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Sometimes we also have

- ▶  **$S$ . merge( $S'$ )**:  $S := S \cup S'$ ;  $S' := \emptyset$ .

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- ▶ **handle  $S$ . insert( $x$ )**: Adds element  $x$  to the data-structure, and returns a **handle** to the object for future reference.
- ▶  **$S$ . delete( $h$ )**: Deletes element specified through handle  $h$ .
- ▶  **$S$ . decrease-key( $h, k$ )**: Decreases the key of the element specified by handle  $h$  to  $k$ . Assumes that the key is at least  $k$  before the operation.

# Dijkstra's Shortest Path Algorithm

## Algorithm 1 Shortest-Path( $G = (V, E, d), s \in V$ )

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.is-empty() = false$  do  
9:      $v \leftarrow S.delete-min()$ ;  
10:    for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:        if  $x.key > v.key + d(v, x)$  then  
12:             $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:             $x.key \leftarrow v.key + d(v, x)$ ;
```

# Prim's Minimum Spanning Tree Algorithm

## Algorithm 2 Prim-MST( $G = (V, E, d), s \in V$ )

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
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13:             $x.key \leftarrow d(v, x)$ ;  
14:             $x.pred \leftarrow v$ ;
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# Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶  $|V|$  insert() operations
- ▶  $|V|$  delete-min() operations
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**How good a running time can we obtain?**

## 8 Priority Queues

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	$n$	$n \log n$	$n \log n$	$n$
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	$n$	$n \log n$	$\log n$	1



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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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Note that most applications use **build()** only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

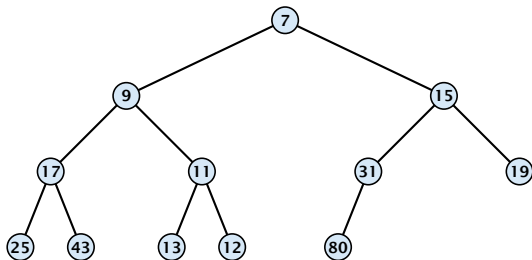
Fibonacci heaps only give an **amortized** guarantee.

## 8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time  $\mathcal{O}((|V| + |E|) \log |V|)$ .

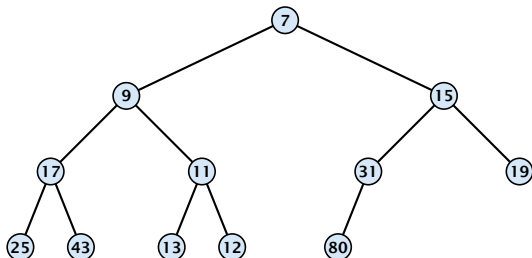
Using Fibonacci Heaps, Prim and Dijkstra run in time  $\mathcal{O}(|V| \log |V| + |E|)$ .

## 8.1 Binary Heaps



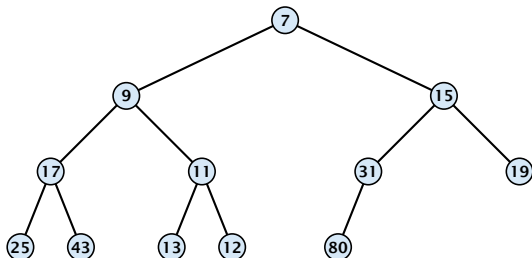
## 8.1 Binary Heaps

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- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- ▶ **Heap property:** A node's key is not larger than the key of one of its children.



**Operations:**



# Binary Heaps

## Operations:

- ▶ **minimum()**: return the root-element. Time  $\mathcal{O}(1)$ .

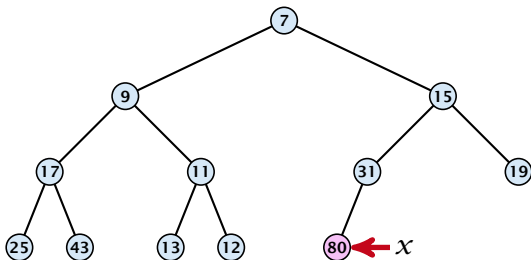
# Binary Heaps

## Operations:

- ▶ **minimum()**: return the root-element. Time  $\mathcal{O}(1)$ .
- ▶ **is-empty()**: check whether root-pointer is **null**. Time  $\mathcal{O}(1)$ .

## 8.1 Binary Heaps

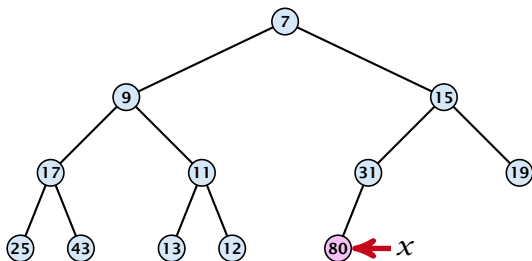
Maintain a pointer to the **last element**  $x$ .



## 8.1 Binary Heaps

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- ▶ We can compute the predecessor of  $x$  (last element when  $x$  is deleted) in time  $\mathcal{O}(\log n)$ .



