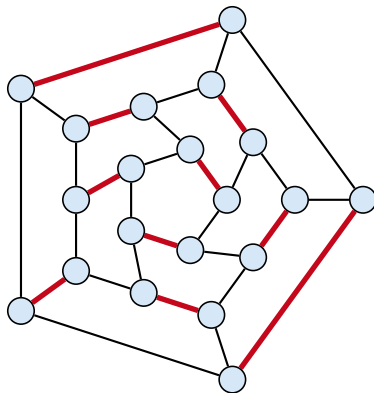
Bipartite Matching

▶ Input: undirected, bipartite graph $G = (L \uplus R, E)$.

▶ $M \subseteq E$ is a matching if each node appears in at most one edge in M.

▶ Maximum Matching: find a matching of maximum cardinality
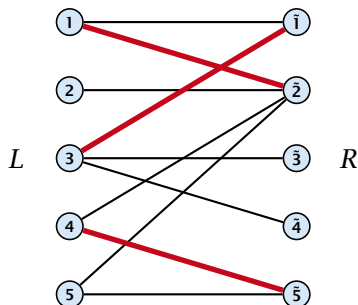
Bipartite Matching

▶ Input: undirected, bipartite graph $G = (L \uplus R, E)$.

▶ $M \subseteq E$ is a matching if each node appears in at most one edge in M.

▶ Maximum Matching: find a matching of maximum cardinality

Maxflow Formulation

▶ Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s, t\}, E')$.

▶ Direct all edges from L to R.

▶ Add source s and connect it to all nodes on the left.

▶ Add t and connect all nodes on the right to t.

▶ All edges have unit capacity.

Proof

Max cardinality matching in G ≤ value of maxflow in G'

- ▶ Given a maximum matching M of cardinality k.
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k.

Proof

Max cardinality matching in G ≤ value of maxflow in G'

▶ Given a maximum matching M of cardinality k.

▶ Consider flow f that sends one unit along each of k paths.

▶ f is a flow and has cardinality k.

Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

▶ Given a maximum matching M of cardinality k.

▶ Consider flow f that sends one unit along each of k paths.

▶ f is a flow and has cardinality k.

Proof

Max cardinality matching in G $\leq$ value of maxflow in G'

▶ Given a maximum matching M of cardinality k.

▶ Consider flow f that sends one unit along each of k paths.

▶ f is a flow and has cardinality k.

Proof

Max cardinality matching in G ≥ value of maxflow in G'

▶ Let f be a maxflow in G' of value k

▶ Integrality theorem ⇒ k integral; we can assume f is 0/1.

▶ Consider M= set of edges from L to R with $f(e) = 1$.

▶ Each node in L and R participates in at most one edge in M.

▶ $|M| = k$, as the flow must use at least k middle edges.

12.1 Matching

Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- ▶ Let f be a maxflow in G' of value k
- ▶ Integrality theorem $\Rightarrow k$ integral; we can assume f is 0/1.
- ▶ Consider $M =$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M.
- ▶ $|M| = k$, as the flow must use at least k middle edges.

G' G

Proof

Max cardinality matching in G ≥ value of maxflow in G'

- Let f be a maxflow in G' of value k

- Integrality theorem ⇒ k integral; we can assume f is 0/1.

- Consider M= set of edges from L to R with $f(e) = 1$.

- Each node in L and R participates in at most one edge in M.

- $|M| = k$, as the flow must use at least k middle edges.

G'

G

12.1 Matching

Which flow algorithm to use?

▶ Generic augmenting path: $\mathcal{O}(m \operatorname{val}(f^*)) = \mathcal{O}(mn)$.

▶ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

▶ Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

Baseball Elimination

| team | wins | losses | remaining games | | | |
i	w_i	ℓ_i	Atl	Phi	NY	Mon
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.

▶ But also Philadelphia is eliminated. Why?

Baseball Elimination

Formal definition of the problem:

▶ Given a set S of teams, and one specific team $z \in S$.

▶ Team x has already won w_x games.

▶ Team x still has to play team y, r_{xy} times.

▶ Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.
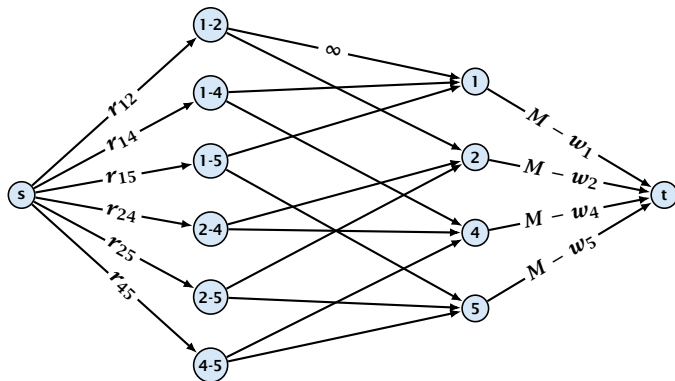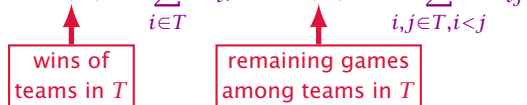
Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i < j} r_{ij}$$

wins of teams in T

remaining games among teams in T

If $\frac{w(T) + r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i<j} r_{ij}$.

Proof ($\Leftarrow$)

▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Proof ($\Leftarrow$)

▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

▶ If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i<j} r_{ij}$.

Proof ($\Leftarrow$)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

- ▶ If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

- ▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\})$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Proof ($\Leftarrow$)

- ▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

- ▶ If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

- ▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\}) > \text{cap}(A, V \setminus A)$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Proof ($\Leftarrow$)

▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

▶ If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\}) > \mathrm{cap}(A, V \setminus A)$$

$$\geq \sum_{i < j : i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i)$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Proof ($\Leftarrow$)

▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

▶ If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\}) > \operatorname{cap}(A, V \setminus A)$$

$$\geq \sum_{i < j : i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i)$$

$$\geq r(S \setminus \{z\}) - r(T) + |T| M - w(T)$$

Theorem 63

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i < j} r_{ij}$.

Proof ($\Leftarrow$)

▶ Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

▶ If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

▶ We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\}) > \operatorname{cap}(A, V \setminus A)$$

$$\geq \sum_{i < j : i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i)$$

$$\geq r(S \setminus \{z\}) - r(T) + |T|M - w(T)$$

▶ This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

▶ Suppose we have a flow that saturates all source edges.

Baseball Elimination

Proof ($\Rightarrow$)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is integral.

Baseball Elimination

Proof (⇒)

▶ Suppose we have a flow that saturates all source edges.

▶ We can assume that this flow is integral.

▶ For every pairing x-y it defines how many games team x and team y should win.

Baseball Elimination

Proof ($\Rightarrow$)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is integral.
- ▶ For every pairing x-y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.

Baseball Elimination

Proof ($\Rightarrow$)

▶ Suppose we have a flow that saturates all source edges.

▶ We can assume that this flow is integral.

▶ For every pairing x-y it defines how many games team x and team y should win.

▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.

▶ This is less than $M - w_x$ because of capacity constraints.

Baseball Elimination

Proof (⇒)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is integral.
- ▶ For every pairing x-y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- ▶ This is less than $M - w_x$ because of capacity constraints.
- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.

Baseball Elimination

Proof (⇒)

▶ Suppose we have a flow that saturates all source edges.

▶ We can assume that this flow is integral.

▶ For every pairing x-y it defines how many games team x and team y should win.

▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.

▶ This is less than $M - w_x$ because of capacity constraints.

▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.

▶ Hence, team z is not eliminated.

Project Selection

Project selection problem:

▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

Project Selection

Project selection problem:

▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

▶ Some projects have requirements (taking course EA2 requires course EA1).

Project Selection

Project selection problem:

▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

▶ Some projects have requirements (taking course EA2 requires course EA1).

▶ Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."

Project Selection

Project selection problem:

- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

- ▶ Some projects have requirements (taking course EA2 requires course EA1).

- ▶ Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."

- ▶ A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Project Selection

Project selection problem:

- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

- ▶ Some projects have requirements (taking course EA2 requires course EA1).

- ▶ Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."

- ▶ A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

- ▶ $\{x, a, z\}$ is a feasible subset.
- ▶ $\{x, a\}$ is infeasible.
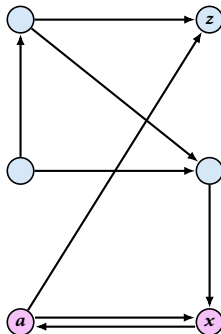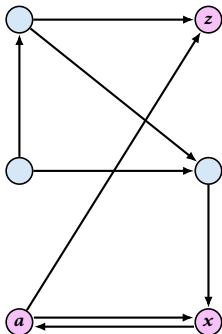
Project Selection

Mincut formulation:

▶ Edges in the prerequisite graph get infinite capacity.

▶ Add edge (s, v) with capacity p_v for nodes v with positive profit.

▶ Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.

prerequisite graph
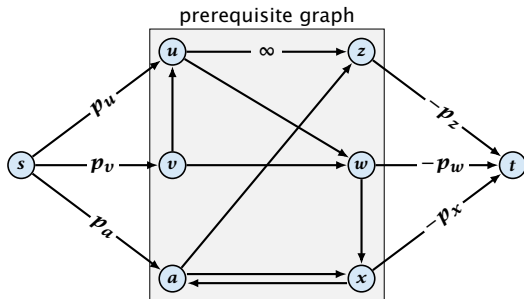
Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

▶ *A* is feasible because of capacity infinity edges.

prerequisite graph

Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

▶ A is feasible because of capacity infinity edges.

▶ $\operatorname{cap}(A, V \setminus A)$

prerequisite graph

Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

- ▶ A is feasible because of capacity infinity edges.
- ▶ $\mathrm{cap}(A, V \setminus A) = \sum_{v \in \bar{A} : p_v > 0} p_v + \sum_{v \in A : p_v < 0} (-p_v)$

prerequisite graph
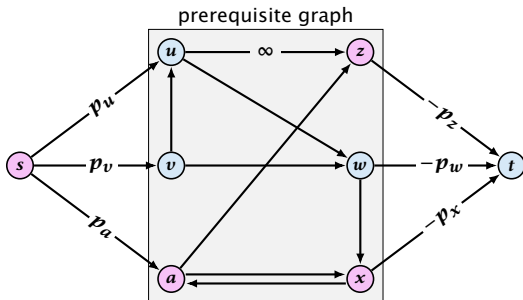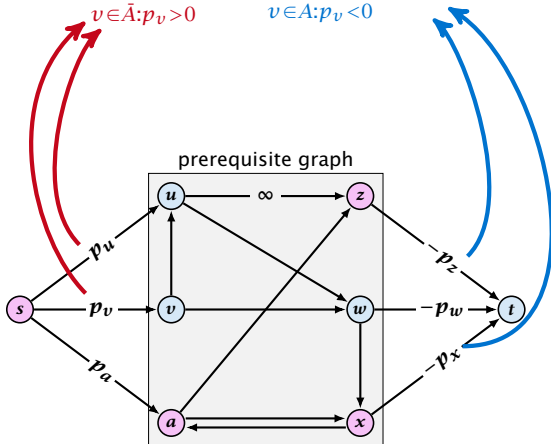
Theorem 64

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

▶ A is feasible because of capacity infinity edges.

▶ $\text{cap}(A, V \setminus A) = \displaystyle\sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$

$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$
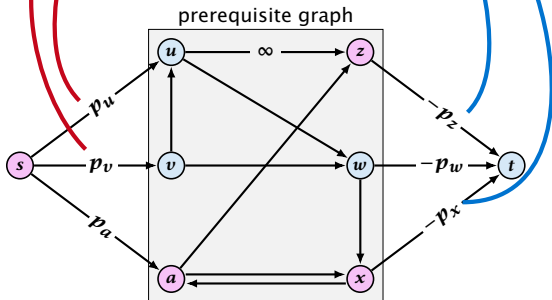
prerequisite graph

Preflows

Preflows

Definition 65

An (s, t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \le f(e) \le c(e) \ .$$

   (capacity constraints)

Preflows

Definition 65

An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
$$0 \le f(e) \le c(e) \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s, t\}$
$$\sum_{e \in \mathrm{out}(v)} f(e) \le \sum_{e \in \mathrm{into}(v)} f(e) \ .$$

Preflows

Example 66
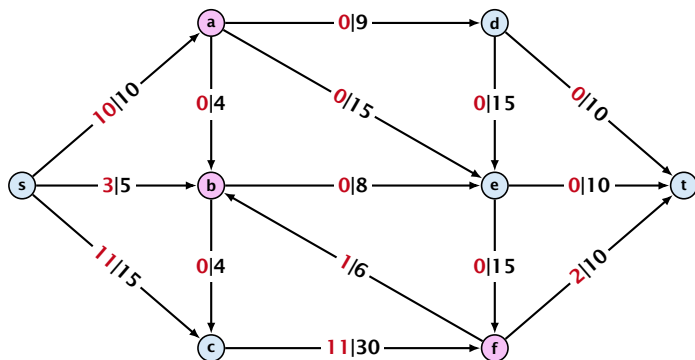
Preflows

Example 66
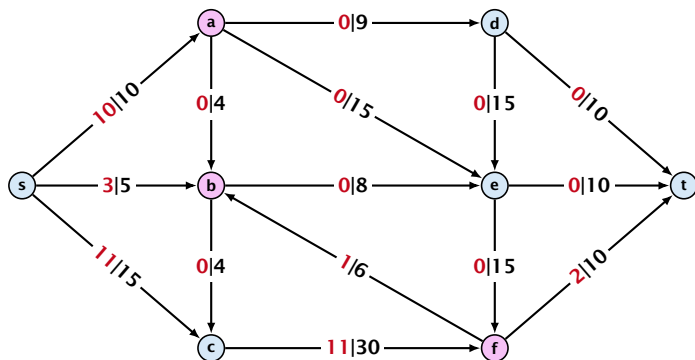
A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

Preflows

Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

▶ $\ell(u) \le \ell(v) + 1$ for all edges (u, v) in the residual graph G_f
  (only non-zero capacity edges!!!)

Preflows

Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- ▶ $\ell(u) \le \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)
- ▶ $\ell(s) = n$

Preflows

Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- ▶ $\ell(u) \leq \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)
- ▶ $\ell(s) = n$
- ▶ $\ell(t) = 0$

Preflows

Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \le \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)

- $\ell(s) = n$

- $\ell(t) = 0$

Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows
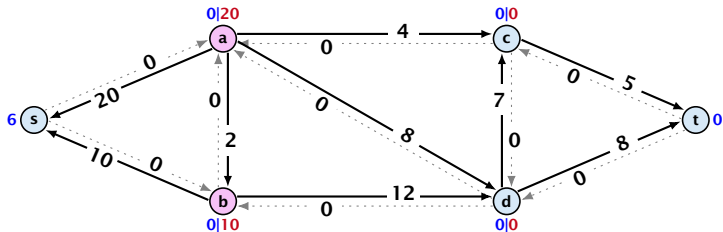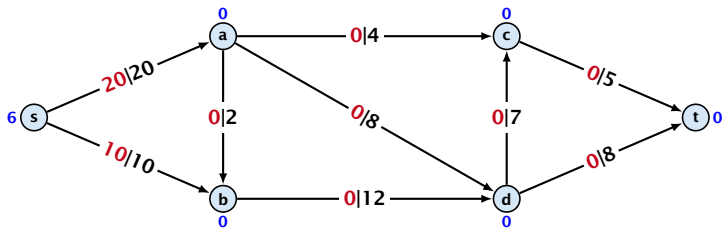
Preflows
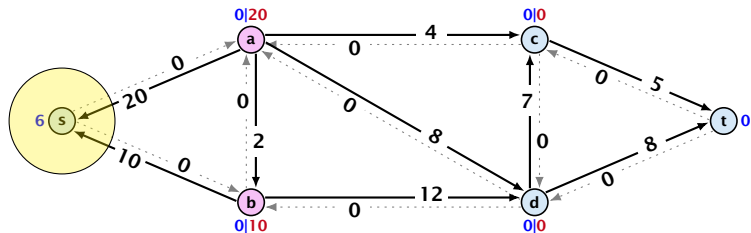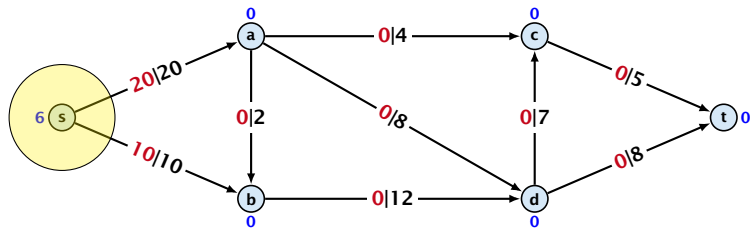
Preflows

Preflows

Lemma 67

A preflow that has a valid labelling saturates a cut.

Preflows

Lemma 67

A preflow that has a valid labelling saturates a cut.

Proof:

▶ There are n nodes but $n + 1$ different labels from $0, \ldots, n$.

Preflows

Lemma 67

A preflow that has a valid labelling saturates a cut.

Proof:

▶ There are n nodes but $n + 1$ different labels from $0, \ldots, n$.

▶ There must exist a label $d \in \{0, \ldots, n\}$ such that none of the nodes carries this label.

Preflows

Lemma 67

A preflow that has a valid labelling saturates a cut.

Proof:

▶ There are n nodes but $n + 1$ different labels from $0, \ldots, n$.

▶ There must exist a label $d \in \{0, \ldots, n\}$ such that none of the nodes carries this label.

▶ Let $A = \{v \in V \mid \ell(v) > d\}$ and $B = \{v \in V \mid \ell(v) < d\}$.

Preflows

Lemma 67

A preflow that has a valid labelling saturates a cut.

Proof:

▶ There are n nodes but $n + 1$ different labels from $0, \ldots, n$.

▶ There must exist a label $d \in \{0, \ldots, n\}$ such that none of the nodes carries this label.

▶ Let $A = \{v \in V \mid \ell(v) > d\}$ and $B = \{v \in V \mid \ell(v) < d\}$.

▶ We have $s \in A$ and $t \in B$ and there is no edge from A to B in the residual graph G_f; this means that (A, B) is a saturated cut.

Preflows

Lemma 67

A preflow that has a valid labelling saturates a cut.

Proof:

▶ There are n nodes but $n + 1$ different labels from $0, \ldots, n$.

▶ There must exist a label $d \in \{0, \ldots, n\}$ such that none of the nodes carries this label.

▶ Let $A = \{v \in V \mid \ell(v) > d\}$ and $B = \{v \in V \mid \ell(v) < d\}$.

▶ We have $s \in A$ and $t \in B$ and there is no edge from A to B in the residual graph G_f; this means that (A, B) is a saturated cut.

Lemma 68

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

Push Relabel Algorithms

Idea:

▶ start with some preflow and some valid labelling

Push Relabel Algorithms

Idea:

- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling

Push Relabel Algorithms

Idea:

- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation
Consider an active node u with excess flow $f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation
Consider an active node u with excess flow
$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation
Consider an active node u with excess flow $f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

- saturating push: $\min\{f(u), c_f(e)\} = c_f(e)$
  the arc e is deleted from the residual graph

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation
Consider an active node u with excess flow
$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$
is an admissible arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

▶ saturating push: $\min\{f(u), c_f(e)\} = c_f(e)$
  the arc e is deleted from the residual graph

▶ deactivating push: $\min\{f(u), c_f(e)\} = f(u)$
  the node u becomes inactive

Push Relabel Algorithms

Push Relabel Algorithms

The relabel operation
Consider an active node u that does not have an outgoing admissible arc.

Push Relabel Algorithms

The relabel operation
Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

Push Relabel Algorithms

The relabel operation
Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- ▶ Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.

Push Relabel Algorithms

The relabel operation
Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.

- An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:

We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.

- Such a node is called active.

- A labelling is valid if for every edge (u, v) in the residual graph $\ell(u) \leq \ell(v) + 1$.

- An arc (u, v) in residual graph is admissible if $\ell(u) = \ell(v) + 1$.

- A saturating push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).

- A deactivating push along $e = (u, v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 1 maxflow(G, s, t, c)

1: find initial preflow f
2: **while** there is active node u **do**
3:     **if** there is admiss. arc e out of u **then**
4:         push(G, e, f, c)
5:     **else**
6:         relabel(u)
7: **return** f

Push Relabel Algorithms

Algorithm 1 maxflow(G, s, t, c)

1: find initial preflow f
2: **while** there is active node u **do**
3:      **if** there is admiss. arc e out of u **then**
4:          push(G, e, f, c)
5:      **else**
6:          relabel(u)
7: **return** f

In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push

Preflow Push

Preflow Push

relabel to 1

Preflow Push

saturating push

Preflow Push

Preflow Push

saturating push

Preflow Push

Preflow Push

saturating push

Preflow Push

Preflow Push

relabel to 7

Preflow Push

Preflow Push

deactivating push
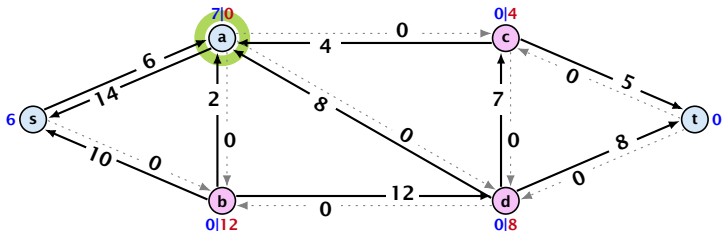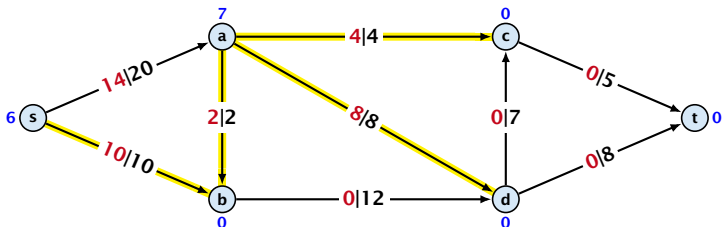
Preflow Push
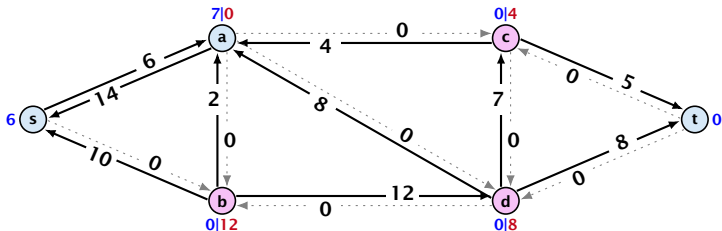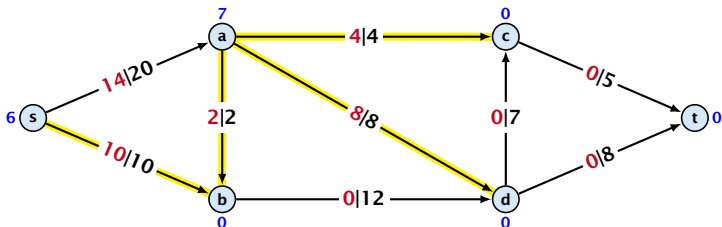
Preflow Push
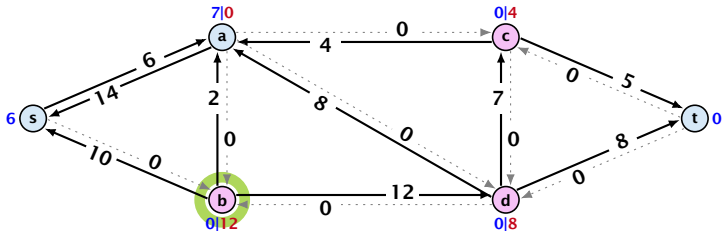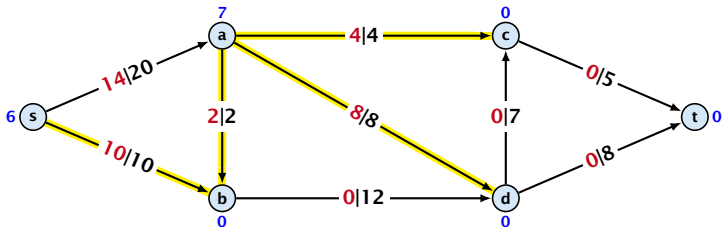
Preflow Push
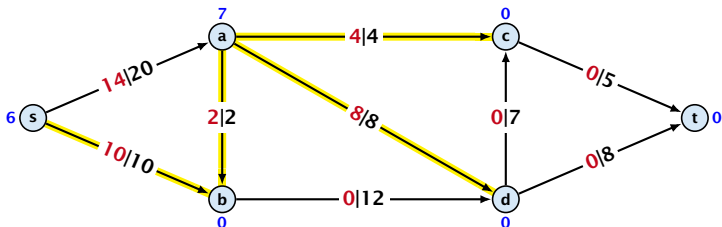
Preflow Push
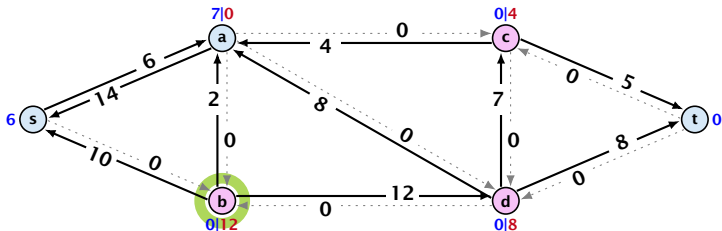
relabel to 1
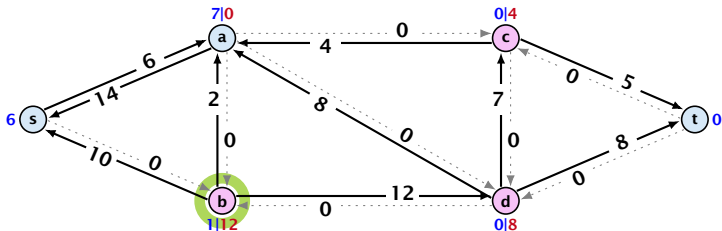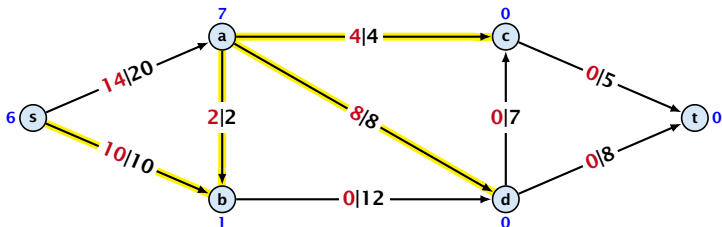
Preflow Push
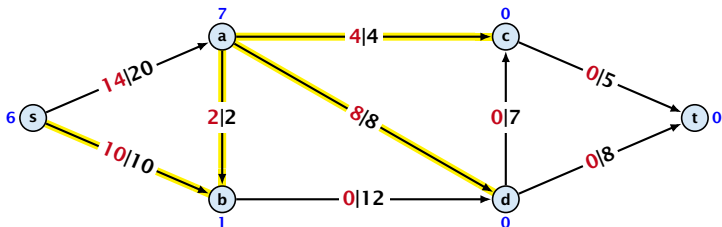
Preflow Push

saturating and deactivating push

Preflow Push

Preflow Push

Preflow Push

Preflow Push

relabel to 1

Preflow Push

Preflow Push

deactivating push

Preflow Push

Preflow Push

Preflow Push

Preflow Push

relabel to 1

Preflow Push

Preflow Push

saturating push

Preflow Push

Preflow Push

relabel to 2

Preflow Push
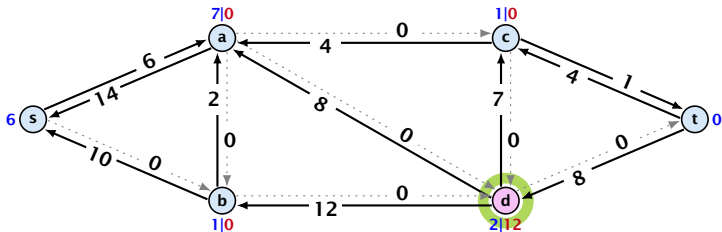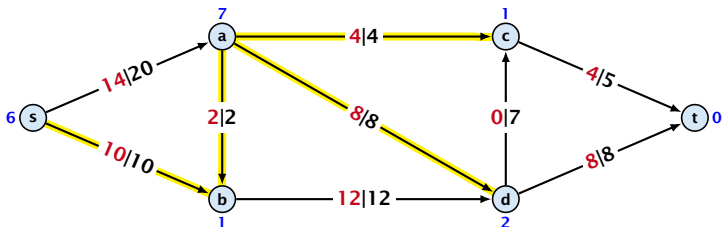
Preflow Push
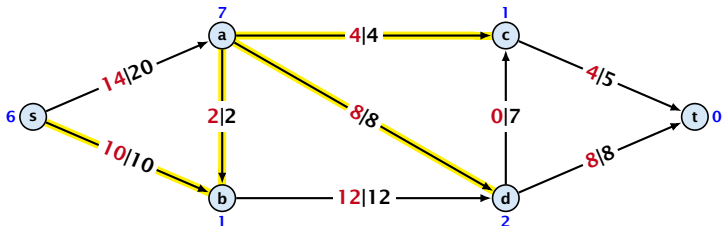
saturating and deactivating push
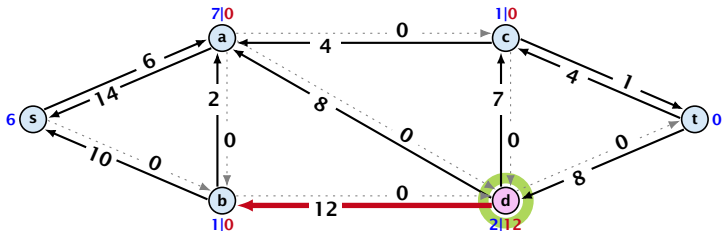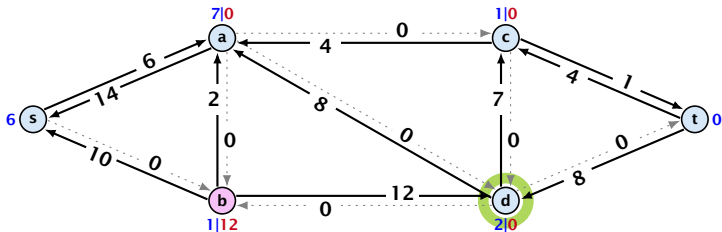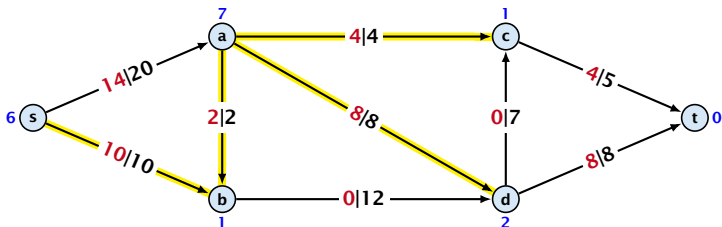
Preflow Push
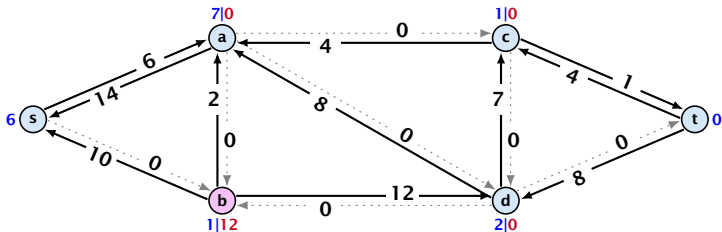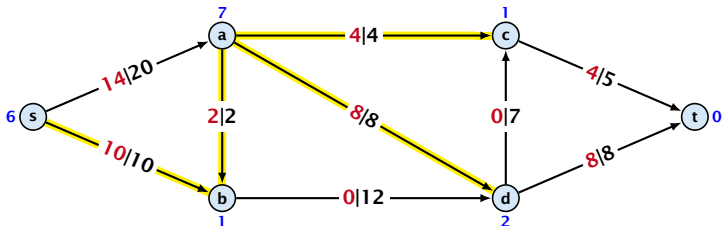
Preflow Push
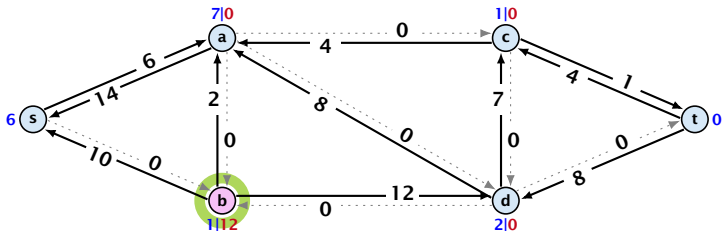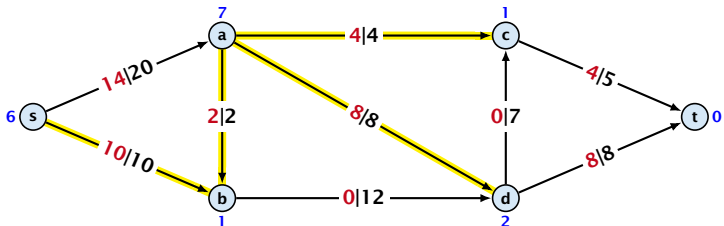
Preflow Push
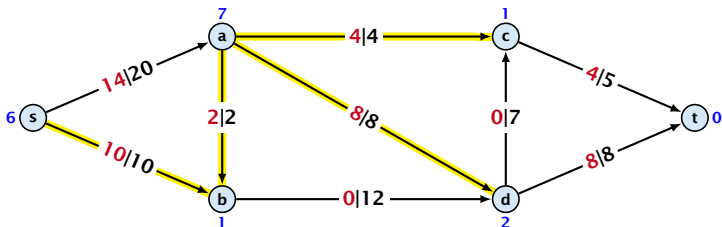
Preflow Push

relabel to 3
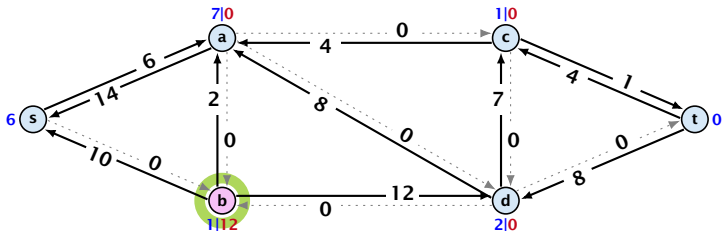
Preflow Push
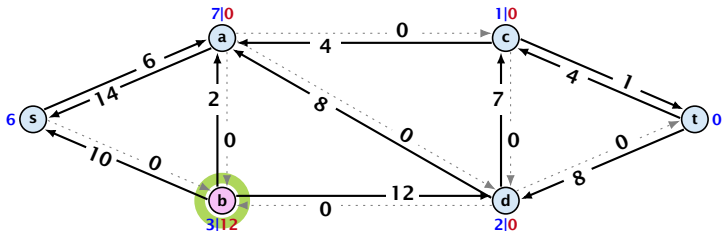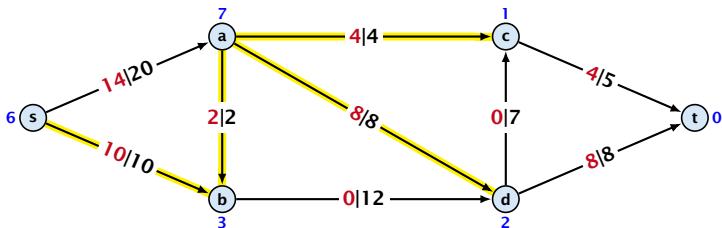
Preflow Push
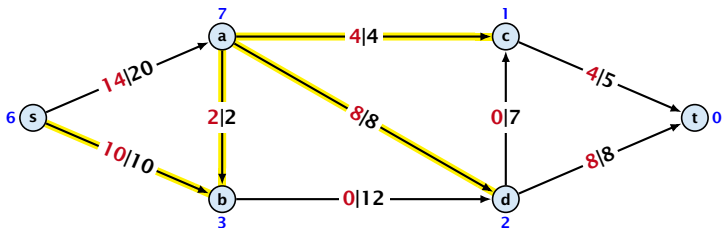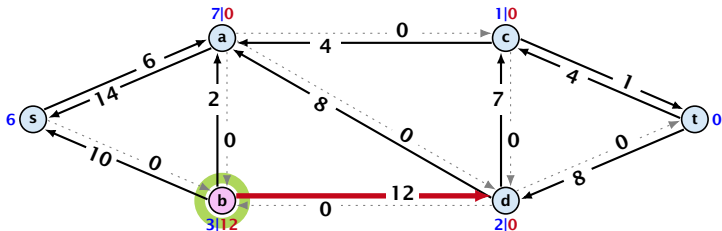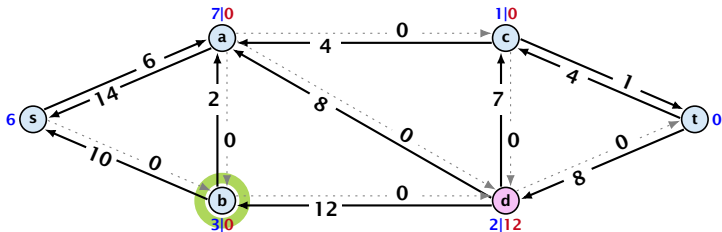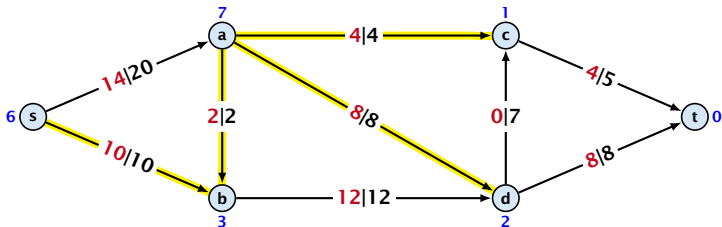
saturating and deactivating push

