21 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set F and capacities w that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f(s, t)$ in G is equal to $f_T(s, t)$.

2. **Cut Property:** A minimum s-t cut in T is also a minimum cut in G.

Here, $f(s, t)$ is the value of a maximum s-t flow in G, and $f_T(s, t)$ is the corresponding value in T.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

▶ In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.

▶ S_i is then removed from T and replaced by X and Y.

▶ X and Y are connected by an edge, and the edges that before the split were incident to S_i are attached to either X or Y.

In the end this gives a tree on the vertex set V.

Details of the Split-operation

▶ Select S_i that contains at least two nodes a and b.

▶ Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.

▶ Consider the graph H obtained from G by contracting these connected components into single nodes.

▶ Compute a minimum a-b cut in H. Let A, and B denote the two sides of this cut.

▶ Split S_i in T into two sets/nodes $S_i^a := S_i \cap A$ and $S_i^b := S_i \cap B$ and add edge $\{S_i^a, S_i^b\}$ with capacity $f_H(a, b)$.

▶ Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

Example: Gomory-Hu Construction

Analysis
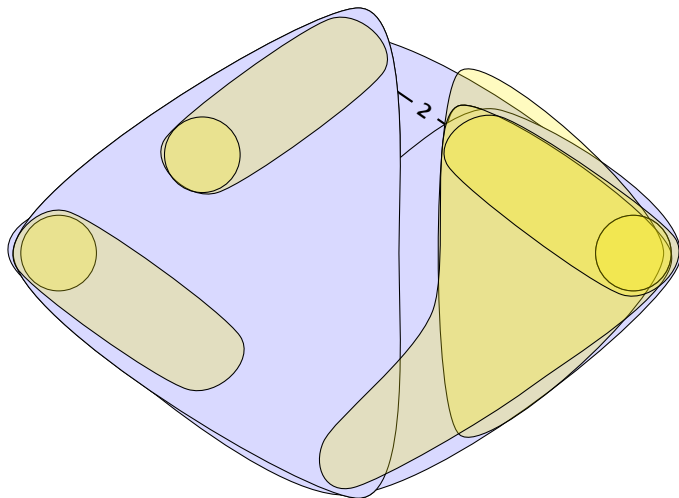
Lemma 6

For nodes $s, t, x \in V$ we have $f(s, t) \geq \min\{f(s, x), f(x, t)\}$

Lemma 7

For nodes $s, t, x_1, \ldots, x_k \in V$ we have
$f(s, t) \geq \min\{f(s, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k), f(x_k, t)\}$

Lemma 8

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \varnothing$ and $X \cap (V \setminus S) \neq \varnothing$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.
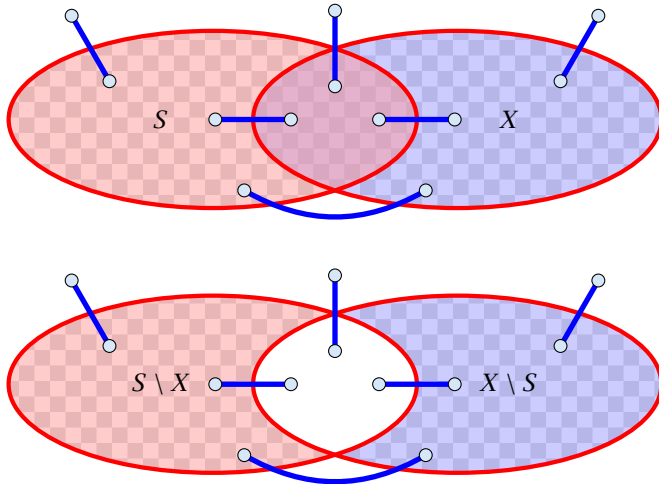
We may assume w.l.o.g. $s \in X$.

First case $r \in X$.