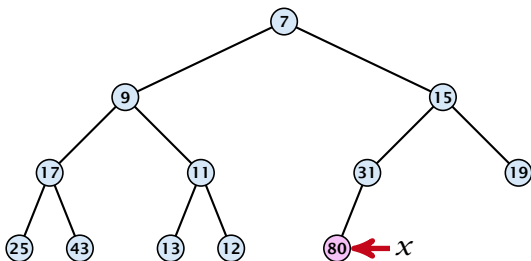
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go left; go right until you reach a leaf



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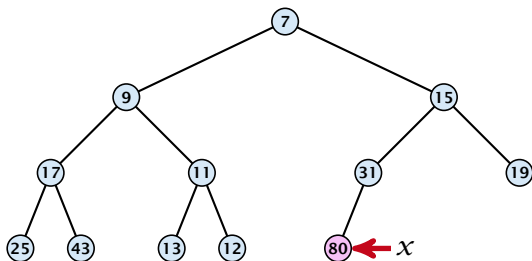
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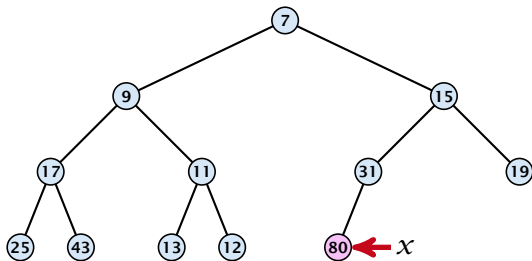
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if you hit the root on the way up, go to the rightmost element



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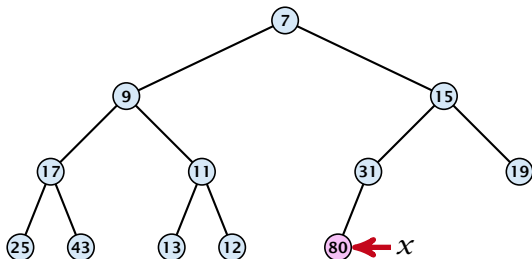
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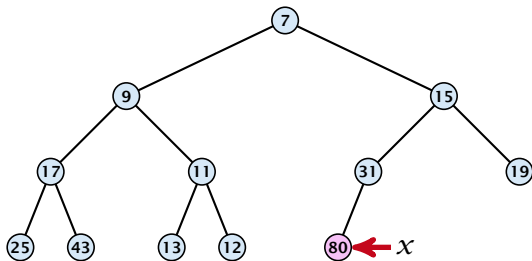
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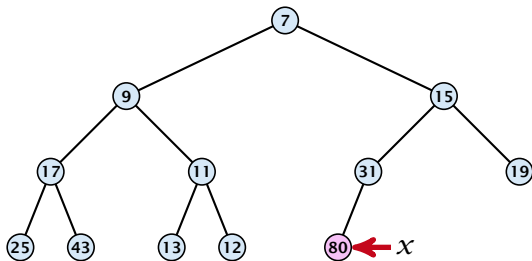
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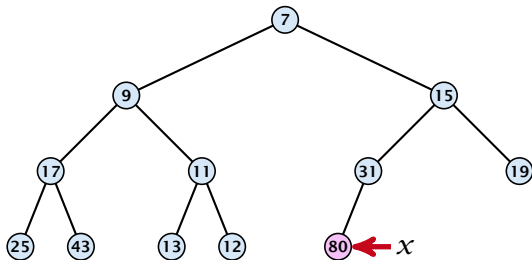
if you hit the root on the way up, go to the leftmost element;

insert a new element as a left child;



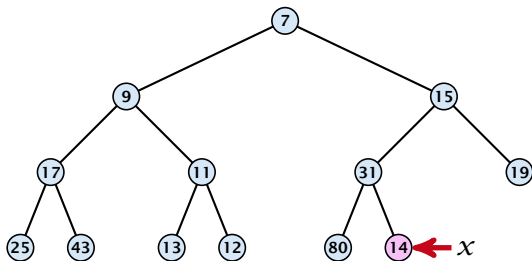
# Insert

1. Insert element at successor of  $x$ .



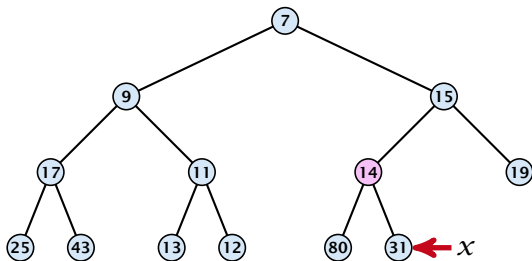
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1. Insert element at successor of  $x$ .
2. Exchange with parent until heap property is fulfilled.



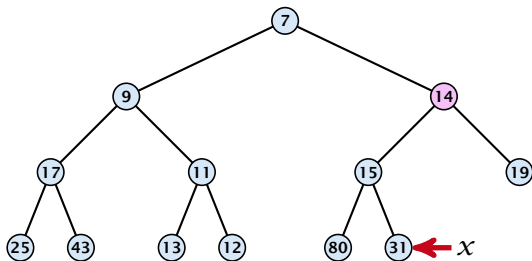
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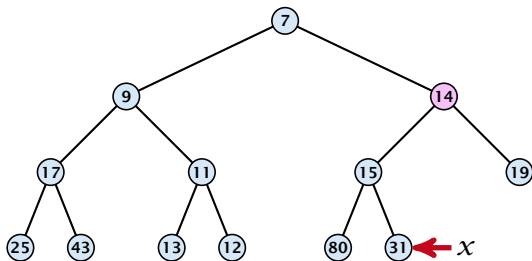
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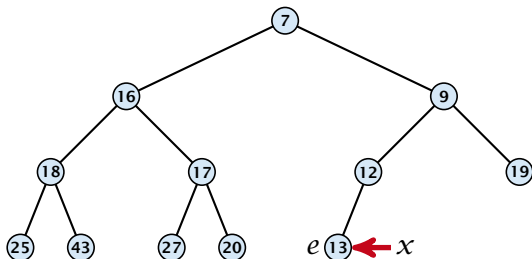
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Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

# Delete

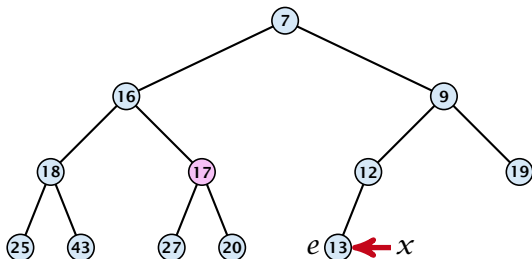
1. Exchange the element to be deleted with the element  $e$  pointed to by  $x$ .





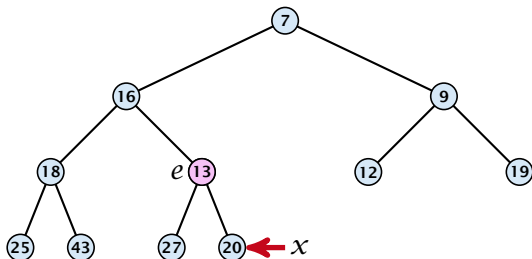
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1. Exchange the element to be deleted with the element  $e$  pointed to by  $x$ .
2. Restore the heap-property for the element  $e$ .



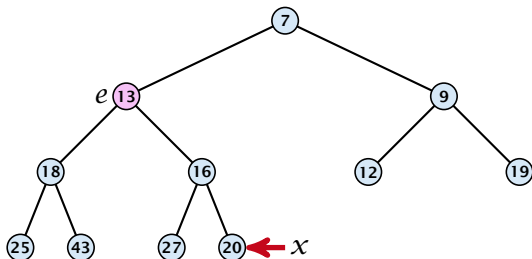
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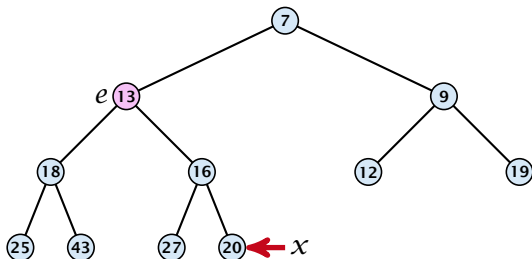
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At its new position  $e$  may either travel up or down in the tree (but not both directions).