Preflow Push
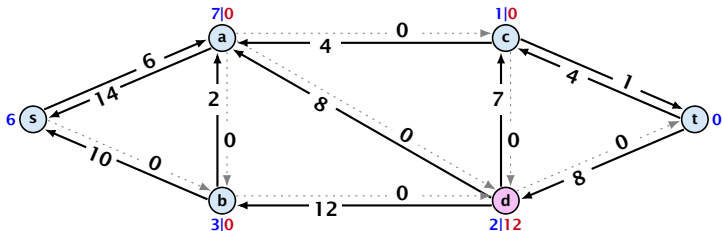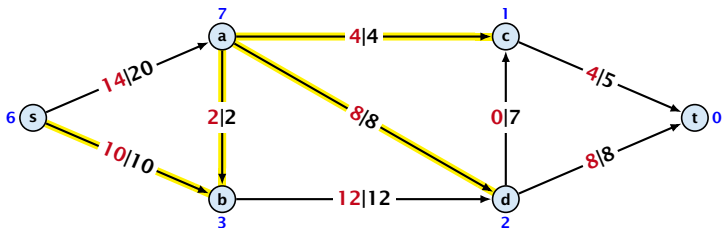
Preflow Push
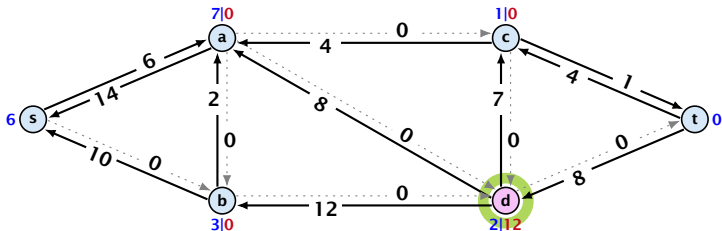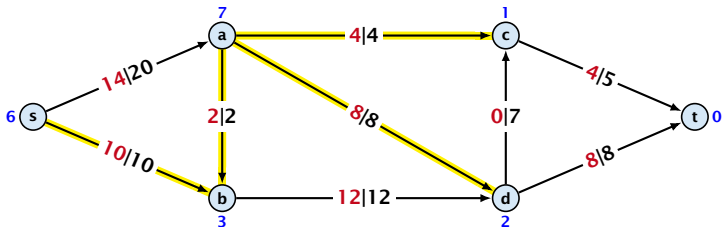
Preflow Push
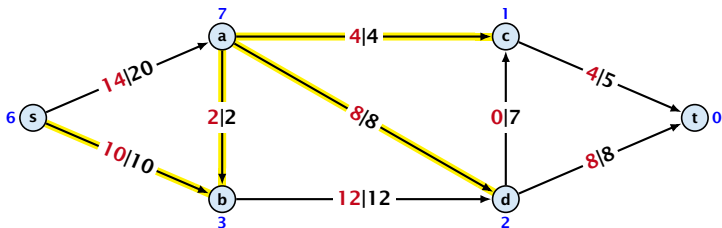
Preflow Push
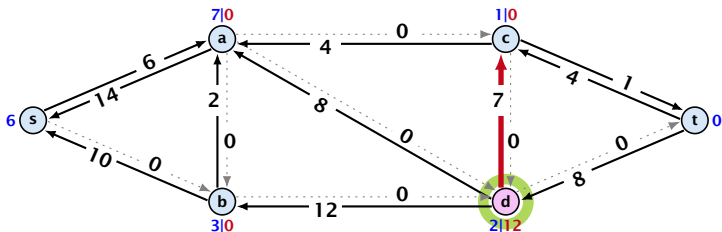
saturating push
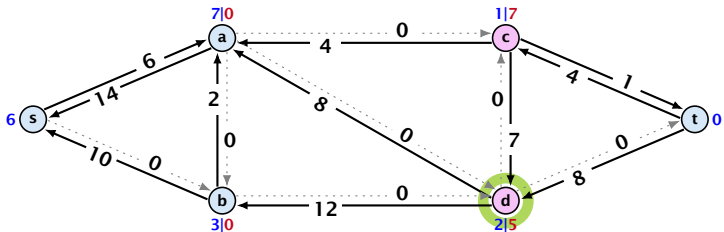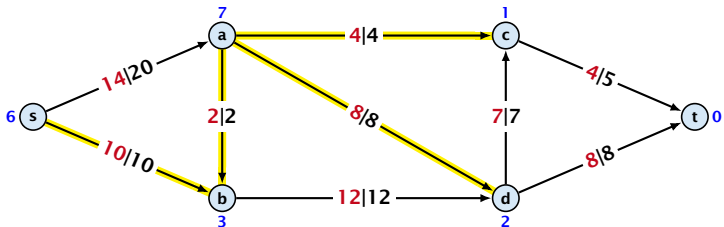
Preflow Push
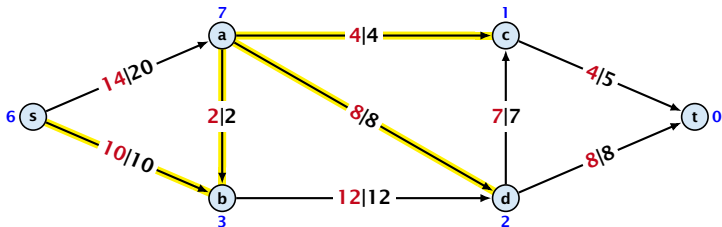
Preflow Push

relabel to 4
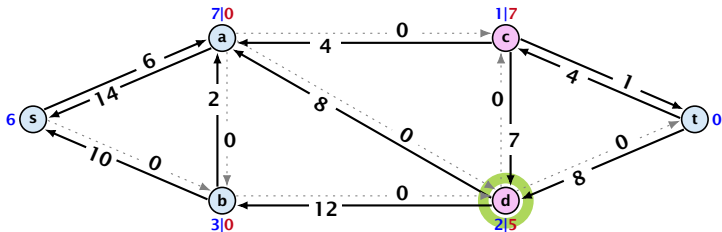
Preflow Push
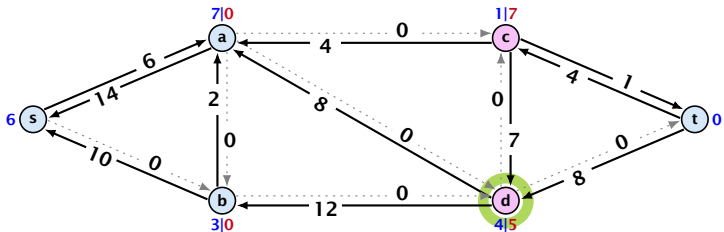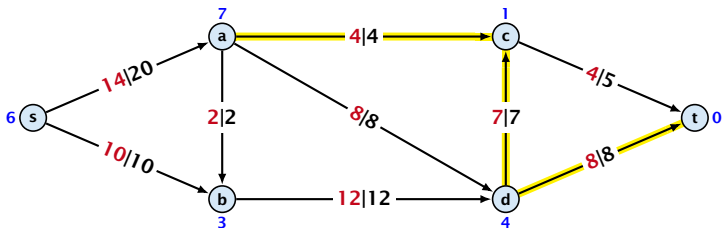
Preflow Push
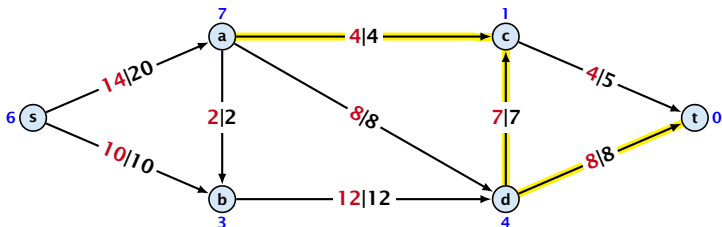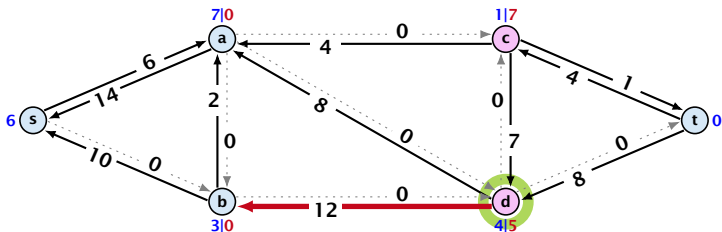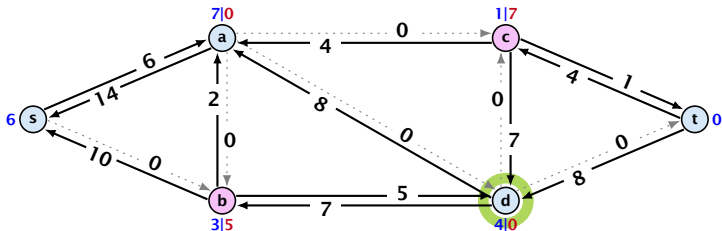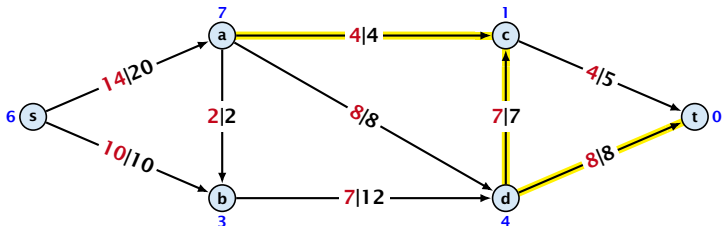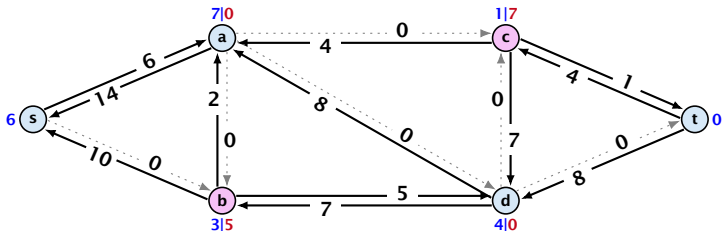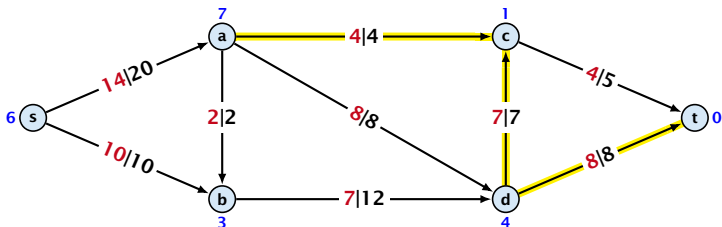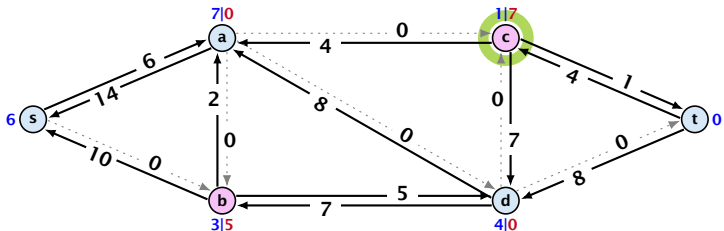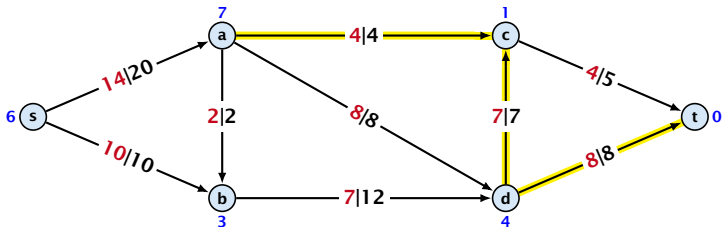
deactivating push

Preflow Push

Preflow Push

Preflow Push

Preflow Push

saturating push

Preflow Push

Preflow Push

relabel to 5

Preflow Push

Preflow Push

deactivating push

Preflow Push

Preflow Push

Preflow Push

Preflow Push

relabel to 5

Preflow Push

Preflow Push

saturating and deactivating push

Preflow Push

13.1 Generic Push Relabel
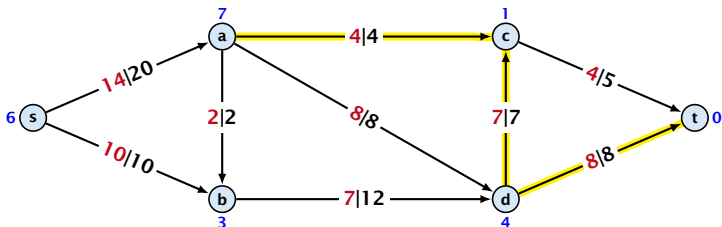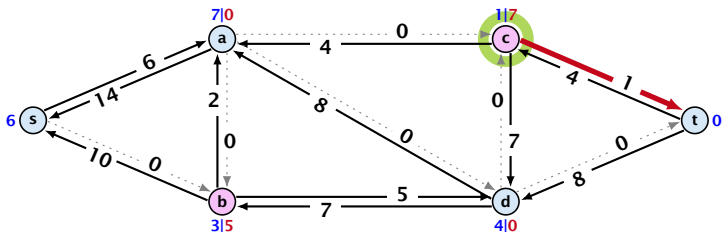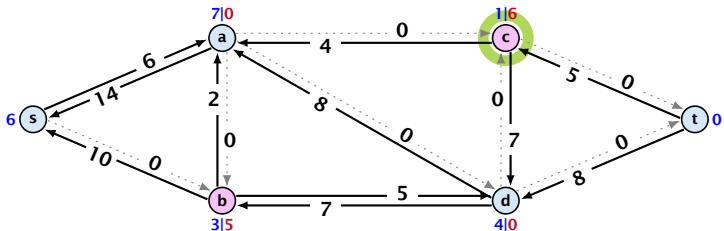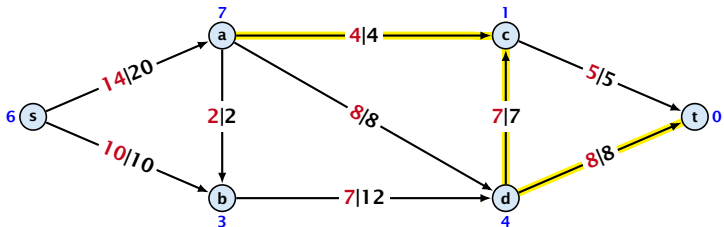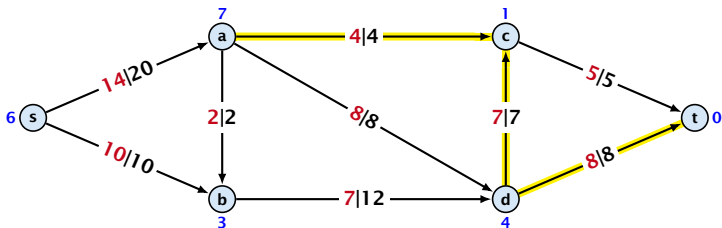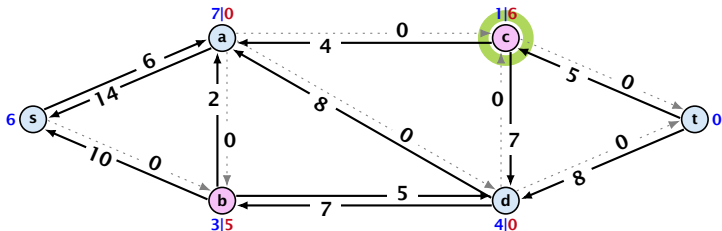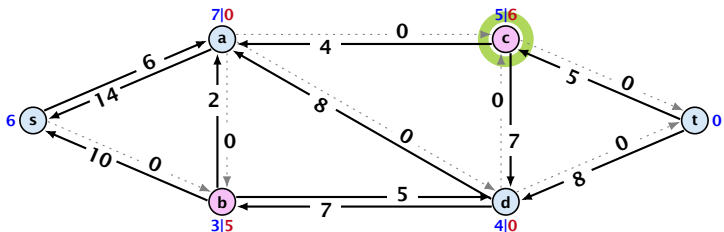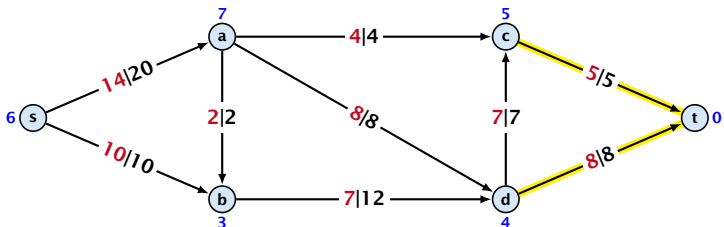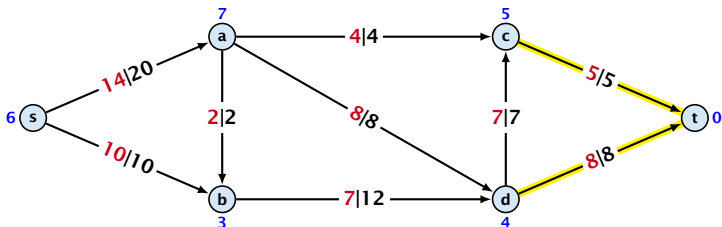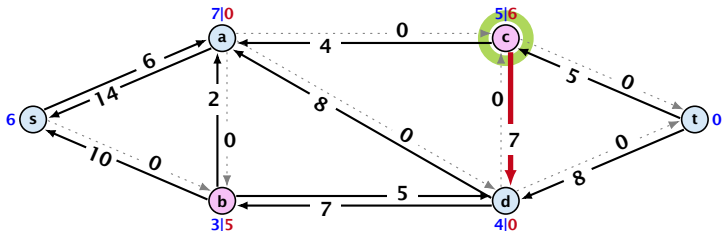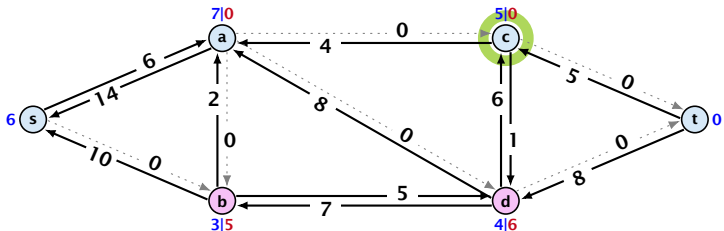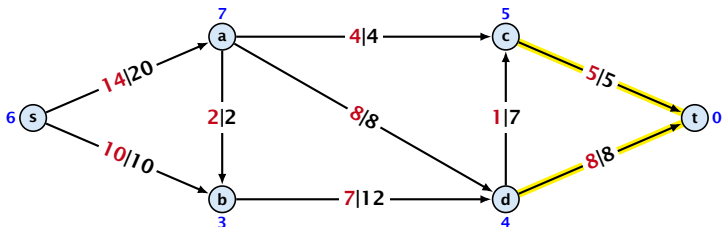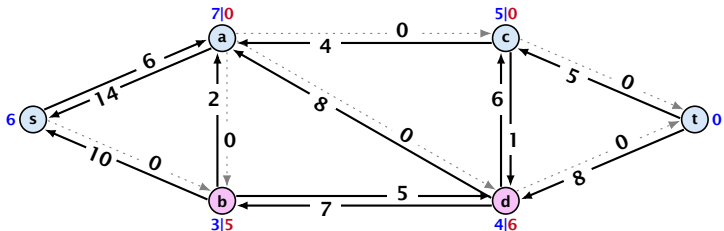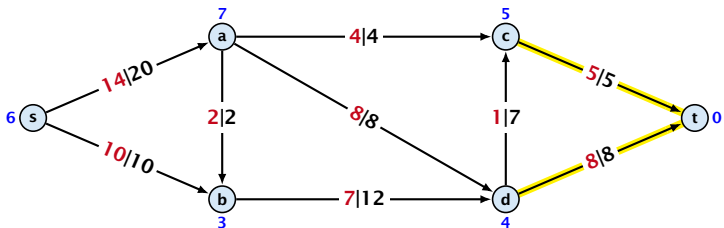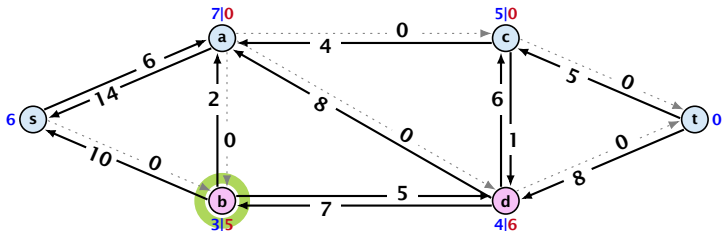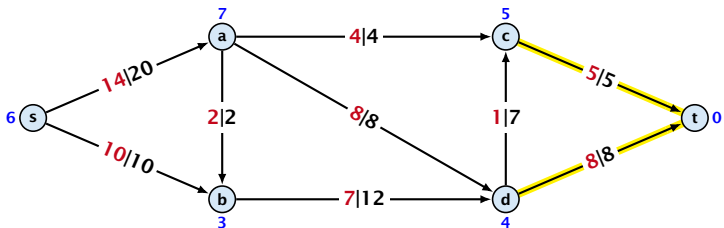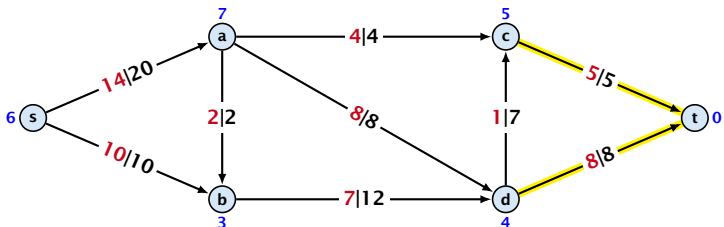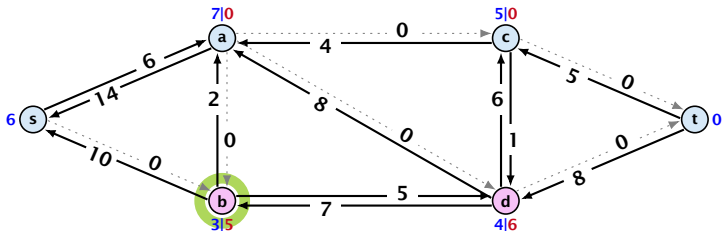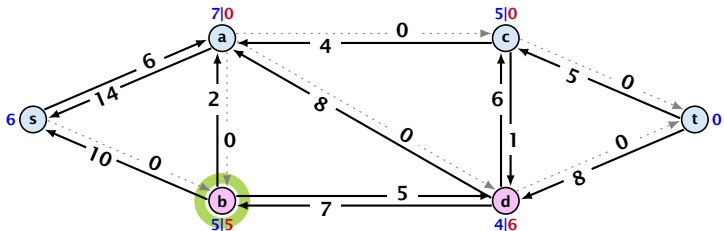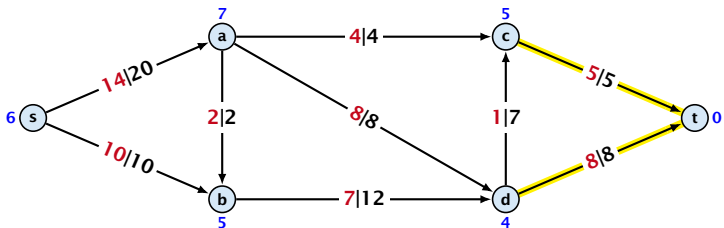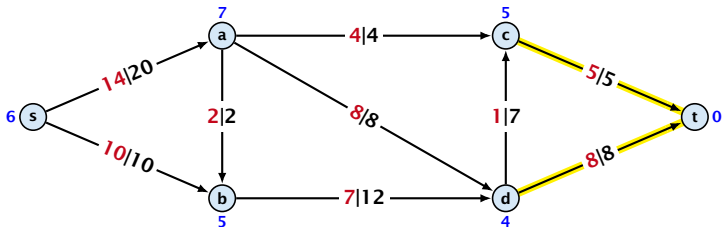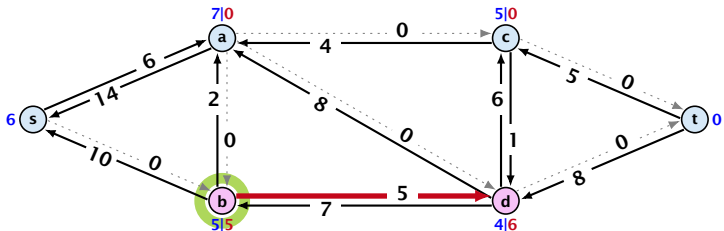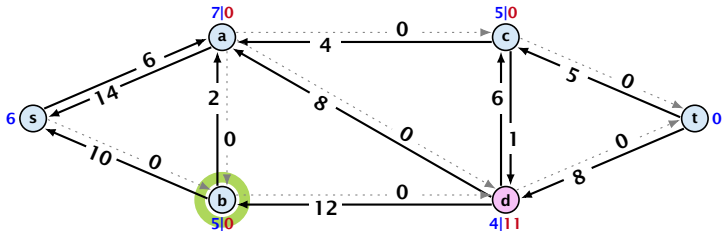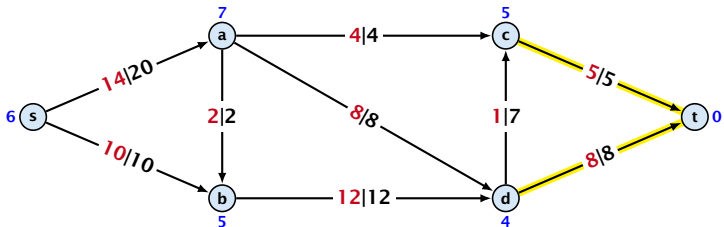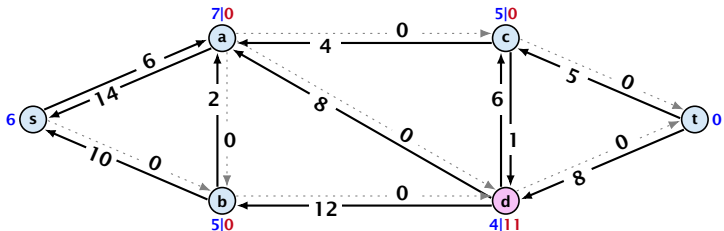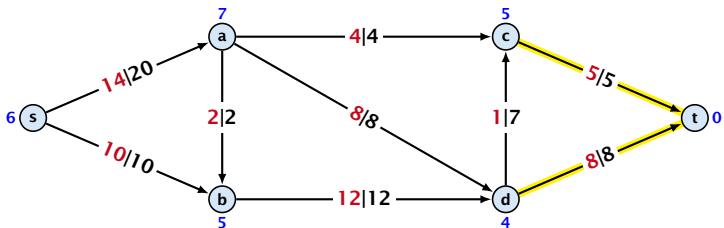
Preflow Push
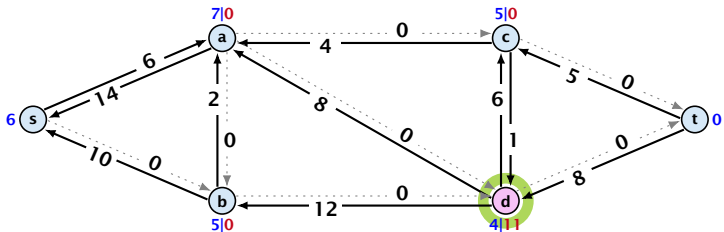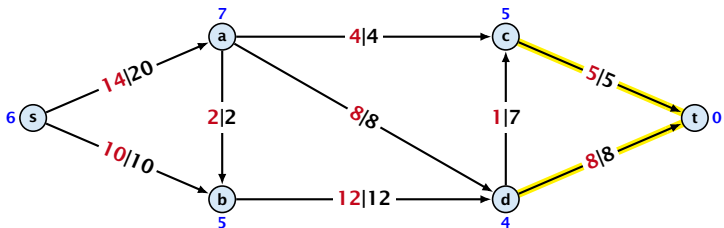
Preflow Push
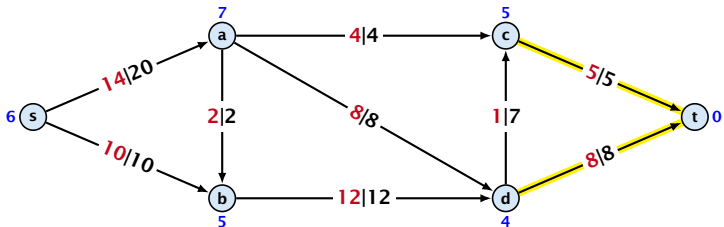
Preflow Push