- ▶ $\text{cap}(X \setminus S) + \text{cap}(S \setminus X) \leq \text{cap}(S) + \text{cap}(X)$.
- ▶ $\text{cap}(X \setminus S) \geq \text{cap}(S)$ because $X \setminus S$ is an r-s cut.
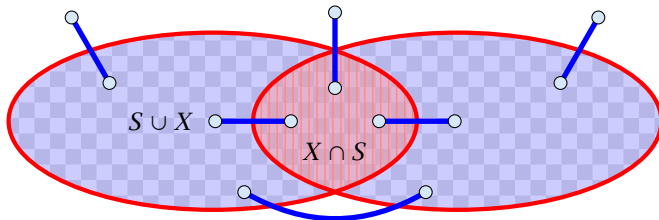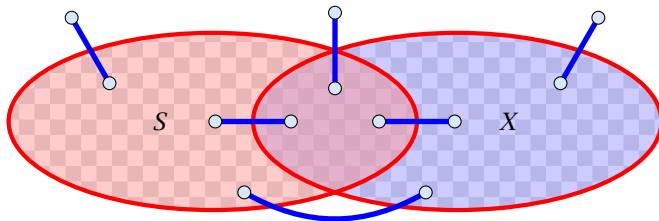- ▶ This gives $\text{cap}(S \setminus X) \leq \text{cap}(X)$.

Second case $r \notin X$.

- ▶ $\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$.
- ▶ $\text{cap}(X \cup S) \geq \text{cap}(S)$ because $X \cup S$ is an r-s cut.
- ▶ This gives $\text{cap}(S \cap X) \leq \text{cap}(X)$.

cap(S \ X) + cap(X \ S) ≤ cap(S) + cap(X)

$cap(X \cup S) + cap(S \cap X) \leq cap(S) + cap(X)$

Analysis

Lemma 8 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s,t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s,t) = f(s,t)$, where $f_H(s,t)$ is the value of a minimum s-t mincut in graph H.

Analysis

Invariant [existence of representatives]:
For any edge $\{S_i, S_j\}$ in T, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum a-b cut in G.

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

▶ Let $s = x_0, x_1, \ldots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T. From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j.

▶ Then

$$
\begin{aligned}
f_T(s, t) &= \min_{i \in \{0, \ldots, k-1\}} \{w(x_i, x_{i+1})\} \\
&= \min_{i \in \{0, \ldots, k-1\}} \{f(x_i, x_{i+1})\} \le f(s, t) \ .
\end{aligned}
$$

▶ Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.

▶ Since by the invariant this edge induces an s-t cut with capacity $f(x_j, x_{j+1})$ we get $f(s, t) \le f(x_j, x_{j+1}) = f_T(s, t)$.

Analysis

- Hence, $f_T(s, t) = f(s, t)$ (flow equivalence).

- The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T.

- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).

- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. s and t, this is an s-t mincut (cut property).

Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let S_i denote our selected cluster with nodes a and b. Because of the invariant all edges leaving $\{S_i\}$ in T correspond to some mincuts.

Therefore, contracting the connected components does not change the mincut btw. a and b due to Lemma 8.

After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a, b)$ we can simply choose a and b as representatives.

Proof of Invariant

For edges that are not incident to S_i we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_i^a\}$ in the new tree (the case when it is replaced by $\{X, S_i^b\}$ is analogous).

If $s \in S_i^a$ we can keep x and s as representatives.

Otherwise, we choose x and a as representatives. We need to show that $f(x, a) = f(x, s)$.

Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x, s)$. Since, x and a are on opposite sides of this cut, we know that $f(x, a) \leq f(x, s)$.

The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 8 we know that $f'(x, a) = f(x, a)$ as $x, a \notin B$.

We further have $f'(x, a) \geq \min\{f'(x, v_B), f'(v_B, a)\}$.

Since $s \in B$ we have $f'(v_B, x) \geq f(s, x)$.

Also, $f'(a, v_B) \geq f(a, b) \geq f(x, s)$ since the a-b cut that splits S_i into S_i^a and S_i^b also separates s and x.

Analysis