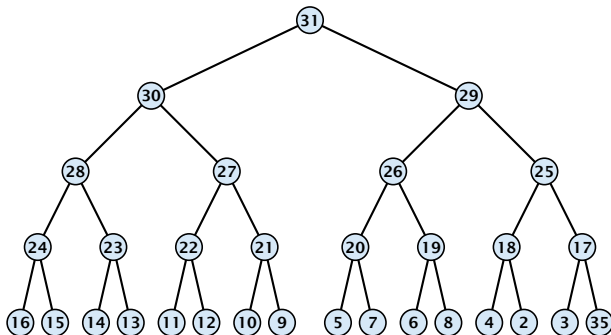
# Binary Heaps

## Operations:

- ▶ **minimum()**: return the root-element. Time  $\mathcal{O}(1)$ .
- ▶ **is-empty()**: check whether root-pointer is **null**. Time  $\mathcal{O}(1)$ .
- ▶ **insert( $k$ )**: insert at successor of  $x$  and bubble up. Time  $\mathcal{O}(\log n)$ .
- ▶ **delete( $h$ )**: swap with  $x$  and bubble up or sift-down. Time  $\mathcal{O}(\log n)$ .

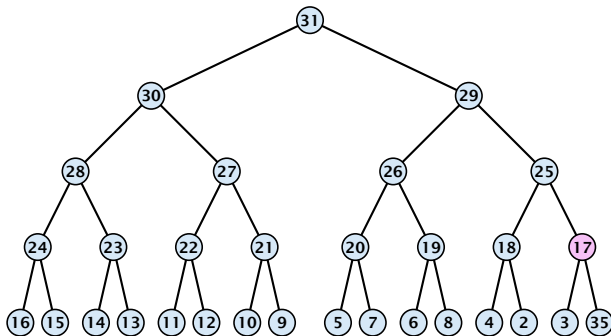
# Build Heap

We can build a heap in linear time:



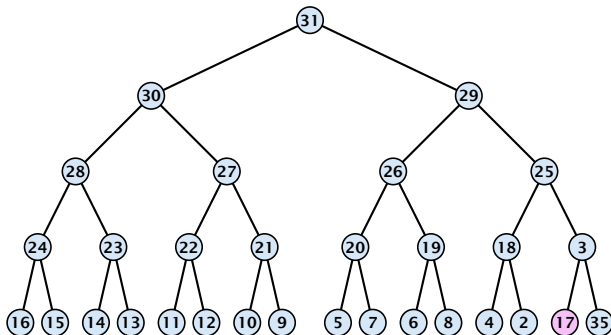
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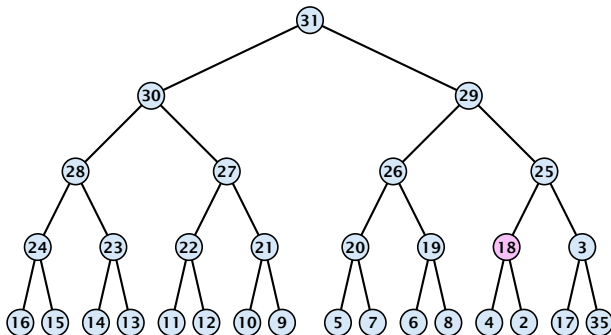
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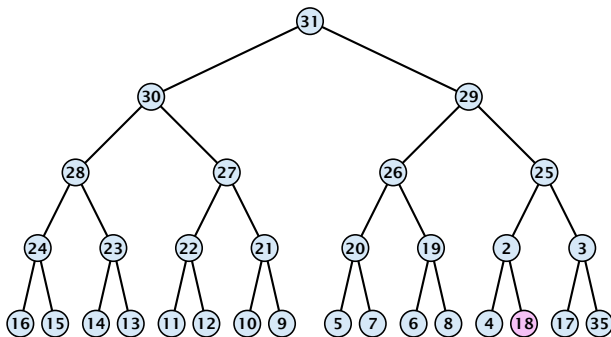
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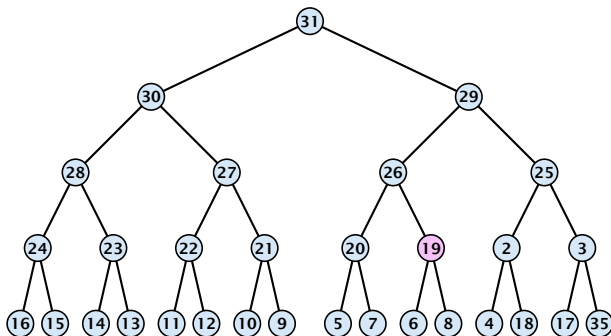
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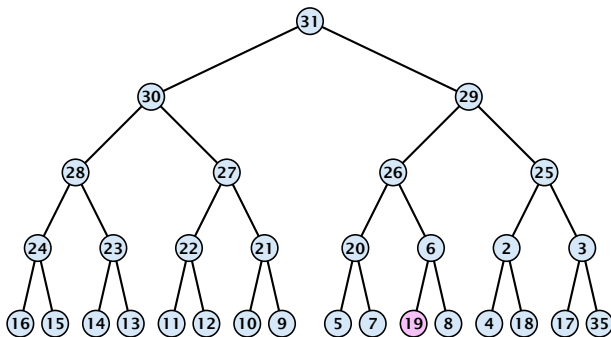
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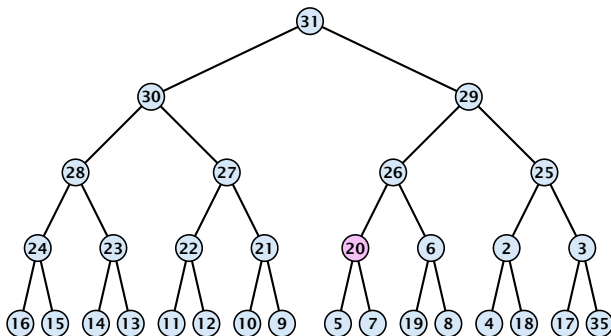
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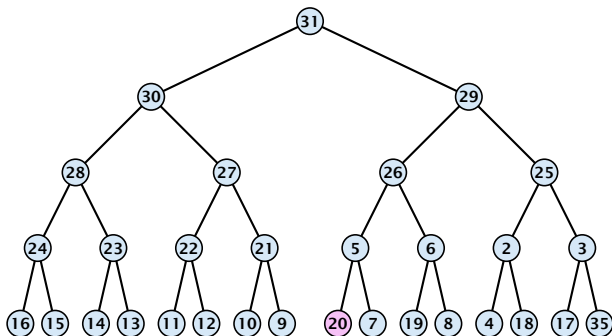
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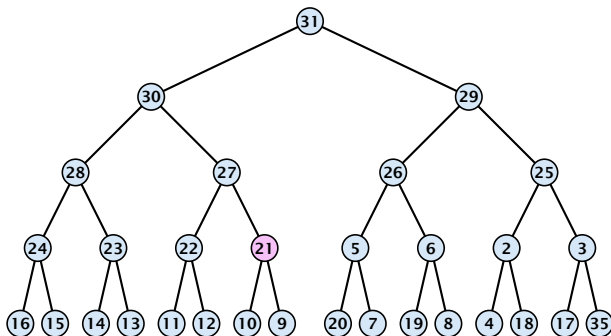