relabel to 6

Preflow Push
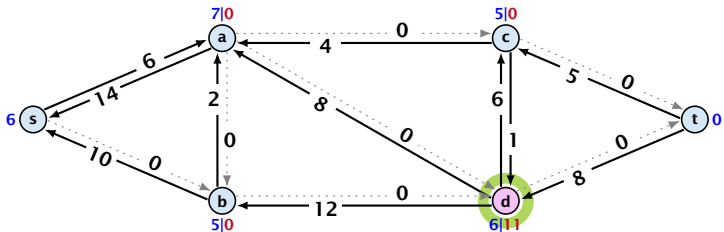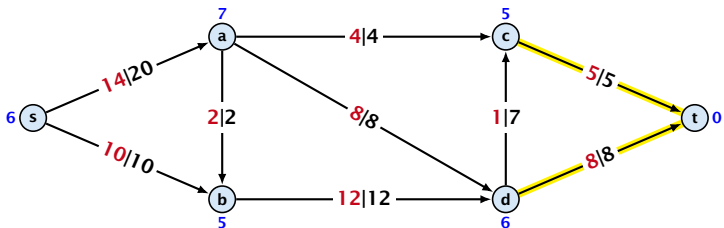
Preflow Push
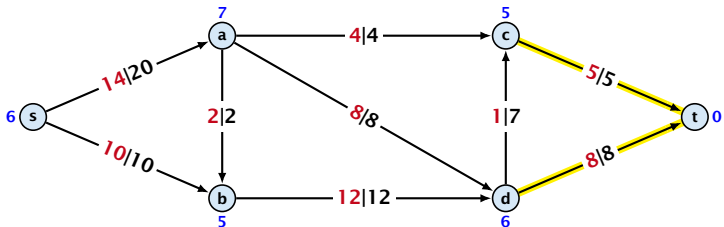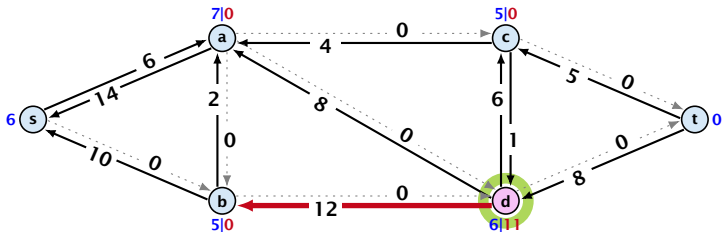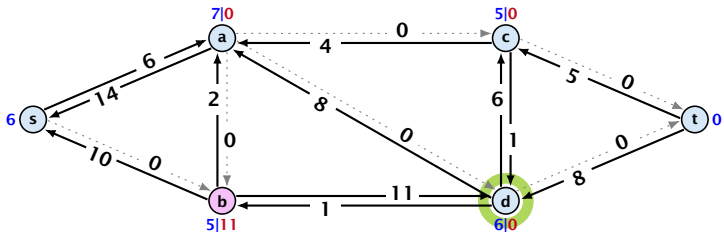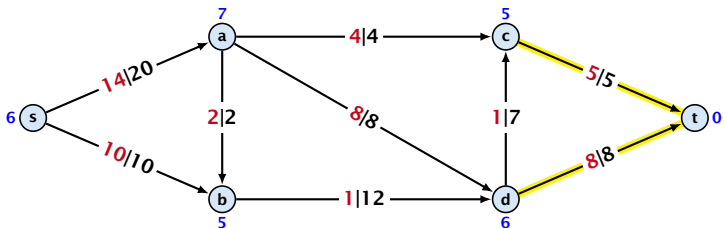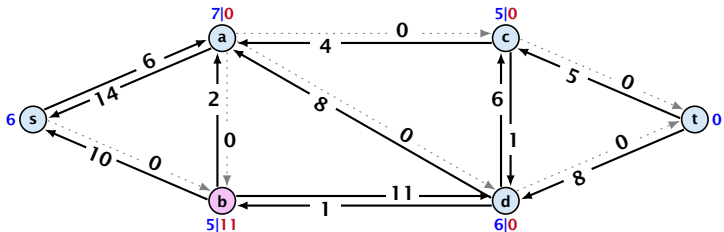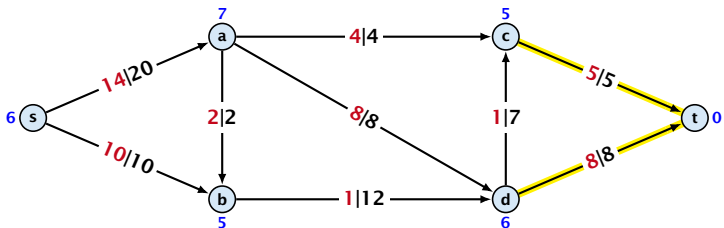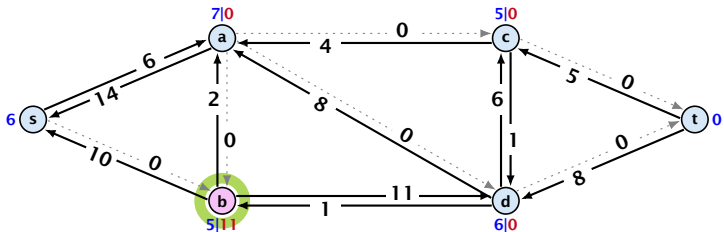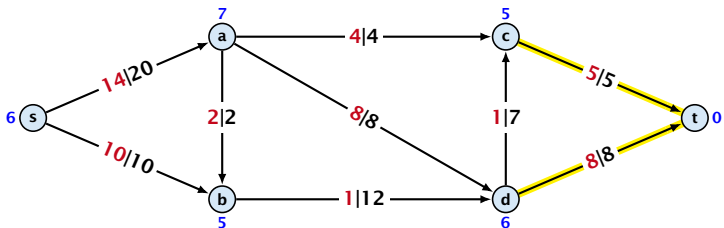
deactivating push

Preflow Push

Preflow Push

Preflow Push
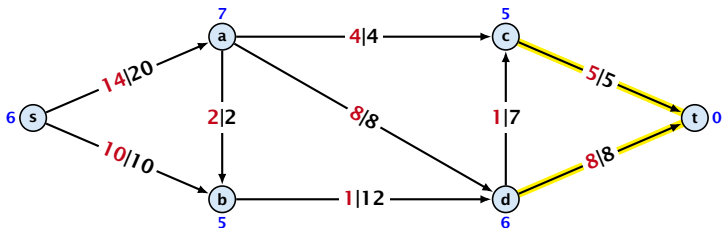
Preflow Push
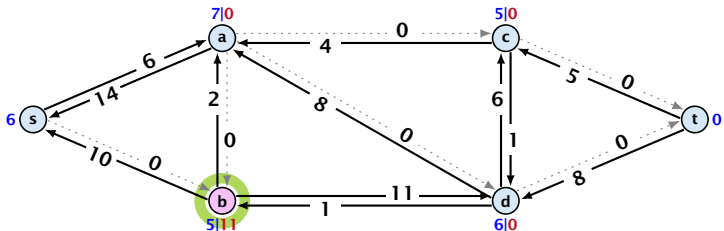
relabel to 7
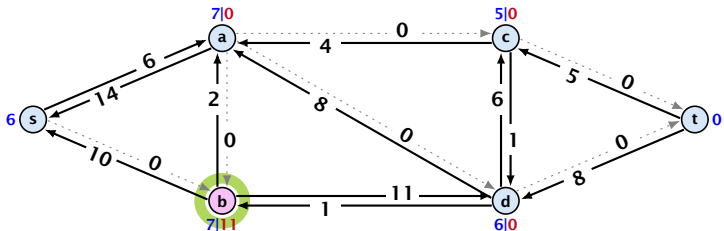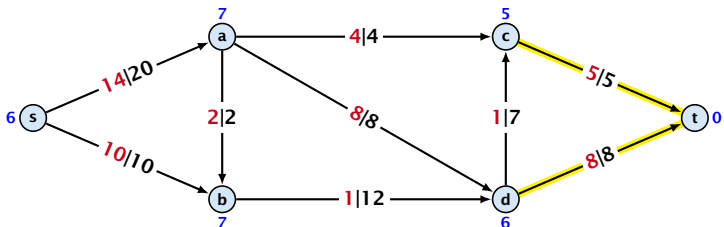
Preflow Push
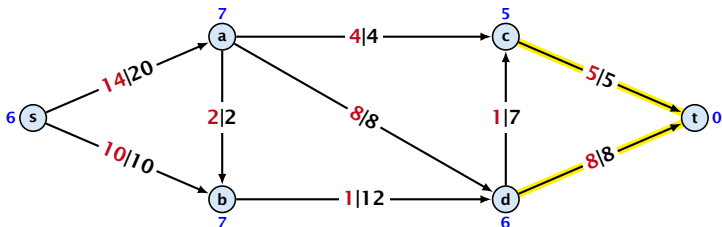
Preflow Push

saturating and deactivating push
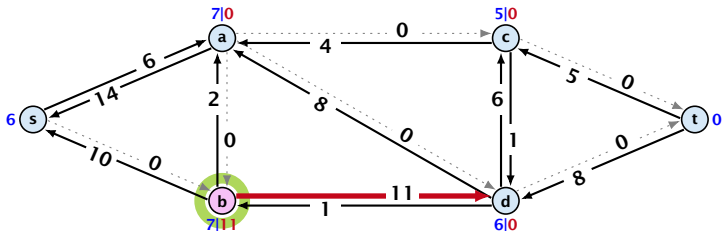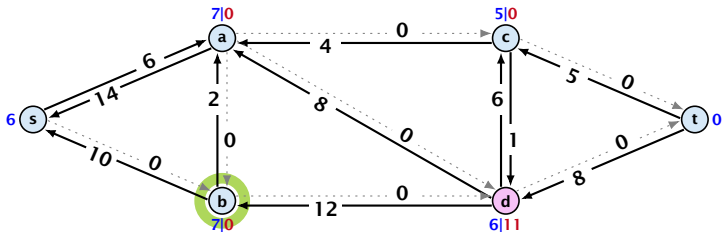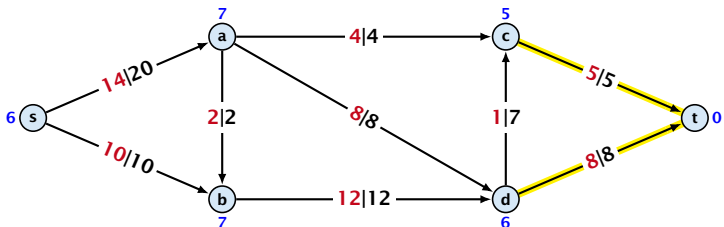
Preflow Push
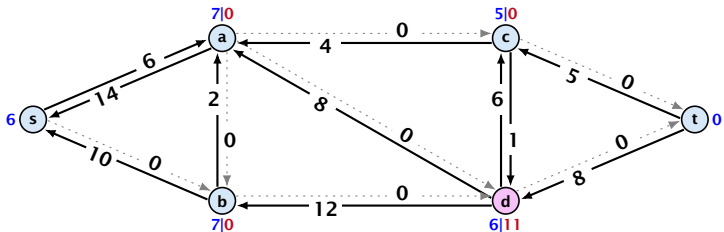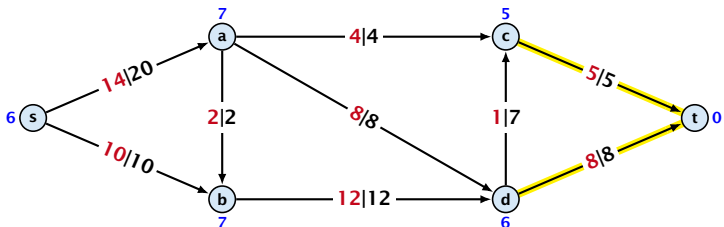
Preflow Push
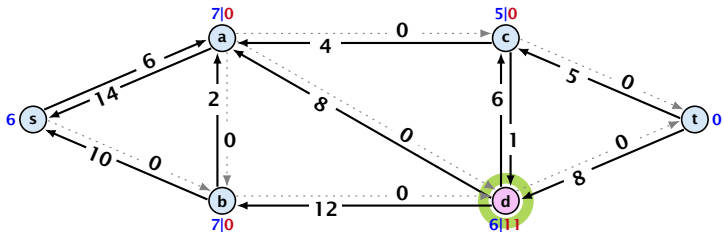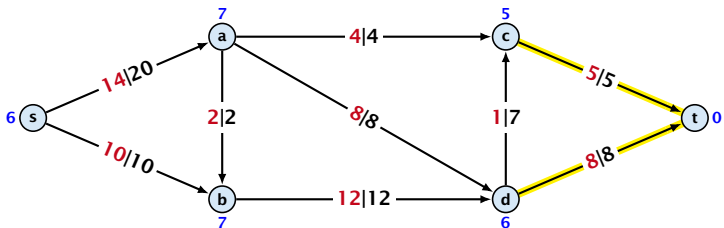
Preflow Push
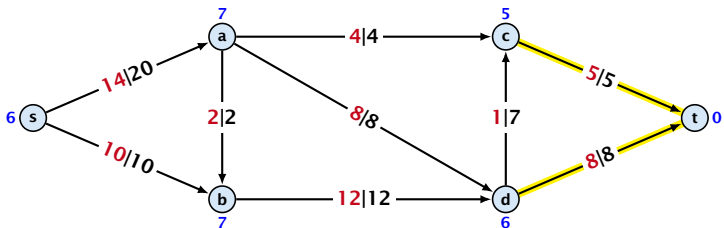
Preflow Push
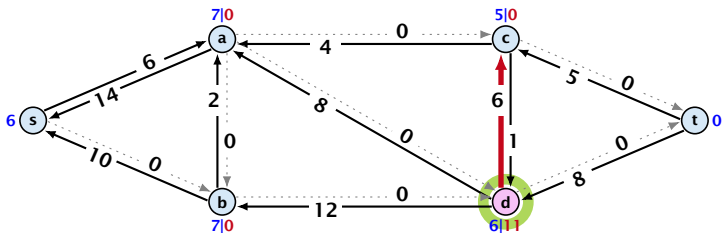
saturating push
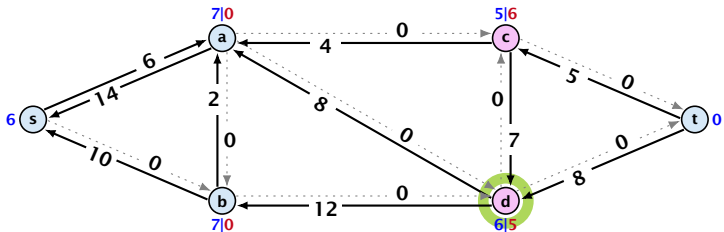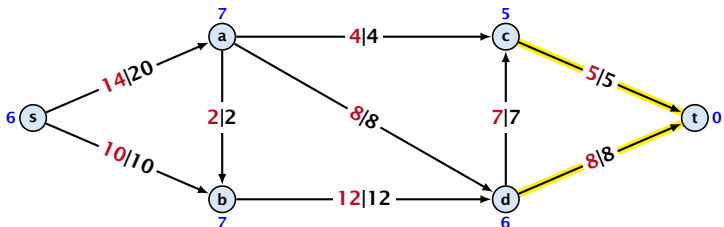
Preflow Push
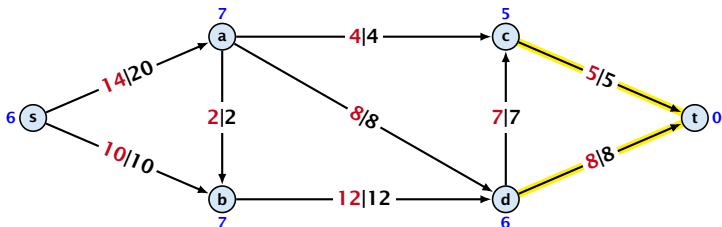
Preflow Push
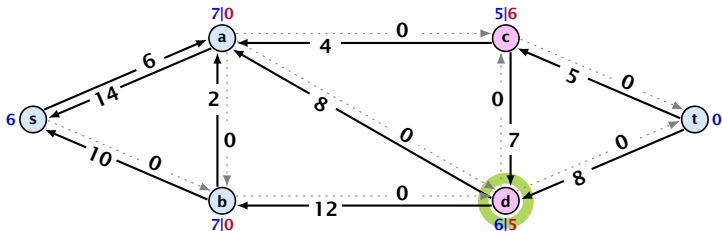
relabel to 8
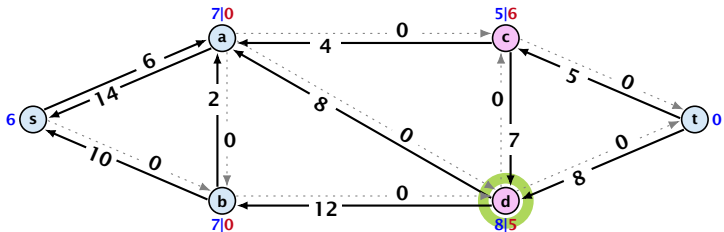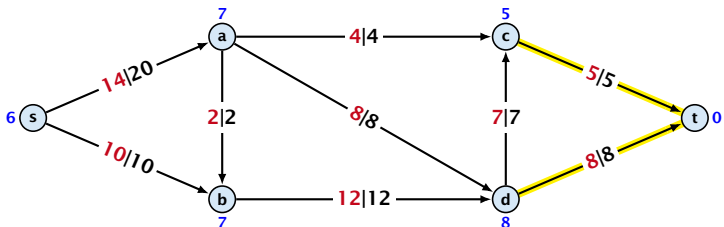
Preflow Push
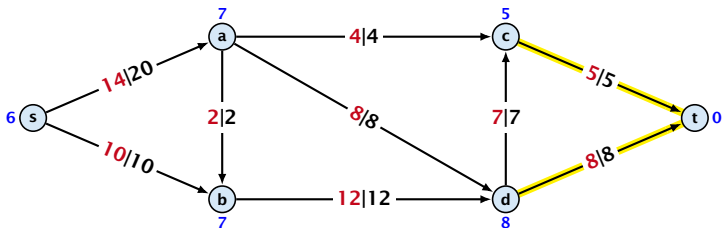
Preflow Push
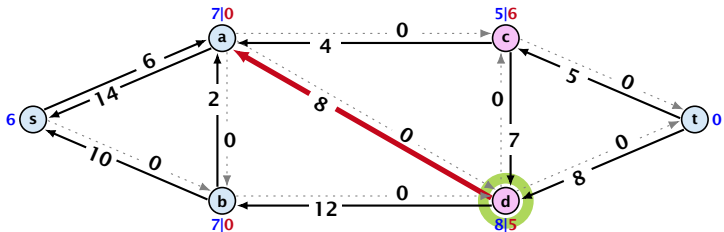
deactivating push
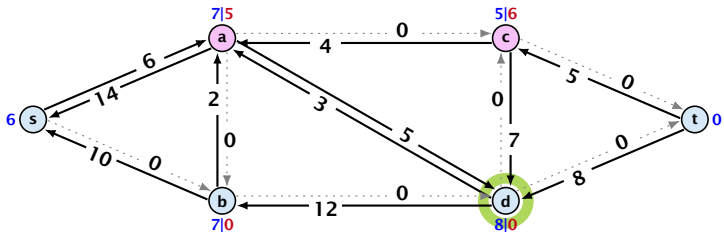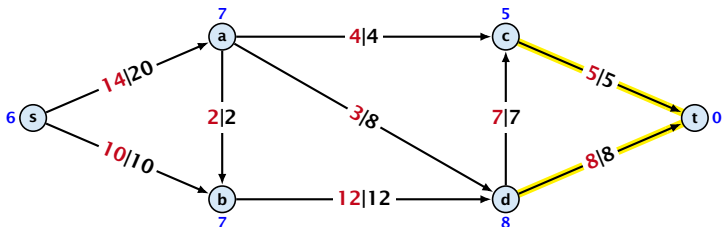
Preflow Push
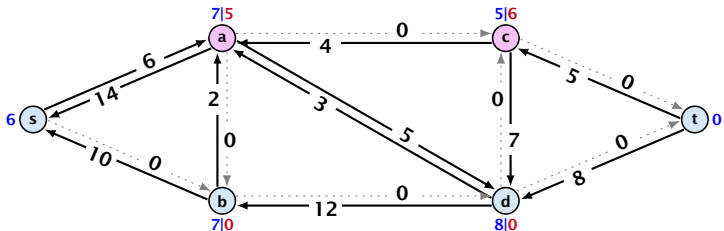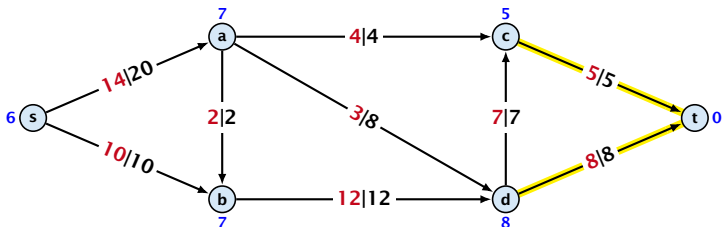
Preflow Push
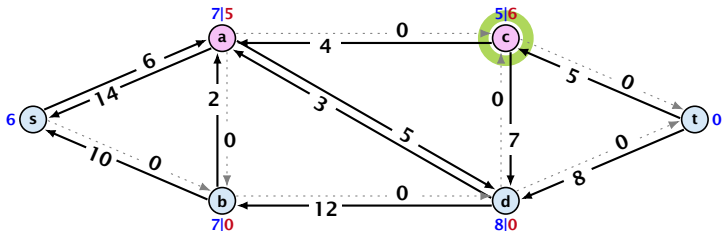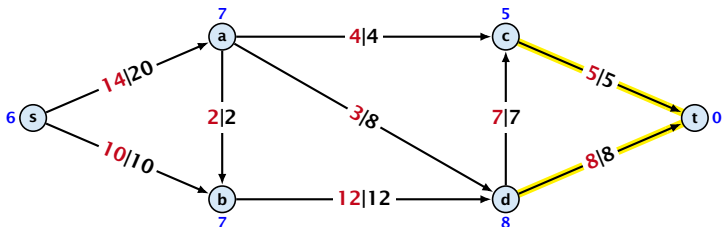
Preflow Push
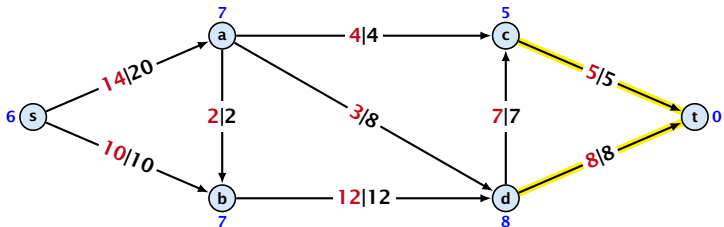
Preflow Push
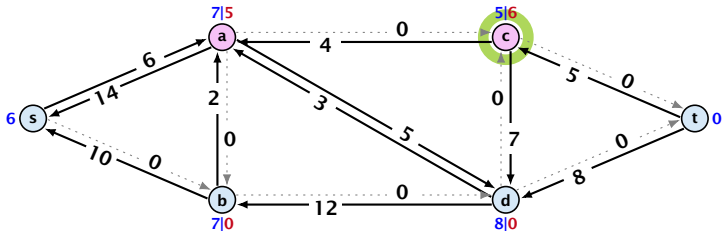
relabel to 8
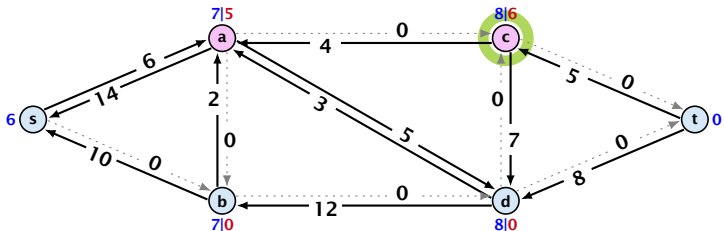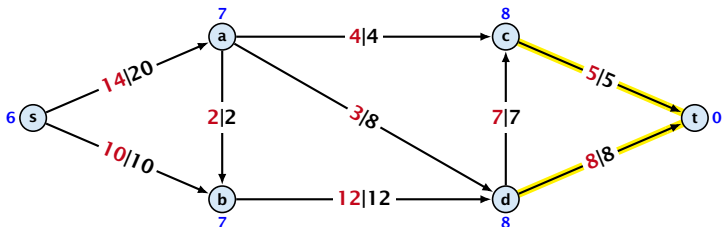
Preflow Push
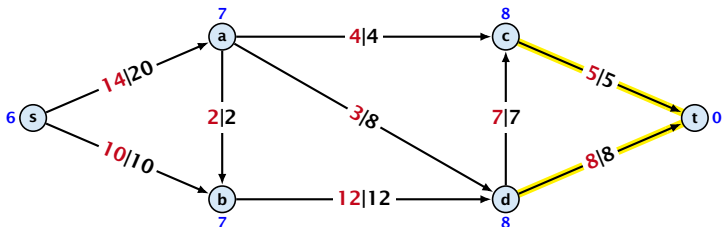
Preflow Push

saturating push
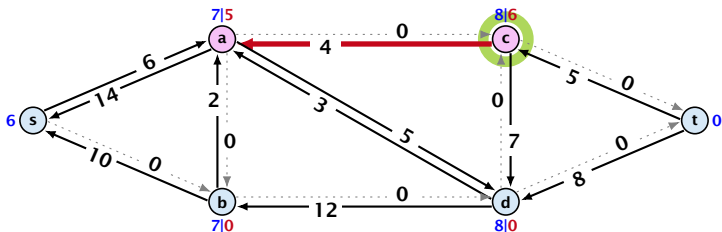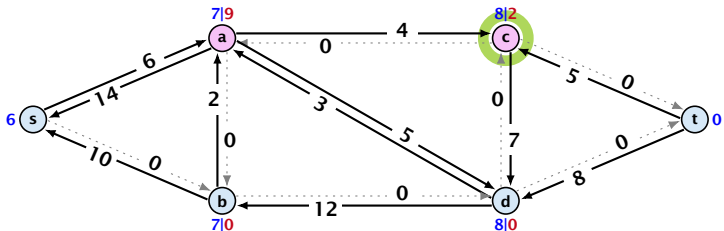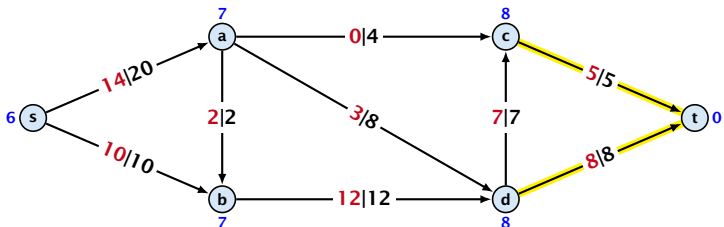
Preflow Push
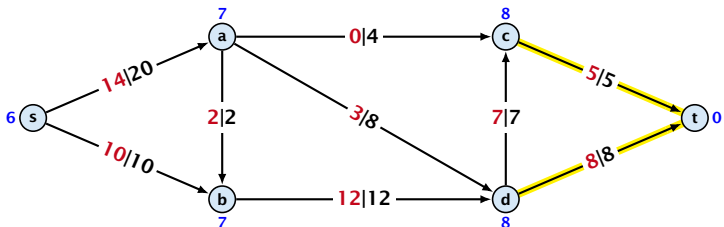
Preflow Push

relabel to 9
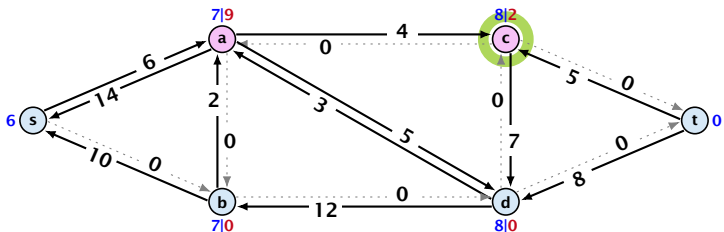
Preflow Push
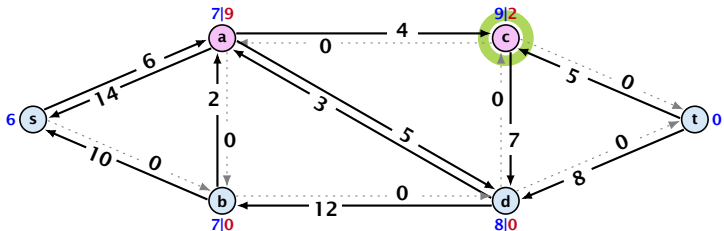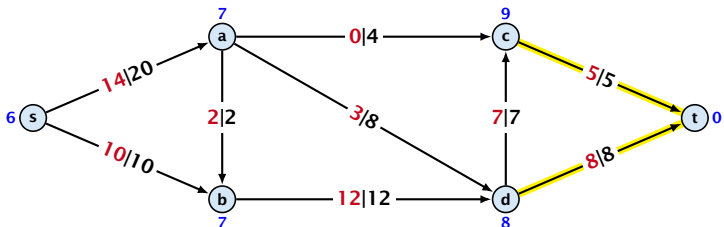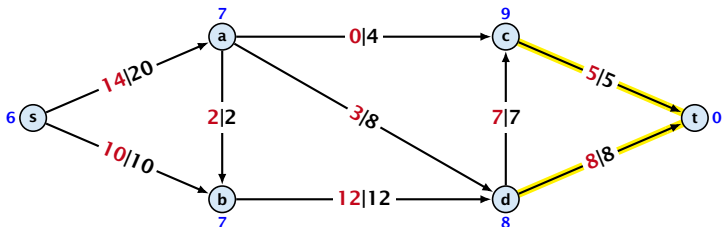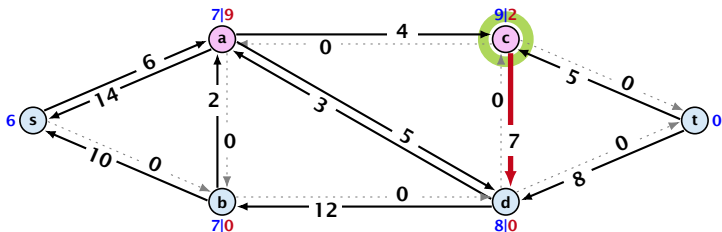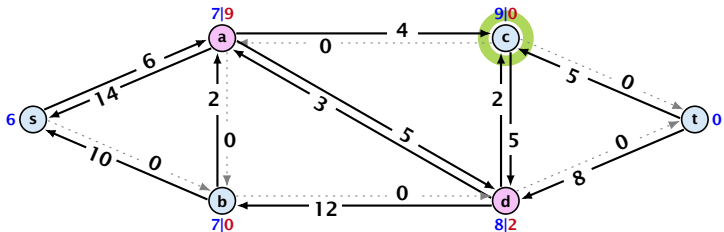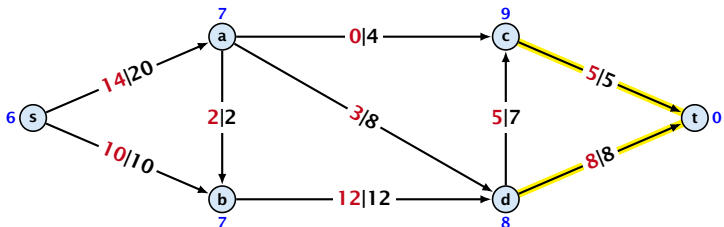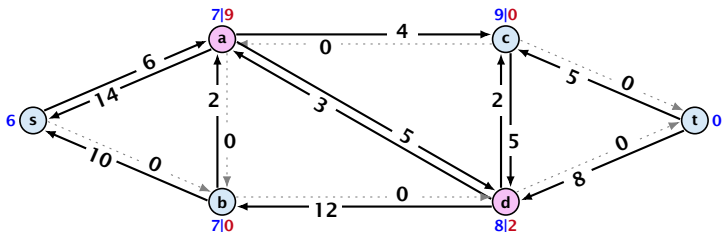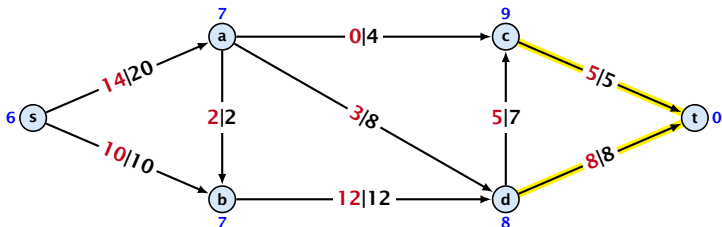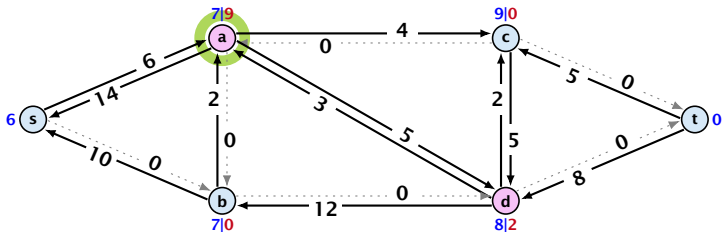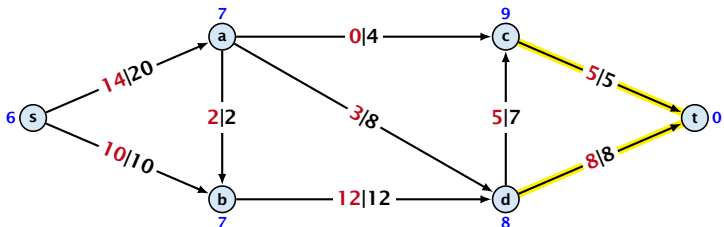
deactivating push

