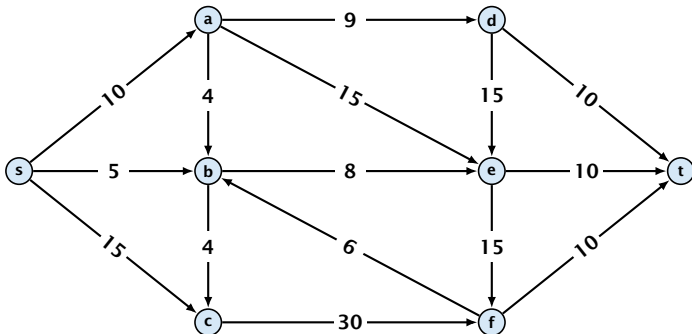


10 Introduction

Flow Network

- ▶ directed graph $G = (V, E)$; edge capacities $c(e)$
- ▶ two special nodes: source s ; target t ;
- ▶ no edges entering s or leaving t ;
- ▶ at least for now: no parallel edges;



Cuts

Definition 1

An (s, t) -cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

Definition 2

The **capacity** of a cut A is defined as

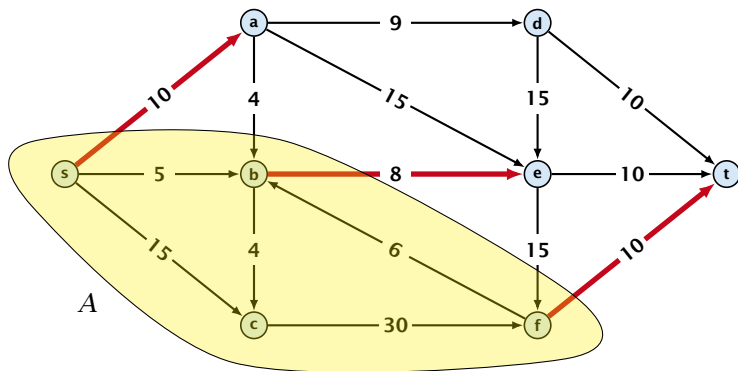
$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e) ,$$

where $\text{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

Minimum Cut Problem: Find an (s, t) -cut with minimum capacity.

Cuts

Example 3



The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$.

Definition 4

An (s, t) -flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) .$$

(flow conservation constraints)

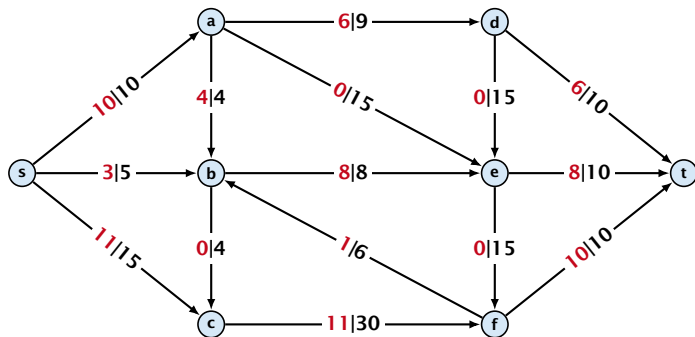
Definition 5

The **value of an (s, t) -flow f** is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

Maximum Flow Problem: Find an (s, t) -flow with maximum value.

Example 6



The value of the flow is $\text{val}(f) = 24$.

Lemma 7 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s, t) -cut. Then the *net-flow* across the cut is equal to the amount of flow leaving s , i.e.,

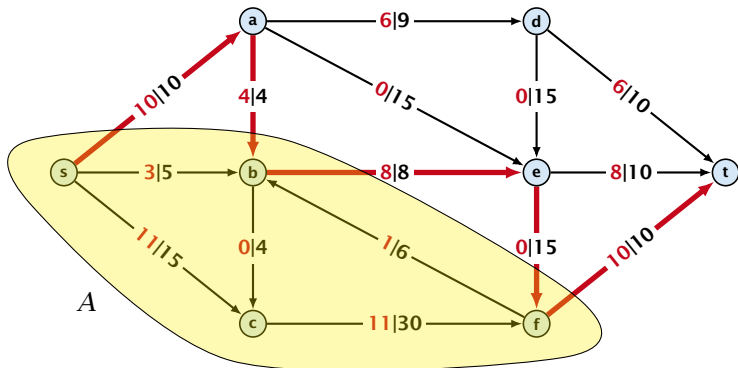
$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) .$$

Proof.

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right) \\ &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)\end{aligned}$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A . \square

Example 8



The net-flow across the cut is $\text{val}(f) = 24$.

Corollary 9

Let f be an (s, t) -flow and let A be an (s, t) -cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

$$\begin{aligned} \text{cap}(A, V \setminus A) &< \text{val}(f') \\ &= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \\ &\leq \sum_{e \in \text{out}(A)} f'(e) \\ &\leq \text{cap}(A, V \setminus A) \end{aligned}$$

□