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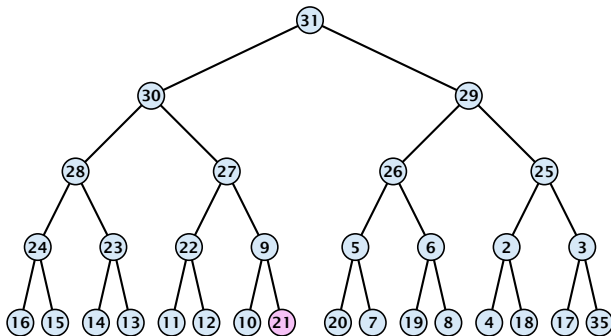
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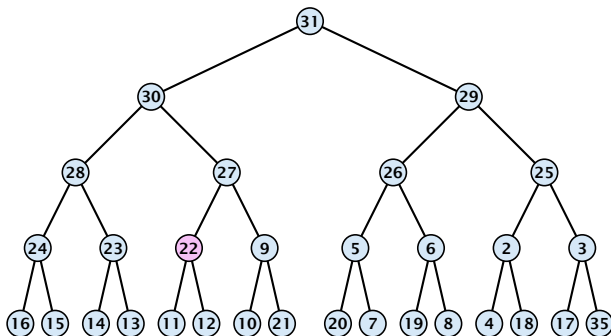
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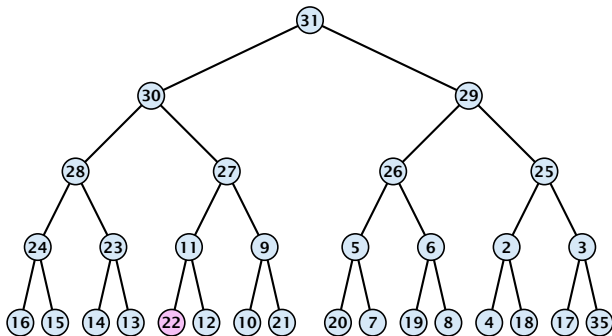
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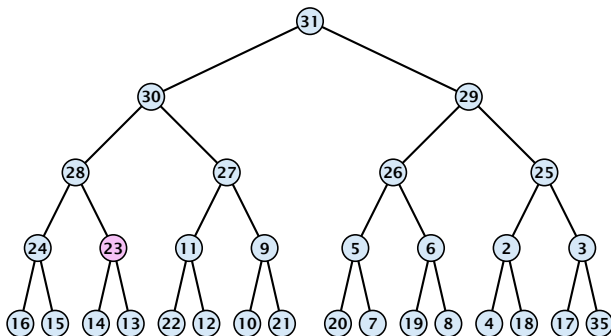
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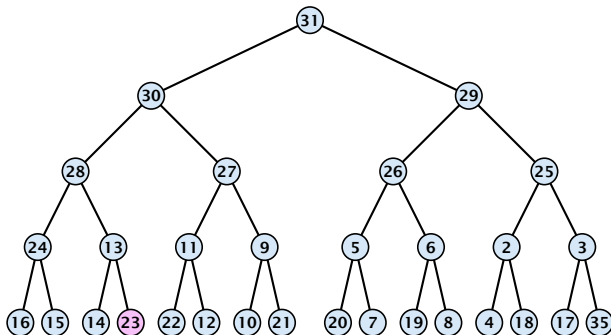
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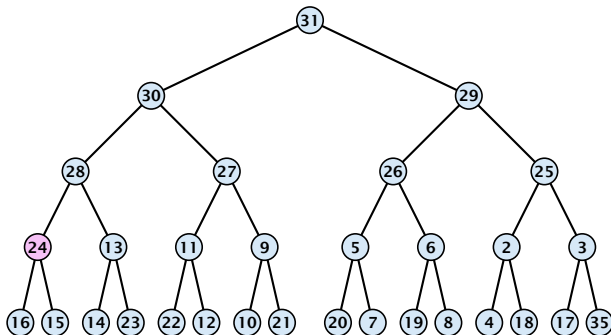
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We can build a heap in linear time:



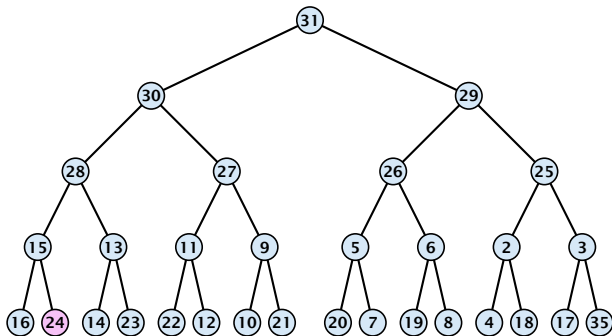
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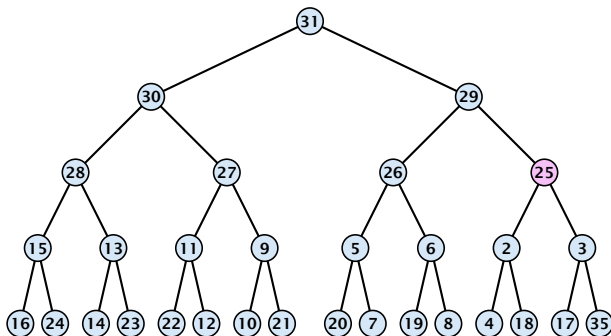
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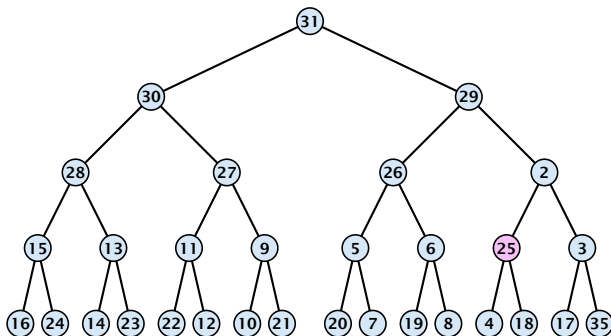
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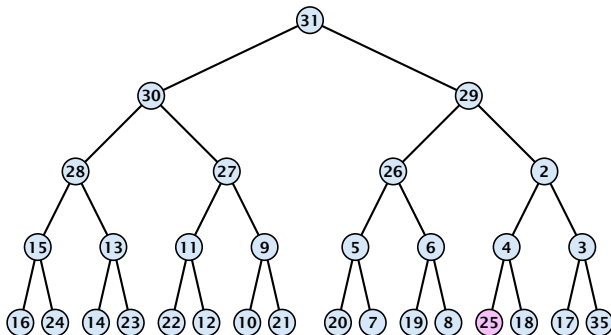
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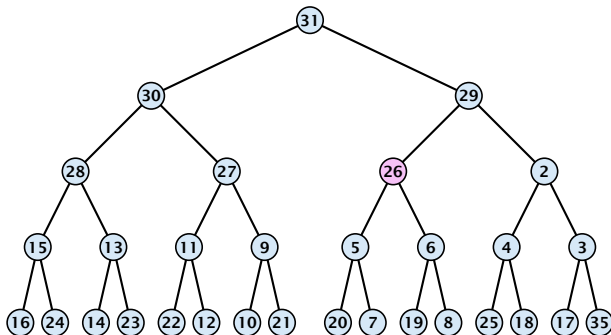
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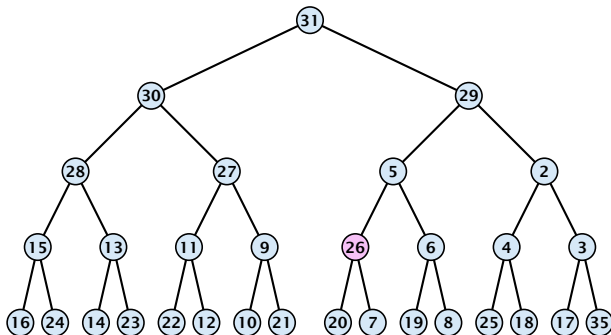