Preflow Push

Preflow Push

Preflow Push
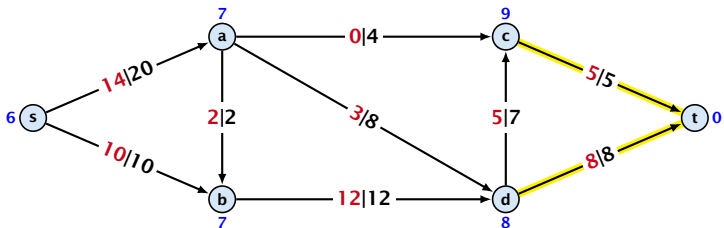
Preflow Push
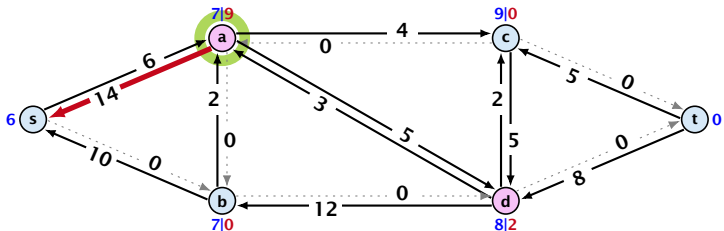
deactivating push
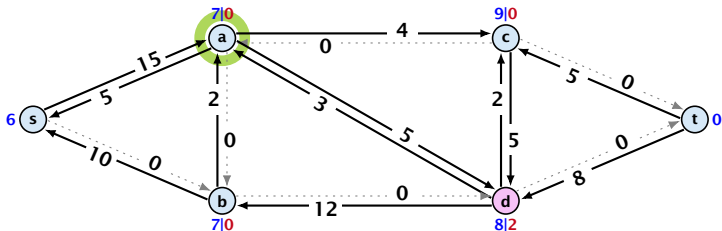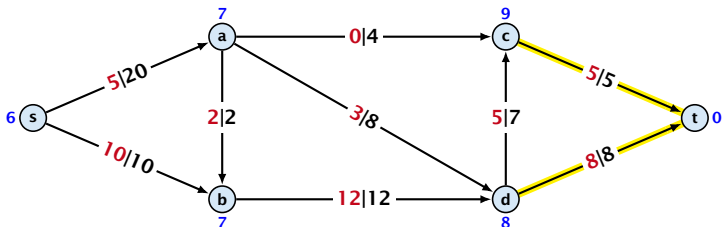
Preflow Push
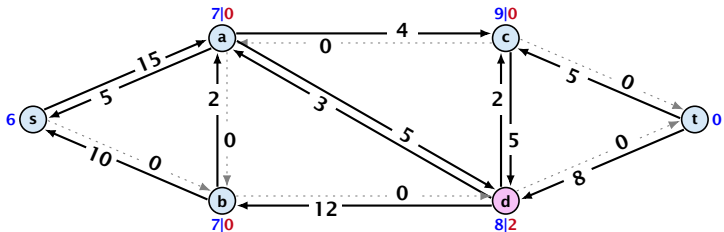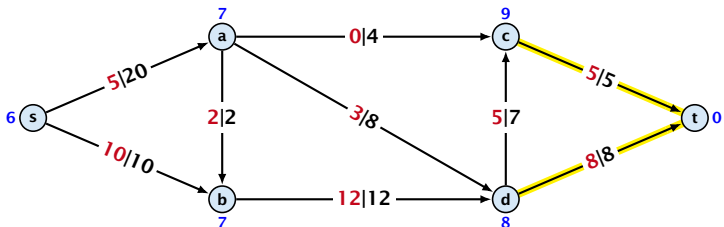
Preflow Push
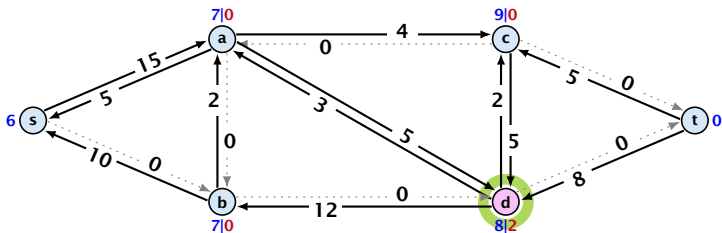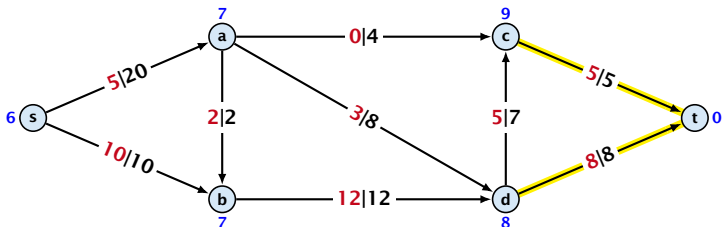
Preflow Push
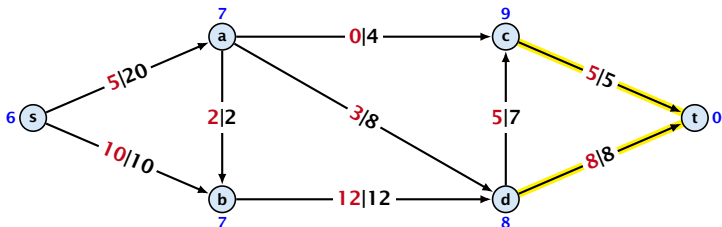
Preflow Push
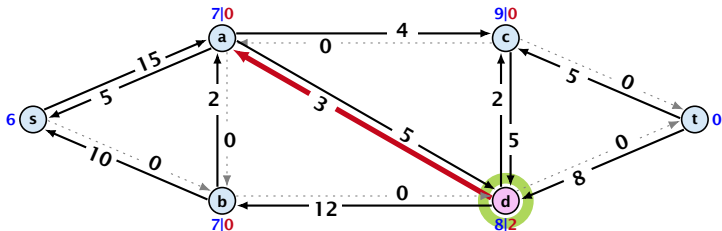
deactivating push
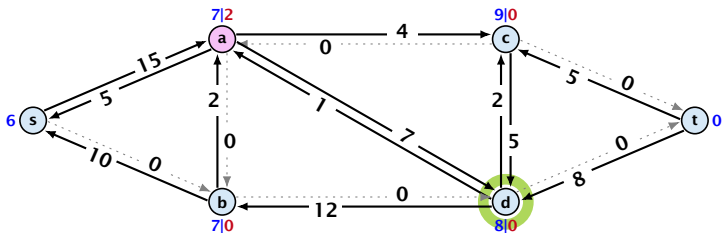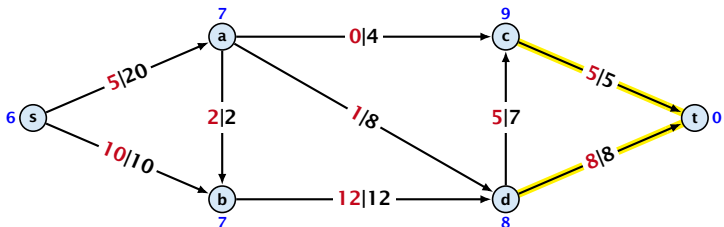
Preflow Push
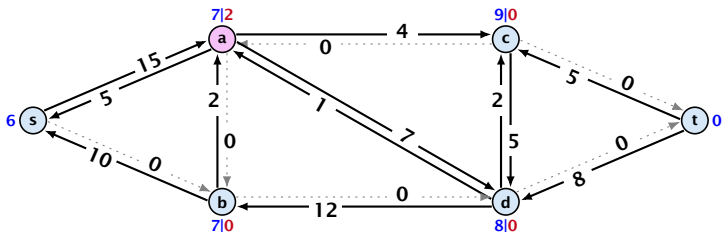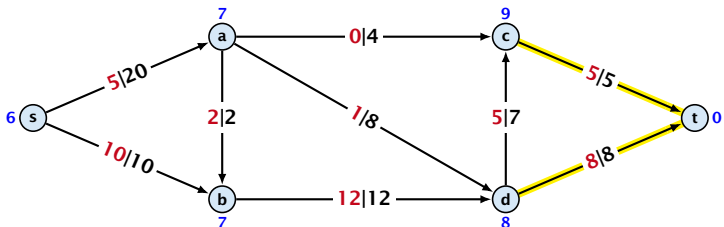
Preflow Push
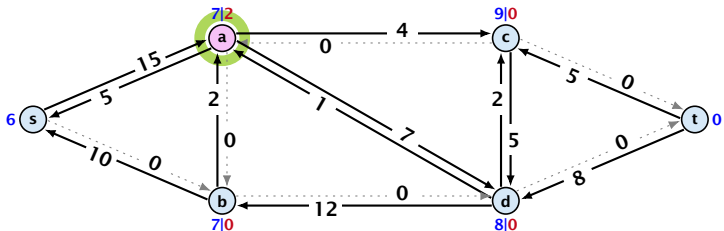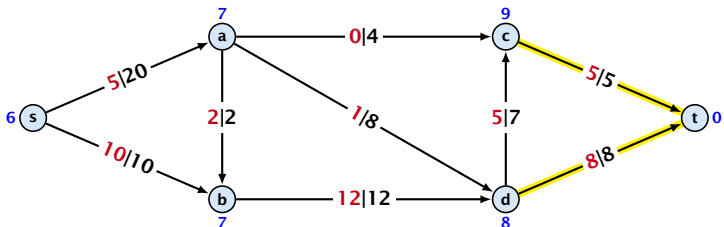
Preflow Push
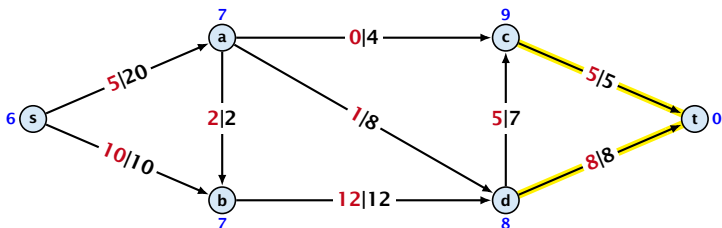
Preflow Push

deactivating push
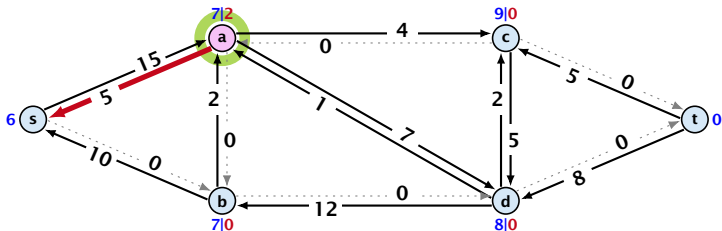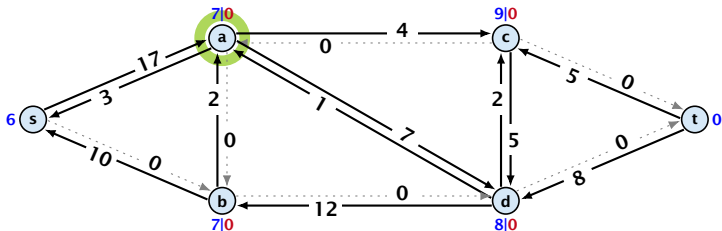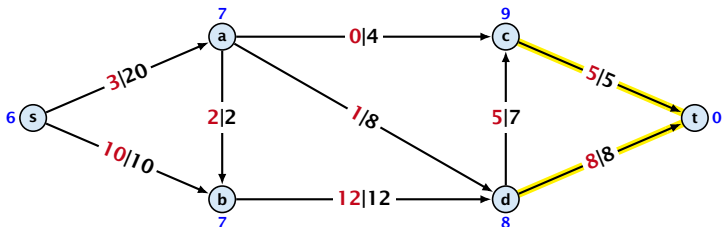
Preflow Push

Preflow Push
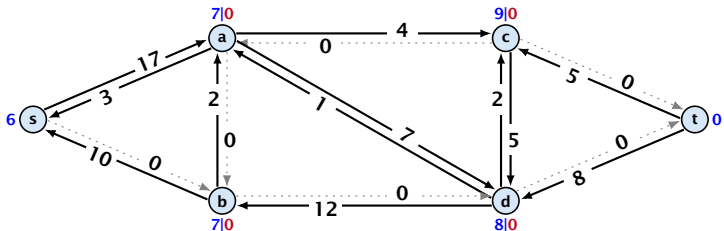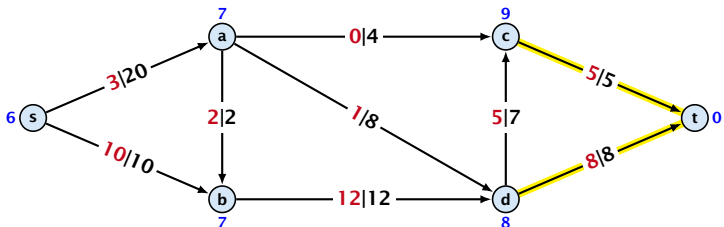
Analysis

Lemma 69

An active node has a path to s in the residual graph.

Analysis

Lemma 69

An active node has a path to s in the residual graph.

Proof.

▶ Let A denote the set of nodes that can reach s, and let B denote the remaining nodes. Note that $s \in A$.

Analysis

Lemma 69

An active node has a path to s in the residual graph.

Proof.

▶ Let A denote the set of nodes that can reach s, and let B denote the remaining nodes. Note that $s \in A$.

▶ In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.

Analysis

Lemma 69

An active node has a path to s in the residual graph.

Proof.

▶ Let A denote the set of nodes that can reach s, and let B denote the remaining nodes. Note that $s \in A$.

▶ In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.

▶ In the residual graph there are no edges into A, and, hence, no edges leaving A/entering B can carry any flow.

Analysis

Lemma 69

An active node has a path to s in the residual graph.

Proof.

▶ Let A denote the set of nodes that can reach s, and let B denote the remaining nodes. Note that $s \in A$.

▶ In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.

▶ In the residual graph there are no edges into A, and, hence, no edges leaving A/entering B can carry any flow.

▶ Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$f(B)$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$f(B) = \sum_{b \in B} f(b)$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$\begin{aligned} f(B) &= \sum_{b \in B} f(b) \\ &= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right)
\end{aligned}
$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \sum_{b \in B} \sum_{v \in A} f(v, b) - \sum_{b \in B} \sum_{v \in A} f(b, v) + \sum_{b \in B} \sum_{v \in B} f(v, b) - \sum_{b \in B} \sum_{v \in B} f(b, v)
\end{aligned}
$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \sum_{b \in B} \sum_{v \in A} f(v, b) - \sum_{b \in B} \sum_{v \in A} f(b, v) + \underbrace{\sum_{b \in B} \sum_{v \in B} f(v, b) - \sum_{b \in B} \sum_{v \in B} f(b, v)}_{= 0}
\end{aligned}
$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x,y) = \begin{cases} 0 & (x,y) \notin E \\ f((x,y)) & (x,y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v,b) - \sum_{v \in V} f(b,v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v,b) + \sum_{v \in B} f(v,b) - \sum_{v \in A} f(b,v) - \sum_{v \in B} f(b,v) \right) \\
&= \sum_{b \in B} \sum_{v \in A} f(v,b) - \sum_{b \in B} \sum_{v \in A} f(b,v)
\end{aligned}
$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \sum_{b \in B} \sum_{v \in A} \boxed{f(v, b)} - \sum_{b \in B} \sum_{v \in A} f(b, v) \\
&\qquad\qquad \mathbf{= 0}
\end{aligned}
$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} f(b, v)
\end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} \boxed{f(b, v)} \\
&\qquad\qquad\qquad\qquad\quad \geq 0
\end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} f(b, v)
\end{aligned}
$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} f(b, v) \\
&\leq 0
\end{aligned}$$

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$
\begin{aligned}
f(B) &= \sum_{b \in B} f(b) \\
&= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\
&= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\
&= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} f(b, v) \\
&\leq 0
\end{aligned}
$$

Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

Lemma 70

The label of a node cannot become larger than $2n - 1$.

Analysis

Lemma 70

The label of a node cannot become larger than $2n - 1$.

Proof.

▶ When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

Analysis

Lemma 70

The label of a node cannot become larger than $2n - 1$.

Proof.

▶ When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

Lemma 71

There are only $\mathcal{O}(n^2)$ relabel operations.

Analysis

Lemma 72
The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Analysis

Lemma 72

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

▶ Suppose that we just made a saturating push along (u, v).

Analysis

Lemma 72

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

- ▶ Suppose that we just made a saturating push along (u, v).
- ▶ Hence, the edge (u, v) is deleted from the residual graph.

Analysis

Lemma 72

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

- ▶ Suppose that we just made a saturating push along (u, v).
- ▶ Hence, the edge (u, v) is deleted from the residual graph.
- ▶ For the edge to appear again, a push from v to u is required.

Analysis

Lemma 72

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

▶ Suppose that we just made a saturating push along (u, v).

▶ Hence, the edge (u, v) is deleted from the residual graph.

▶ For the edge to appear again, a push from v to u is required.

▶ Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.

Analysis

Lemma 72

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

- ▶ Suppose that we just made a saturating push along (u, v).
- ▶ Hence, the edge (u, v) is deleted from the residual graph.
- ▶ For the edge to appear again, a push from v to u is required.
- ▶ Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.
- ▶ For a push from v to u the edge (v, u) must become admissible. The label of v must increase by at least 2.

Analysis

Lemma 72

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

- ▶ Suppose that we just made a saturating push along (u, v).
- ▶ Hence, the edge (u, v) is deleted from the residual graph.
- ▶ For the edge to appear again, a push from v to u is required.
- ▶ Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.
- ▶ For a push from v to u the edge (v, u) must become admissible. The label of v must increase by at least 2.
- ▶ Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

Lemma 73

The number of deactivating pushes performed is at most $\mathcal{O}(n^2 m)$.

Lemma 73

The number of deactivating pushes performed is at most
$\mathcal{O}(n^2 m)$.

Proof.

▶ Define a potential function $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$

Lemma 73

The number of deactivating pushes performed is at most $\mathcal{O}(n^2 m)$.

Proof.

▶ Define a potential function $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$

▶ A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).

Lemma 73

The number of deactivating pushes performed is at most
$\mathcal{O}(n^2 m)$.

Proof.

▶ Define a potential function $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$

▶ A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).

▶ A relabel increases Φ by at most 1.

Lemma 73

The number of deactivating pushes performed is at most
$\mathcal{O}(n^2 m)$.

Proof.

▶ Define a potential function $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$

▶ A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).

▶ A relabel increases Φ by at most 1.

▶ A deactivating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.

Lemma 73

The number of deactivating pushes performed is at most $\mathcal{O}(n^2 m)$.

Proof.

▶ Define a potential function $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$

▶ A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).

▶ A relabel increases Φ by at most 1.

▶ A deactivating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.

▶ Hence,

#deactivating_pushes $\leq$ #relabels $+ 2n \cdot$ #saturating_pushes

$$\leq \mathcal{O}(n^2 m) \ .$$

Analysis

Theorem 74

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2 m)$.

Analysis

Proof:

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible

Analysis

Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible
- ▶ check for all incoming edges if they become non-admissible

Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 2 discharge(u)

1: **while** u is active **do**
2:     $v \leftarrow u.current\text{-}neighbour$
3:     **if** $v = $ null **then**
4:         relabel(u)
5:         $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
6:     **else**
7:         **if** (u, v) admissible **then** push(u, v)
8:         **else** $u.current\text{-}neighbour \leftarrow v.next\text{-}in\text{-}list$

Note that $u.current\text{-}neighbour$ is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 75

If $v =$ null in Line 3, then there is no
outgoing admissible edge from u.

Proof.

▶ While pushing from u the current-neighbour pointer is only
  advanced if the current edge is not admissible.

▶ The only thing that could make the edge admissible again
  would be a relabel at u.

▶ If we reach the end of the list ($v =$ null) all edges are not
  admissible.   □

This shows that discharge(u) is correct, and that we can perform
a relabel in Line 4.

13.2 Relabel to Front

Algorithm 1 relabel-to-front(G, s, t)

1: initialize preflow
2: initialize node list L containing $V \setminus \{s, t\}$ in any order
3: **foreach** $u \in V \setminus \{s, t\}$ **do**
4:      $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
5: $u \leftarrow L.head$
6: **while** $u \neq$ null **do**
7:      $old\text{-}height \leftarrow \ell(u)$
8:      discharge(u)
9:      **if** $\ell(u) > old\text{-}height$ **then** // relabel happened
10:          move u to the front of L
11:      $u \leftarrow u.next$

13.2 Relabel to Front

Lemma 76 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L.*

2. *No node before u in the list L is active.*

Proof:

- ▶ Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active

- ▶ Maintenance:
  1. ▶ Pushes do no create any new admissible edges. Therefore, if discharge() does not relabel u, L is still topologically sorted.
     ▶ After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
     Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

13.2 Relabel to Front

Proof:

▶ Maintenance:

  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

  For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to Front

Lemma 77

There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to Front

Lemma 78

The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the
label and resetting *u.current-neighbour*). In total we have $\mathcal{O}(n^2)$
relabel-operations.

13.2 Relabel to Front

Recall that a saturating push operation
($\min\{c_f(e), f(u)\} = c_f(e)$) can also be a deactivating push
operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 79
*The cost for all saturating push-operations that are **not**
deactivating is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but
makes the edge e disappear from the residual graph. Therefore
the push-operation is immediately followed by an increase of the
pointer $u.current\text{-}neighbour$.
This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times
(upper bound on number of relabels) and the neighbour-list has
only $degree(u) + 1$ many entries ($+1$ for null-entry).

13.2 Relabel to Front

Lemma 80

The cost for all deactivating push-operations is only $\mathcal{O}(n^3)$.

A deactivating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 81

The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Algorithm 1 highest-label(G, s, t)

1: initialize preflow
2: **foreach** $u \in V \setminus \{s, t\}$ **do**
3:     $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$
4: **while** $\exists$ active node u **do**
5:     select active node u with highest label
6:     discharge(u)

13.3 Highest Label

Lemma 82
When using highest label the number of deactivating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only "activate" nodes on levels strictly less than ℓ.

This means, after a deactivating push from u a relabel is required to make u active again.

Hence, after n deactivating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of deactivating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest Label

Since a discharge-operation is terminated by a deactivating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:

How do we find the next node for a discharge operation?

13.3 Highest Label

Maintain lists L_i, $i \in \{0, \ldots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \ldots, L_0$, (in that order) until you find a non-empty list.

Unless the last (deactivating) push was to s or t the list $k - 1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#\textit{deactivating-pushes-to-s-or-t})$$

Lemma 83

The number of deactivating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 84

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

13.3 Highest Label

Proof of the Lemma.

▶ We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.

▶ After a node v (which must have $\ell(v) = n + 1$) made a deactivating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.

▶ This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.

▶ As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

Mincost Flow

Problem Definition:

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E: \ 0 \le f(e) \le u(e)$$
$$\quad \forall v \in V: \ f(v) = b(v)$$

Mincost Flow

Problem Definition:

$$
\begin{aligned}
\min \quad & \sum_e c(e)f(e) \\
\text{s.t.} \quad & \forall e \in E: \ 0 \le f(e) \le u(e) \\
& \forall v \in V: \ f(v) = b(v)
\end{aligned}
$$

▶ $G = (V, E)$ is a directed graph.

Mincost Flow

Problem Definition:

$$
\begin{aligned}
\min \quad & \sum_e c(e) f(e) \\
\text{s.t.} \quad & \forall e \in E : \ 0 \le f(e) \le u(e) \\
& \forall v \in V : \ f(v) = b(v)
\end{aligned}
$$

- ▶ $G = (V, E)$ is a directed graph.
- ▶ $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.

Mincost Flow

Problem Definition:

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E: \ 0 \le f(e) \le u(e)$$
$$\forall v \in V: \ f(v) = b(v)$$

▶ $G = (V, E)$ is a directed graph.

▶ $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.

▶ $c : E \to \mathbb{R}$ is the cost function
  (note that $c(e)$ may be negative).

Mincost Flow

Problem Definition:

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E: \ 0 \le f(e) \le u(e)$$
$$\forall v \in V: \ f(v) = b(v)$$

▶ $G = (V, E)$ is a directed graph.

▶ $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.

▶ $c : E \to \mathbb{R}$ is the cost function
   (note that $c(e)$ may be negative).

▶ $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow
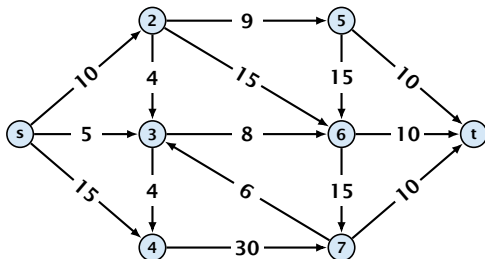
Solve Maxflow Using Mincost Flow
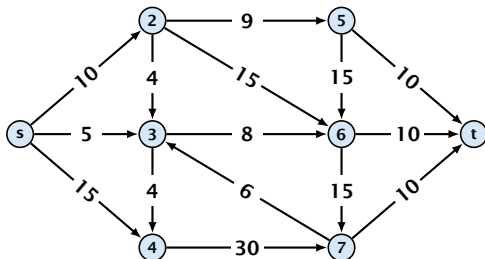
▶ Given a flow network for a standard maxflow problem.

Solve Maxflow Using Mincost Flow
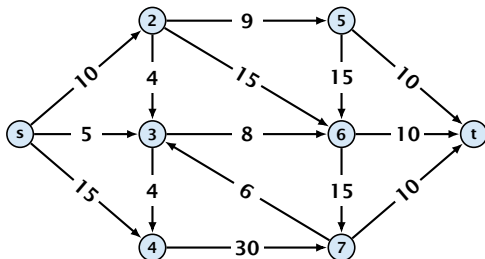
- ▶ Given a flow network for a standard maxflow problem.
- ▶ Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

Solve Maxflow Using Mincost Flow
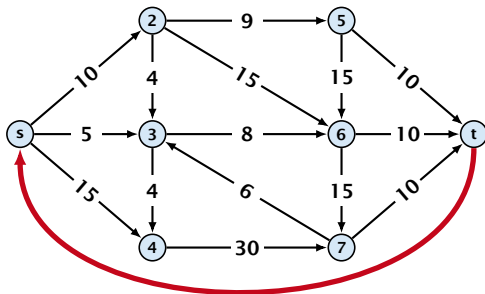
- ▶ Given a flow network for a standard maxflow problem.
- ▶ Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- ▶ Add an edge from t to s with infinite capacity and cost -1.

Solve Maxflow Using Mincost Flow
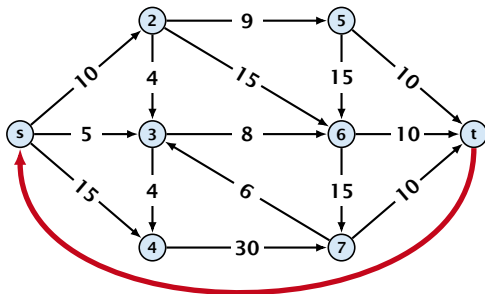
- ▶ Given a flow network for a standard maxflow problem.
- ▶ Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- ▶ Add an edge from t to s with infinite capacity and cost -1.
- ▶ Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

▶ Given a flow network for a standard maxflow problem, and a value k.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

▶ Given a flow network for a standard maxflow problem, and a value k.

▶ Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

▶ Given a flow network for a standard maxflow problem, and a value k.

▶ Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.

▶ Set edge-costs to zero, and keep the capacities.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

▶ Given a flow network for a standard maxflow problem, and a value k.

▶ Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.

▶ Set edge-costs to zero, and keep the capacities.

▶ There exists a maxflow of value at least k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$
\begin{aligned}
\min \quad & \textstyle\sum_e c(e) f(e) \\
\text{s.t.} \quad & \forall e \in E : \ 0 \le f(e) \le u(e) \\
& \forall v \in V : \ f(v) = b(v)
\end{aligned}
$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

Generalization

Our model:

$$\begin{aligned}
\min \quad & \sum_e c(e) f(e) \\
\text{s.t.} \quad & \forall e \in E : \ 0 \le f(e) \le u(e) \\
& \forall v \in V : \ f(v) = b(v)
\end{aligned}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

A more general model?

$$\begin{aligned}
\min \quad & \sum_e c(e) f(e) \\
\text{s.t.} \quad & \forall e \in E : \ \ell(e) \le f(e) \le u(e) \\
& \forall v \in V : \ a(v) \le f(v) \le b(v)
\end{aligned}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$
$c : E \to \mathbb{R}$;

Generalization

Differences

▶ Flow along an edge e may have non-zero lower bound $\ell(e)$.

▶ Flow along e may have negative upper bound $u(e)$.

▶ The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

$$\min \quad \sum_e c(e) f(e)$$

$$\text{s.t.} \quad \forall e \in E: \ \ell(e) \le f(e) \le u(e)$$

$$\forall v \in V: \ a(v) \le f(v) \le b(v)$$

Reduction I

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V: \quad a(v) \le f(v) \le b(v)$$

We can assume that $a(v) = b(v)$:

Reduction I

min $\sum_e c(e)f(e)$

s.t. $\forall e \in E: \ \ell(e) \leq f(e) \leq u(e)$

$\forall v \in V: \ a(v) \leq f(v) \leq b(v)$

We can assume that $a(v) = b(v)$:

Reduction I

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V: \quad a(v) \le f(v) \le b(v)$$

We can assume that $a(v) = b(v)$:

Add new node r.

$$u(e) = b(v) - a(v)$$
$$\ell(e) = 0$$
$$c(e) = 0$$

Reduction I

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V: \quad a(v) \le f(v) \le b(v)$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

$$u(e) = b(v) - a(v)$$
$$\ell(e) = 0$$
$$c(e) = 0$$

Reduction I

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\quad \forall v \in V: \quad a(v) \le f(v) \le b(v)$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

$$u(e) = b(v) - a(v)$$
$$\ell(e) = 0$$
$$c(e) = 0$$

Reduction I

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V: \quad a(v) \le f(v) \le b(v)$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

$u(e) = b(v) - a(v)$
$\ell(e) = 0$
$c(e) = 0$

Reduction I

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V: \quad a(v) \le f(v) \le b(v)$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

$$u(e) = b(v) - a(v)$$
$$\ell(e) = 0$$
$$c(e) = 0$$

Reduction I

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E : \quad \ell(e) \le f(e) \le u(e)$$
$$\qquad \forall v \in V : \quad a(v) \le f(v) \le b(v)$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these
edges.

Set $u(e) = b(v) - a(v)$ for
edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = - \sum_{v \in V} b(v)$.

Reduction I

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E : \quad \ell(e) \leq f(e) \leq u(e)$$
$$\forall v \in V : \quad a(v) \leq f(v) \leq b(v)$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.

$u(e) = b(v) - a(v)$
$\ell(e) = 0$
$c(e) = 0$

Reduction II

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\quad \forall v \in V: \quad f(v) = b(v)$$

We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:

$$u(e) = \infty$$
$$\ell(e) = -\infty$$
$$c(e) = 0$$

Reduction II

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V: \quad f(v) = b(v)$$

We can assume that either $\ell(e) \ne -\infty$ or $u(e) \ne \infty$:

$$u(e) = \infty$$
$$\ell(e) = -\infty$$
$$c(e) = 0$$

If $c(e) = 0$ we can contract the edge/identify nodes u and v.
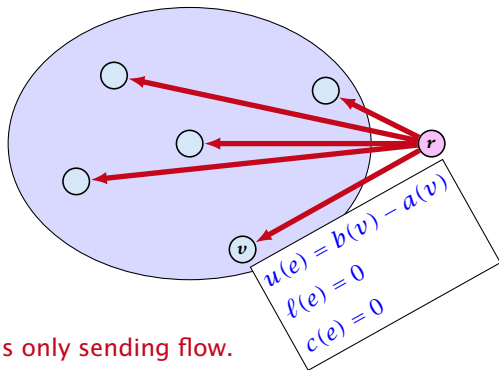
Reduction II

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{aligned}$$

We can assume that either $\ell(e) \ne -\infty$ or $u(e) \ne \infty$:

$u(e) = \infty$
$\ell(e) = -\infty$
$c(e) = 0$
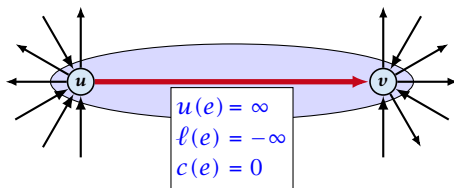
If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \ne 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:

$u(e) = \infty$
$\ell(e) = -\infty$
$c(e) = \delta \neq 0$
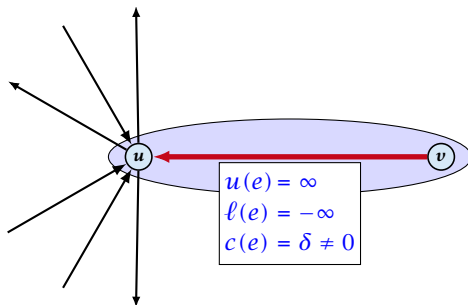
Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:

Reduction II
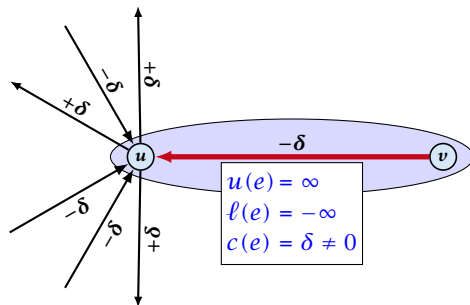
We can transform any network so that a particular edge has cost $c(e) = 0$:
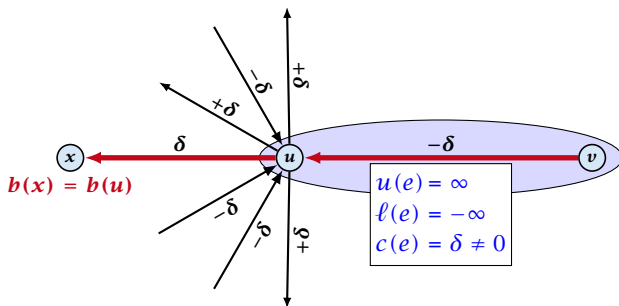
Additionally we set $b(u) = 0$.

