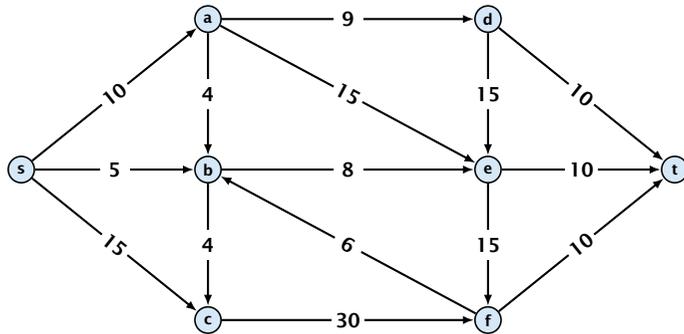


## 10 Introduction

### Flow Network

- ▶ directed graph  $G = (V, E)$ ; edge capacities  $c(e)$
- ▶ two special nodes: source  $s$ ; target  $t$ ;
- ▶ no edges entering  $s$  or leaving  $t$ ;
- ▶ at least for now: no parallel edges;



## Cuts

### Definition 1

An  $(s, t)$ -cut in the graph  $G$  is given by a set  $A \subset V$  with  $s \in A$  and  $t \in V \setminus A$ .

### Definition 2

The **capacity** of a cut  $A$  is defined as

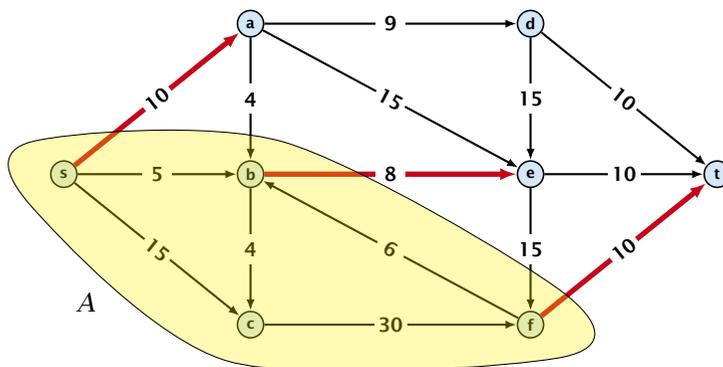
$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e),$$

where  $\text{out}(A)$  denotes the set of edges of the form  $A \times V \setminus A$  (i.e. edges leaving  $A$ ).

**Minimum Cut Problem:** Find an  $(s, t)$ -cut with minimum capacity.

## Cuts

### Example 3



The capacity of the cut is  $\text{cap}(A, V \setminus A) = 28$ .

## Flows

### Definition 4

An  $(s, t)$ -flow is a function  $f : E \rightarrow \mathbb{R}^+$  that satisfies

1. For each edge  $e$

$$0 \leq f(e) \leq c(e).$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e).$$

(flow conservation constraints)

## Flows

### Definition 5

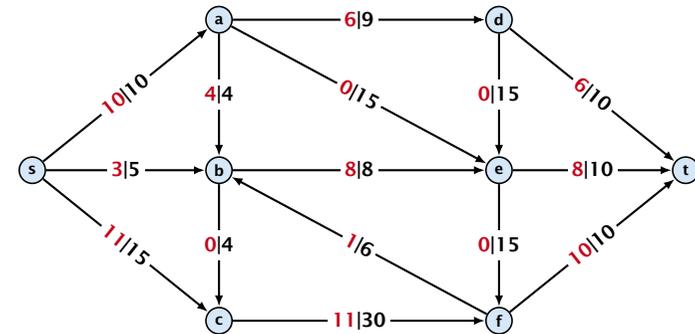
The value of an  $(s, t)$ -flow  $f$  is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e).$$

**Maximum Flow Problem:** Find an  $(s, t)$ -flow with maximum value.

## Flows

### Example 6



The value of the flow is  $\text{val}(f) = 24$ .

## Flows

### Lemma 7 (Flow value lemma)

Let  $f$  be a flow, and let  $A \subseteq V$  be an  $(s, t)$ -cut. Then the net-flow across the cut is equal to the amount of flow leaving  $s$ , i.e.,

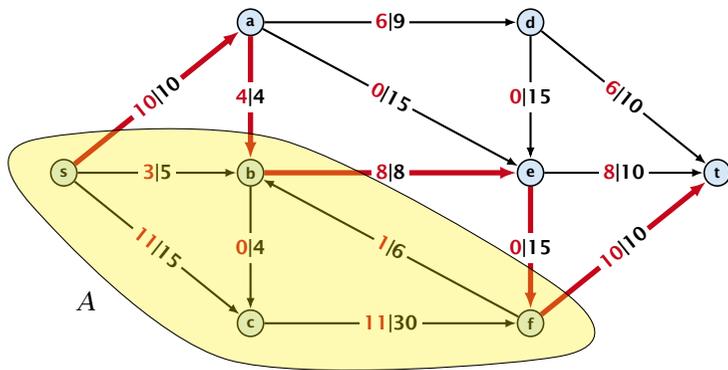
$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e).$$

### Proof.

$$\begin{aligned} \text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right) \\ &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \end{aligned}$$

The last equality holds since every edge with both end-points in  $A$  contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering  $A$ .  $\square$

### Example 8



The net-flow across the cut is  $\text{val}(f) = 24$ .

### Corollary 9

Let  $f$  be an  $(s, t)$ -flow and let  $A$  be an  $(s, t)$ -cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then  $f$  is a maximum flow.

#### Proof.

Suppose that there is a flow  $f'$  with larger value. Then

$$\begin{aligned} \text{cap}(A, V \setminus A) &< \text{val}(f') \\ &= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \\ &\leq \sum_{e \in \text{out}(A)} f'(e) \\ &\leq \text{cap}(A, V \setminus A) \end{aligned}$$

□