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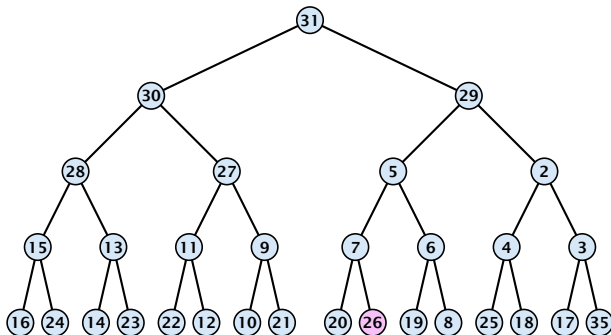
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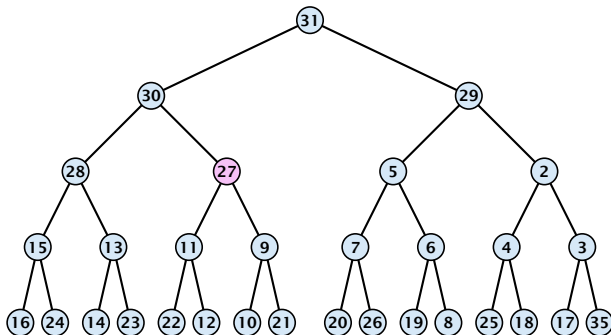
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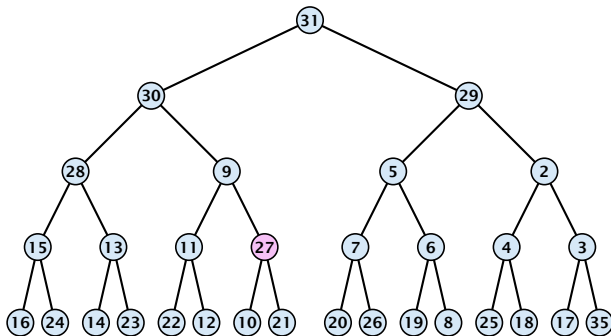
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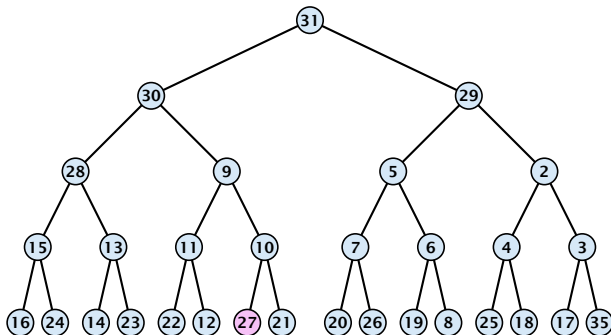
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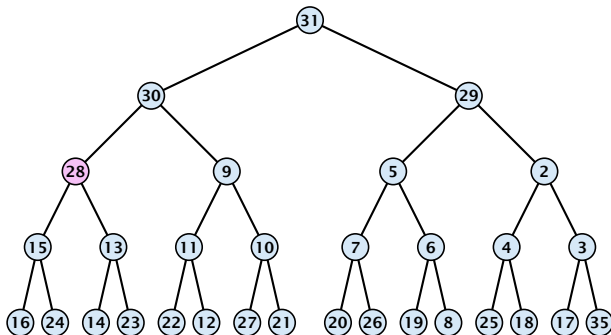
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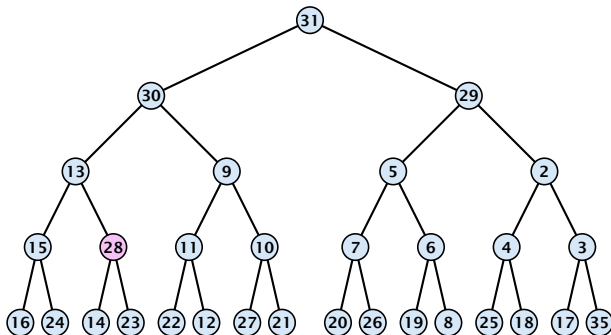
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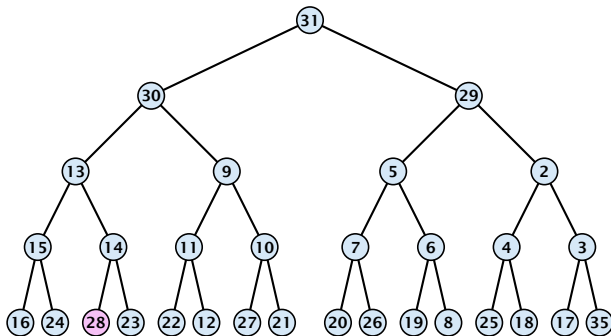
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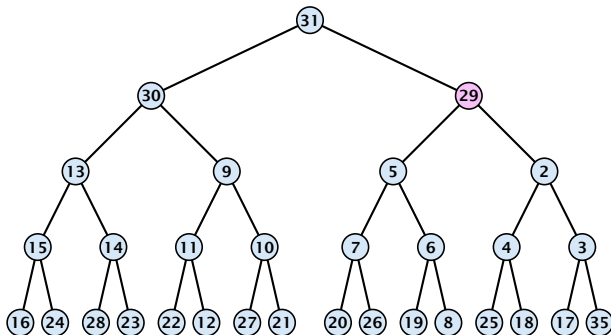
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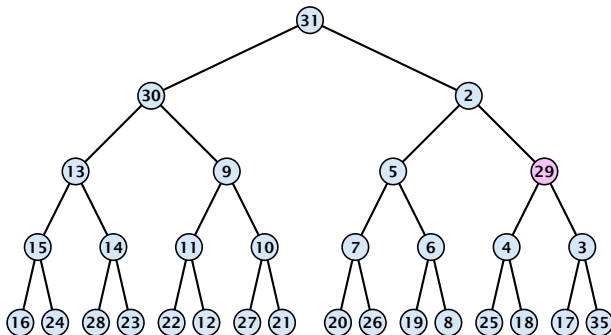
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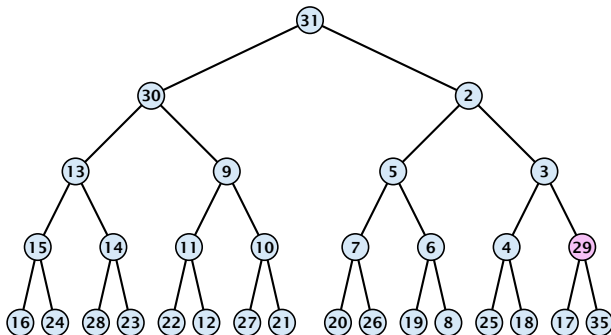
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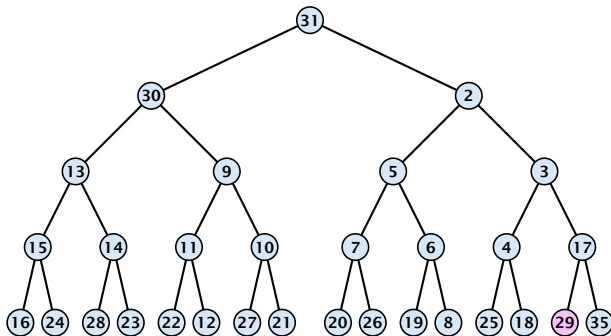
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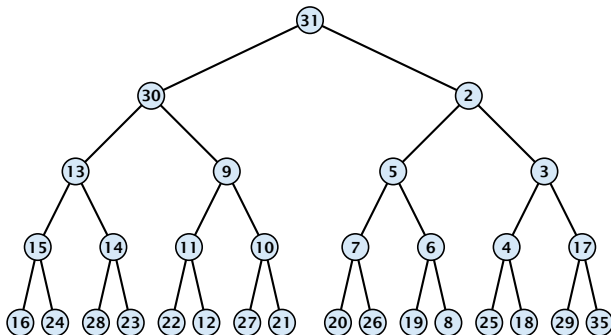