Reduction III

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \leq f(e) \leq u(e)$$
$$\qquad \forall v \in V: \quad f(v) = b(v)$$

We can assume that $\ell(e) \neq -\infty$:

$$u(e) = d \neq \infty$$
$$\ell(e) = -\infty$$
$$c(e) = a$$

$$u(e) = \infty$$
$$\ell(e) = -d$$
$$c(e) = -a$$

Replace the edge by an edge in opposite direction.

Reduction IV

$$\min \quad \sum_e c(e) f(e)$$
$$\text{s.t.} \quad \forall e \in E: \quad \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V: \quad f(v) = b(v)$$

We can assume that $\ell(e) = 0$:
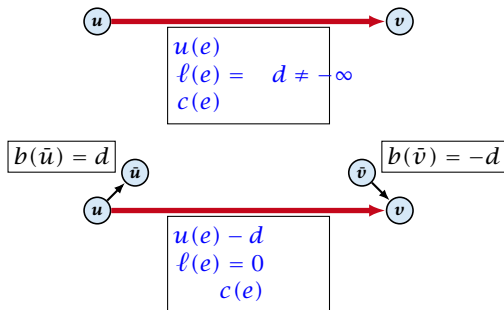
The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

▶ She needs to supply r_i napkins on N successive days.

Applications

Caterer Problem

▶ She needs to supply r_i napkins on N successive days.

▶ She can buy new napkins at p cents each.

Applications

Caterer Problem

▶ She needs to supply r_i napkins on N successive days.
▶ She can buy new napkins at p cents each.
▶ She can launder them at a fast laundry that takes m days and cost f cents a napkin.

Applications

Caterer Problem

▶ She needs to supply r_i napkins on N successive days.

▶ She can buy new napkins at p cents each.

▶ She can launder them at a fast laundry that takes m days and cost f cents a napkin.

▶ She can use a slow laundry that takes $k > m$ days and costs s cents each.

Applications

Caterer Problem

▶ She needs to supply r_i napkins on N successive days.

▶ She can buy new napkins at p cents each.

▶ She can launder them at a fast laundry that takes m days and cost f cents a napkin.

▶ She can use a slow laundry that takes $k > m$ days and costs s cents each.

▶ At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.

Applications

Caterer Problem

- ▶ She needs to supply r_i napkins on N successive days.
- ▶ She can buy new napkins at p cents each.
- ▶ She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- ▶ She can use a slow laundry that takes $k > m$ days and costs s cents each.
- ▶ At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- ▶ Minimize cost.
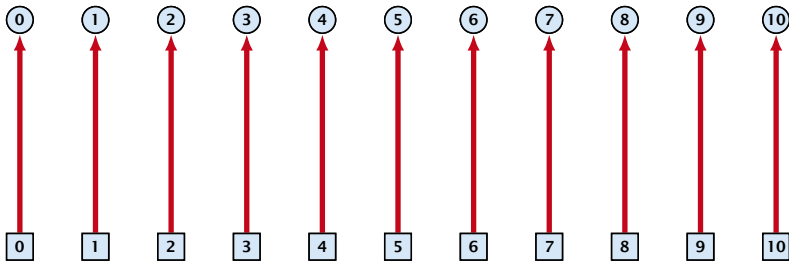
day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$
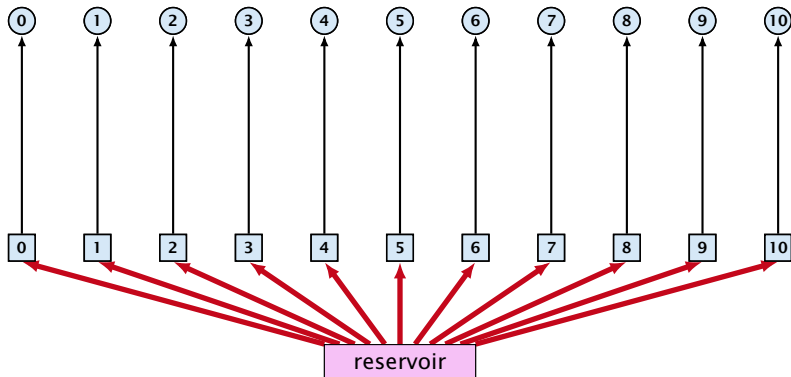
buy edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = p$
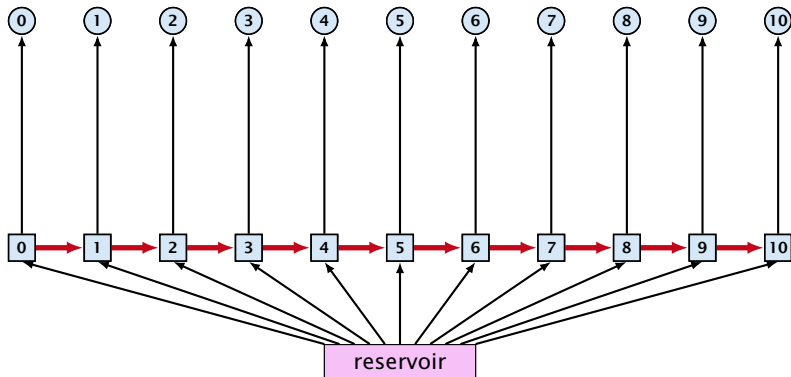
forward edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$
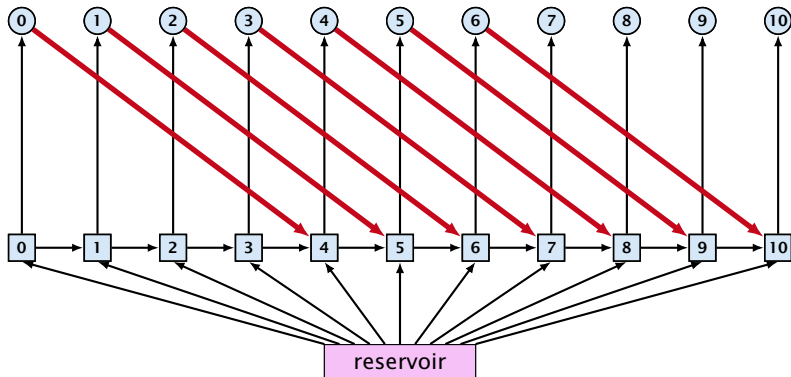
slow edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = s$
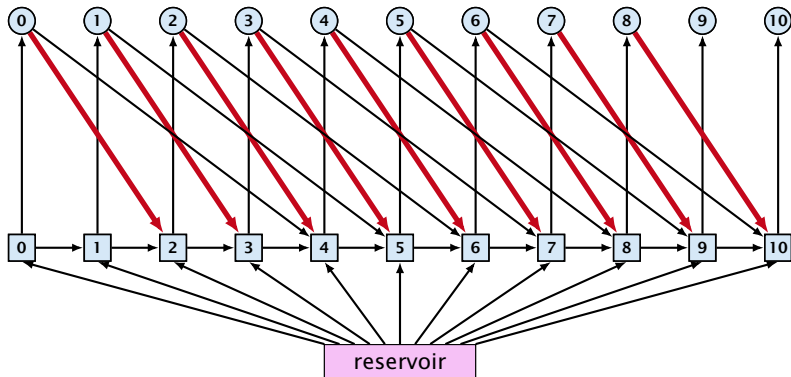
fast edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = f$
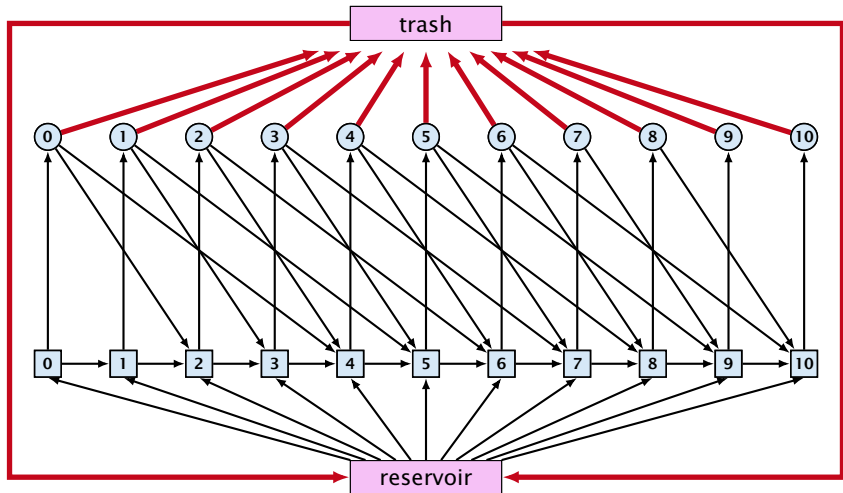
trash edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$
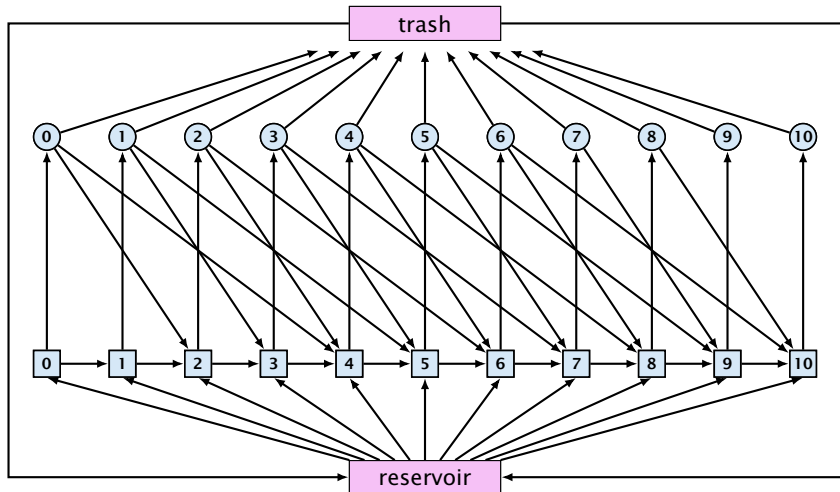
Residual Graph

Version A:
The residual graph G' for a mincost flow is just a copy of the graph G.

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

Residual Graph

Version A:

The residual graph G' for a mincost flow is just a copy of the graph G.

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

Version B:

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

14 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

14 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \leq u(e)$ for every edge of G.

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

$\Rightarrow$ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\mathrm{cost}(f) + \mathrm{cost}(g) < \mathrm{cost}(f)$. Hence, f is not minimum cost.

$\Leftarrow$ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

▶ Suppose that we have a negative cost circulation.

14 Mincost Flow

Lemma 86
A graph (without zero-capacity edges) has a feasible circulation of
negative cost if and only if it has a negative cycle w.r.t.
edge-weights $c : E \to \mathbb{R}$.

Proof.

▶ Suppose that we have a negative cost circulation.

▶ Find directed cycle only using edges that have non-zero flow.

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

▶ Suppose that we have a negative cost circulation.

▶ Find directed cycle only using edges that have non-zero flow.

▶ If this cycle has negative cost you are done.

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- ▶ Suppose that we have a negative cost circulation.
- ▶ Find directed cycle only using edges that have non-zero flow.
- ▶ If this cycle has negative cost you are done.
- ▶ Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

▶ Suppose that we have a negative cost circulation.

▶ Find directed cycle only using edges that have non-zero flow.

▶ If this cycle has negative cost you are done.

▶ Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.

▶ You still have a circulation with negative cost.

14 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

▶ Suppose that we have a negative cost circulation.

▶ Find directed cycle only using edges that have non-zero flow.

▶ If this cycle has negative cost you are done.

▶ Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.

▶ You still have a circulation with negative cost.

▶ Repeat.

14 Mincost Flow

Algorithm 48 CycleCanceling($G = (V, E), c, u, b$)

1: establish a feasible flow f in G
2: **while** G_f contains negative cycle **do**
3:      use Bellman-Ford to find a negative circuit Z
4:      $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
5:      augment δ units along Z and update G_f

How do we find the initial feasible flow?
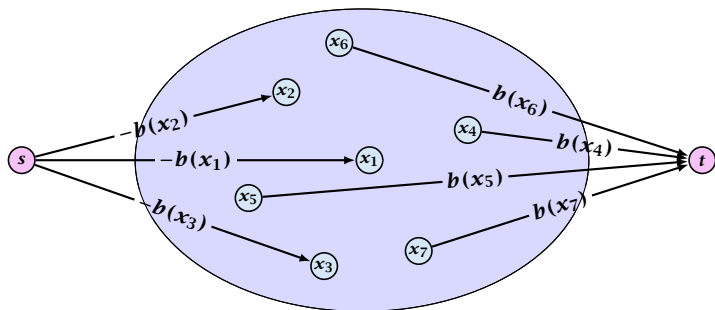
▶ Connect new node s to all nodes with negative $b(v)$-value.

▶ Connect nodes with positive $b(v)$-value to a new node t.

▶ There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \ .$$

14 Mincost Flow
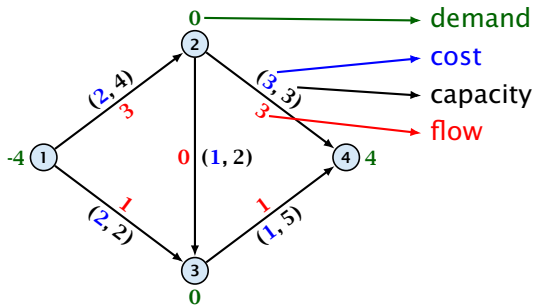
14 Mincost Flow
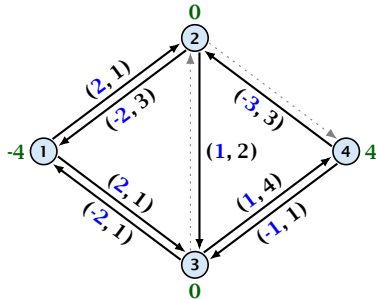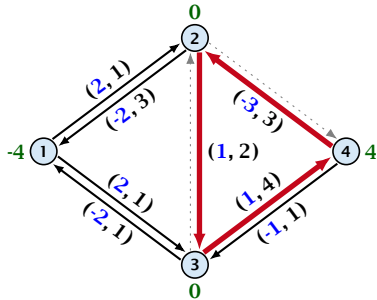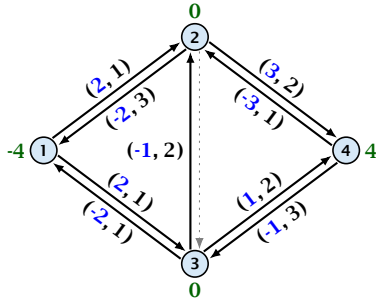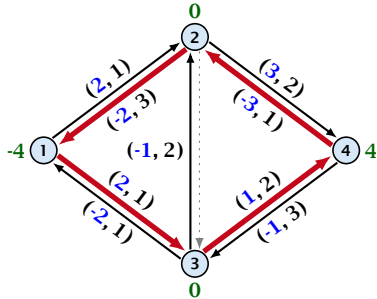
14 Mincost Flow
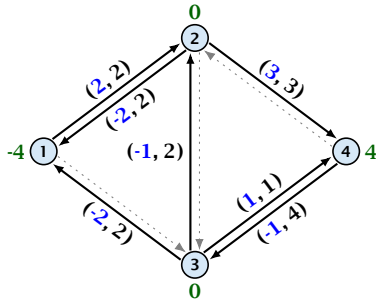
14 Mincost Flow

14 Mincost Flow

Lemma 87

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \leq C$ and $|u(e)| \leq U$.

- ▶ Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- ▶ Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- ▶ Each iteration decreases the total cost by at least 1.
- ▶ The true optimum cost must lie in the interval $[-mCU, \ldots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

14 Mincost Flow

A general mincost flow problem is of the following form:

$$\min \quad \sum_e c(e)f(e)$$
$$\text{s.t.} \quad \forall e \in E : \ \ell(e) \le f(e) \le u(e)$$
$$\forall v \in V : \ a(v) \le f(v) \le b(v)$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$
$c : E \to \mathbb{R}$;

Lemma 88 (without proof)
A general mincost flow problem can be solved in polynomial time.

Part V

Matchings

Matching

- ▶ Input: undirected graph $G = (V, E)$.
- ▶ $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- ▶ Maximum Matching: find a matching of maximum cardinality
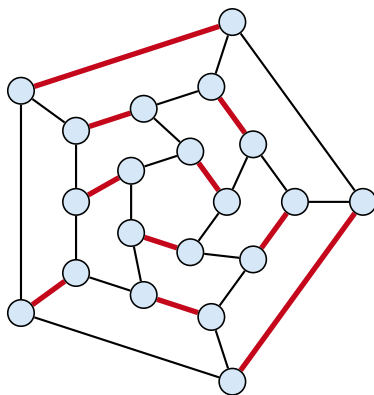
16 Bipartite Matching via Flows

Which flow algorithm to use?

- ▶ Generic augmenting path: $\mathcal{O}(m \, \mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- ▶ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- ▶ Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

17 Augmenting Paths for Matchings

Definitions.

▶ Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.

17 Augmenting Paths for Matchings

Definitions.

▶ Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.

▶ For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.

17 Augmenting Paths for Matchings

Definitions.

▶ Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.

▶ For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.

▶ An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

17 Augmenting Paths for Matchings

Definitions.

- ▶ Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.

- ▶ For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.

- ▶ An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

Theorem 89
A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

Augmenting Paths in Action
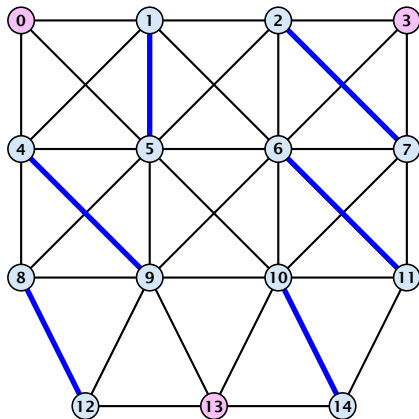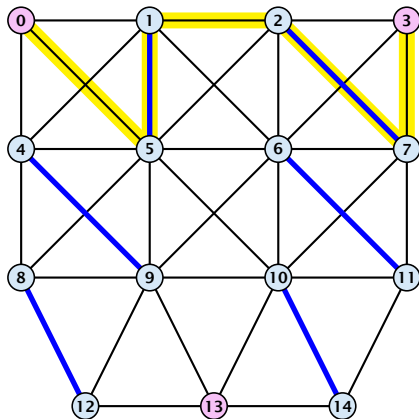
Augmenting Paths in Action
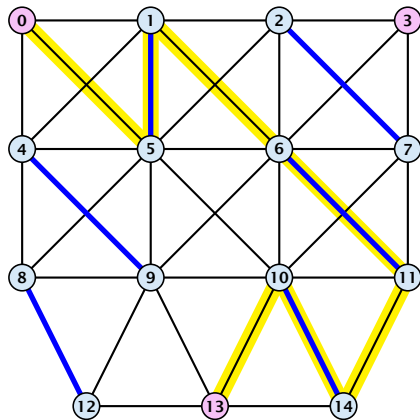
Augmenting Paths in Action
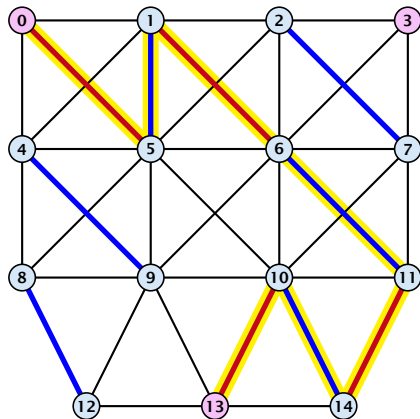
Augmenting Paths in Action
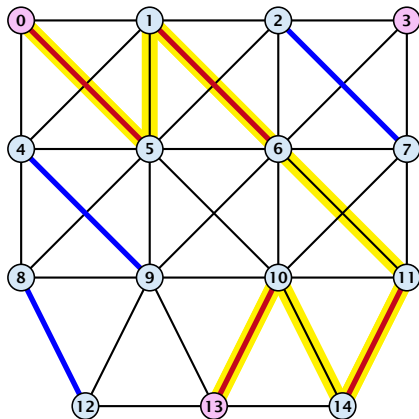
Augmenting Paths in Action
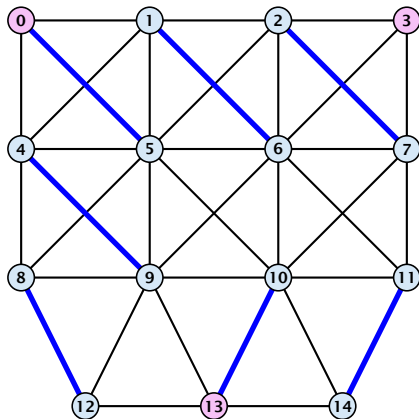
17 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

17 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

17 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

17 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

17 Augmenting Paths for Matchings

Algorithmic idea:
As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

17 Augmenting Paths for Matchings

Algorithmic idea:
As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 90
Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

Proof

Proof

▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

17 Augmenting Paths for Matchings

Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.

- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M ($\lightning$).

17 Augmenting Paths for Matchings

Proof

▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).

17 Augmenting Paths for Matchings

Proof

▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M ($\lightning$).

▶ Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.

17 Augmenting Paths for Matchings

Proof

- ▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

- ▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M ($\frac{1}{2}$).

- ▶ Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.

17 Augmenting Paths for Matchings

Proof

▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M ($\lightning$).

▶ Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.

17 Augmenting Paths for Matchings

Proof

- ▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

- ▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (⨎).

- ▶ Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.

- ▶ u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P'_1.

17 Augmenting Paths for Matchings

Proof

▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M ($\notlightning$).

▶ Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.

▶ u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P_1'.

17 Augmenting Paths for Matchings

Proof

- ▶ Assume there is an augmenting path P' w.r.t. M' starting at u.

- ▶ If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).

- ▶ Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.

- ▶ u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P_1'.

- ▶ $P_1 \circ P_1'$ is augmenting path in M (↯).
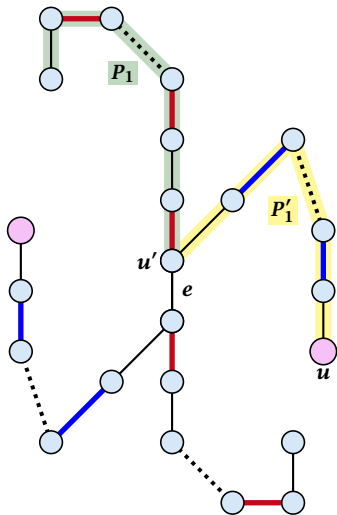
How to find an augmenting path?
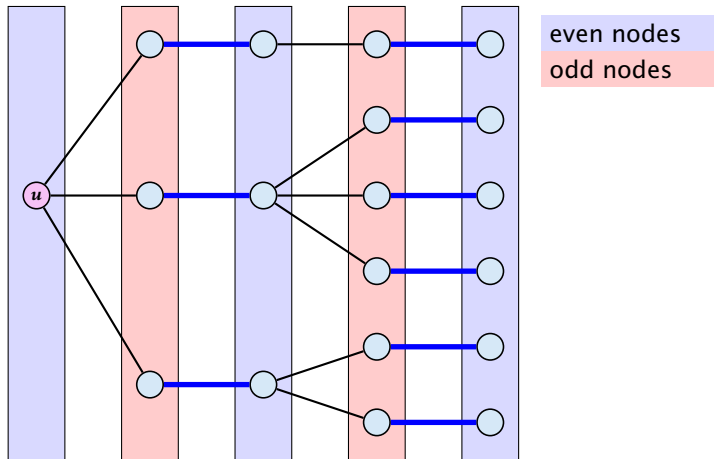
Construct an alternating tree.

even nodes
odd nodes

How to find an augmenting path?

Construct an alternating tree.

even nodes
odd nodes

Case 1:
y is free vertex not contained in T

you found alternating path

How to find an augmenting path?

Construct an alternating tree.

even nodes
odd nodes

Case 2:
y is matched vertex
not in T; then
mate$[y] \notin T$

grow the tree
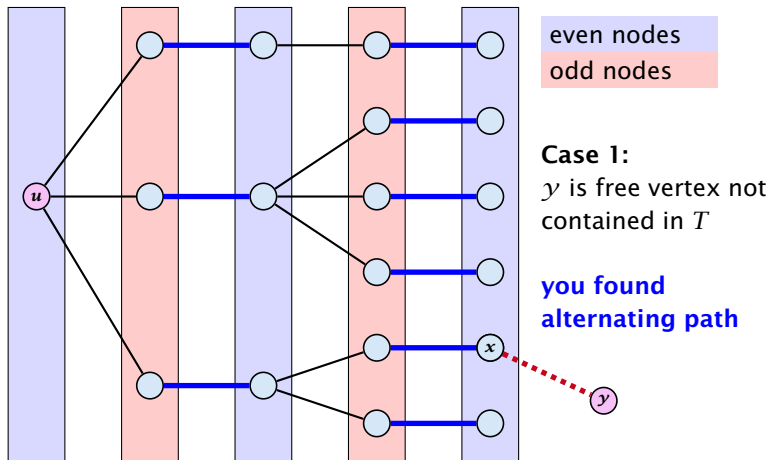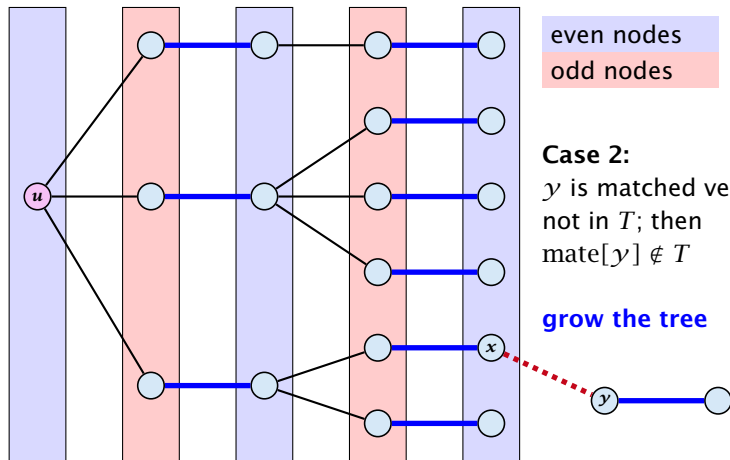
How to find an augmenting path?

Construct an alternating tree.
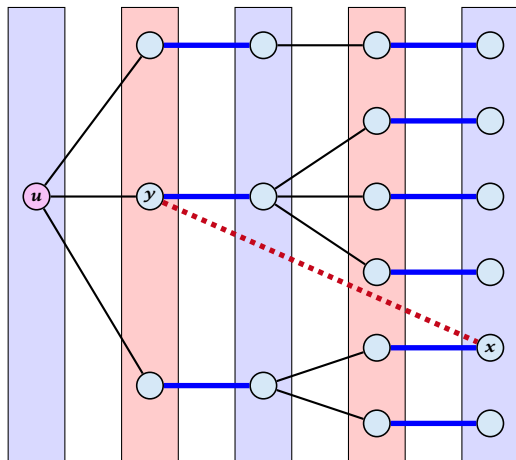
even nodes
odd nodes

Case 3:
y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.
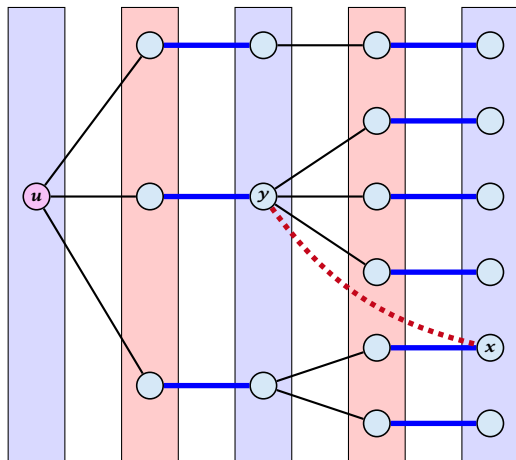
even nodes
odd nodes

Case 4:
y is already contained
in T as an even vertex

can't ignore y

**does not happen in
bipartite graphs**

Algorithm 49 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i'] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q.dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

graph $G = (S \cup S', E)$

$$S = \{1, \ldots, n\}$$
$$S' = \{1', \ldots, n'\}$$

Algorithm 49 BiMatch(G, match)

```
1:  for x ∈ V do mate[x] ← 0;
2:  r ← 0; free ← n;
3:  while free ≥ 1 and r < n  do
4:      r ← r + 1
5:      if mate[r] = 0  then
6:          for i = 1 to n do parent[i'] ← 0
7:          Q ← ∅; Q. append(r); aug ← false;
8:          while aug = false and Q ≠ ∅  do
9:              x ← Q. dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

start with an
empty matching

Algorithm 49 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n do
 4:     r ← r + 1
 5:     if mate[r] = 0 then
 6:         for i = 1 to n do parent[i'] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅ do
 9:             x ← Q.dequeue();
10:             for y ∈ Aₓ do
11:                 if mate[y] = 0 then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0 then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

free: number of unmatched nodes in S

r: root of current tree

Algorithm 49 BiMatch(*G, match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i′] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q.dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

Algorithm 49 BiMatch(*G*, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i′] ← 0
 7:         Q ← ∅; Q. append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q. dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

r is the new node that we grow from.

Algorithm 49 BiMatch(G, $match$)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i'] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q.dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

If r is free start tree construction

Algorithm 49 BiMatch(G, match)

```
1:  for x ∈ V do mate[x] ← 0;
2:  r ← 0; free ← n;
3:  while free ≥ 1 and r < n  do
4:      r ← r + 1
5:      if mate[r] = 0  then
6:          for i = 1 to n do parent[i'] ← 0
7:          Q ← ∅; Q.append(r); aug ← false;
8:          while aug = false and Q ≠ ∅  do
9:              x ← Q.dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

Initialize an empty tree. Note that only nodes i' have parent pointers.

Algorithm 49 BiMatch(G, $match$)

1: **for** $x \in V$ **do** $mate[x] \leftarrow 0$;
2: $r \leftarrow 0$; $free \leftarrow n$;
3: **while** $free \geq 1$ **and** $r < n$ **do**
4:     $r \leftarrow r + 1$
5:     **if** $mate[r] = 0$ **then**
6:         **for** $i = 1$ **to** n **do** $parent[i'] \leftarrow 0$
7:         $Q \leftarrow \varnothing$; $Q.\,\mathrm{append}(r)$; $aug \leftarrow$ false;
8:         **while** $aug =$ false **and** $Q \neq \varnothing$ **do**
9:             $x \leftarrow Q.\,\mathrm{dequeue}()$;
10:             **for** $y \in A_x$ **do**
11:                 **if** $mate[y] = 0$ **then**
12:                     $\mathrm{augm}(mate, parent, y)$;
13:                     $aug \leftarrow$ true;
14:                     $free \leftarrow free - 1$;
15:                 **else**
16:                     **if** $parent[y] = 0$ **then**
17:                         $parent[y] \leftarrow x$;
18:                         $Q.\,\mathrm{enqueue}(mate[y])$;

Q is a queue (BFS!!!).

aug is a Boolean that stores whether we already found an augmenting path.