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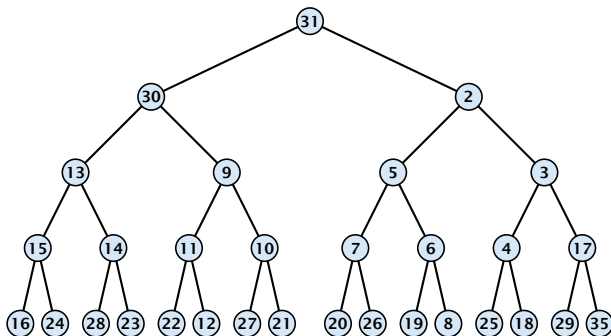
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We can build a heap in linear time:



$$\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \sum_i i 2^{h-i} = \mathcal{O}(2^h) = \mathcal{O}(n)$$



# Binary Heaps

## Operations:

- ▶ **minimum()**: Return the root-element. Time  $\mathcal{O}(1)$ .
- ▶ **is-empty()**: Check whether root-pointer is **null**. Time  $\mathcal{O}(1)$ .
- ▶ **insert( $k$ )**: Insert at  $x$  and bubble up. Time  $\mathcal{O}(\log n)$ .
- ▶ **delete( $h$ )**: Swap with  $x$  and bubble up or sift-down. Time  $\mathcal{O}(\log n)$ .
- ▶ **build( $x_1, \dots, x_n$ )**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time  $\mathcal{O}(n)$ .

# Binary Heaps

# Binary Heaps

The standard implementation of binary heaps is via arrays. Let  $A[0, \dots, n - 1]$  be an array

- ▶ The parent of  $i$ -th element is at position  $\lfloor \frac{i-1}{2} \rfloor$ .
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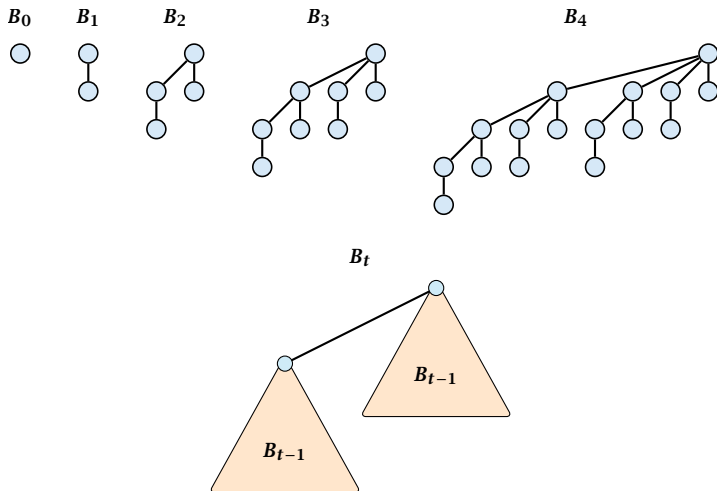
Finding the successor of  $x$  is much easier than in the description on the previous slide. Simply increase or decrease  $x$ .

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

## 8.2 Binomial Heaps

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	$n$	$n \log n$	$n \log n$	$n$
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	$n$	$n \log n$	<b><math>\log n</math></b>	1

# Binomial Trees



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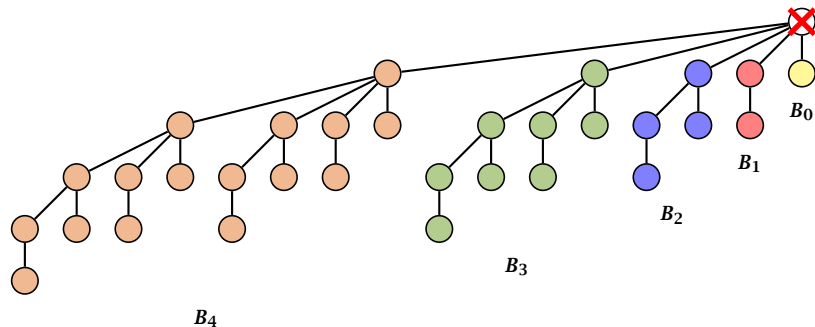
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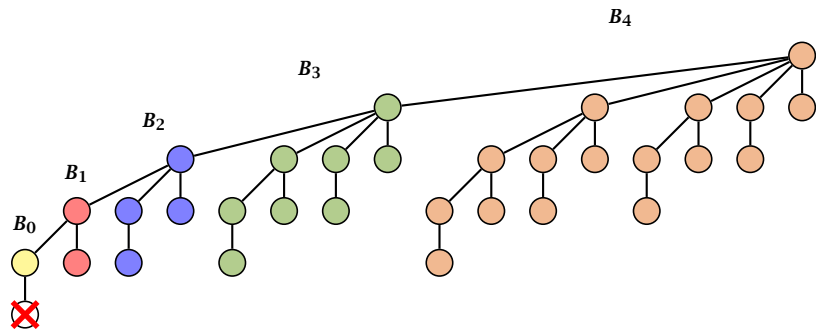
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- ▶  $B_k$  has  $\binom{k}{\ell}$  nodes on level  $\ell$ .
- ▶ Deleting the root of  $B_k$  gives trees  $B_0, B_1, \dots, B_{k-1}$ .

# Binomial Trees



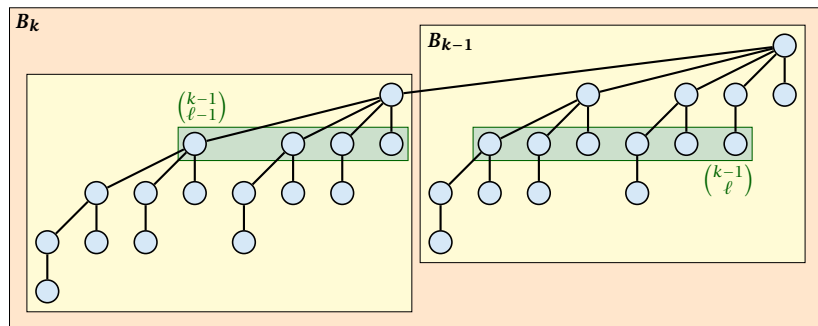
Deleting the root of  $B_5$  leaves sub-trees  $B_4$ ,  $B_3$ ,  $B_2$ ,  $B_1$ , and  $B_0$ .

# Binomial Trees



Deleting the leaf furthest from the root (in  $B_5$ ) leaves a path that connects the roots of sub-trees  $B_4$ ,  $B_3$ ,  $B_2$ ,  $B_1$ , and  $B_0$ .

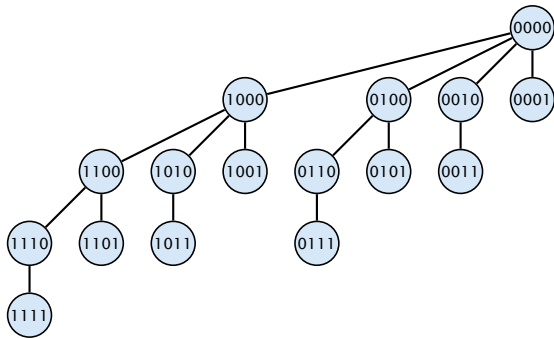
# Binomial Trees



The number of nodes on level  $\ell$  in tree  $B_k$  is therefore

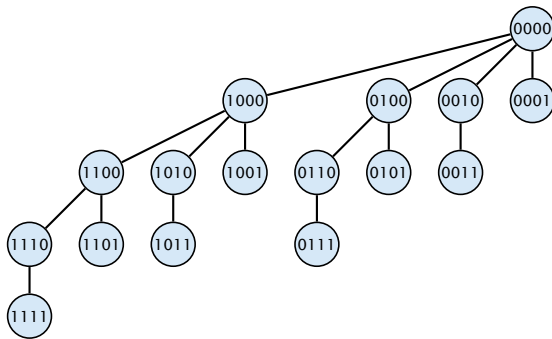
$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

# Binomial Trees



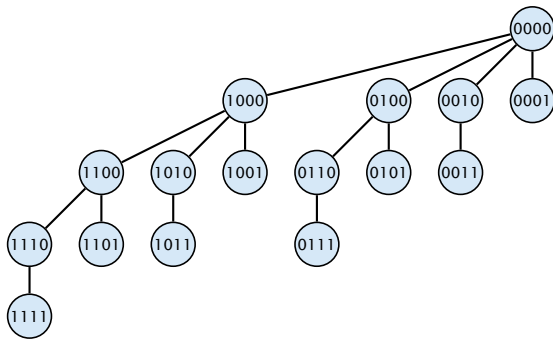


# Binomial Trees



The binomial tree  $B_k$  is a sub-graph of the hypercube  $H_k$ .