Algorithm 49 BiMatch(G, match)

1: **for** $x \in V$ **do** mate$[x] \leftarrow 0$;
2: $r \leftarrow 0$; free $\leftarrow n$;
3: **while** free ≥ 1 **and** $r < n$ **do**
4:    $r \leftarrow r + 1$
5:    **if** mate$[r] = 0$ **then**
6:       **for** $i = 1$ **to** n **do** parent$[i'] \leftarrow 0$
7:       $Q \leftarrow \varnothing$; Q. append(r); aug $\leftarrow$ false;
8:       **while** aug = false **and** $Q \neq \varnothing$ **do**
9:          $x \leftarrow Q$. dequeue();
10:          **for** $y \in A_x$ **do**
11:             **if** mate$[y] = 0$ **then**
12:                augm(mate, parent, y);
13:                aug $\leftarrow$ true;
14:                free $\leftarrow$ free $- 1$;
15:             **else**
16:                **if** parent$[y] = 0$ **then**
17:                    parent$[y] \leftarrow x$;
18:                    Q. enqueue(mate$[y]$);

as long as we did not augment and there are still unexamined leaves continue...

Algorithm 49 BiMatch(G, $match$)

```
1: for x ∈ V do mate[x] ← 0;
2: r ← 0; free ← n;
3: while free ≥ 1 and r < n do
4:     r ← r + 1
5:     if mate[r] = 0 then
6:         for i = 1 to n do parent[i'] ← 0
7:         Q ← ∅; Q.append(r); aug ← false;
8:         while aug = false and Q ≠ ∅ do
9:             x ← Q.dequeue();
10:            for y ∈ A_x do
11:                if mate[y] = 0 then
12:                    augm(mate, parent, y);
13:                    aug ← true;
14:                    free ← free − 1;
15:                else
16:                    if parent[y] = 0 then
17:                        parent[y] ← x;
18:                        Q.enqueue(mate[y]);
```

take next unexamined leaf

Algorithm 49 BiMatch(*G*, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i′] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q.dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

if x has unmatched neighbour we found an augmenting path (note that $y \neq r$ because we are in a bipartite graph)

Algorithm 49 BiMatch(*G*, *match*)

1: **for** $x \in V$ **do** $mate[x] \leftarrow 0$;
2: $r \leftarrow 0$; $free \leftarrow n$;
3: **while** $free \geq 1$ **and** $r < n$ **do**
4:     $r \leftarrow r + 1$
5:     **if** $mate[r] = 0$ **then**
6:         **for** $i = 1$ **to** n **do** $parent[i'] \leftarrow 0$
7:         $Q \leftarrow \varnothing$; Q. append(r); $aug \leftarrow$ false;
8:         **while** $aug =$ false **and** $Q \neq \varnothing$ **do**
9:             $x \leftarrow Q$. dequeue();
10:             **for** $y \in A_x$ **do**
11:                 **if** $mate[y] = 0$ **then**
12:                     augm(*mate*, *parent*, *y*);
13:                     $aug \leftarrow$ true;
14:                     $free \leftarrow free - 1$;
15:                 **else**
16:                     **if** $parent[y] = 0$ **then**
17:                         $parent[y] \leftarrow x$;
18:                         Q. enqueue($mate[y]$);

do an augmentation...

Algorithm 49 BiMatch(G, match)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i'] ← 0
 7:         Q ← ∅; Q. append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q. dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

setting aug = true ensures that the tree construction will not continue

Algorithm 49 BiMatch(*G*, *match*)

```
1: for x ∈ V do mate[x] ← 0;
2: r ← 0; free ← n;
3: while free ≥ 1 and r < n  do
4:     r ← r + 1
5:     if mate[r] = 0  then
6:         for i = 1 to n do parent[i'] ← 0
7:         Q ← ∅; Q.append(r); aug ← false;
8:         while aug = false and Q ≠ ∅  do
9:             x ← Q.dequeue();
10:            for y ∈ A_x do
11:                if mate[y] = 0  then
12:                    augm(mate, parent, y);
13:                    aug ← true;
14:                    free ← free − 1;
15:                else
16:                    if parent[y] = 0  then
17:                        parent[y] ← x;
18:                        Q.enqueue(mate[y]);
```

reduce number of free nodes

Algorithm 49 BiMatch(G, $match$)

1: **for** $x \in V$ **do** $mate[x] \leftarrow 0$;
2: $r \leftarrow 0$; $free \leftarrow n$;
3: **while** $free \geq 1$ **and** $r < n$ **do**
4:      $r \leftarrow r + 1$
5:      **if** $mate[r] = 0$ **then**
6:          **for** $i = 1$ **to** n **do** $parent[i'] \leftarrow 0$
7:          $Q \leftarrow \varnothing$; $Q.\text{append}(r)$; $aug \leftarrow$ false;
8:          **while** $aug =$ false **and** $Q \neq \varnothing$ **do**
9:              $x \leftarrow Q.\text{dequeue}()$;
10:              **for** $y \in A_x$ **do**
11:                  **if** $mate[y] = 0$ **then**
12:                      $\text{augm}(mate, parent, y)$;
13:                      $aug \leftarrow$ true;
14:                      $free \leftarrow free - 1$;
15:                  **else**
16:                      **if** $parent[y] = 0$ **then**
17:                          $parent[y] \leftarrow x$;
18:                          $Q.\text{enqueue}(mate[y])$;

if y is not in the tree yet

Algorithm 49 BiMatch(G, match)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n do
 4:     r ← r + 1
 5:     if mate[r] = 0 then
 6:         for i = 1 to n do parent[i′] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅ do
 9:             x ← Q.dequeue();
10:             for y ∈ Aₓ do
11:                 if mate[y] = 0 then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0 then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

...put it into the tree

Algorithm 49 BiMatch(*G*, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to n do parent[i′] ← 0
 7:         Q ← ∅; Q. append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q. dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q. enqueue(mate[y]);
```

add its buddy to the set
of unexamined leaves

18 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

▶ Input: undirected, bipartite graph $G = L \cup R, E$.

▶ an edge $e = (\ell, r)$ has weight $w_e \geq 0$

▶ find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

Simplifying Assumptions (wlog [why?]):

▶ assume that $|L| = |R| = n$

▶ assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$

▶ can assume goal is to construct maximum weight perfect matching

Weighted Bipartite Matching

Theorem 91 (Halls Theorem)

A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S.

18 Weighted Bipartite Matching
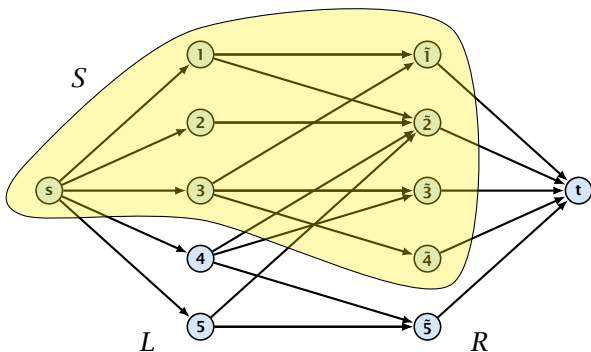
Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

- ▶ Let S denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

   ▶ Let S denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.

   ▶ Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

- ▶ Let S denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.

- ▶ Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.

- ▶ This gives $R_S \geq |\Gamma(L_S)|$.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

- ▶ Let S denote a minimum cut and let $L_S \overset{\text{def}}{=} L \cap S$ and $R_S \overset{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.
- ▶ Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.
- ▶ This gives $R_S \geq |\Gamma(L_S)|$.
- ▶ The size of the cut is $|L| - |L_S| + |R_S|$.

Halls Theorem

Proof:

⇐ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.

⇒ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

- ▶ Let S denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.
- ▶ Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.
- ▶ This gives $R_S \geq |\Gamma(L_S)|$.
- ▶ The size of the cut is $|L| - |L_S| + |R_S|$.
- ▶ Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

Algorithm Outline

Idea:

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node v.

Algorithm Outline

Idea:

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node v.

▶ Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

Algorithm Outline

Idea:

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node v.

▶ Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

▶ Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.

Algorithm Outline

Idea:

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node v.

▶ Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

▶ Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.

▶ Try to compute a perfect matching in the subgraph $H(\vec{x})$. If you are successful you found an optimal matching.

Algorithm Outline

Reason:

▶ The weight of your matching M^* is

$$\sum_{(u,v) \in M^*} w_{(u,v)} = \sum_{(u,v) \in M^*} (x_u + x_v) = \sum_v x_v \ .$$

▶ Any other perfect matching M (in G, not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v) \in M} w_{(u,v)} \leq \sum_{(u,v) \in M} (x_u + x_v) = \sum_v x_v \ .$$

Algorithm Outline

What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Algorithm Outline

What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Idea: reweight such that:

▶ the total weight assigned to nodes decreases

▶ the weight function still dominates the edge-weights

Algorithm Outline

What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Idea: reweight such that:

▶ the total weight assigned to nodes decreases

▶ the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

Changing Node Weights
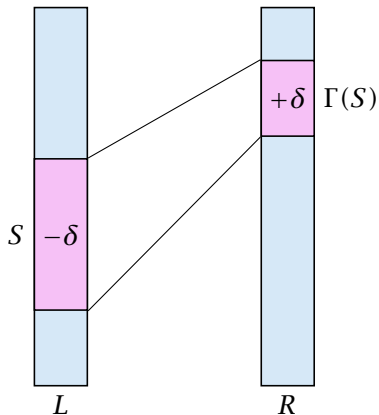
Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

▶ Total node-weight decreases.
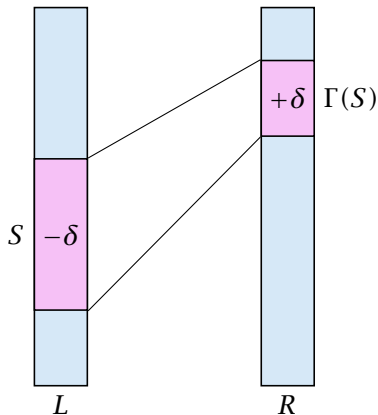
Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

- ▶ Total node-weight decreases.
- ▶ Only edges from S to $R - \Gamma(S)$ decrease in their weight.
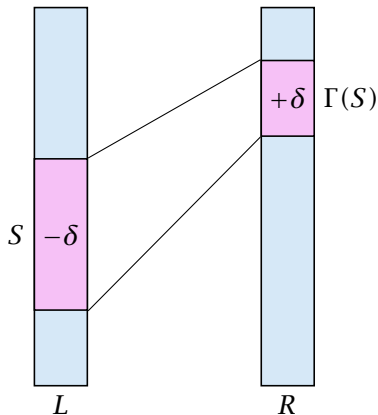
Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

- ▶ Total node-weight decreases.
- ▶ Only edges from S to $R - \Gamma(S)$ decrease in their weight.
- ▶ Since, none of these edges is tight (otw. the edge would be contained in $H(\vec{x})$, and hence would go between S and $\Gamma(S)$) we can do this decrement for small enough $\delta > 0$ until a new edge gets tight.

Weighted Bipartite Matching
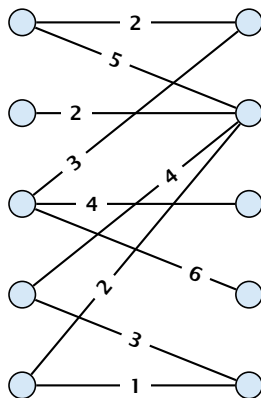
Edges not drawn have weight 0.

Weighted Bipartite Matching

Edges not drawn have weight 0.

Weighted Bipartite Matching

Edges not drawn have weight 0.

Weighted Bipartite Matching

Edges not drawn have weight 0.

Weighted Bipartite Matching

Edges not drawn have weight 0.

$\delta = 1$

Weighted Bipartite Matching

Edges not drawn have weight 0.

Weighted Bipartite Matching

Edges not drawn have weight 0.

Weighted Bipartite Matching

Edges not drawn have weight 0.

$\delta = 1$
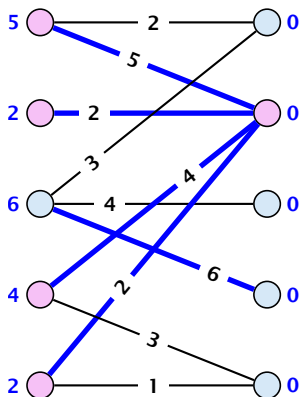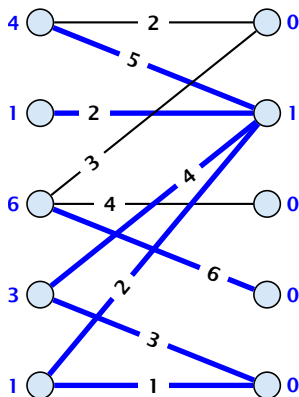
Weighted Bipartite Matching

Edges not drawn have weight 0.

Weighted Bipartite Matching

Edges not drawn have weight 0.
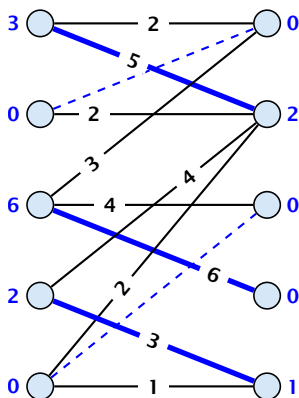
Weighted Bipartite Matching

Edges not drawn have weight 0.

Analysis

How many iterations do we need?

▶ One reweighting step increases the number of edges out of S by at least one.

Analysis

How many iterations do we need?

▶ One reweighting step increases the number of edges out of S by at least one.

▶ Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).

Analysis

How many iterations do we need?

▶ One reweighting step increases the number of edges out of S by at least one.

▶ Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).

▶ This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.

Analysis

How many iterations do we need?

▶ One reweighting step increases the number of edges out of S by at least one.

▶ Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).

▶ This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.

▶ Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

Analysis

▶ We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.

▶ This gives a polynomial running time.

How to find an augmenting path?

Construct an alternating tree.
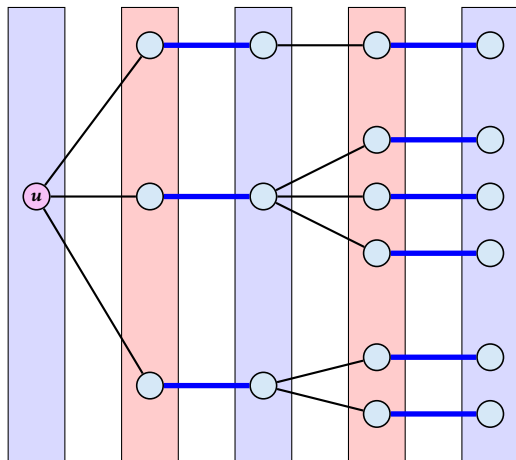
How to find an augmenting path?
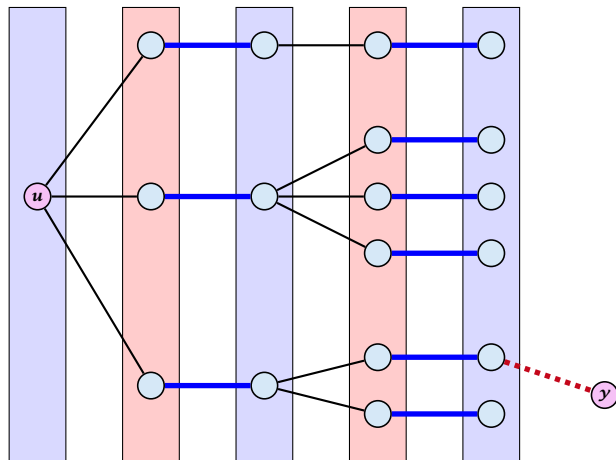
Construct an alternating tree.

Analysis

How do we find S?

▶ Start on the left and compute an alternating tree, starting at any free node u.

Analysis

How do we find S?

▶ Start on the left and compute an alternating tree, starting at any free node u.

▶ If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).

Analysis

How do we find S?

▶ Start on the left and compute an alternating tree, starting at any free node u.

▶ If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).

▶ The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.

Analysis

How do we find S?

- Start on the left and compute an alternating tree, starting at any free node u.

- If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).

- The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.

- All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

Analysis

- The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

Analysis

- The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.

Analysis

- The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.

- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).

Analysis

- The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.

- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).

- An augmentation takes at most $\mathcal{O}(n)$ time.

Analysis

▶ The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

▶ After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.

▶ A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).

▶ An augmentation takes at most $\mathcal{O}(n)$ time.

▶ In total we obtain a running time of $\mathcal{O}(n^4)$.

Analysis

- ▶ The current matching does not have any edges from V_{odd} to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

- ▶ After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.

- ▶ A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).

- ▶ An augmentation takes at most $\mathcal{O}(n)$ time.

- ▶ In total we obtain a running time of $\mathcal{O}(n^4)$.

- ▶ A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.

How to find an augmenting path?
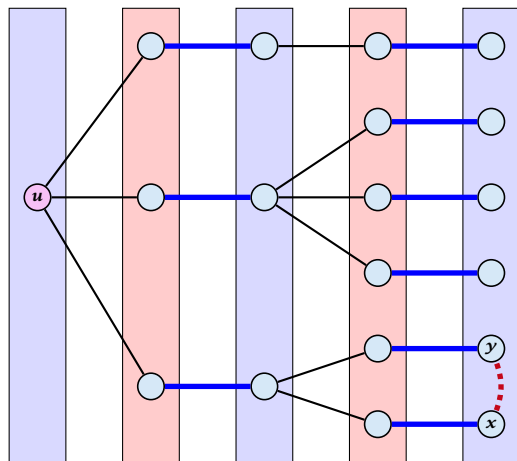
Construct an alternating tree.

even nodes
odd nodes

Case 4:
y is already contained
in T as an even vertex

can't ignore y

How to find an augmenting path?

Construct an alternating tree.
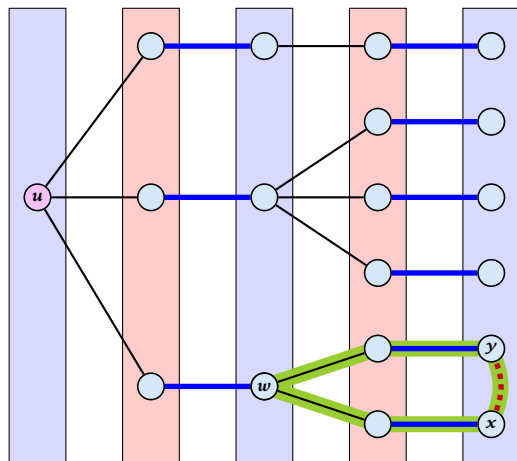
even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

How to find an augmenting path?

Construct an alternating tree.
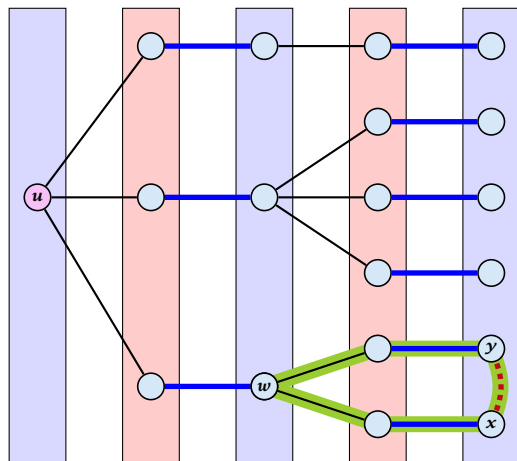
even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w is called the stem of the blossom.

Flowers and Blossoms

Definition 92

A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

Flowers and Blossoms

Definition 92

A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

- ▶ A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).

Flowers and Blossoms

Definition 92

A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

▶ A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).

▶ A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

Flowers and Blossoms
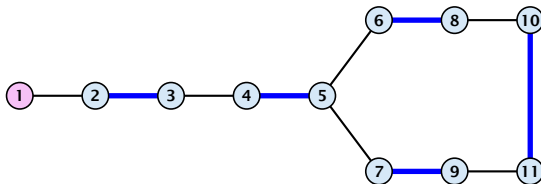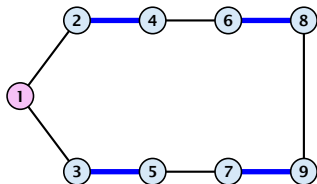
Flowers and Blossoms

Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.

Flowers and Blossoms

Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.

2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.

Flowers and Blossoms

Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.

2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.

3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

5. The even alternating path to x terminates with a matched edge and the odd path with an unmatched edge.

Flowers and Blossoms
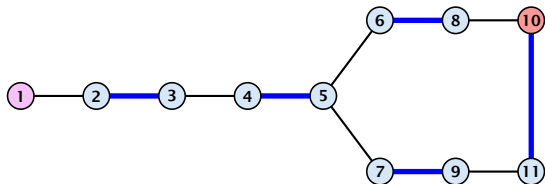
Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- ▶ Delete all vertices in B (and its incident edges) from G.

Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- ▶ Delete all vertices in B (and its incident edges) from G.
- ▶ Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

Shrinking Blossoms

▶ Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.

▶ Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.

▶ Nodes that are connected in G to at least one node in B become connected to b in G'.
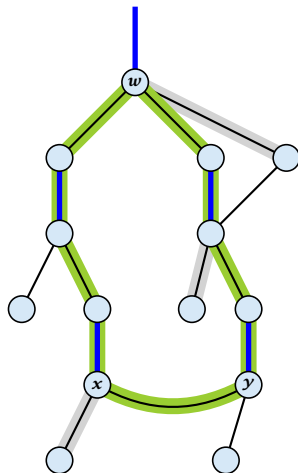
Shrinking Blossoms

▶ Edges of T that connect a node u
  not in B to a node in B become
  tree edges in T' connecting u to
  b.

▶ Matching edges (there is at most
  one) that connect a node u not in
  B to a node in B become matching
  edges in M'.

▶ Nodes that are connected in G to
  at least one node in B become
  connected to b in G'.

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

different choices
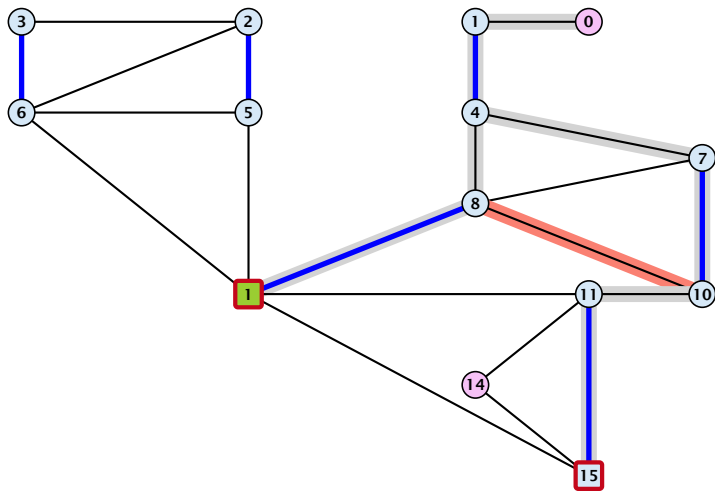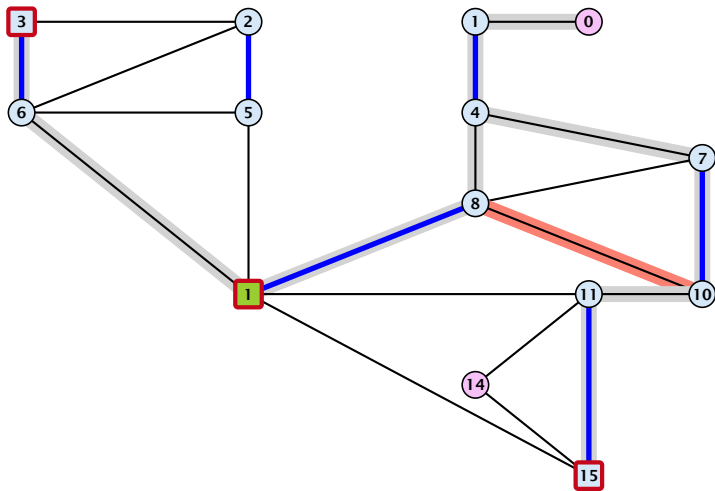
Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 93

If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

Case 1: non-empty stem

▶ Next suppose that the stem is non-empty.

Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

Case 1: non-empty stem

▶ Next suppose that the stem is non-empty.
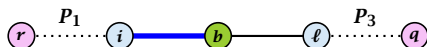
Correctness

Proof.

If P' does not contain b it is also an augmenting path in G.

Case 1: non-empty stem

▶ Next suppose that the stem is non-empty.
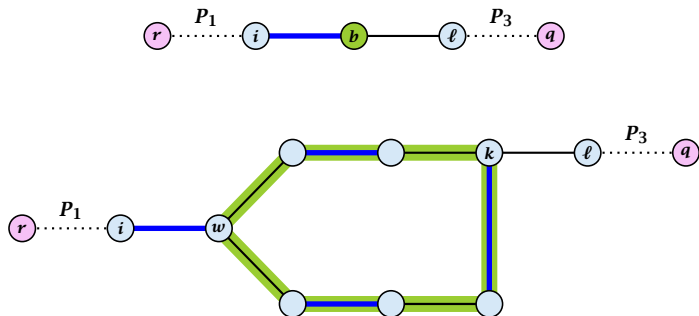
Correctness

- ▶ After the expansion ℓ must be incident to some node in the blossom. Let this node be k.

- ▶ If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.

- ▶ $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

- ▶ If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

Correctness

Proof.

Case 2: empty stem

▶ If the stem is empty then after expanding the blossom, $w = r$.

Correctness

Proof.

Case 2: empty stem

▶ If the stem is empty then after expanding the blossom, $w = r$.

Correctness

Proof.

Case 2: empty stem

▶ If the stem is empty then after expanding the blossom,
$w = r$.
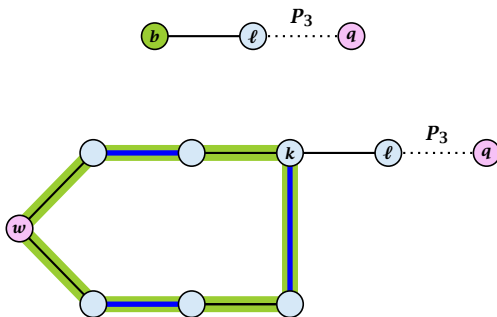
Correctness

Proof.

Case 2: empty stem

▶ If the stem is empty then after expanding the blossom, $w = r$.
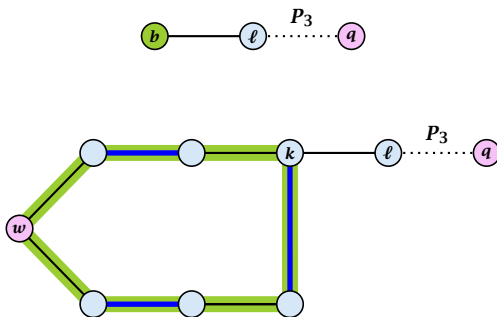
▶ The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 94

If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Correctness

Proof.

▶ If P does not contain a node from B there is nothing to prove.

Correctness

Proof.

- ▶ If P does not contain a node from B there is nothing to prove.
- ▶ We can assume that r and q are the only free nodes in G.

Correctness

Proof.

- ▶ If P does not contain a node from B there is nothing to prove.
- ▶ We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Correctness

Proof.

- ▶ If P does not contain a node from B there is nothing to prove.
- ▶ We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

Correctness

Proof.

► If P does not contain a node from B there is nothing to prove.

► We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

Correctness

Proof.

- ▶ If P does not contain a node from B there is nothing to prove.
- ▶ We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i, j) \circ P_2$, for some node j and (i, j) is unmatched.

$(b, j) \circ P_2$ is an augmenting path in the contracted network.

Correctness

Illustration for Case 1:
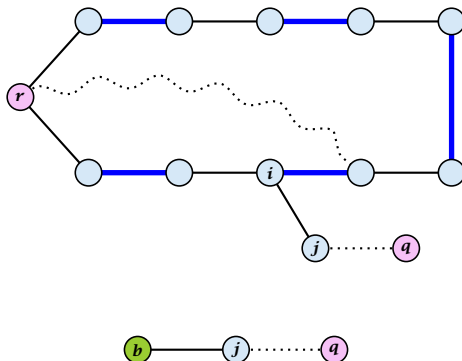
Correctness

Case 2: non-empty stem

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M_+' the blossom has an empty stem. Case 1 applies.

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M_+' the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M_+'. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

This path must go between r and q.

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \varnothing$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

Search for an augmenting path
starting at r.

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \varnothing$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

$A(i)$ contains neighbours of node i.

We create a copy $\bar{A}(i)$ so that we later can shrink blossoms.

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i

2: *found* $\leftarrow$ false

3: unlabel all nodes;

4: give an even label to r and initialize *list* $\leftarrow \{r\}$

5: **while** *list* $\neq \varnothing$ **do**

6:     delete a node i from *list*

7:     examine(i, *found*)

8:     **if** *found* $=$ true **then return**

---

found is just a Boolean that allows
to abort the search process...

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \varnothing$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

In the beginning no node is in the tree.

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \varnothing$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

Put the root in the tree.

list could also be a set or a stack.

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \varnothing$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

As long as there are nodes with
unexamined neighbours...

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \varnothing$ **do**
6:      delete a node i from *list*
7:      examine(i, *found*)
8:      **if** *found* = true **then return**

...examine the next one

Algorithm 50 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: *found* $\leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize *list* $\leftarrow \{r\}$
5: **while** *list* $\neq \varnothing$ **do**
6:     delete a node i from *list*
7:     examine(i, *found*)
8:     **if** *found* = true **then return**

If you found augmenting path
abort and start from next root.

Algorithm 51 examine(i, *found*)

1: **for** all $j \in \bar{A}(i)$ **do**
2:     **if** j is even **then** contract(i, j) and **return**
3:     **if** j is unmatched **then**
4:         $q \leftarrow j$;
5:         pred(q) $\leftarrow i$;
6:         *found* $\leftarrow$ true;
7:         **return**
8:     **if** j is matched and unlabeled **then**
9:         pred(j) $\leftarrow i$;
10:        pred(mate(j)) $\leftarrow j$;
11:        add mate(j) to *list*

Examine the neighbours of a node i

Algorithm 51 examine(i, *found*)

1: **for** all $j \in \bar{A}(i)$ **do**
2:      **if** j is even **then** contract(i, j) and **return**
3:      **if** j is unmatched **then**
4:          $q \leftarrow j$;
5:          pred(q) $\leftarrow i$;
6:          *found* $\leftarrow$ true;
7:          **return**
8:      **if** j is matched and unlabeled **then**
9:          pred(j) $\leftarrow i$;
10:         pred(mate(j)) $\leftarrow j$;
11:         add mate(j) to *list*

For all neighbours j do...

Algorithm 51 examine(i, *found*)

1: **for** all $j \in \bar{A}(i)$ **do**
2:     **if** j is even **then** contract(i, j) and **return**
3:     **if** j is unmatched **then**
4:         $q \leftarrow j$;
5:         pred(q) $\leftarrow i$;
6:         *found* $\leftarrow$ true;
7:         **return**
8:     **if** j is matched and unlabeled **then**
9:         pred(j) $\leftarrow i$;
10:         pred(mate(j)) $\leftarrow j$;
11:         add mate(j) to *list*

You have found a blossom...

Algorithm 51 examine(i, *found*)

1: **for** all $j \in \bar{A}(i)$ **do**
2:     **if** j is even **then** contract(i, j) and **return**
3:     **if** j is unmatched **then**
4:         $q \leftarrow j$;
5:         pred(q) $\leftarrow i$;
6:         *found* $\leftarrow$ true;
7:         **return**
8:     **if** j is matched and unlabeled **then**
9:         pred(j) $\leftarrow i$;
10:        pred(mate(j)) $\leftarrow j$;
11:       add mate(j) to *list*

You have found a free node which
gives you an augmenting path.

Algorithm 51 examine(i, *found*)

1: **for** all $j \in \bar{A}(i)$ **do**
2:      **if** j is even **then** contract(i, j) and **return**
3:      **if** j is unmatched **then**
4:          $q \leftarrow j$;
5:          pred(q) $\leftarrow i$;
6:          *found* $\leftarrow$ true;
7:          **return**
8:      **if** j is matched and unlabeled **then**
9:          pred(j) $\leftarrow i$;
10:         pred(mate(j)) $\leftarrow j$;
11:        add mate(j) to *list*

If you find a matched node that is not in the tree you grow...

Algorithm 51 examine(i, *found*)

1:	**for** all $j \in \bar{A}(i)$ **do**
2:	**if** j is even **then** contract(i, j) and **return**
3:	**if** j is unmatched **then**
4:	$q \leftarrow j$;
5:	pred(q) $\leftarrow i$;
6:	*found* $\leftarrow$ true;
7:	**return**
8:	**if** j is matched and unlabeled **then**
9:	pred(j) $\leftarrow i$;
10:	pred(mate(j)) $\leftarrow j$;
11:	add mate(j) to *list*

mate(j) is a new node from
which you can grow further.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Contract blossom identified by
nodes i and j

19 Maximum Matching in General Graphs

6. Feb. 2022

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Identify all neighbours of b.

Time: $\mathcal{O}(m)$ (how?)

19 Maximum Matching in General Graphs

6. Feb. 2022

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

b will be an even node, and it has
unexamined neighbours.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Every node that was adjacent to a node
in B is now adjacent to b

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only for making a blossom
expansion easier.

Algorithm 52 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only delete links from nodes not in B to B.

When expanding the blossom again we can
recreate these links in time $\mathcal{O}(m)$.

19 Maximum Matching in General Graphs

6. Feb. 2022

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$.
  Note, that any graph created will have at most m edges.

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.

- There are at most n contractions as each contraction reduces the number of vertices.

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.

- There are at most n contractions as each contraction reduces the number of vertices.

- The expansion can trivially be done in the same time as needed for all contractions.

Analysis

▶ A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

▶ The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.

▶ There are at most n contractions as each contraction reduces the number of vertices.

▶ The expansion can trivially be done in the same time as needed for all contractions.

▶ An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.

Analysis

- ▶ A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.

- ▶ The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.

- ▶ There are at most n contractions as each contraction reduces the number of vertices.

- ▶ The expansion can trivially be done in the same time as needed for all contractions.

- ▶ An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.

- ▶ In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

Example: Blossom Algorithm

A Fast Matching Algorithm

Algorithm 53 Bimatch-Hopcroft-Karp(G)

1: $M \leftarrow \varnothing$
2: **repeat**
3:      let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be maximal set of
4:      vertex-disjoint, shortest augmenting path w.r.t. M.
5:      $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
6: **until** $\mathcal{P} = \varnothing$
7: **return** M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis Hopcroft-Karp

Lemma 95

Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$.
There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Analysis Hopcroft-Karp

Lemma 95

Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$.
There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.

Analysis Hopcroft-Karp

Lemma 95

Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$.
There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.

▶ Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.

Analysis Hopcroft-Karp

Lemma 95

Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$.
There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

- ▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- ▶ Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- ▶ The connected components of G are cycles and paths.

Analysis Hopcroft-Karp

Lemma 95

Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$.
There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.

▶ Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.

▶ The connected components of G are cycles and paths.

▶ The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.

Analysis Hopcroft-Karp

Lemma 95

Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$.
There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

- ▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- ▶ Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- ▶ The connected components of G are cycles and paths.
- ▶ The graph contains $k \overset{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.
- ▶ Hence, there are at least k components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. M.

Analysis Hopcroft-Karp

▶ Let $P_1, \ldots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).

Analysis Hopcroft-Karp

▶ Let $P_1, \dots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).

▶ $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.

Analysis Hopcroft-Karp

▶ Let $P_1, \ldots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).

▶ $M' \overset{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.

▶ Let P be an augmenting path in M'.

Analysis Hopcroft-Karp

▶ Let $P_1, \ldots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).

▶ $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.

▶ Let P be an augmenting path in M'.

Lemma 96
The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \cdots \cup P_k) \oplus P$ contains at least $(k+1)\ell$ edges.

Proof.

▶ The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.

Analysis Hopcroft-Karp

Proof.

▶ The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.

▶ Hence, the set contains at least $k + 1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.

Analysis Hopcroft-Karp

Proof.

▶ The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.

▶ Hence, the set contains at least $k + 1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.

▶ Each of these paths is of length at least ℓ.

Analysis Hopcroft-Karp

Lemma 97

P *is of length at least* $\ell + 1$. *This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.*

Analysis Hopcroft-Karp

Lemma 97

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

▶ If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.

Analysis Hopcroft-Karp

Lemma 97

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

▶ If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.

▶ Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.

Analysis Hopcroft-Karp

Lemma 97

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

▶ If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.

▶ Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.

▶ This edge is not contained in A.

Analysis Hopcroft-Karp

Lemma 97

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

▶ If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.

▶ Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.

▶ This edge is not contained in A.

▶ Hence, $|A| \leq k\ell + |P| - 1$.

Analysis Hopcroft-Karp

Lemma 97

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

▶ If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.

▶ Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.

▶ This edge is not contained in A.

▶ Hence, $|A| \leq k\ell + |P| - 1$.

▶ The lower bound on $|A|$ gives $(k + 1)\ell \leq |A| \leq k\ell + |P| - 1$, and hence $|P| \geq \ell + 1$.

Analysis Hopcroft-Karp

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Analysis Hopcroft-Karp

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Proof.
The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

Analysis Hopcroft-Karp

Lemma 98

The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

Analysis Hopcroft-Karp

Lemma 98

The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

Proof.

▶ After iteration $\lfloor \sqrt{|V|} \rfloor$ the length of a shortest augmenting path must be at least $\lfloor \sqrt{|V|} \rfloor + 1 \geq \sqrt{|V|}$.

▶ Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.

Analysis Hopcroft-Karp

Lemma 99

One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

construct a "level graph" G':

- ▶ construct Level 0 that includes all free vertices on left side L
- ▶ construct Level 1 containing all neighbors of Level 0
- ▶ construct Level 2 containing matching neighbors of Level 1
- ▶ construct Level 3 containing all neighbors of Level 2
- ▶ . . .
- ▶ stop when a level (apart from Level 0) contains a free vertex

can be done in time $\mathcal{O}(m)$ by a modified BFS

Analysis Hopcroft-Karp

- ▶ a shortest augmenting path must go from Level 0 to the last layer constructed
- ▶ it can only use edges between layers
- ▶ construct a maximal set of vertex disjoint augmenting path connecting the layers
- ▶ for this, go forward until you either reach a free vertex or you reach a "dead end" v
- ▶ if you reach a free vertex delete the augmenting path and all incident edges from the graph
- ▶ if you reach a dead end backtrack and delete v together with its incident edges

Analysis Hopcroft-Karp
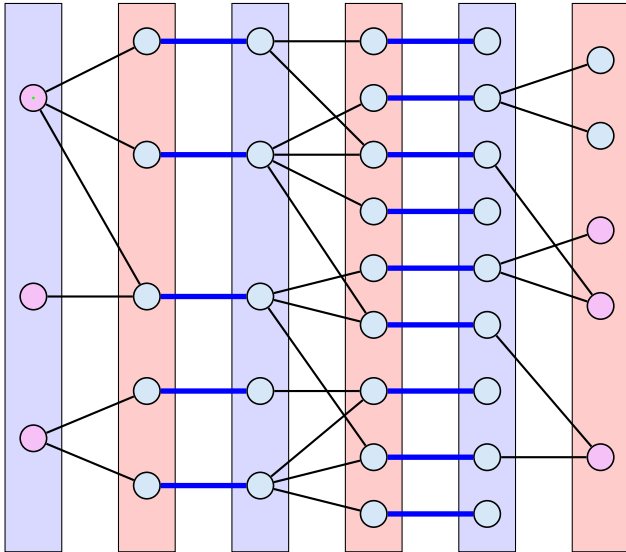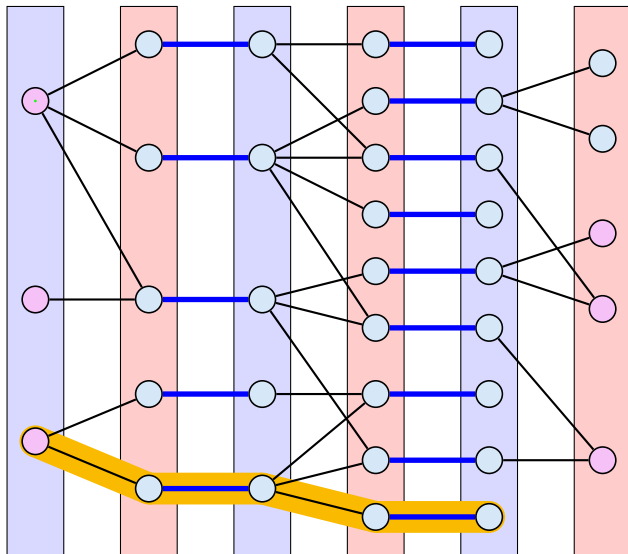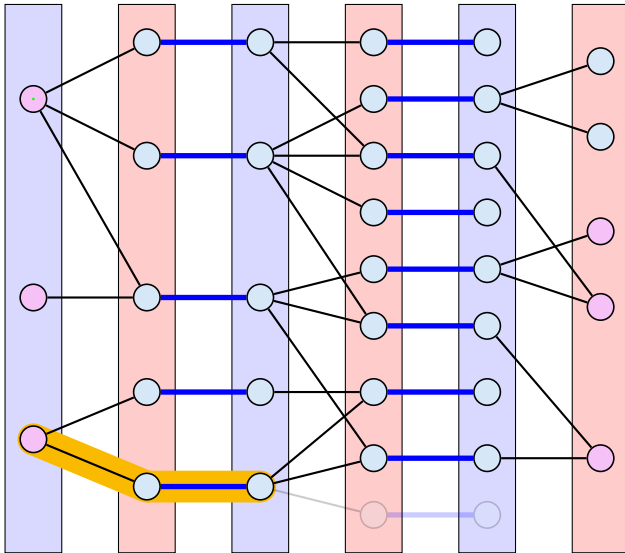
Analysis Hopcroft-Karp

Analysis Hopcroft-Karp

Analysis Hopcroft-Karp
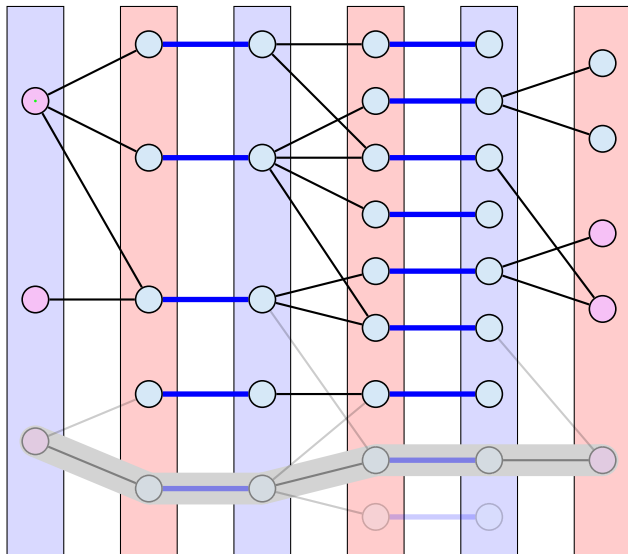
Analysis Hopcroft-Karp
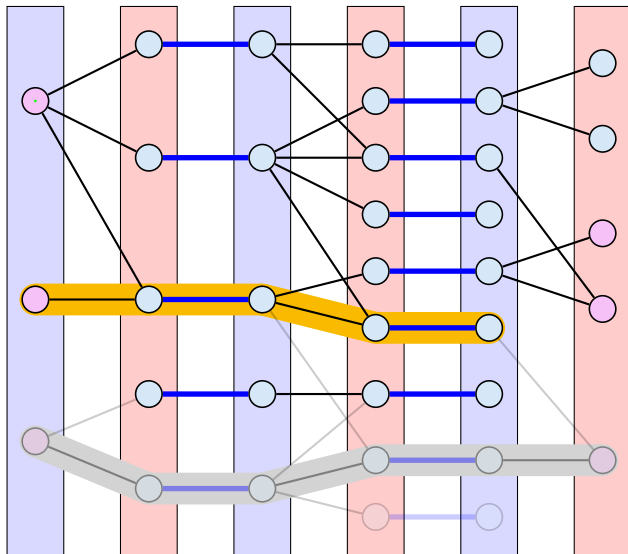
Analysis Hopcroft-Karp
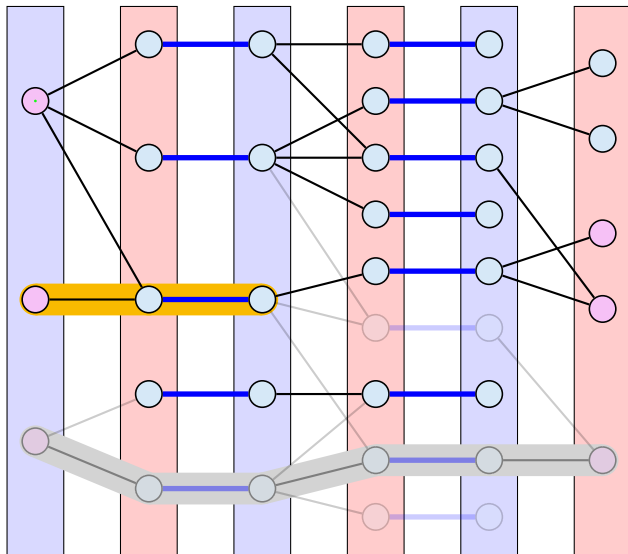
Analysis Hopcroft-Karp
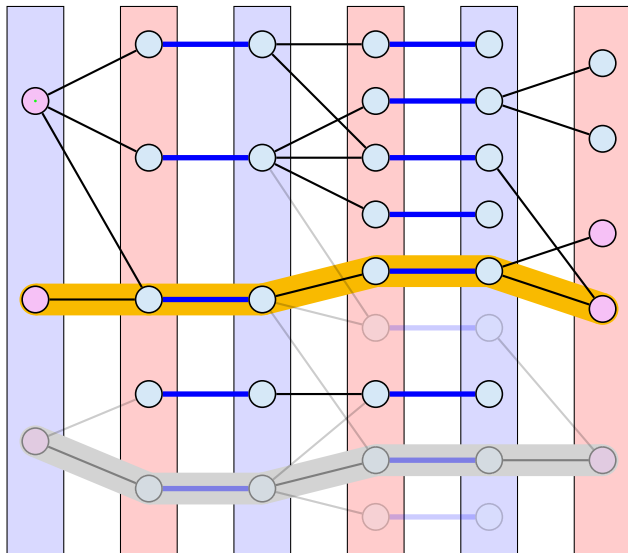
Analysis Hopcroft-Karp
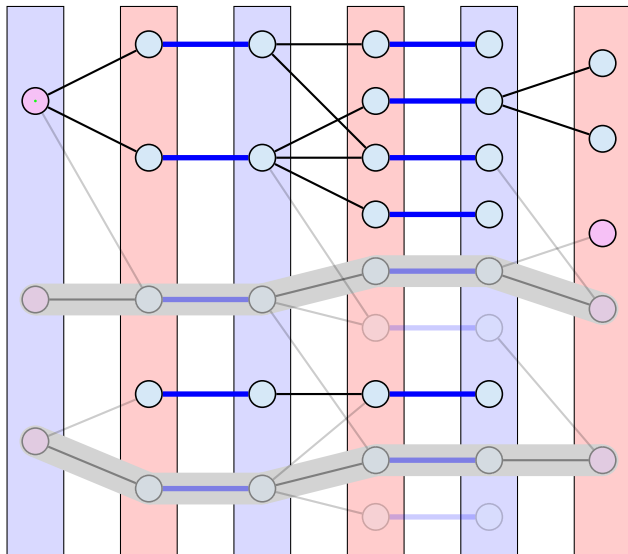
Analysis Hopcroft-Karp
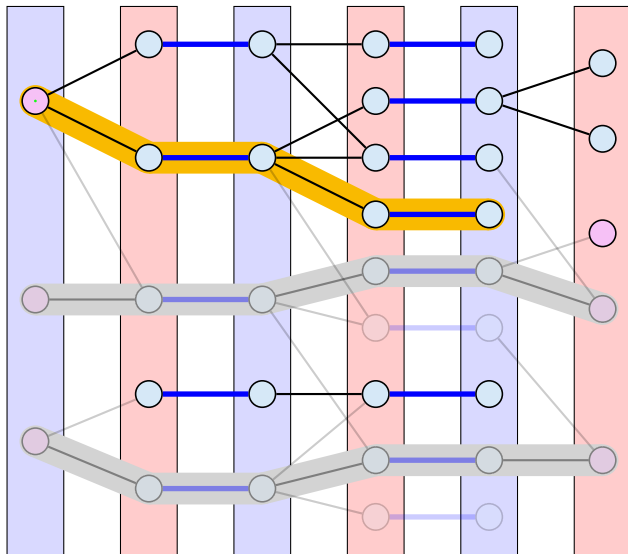
Analysis Hopcroft-Karp
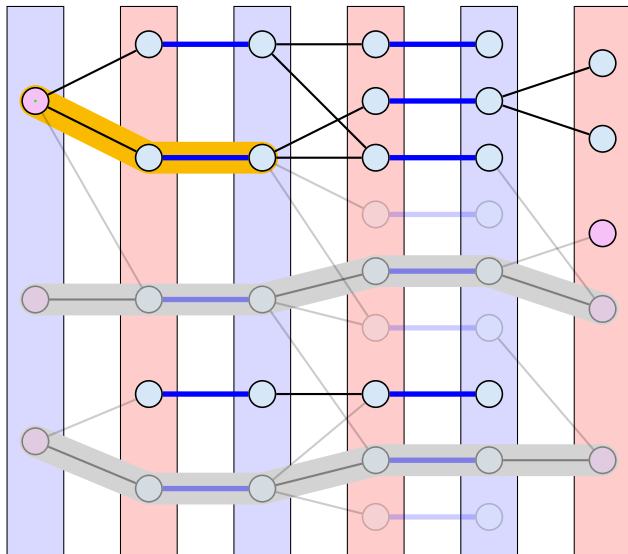
Analysis Hopcroft-Karp
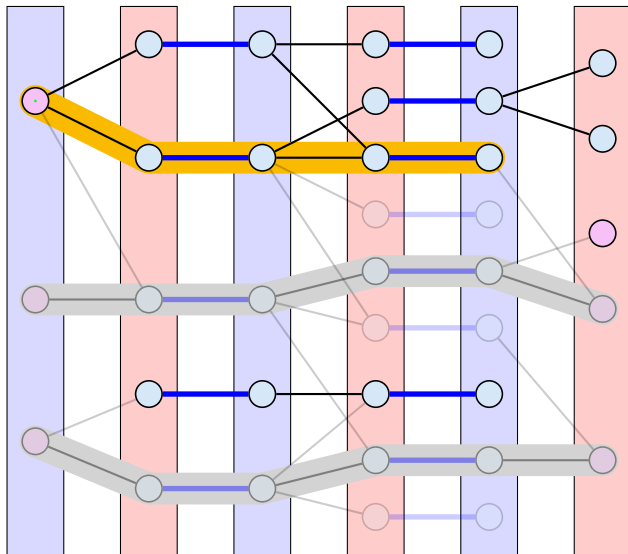
Analysis Hopcroft-Karp
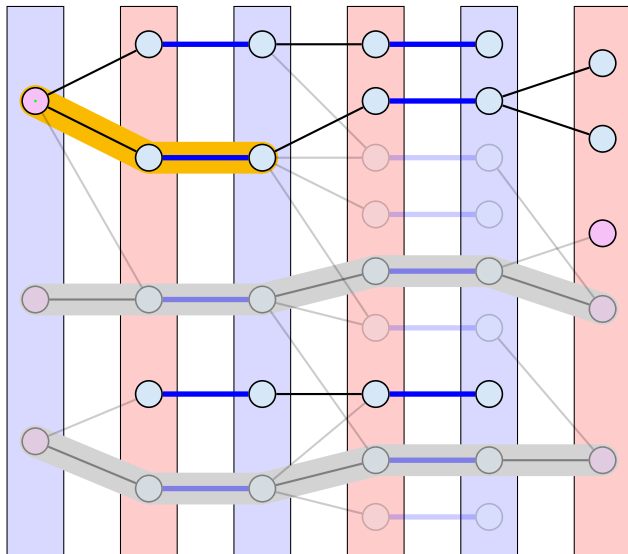
Analysis Hopcroft-Karp
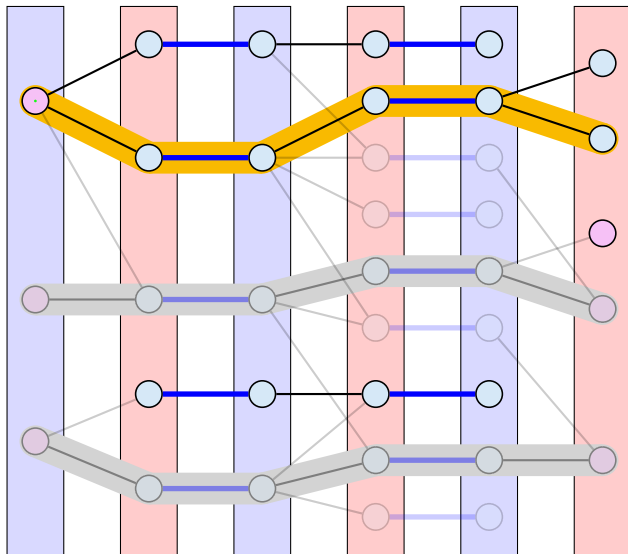
Analysis Hopcroft-Karp
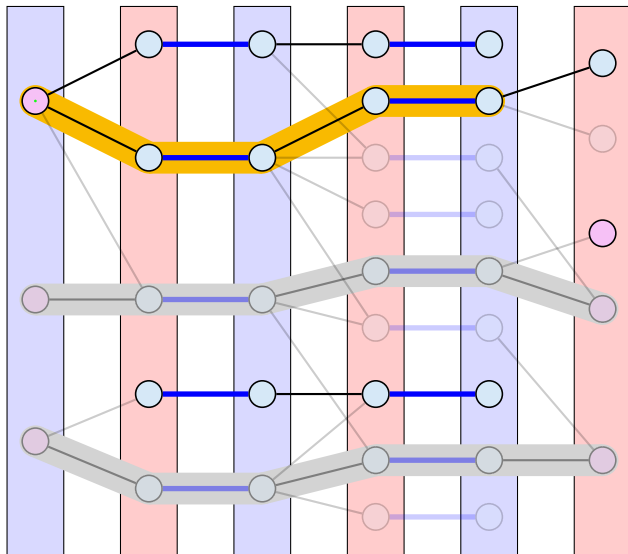
Analysis Hopcroft-Karp

Analysis Hopcroft-Karp
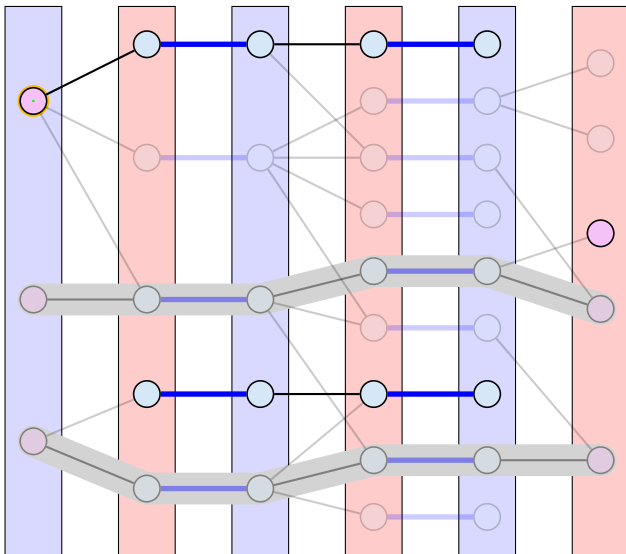
Analysis Hopcroft-Karp
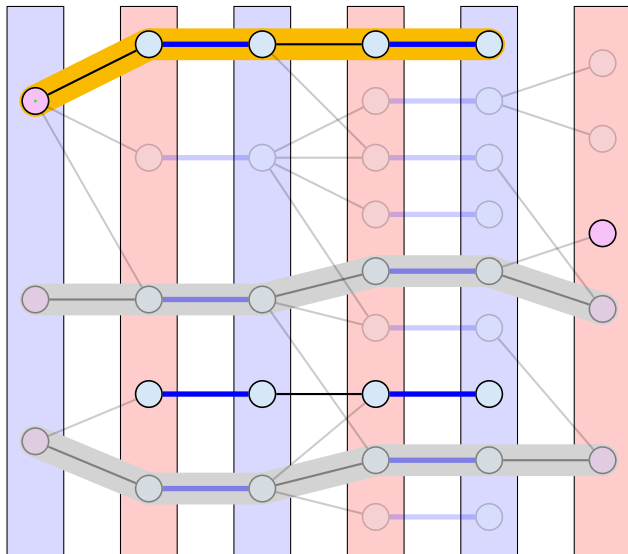
Analysis Hopcroft-Karp
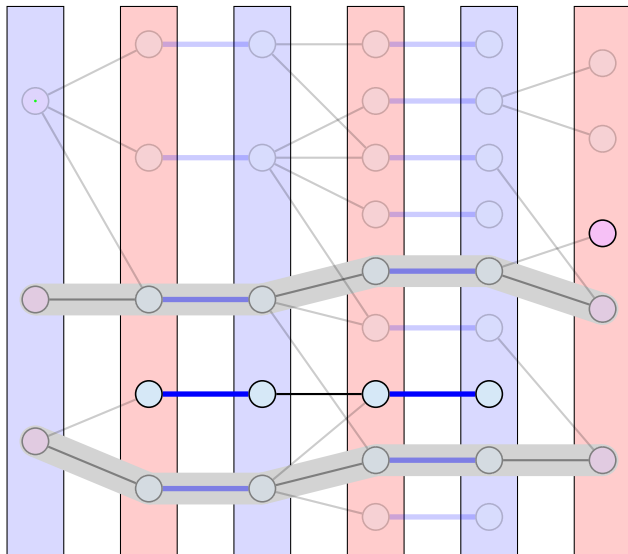
Analysis Hopcroft-Karp
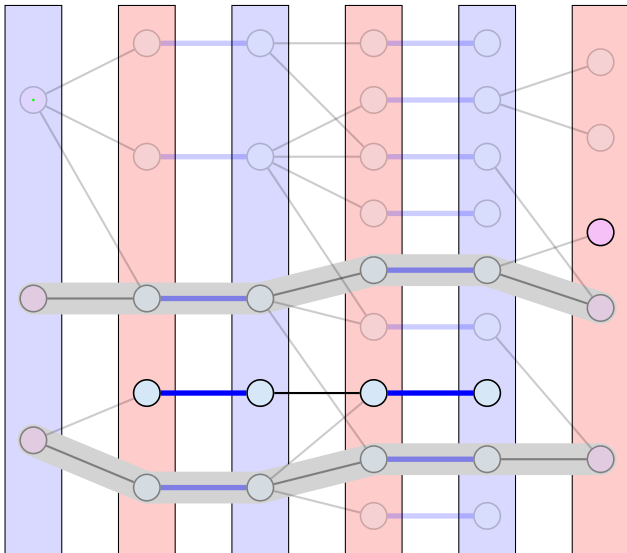
Analysis: Shortest Augmenting Path for Flows

cost for searches during a phase is $\mathcal{O}(mn)$

▶ a search (successful or unsuccessful) takes time $\mathcal{O}(n)$

▶ a search deletes at least one edge from the level graph

there are at most n phases

Time: $\mathcal{O}(mn^2)$.

Analysis for Unit-capacity Simple Networks

cost for searches during a phase is $\mathcal{O}(m)$

▶ an edge/vertex is traversed at most twice

need at most $\mathcal{O}(\sqrt{n})$ phases

▶ after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph

▶ hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$.

21 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree
$T = (V, F, w)$ is a tree with edge-set F and capacities w that
fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$,
   $f(s, t)$ in G is equal to $f_T(s, t)$.

2. **Cut Property:** A minimum s-t cut in T is also a minimum
   cut in G.

Here, $f(s, t)$ is the value of a maximum s-t flow in G, and $f_T(s, t)$
is the corresponding value in T.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

▶ In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

► In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.

► S_i is then removed from T and replaced by X and Y.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

▶ In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.

▶ S_i is then removed from T and replaced by X and Y.

▶ X and Y are connected by an edge, and the edges that before the split were incident to S_i are attached to either X or Y.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

▶ In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.

▶ S_i is then removed from T and replaced by X and Y.

▶ X and Y are connected by an edge, and the edges that before the split were incident to S_i are attached to either X or Y.

In the end this gives a tree on the vertex set V.

Details of the Split-operation

▶ Select S_i that contains at least two nodes a and b.

Details of the Split-operation

- Select S_i that contains at least two nodes a and b.
- Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.

Details of the Split-operation

▶ Select S_i that contains at least two nodes a and b.

▶ Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.

▶ Consider the graph H obtained from G by contracting these connected components into single nodes.

Details of the Split-operation

▶ Select S_i that contains at least two nodes a and b.

▶ Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.

▶ Consider the graph H obtained from G by contracting these connected components into single nodes.

▶ Compute a minimum a-b cut in H. Let A, and B denote the two sides of this cut.

Details of the Split-operation

▶ Select S_i that contains at least two nodes a and b.

▶ Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.

▶ Consider the graph H obtained from G by contracting these connected components into single nodes.

▶ Compute a minimum a-b cut in H. Let A, and B denote the two sides of this cut.

▶ Split S_i in T into two sets/nodes $S_i^a := S_i \cap A$ and $S_i^b := S_i \cap B$ and add edge $\{S_i^a, S_i^b\}$ with capacity $f_H(a, b)$.

Details of the Split-operation

▶ Select S_i that contains at least two nodes a and b.

▶ Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.

▶ Consider the graph H obtained from G by contracting these connected components into single nodes.

▶ Compute a minimum a-b cut in H. Let A, and B denote the two sides of this cut.

▶ Split S_i in T into two sets/nodes $S_i^a := S_i \cap A$ and $S_i^b := S_i \cap B$ and add edge $\{S_i^a, S_i^b\}$ with capacity $f_H(a, b)$.

▶ Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Example: Gomory-Hu Construction

Analysis

Lemma 100

For nodes $s, t, x \in V$ we have $f(s,t) \geq \min\{f(s,x), f(x,t)\}$

Analysis

Lemma 100
For nodes $s, t, x \in V$ we have $f(s, t) \geq \min\{f(s, x), f(x, t)\}$

Lemma 101
For nodes $s, t, x_1, \ldots, x_k \in V$ we have
$f(s, t) \geq \min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k), f(x_k, t)\}$

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.
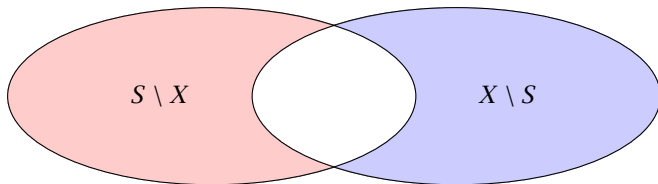
We may assume w.l.o.g. $s \in X$.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \emptyset$ and $X \cap (V \setminus S) \neq \emptyset$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.
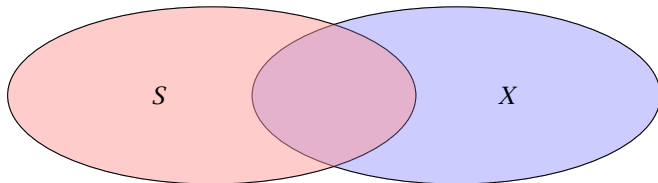
We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.

We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

▶ $\text{cap}(X \setminus S) + \text{cap}(S \setminus X) \leq \text{cap}(S) + \text{cap}(X)$.

Lemma 102

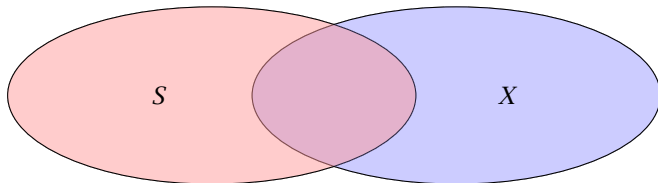*Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.*

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.
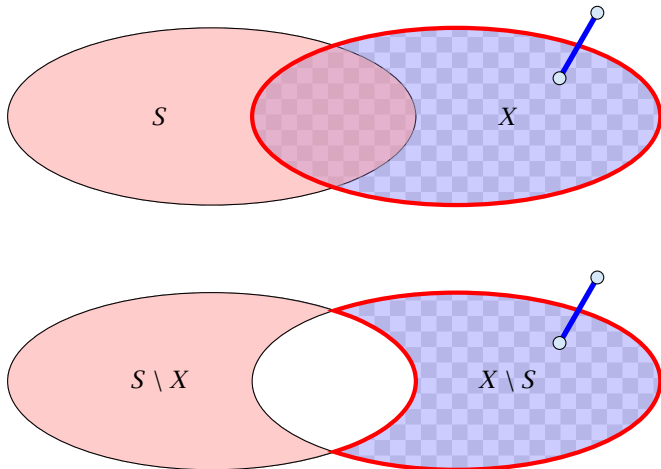
We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

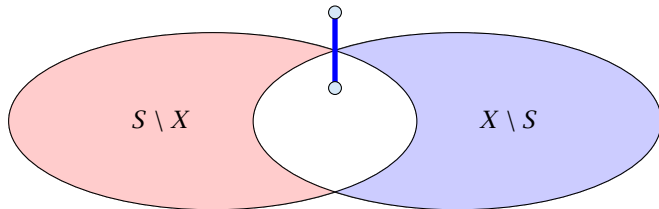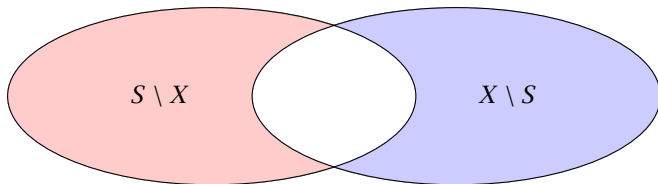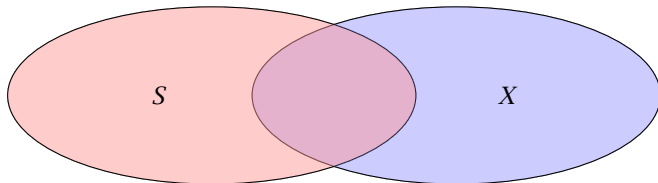▶ $\text{cap}(X \setminus S) + \text{cap}(S \setminus X) \leq \text{cap}(S) + \text{cap}(X)$.

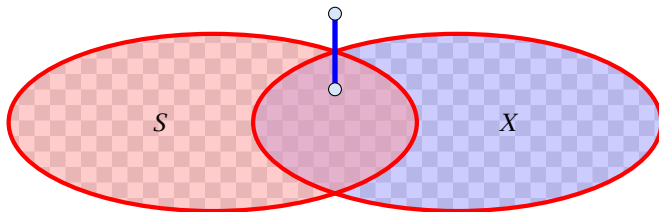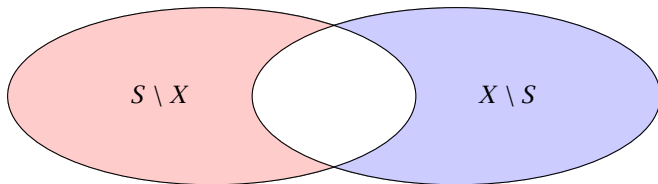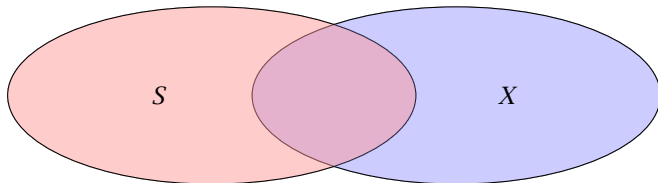▶ $\text{cap}(X \setminus S) \geq \text{cap}(S)$ because $X \setminus S$ is an r-s cut.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.

We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

▶ $\mathrm{cap}(X \setminus S) + \mathrm{cap}(S \setminus X) \leq \mathrm{cap}(S) + \mathrm{cap}(X)$.

▶ $\mathrm{cap}(X \setminus S) \geq \mathrm{cap}(S)$ because $X \setminus S$ is an r-s cut.

▶ This gives $\mathrm{cap}(S \setminus X) \leq \mathrm{cap}(X)$.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.

We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

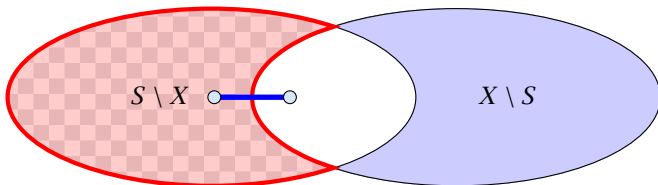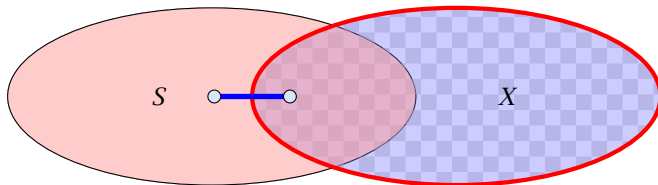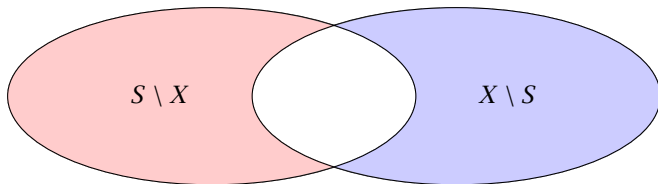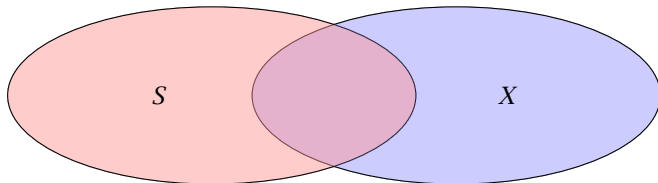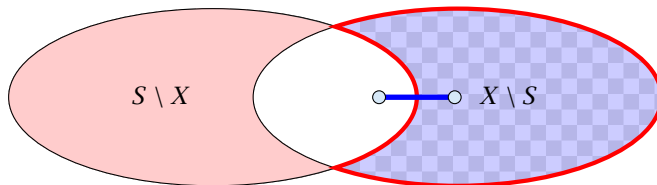▶ $\text{cap}(X \setminus S) + \text{cap}(S \setminus X) \leq \text{cap}(S) + \text{cap}(X)$.

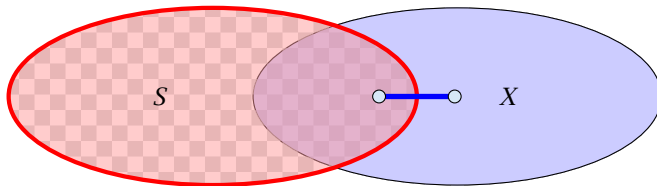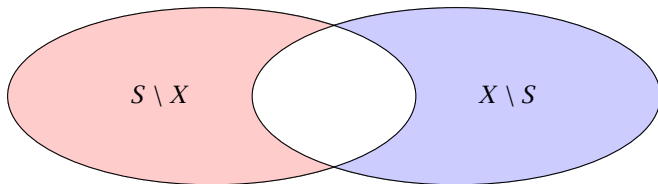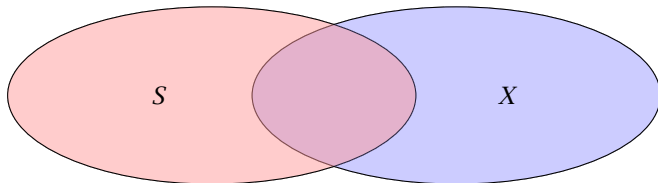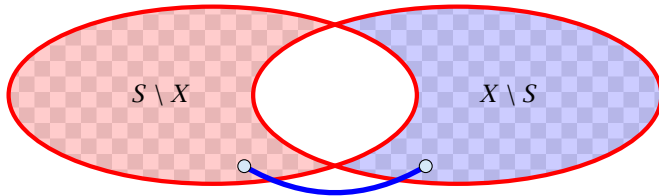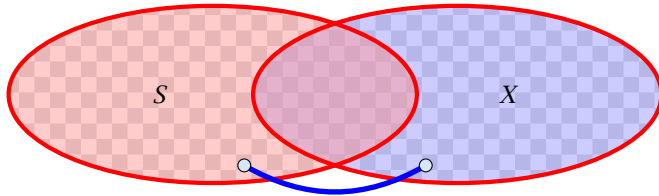▶ $\text{cap}(X \setminus S) \geq \text{cap}(S)$ because $X \setminus S$ is an r-s cut.

▶ This gives $\text{cap}(S \setminus X) \leq \text{cap}(X)$.

Second case $r \notin X$.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S. We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

▶ $\operatorname{cap}(X \setminus S) + \operatorname{cap}(S \setminus X) \leq \operatorname{cap}(S) + \operatorname{cap}(X)$.

▶ $\operatorname{cap}(X \setminus S) \geq \operatorname{cap}(S)$ because $X \setminus S$ is an r-s cut.

▶ This gives $\operatorname{cap}(S \setminus X) \leq \operatorname{cap}(X)$.

Second case $r \notin X$.

▶ $\operatorname{cap}(X \cup S) + \operatorname{cap}(S \cap X) \leq \operatorname{cap}(S) + \operatorname{cap}(X)$.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.

We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

▶ $\mathrm{cap}(X \setminus S) + \mathrm{cap}(S \setminus X) \leq \mathrm{cap}(S) + \mathrm{cap}(X)$.

▶ $\mathrm{cap}(X \setminus S) \geq \mathrm{cap}(S)$ because $X \setminus S$ is an r-s cut.

▶ This gives $\mathrm{cap}(S \setminus X) \leq \mathrm{cap}(X)$.

Second case $r \notin X$.

▶ $\mathrm{cap}(X \cup S) + \mathrm{cap}(S \cap X) \leq \mathrm{cap}(S) + \mathrm{cap}(X)$.

▶ $\mathrm{cap}(X \cup S) \geq \mathrm{cap}(S)$ because $X \cup S$ is an r-s cut.

Lemma 102

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S. We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

▶ $\text{cap}(X \setminus S) + \text{cap}(S \setminus X) \leq \text{cap}(S) + \text{cap}(X)$.

▶ $\text{cap}(X \setminus S) \geq \text{cap}(S)$ because $X \setminus S$ is an r-s cut.

▶ This gives $\text{cap}(S \setminus X) \leq \text{cap}(X)$.

Second case $r \notin X$.

▶ $\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$.

▶ $\text{cap}(X \cup S) \geq \text{cap}(S)$ because $X \cup S$ is an r-s cut.

▶ This gives $\text{cap}(S \cap X) \leq \text{cap}(X)$.

$$\text{cap}(S \setminus X) + \text{cap}(X \setminus S) \leq \text{cap}(S) + \text{cap}(X)$$

$\mathbf{cap}(S \setminus X) + \mathbf{cap}(X \setminus S) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

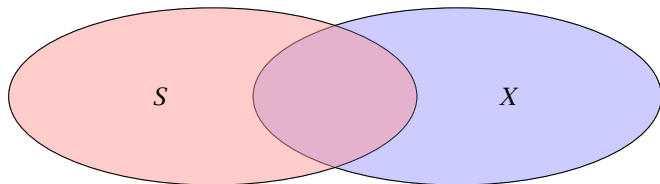$$cap(S \setminus X) + cap(X \setminus S) \leq cap(S) + cap(X)$$

$\text{cap}(S \setminus X) + \text{cap}(X \setminus S) \le \text{cap}(S) + \text{cap}(X)$

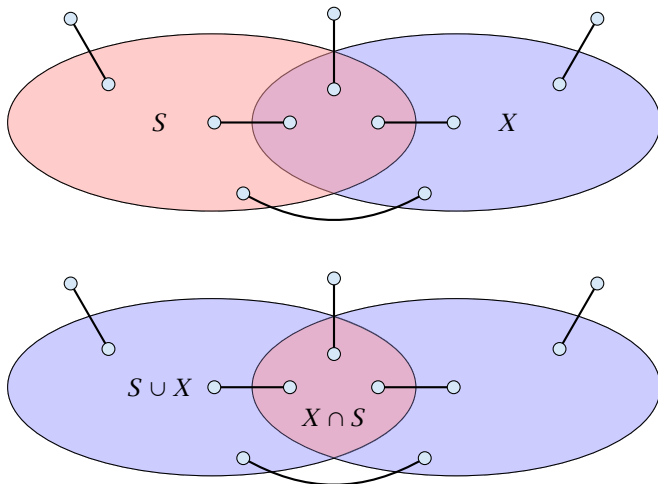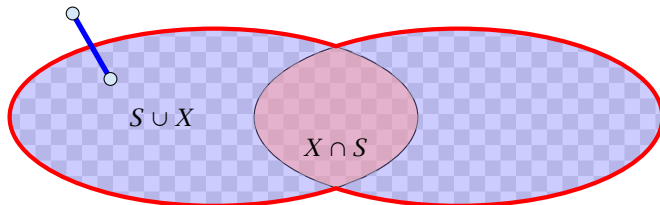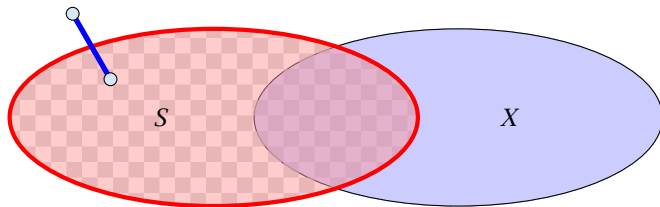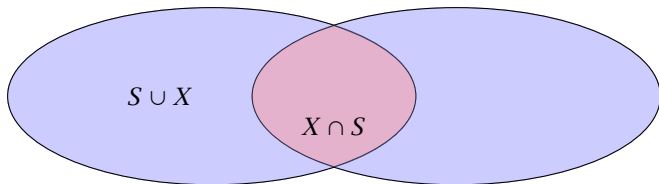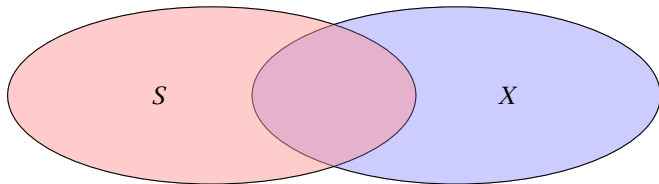$$\text{cap}(S \setminus X) + \text{cap}(X \setminus S) \leq \text{cap}(S) + \text{cap}(X)$$

cap$(S \setminus X)$ + cap$(X \setminus S)$ ≤ cap(S) + cap(X)

$\text{cap}(S \setminus X) + \text{cap}(X \setminus S) \leq \text{cap}(S) + \text{cap}(X)$

cap($S \setminus X$) + cap($X \setminus S$) $\leq$ cap(S) + cap(X)

cap($S \setminus X$) + cap($X \setminus S$) ≤ cap(S) + cap(X)

$\mathbf{cap}(S \setminus X) + \mathbf{cap}(X \setminus S) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

$\mathbf{cap}(S \setminus X) + \mathbf{cap}(X \setminus S) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

$\mathbf{cap}(S \setminus X) + \mathbf{cap}(X \setminus S) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

$\text{cap}(S \setminus X) + \text{cap}(X \setminus S) \leq \text{cap}(S) + \text{cap}(X)$
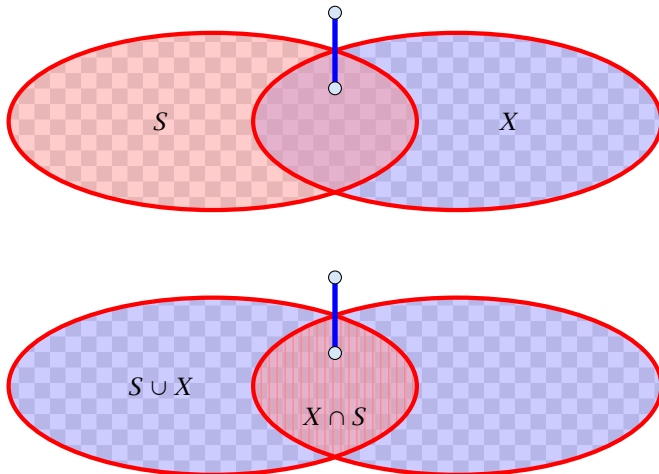
cap($S \setminus X$) + cap($X \setminus S$) ≤ cap(S) + cap(X)

$\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$

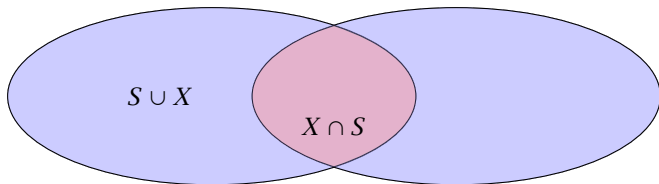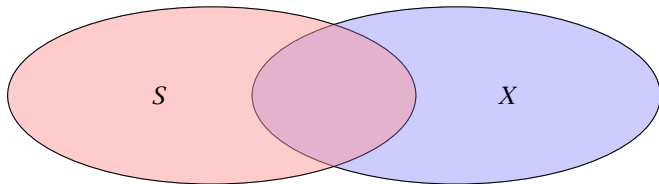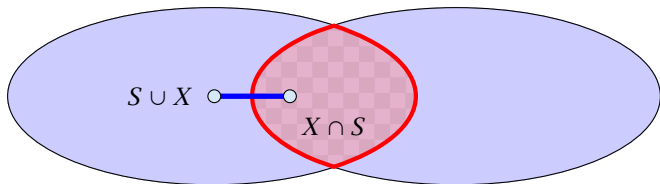# $\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$

$\mathbf{cap}(X \cup S) + \mathbf{cap}(S \cap X) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

$\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$

$\mathbf{cap}(X \cup S) + \mathbf{cap}(S \cap X) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

cap$(X \cup S)$ + cap$(S \cap X)$ ≤ cap(S) + cap(X)

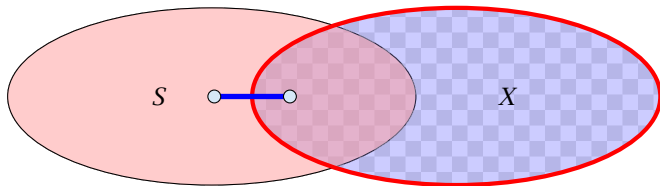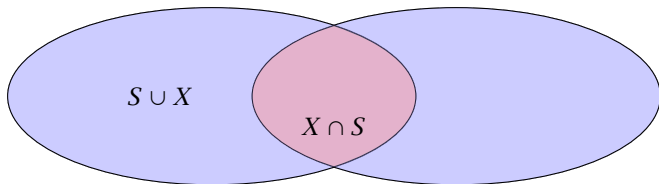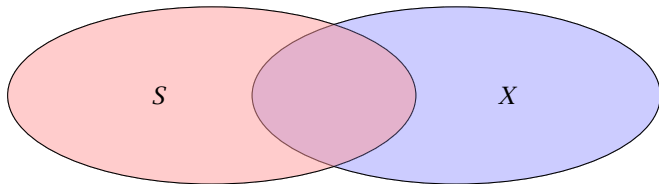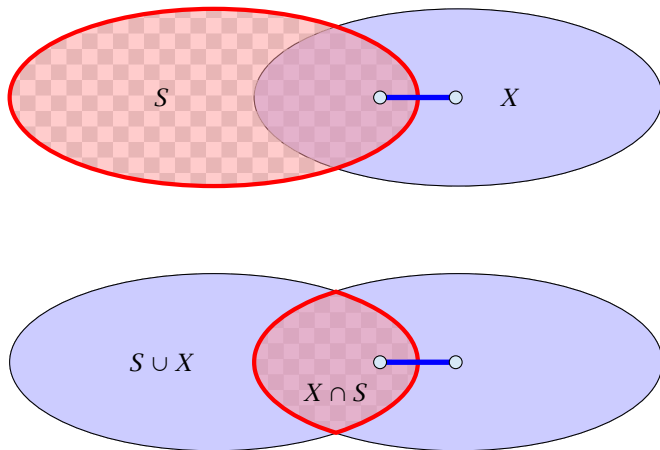$$\text{cap}(X \cup S) + \text{cap}(S \cap X) \le \text{cap}(S) + \text{cap}(X)$$

$\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$

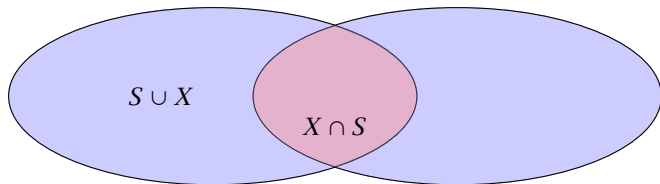# $\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$

$\operatorname{cap}(X \cup S) + \operatorname{cap}(S \cap X) \le \operatorname{cap}(S) + \operatorname{cap}(X)$

$\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$

$\mathbf{cap}(X \cup S) + \mathbf{cap}(S \cap X) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

$\mathbf{cap}(X \cup S) + \mathbf{cap}(S \cap X) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

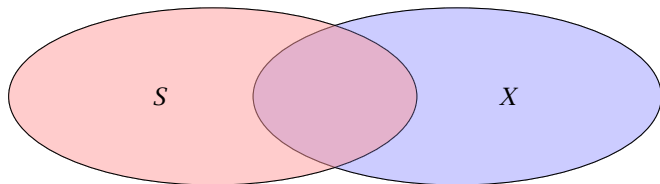# $\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$
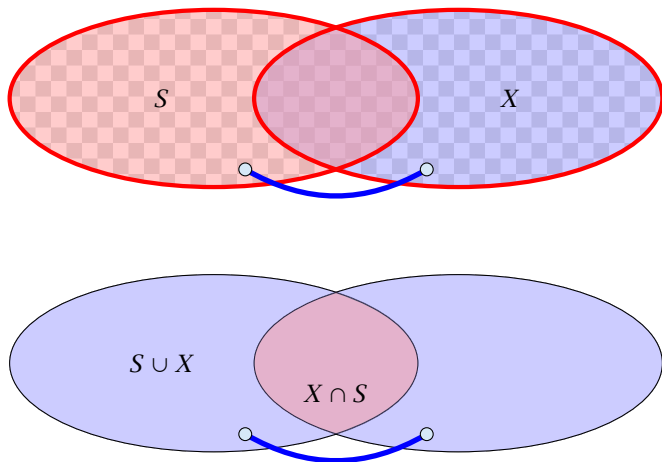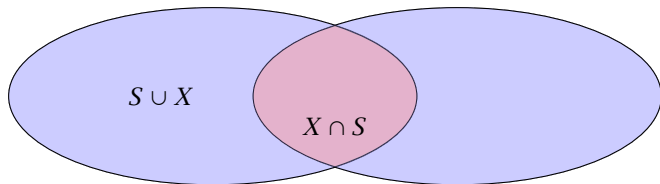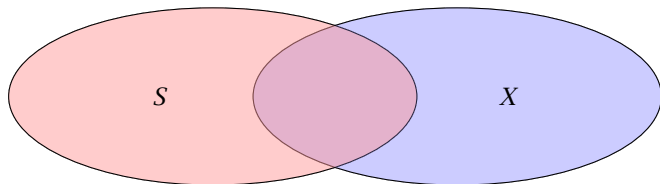
$\mathbf{cap}(X \cup S) + \mathbf{cap}(S \cap X) \leq \mathbf{cap}(S) + \mathbf{cap}(X)$

Analysis

Lemma 102 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s, t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s, t) = f(s, t)$, where $f_H(s, t)$ is the value of a minimum s-t mincut in graph H.

Analysis

Invariant [existence of representatives]:
For any edge $\{S_i, S_j\}$ in T, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum a-b cut in G.

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

- ▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

▶ Then

$$f_T(s, t)$$

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

▶ Then

$$f_T(s, t) = \min_{i \in \{0, \ldots, k-1\}} \{w(x_i, x_{i+1})\}$$

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

▶ Then

$$f_T(s, t) = \min_{i \in \{0, \ldots, k-1\}} \{w(x_i, x_{i+1})\}$$
$$= \min_{i \in \{0, \ldots, k-1\}} \{f(x_i, x_{i+1})\}$$

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

▶ Then

$$
\begin{aligned}
f_T(s,t) &= \min_{i \in \{0,\ldots,k-1\}} \{w(x_i, x_{i+1})\} \\
&= \min_{i \in \{0,\ldots,k-1\}} \{f(x_i, x_{i+1})\} \leq f(s,t) \ .
\end{aligned}
$$

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

- ▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

- ▶ Then

$$
\begin{aligned}
f_T(s, t) &= \min_{i \in \{0, \ldots, k-1\}} \{w(x_i, x_{i+1})\} \\
&= \min_{i \in \{0, \ldots, k-1\}} \{f(x_i, x_{i+1})\} \le f(s, t) \ .
\end{aligned}
$$

- ▶ Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

▶ Then

$$f_T(s, t) = \min_{i \in \{0, \ldots, k-1\}} \{w(x_i, x_{i+1})\}$$

$$= \min_{i \in \{0, \ldots, k-1\}} \{f(x_i, x_{i+1})\} \leq f(s, t) \ .$$

▶ Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.

▶ Since by the invariant this edge induces an s-t cut with capacity $f(x_j, x_{j+1})$ we get $f(s, t) \leq f(x_j, x_{j+1}) = f_T(s, t)$.

Analysis

▶ Hence, $f_T(s, t) = f(s, t)$ (flow equivalence).

Analysis

- Hence, $f_T(s,t) = f(s,t)$ (flow equivalence).
- The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T.

Analysis

- Hence, $f_T(s, t) = f(s, t)$ (flow equivalence).

- The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T.

- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).

Analysis

- Hence, $f_T(s,t) = f(s,t)$ (flow equivalence).
- The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T.
- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).
- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. s and t, this is an s-t mincut (cut property).

Proof of Invariant

Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let S_i denote our selected cluster with nodes a and b. Because of the invariant all edges leaving $\{S_i\}$ in T correspond to some mincuts.

Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let S_i denote our selected cluster with nodes a and b. Because of the invariant all edges leaving $\{S_i\}$ in T correspond to some mincuts.

Therefore, contracting the connected components does not change the mincut btw. a and b due to Lemma 102.

Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let S_i denote our selected cluster with nodes a and b. Because of the invariant all edges leaving $\{S_i\}$ in T correspond to some mincuts.

Therefore, contracting the connected components does not change the mincut btw. a and b due to Lemma 102.

After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a, b)$ we can simply choose a and b as representatives.

Proof of Invariant

Proof of Invariant

For edges that are not incident to S_i we do not need to change representatives as the neighbouring sets do not change.

Proof of Invariant

For edges that are not incident to S_i we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_i^a\}$ in the new tree (the case when it is replaced by $\{X, S_i^b\}$ is analogous).

Proof of Invariant

For edges that are not incident to S_i we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_i^a\}$ in the new tree (the case when it is replaced by $\{X, S_i^b\}$ is analogous).

If $s \in S_i^a$ we can keep x and s as representatives.

Proof of Invariant

For edges that are not incident to S_i we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_i^a\}$ in the new tree (the case when it is replaced by $\{X, S_i^b\}$ is analogous).

If $s \in S_i^a$ we can keep x and s as representatives.

Otherwise, we choose x and a as representatives. We need to show that $f(x, a) = f(x, s)$.

Proof of Invariant

Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x, s)$. Since, x and a are on opposite sides of this cut, we know that $f(x, a) \leq f(x, s)$.

Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x, s)$. Since, x and a are on opposite sides of this cut, we know that $f(x, a) \leq f(x, s)$.

The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 102 we know that $f'(x, a) = f(x, a)$ as $x, a \notin B$.
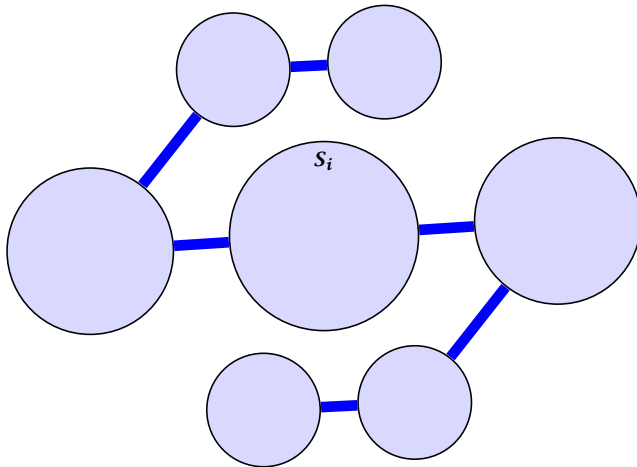
Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x, s)$. Since, x and a are on opposite sides of this cut, we know that $f(x, a) \leq f(x, s)$.

The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 102 we know that $f'(x, a) = f(x, a)$ as $x, a \notin B$.

We further have $f'(x, a) \geq \min\{f'(x, v_B), f'(v_B, a)\}$.

Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x, s)$. Since, x and a are on opposite sides of this cut, we know that $f(x, a) \leq f(x, s)$.
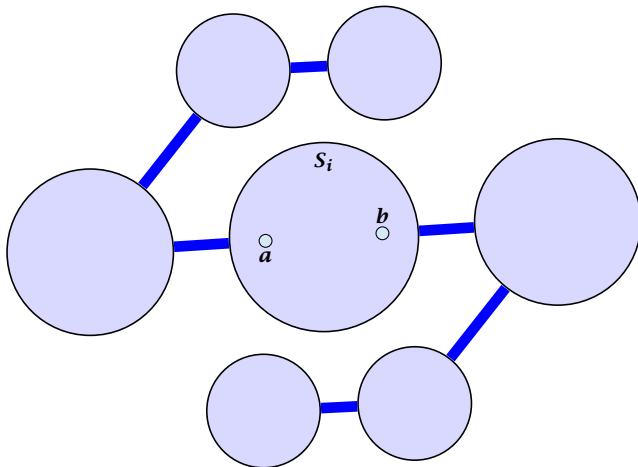
The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 102 we know that $f'(x, a) = f(x, a)$ as $x, a \notin B$.

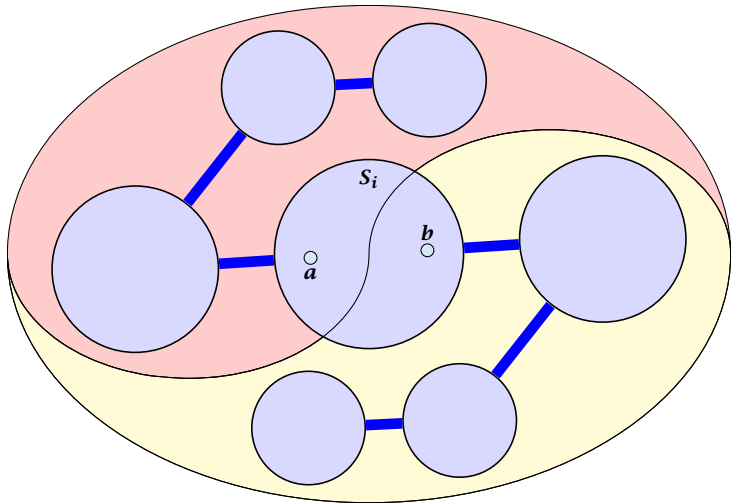We further have $f'(x, a) \geq \min\{f'(x, v_B), f'(v_B, a)\}$.

Since $s \in B$ we have $f'(v_B, x) \geq f(s, x)$.

Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x, s)$. Since, x and a are on opposite sides of this cut, we know that $f(x, a) \leq f(x, s)$.

The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 102 we know that $f'(x, a) = f(x, a)$ as $x, a \notin B$.

We further have $f'(x, a) \geq \min\{f'(x, v_B), f'(v_B, a)\}$.

Since $s \in B$ we have $f'(v_B, x) \geq f(s, x)$.

Also, $f'(a, v_B) \geq f(a, b) \geq f(x, s)$ since the a-b cut that splits S_i into S_i^a and S_i^b also separates s and x.
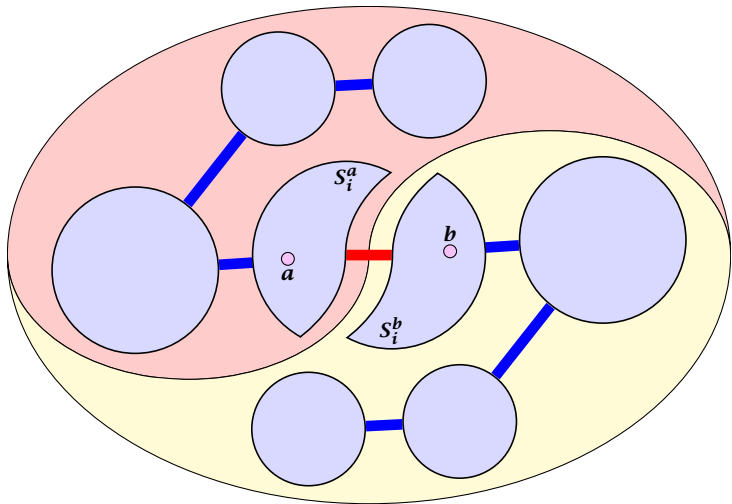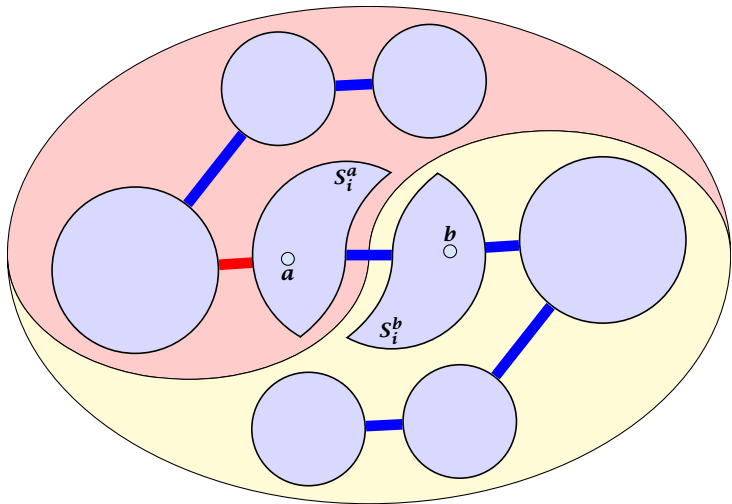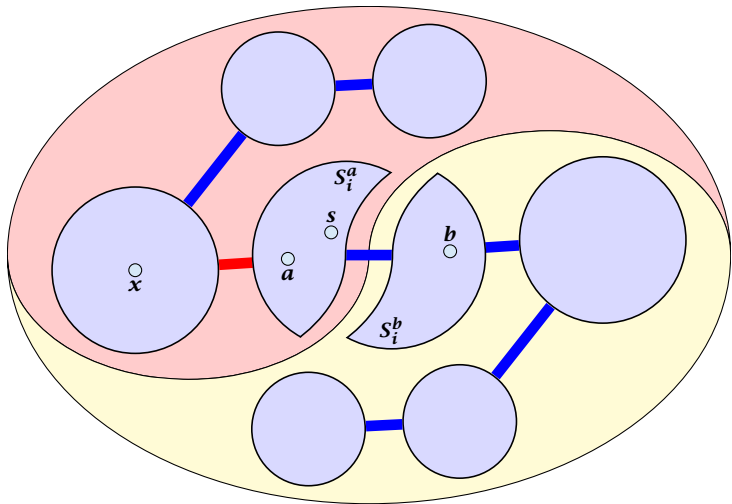
Analysis

Analysis

Analysis

Analysis

Analysis

Analysis

Analysis

Analysis

Analysis

Analysis

Harald Räcke