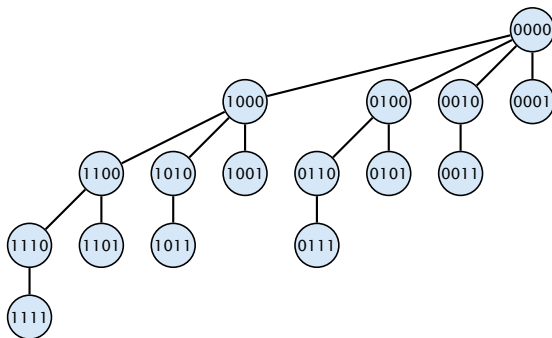
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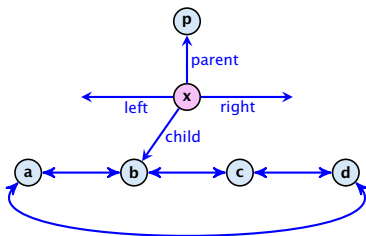
The parent of a node with label  $b_k, \dots, b_1$  is obtained by setting the least significant 1-bit to 0.

The  $\ell$ -th level contains nodes that have  $\ell$  1's in their label.

## 8.2 Binomial Heaps

How do we implement trees with non-constant degree?

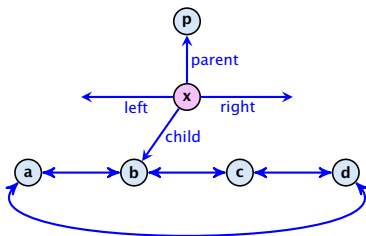
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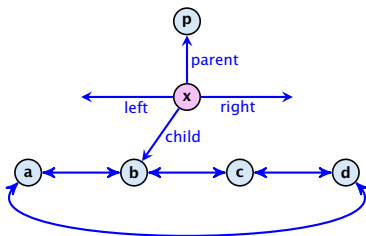
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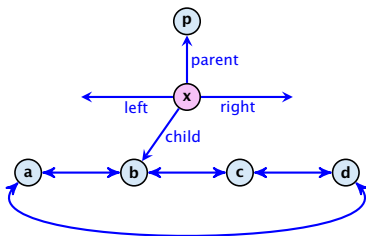
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- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers  $x.left$  and  $x.right$  point to the left and right sibling of  $x$  (if  $x$  does not have siblings then  $x.left = x.right = x$ ).

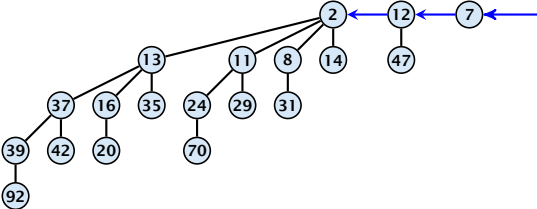


## 8.2 Binomial Heaps

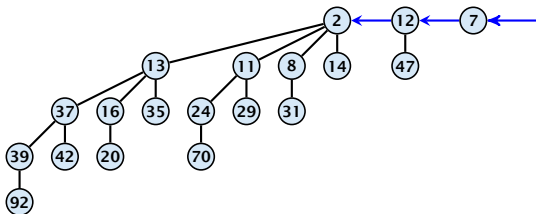
- ▶ Given a pointer to a node  $x$  we can splice out the sub-tree rooted at  $x$  in constant time.
- ▶ We can add a child-tree  $T$  to a node  $x$  in constant time if we are given a pointer to  $x$  and a pointer to the root of  $T$ .



# Binomial Heap

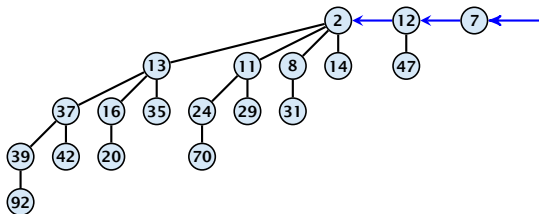


# Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

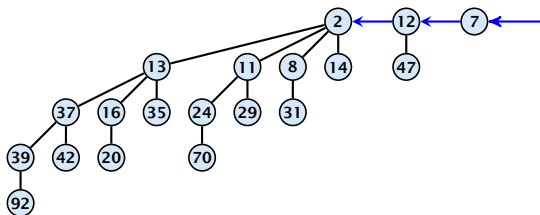
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There is at most one tree for every dimension/order. For example the above heap contains trees  $B_0$ ,  $B_1$ , and  $B_4$ .

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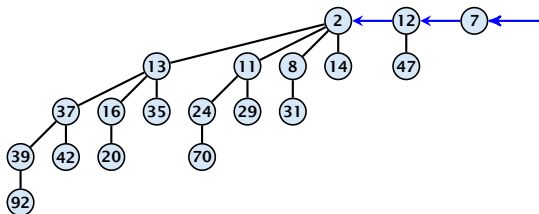
Let  $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$  denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then  $n = \sum_i 2^{k_i}$  must hold. But since the  $k_i$  are all distinct this means that the  $k_i$  define the non-zero bit-positions in the binary representation of  $n$ .



# Binomial Heap

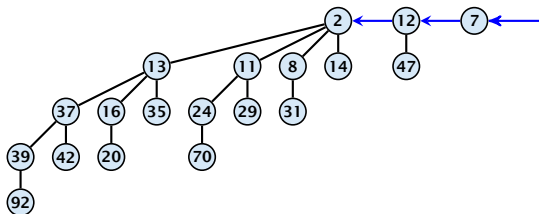
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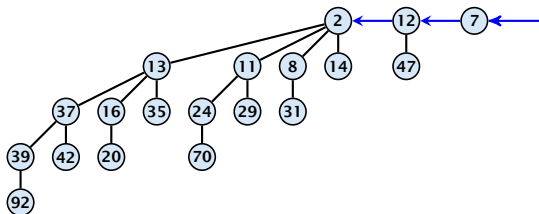
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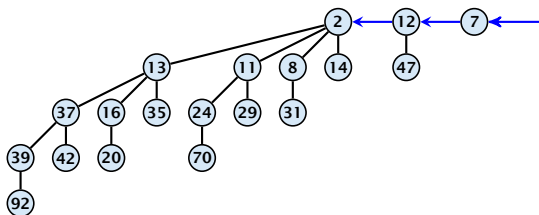
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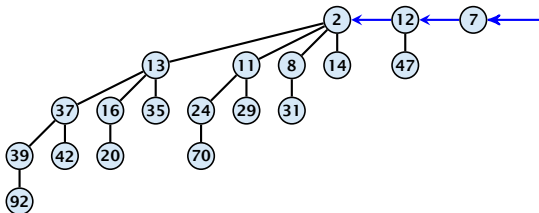
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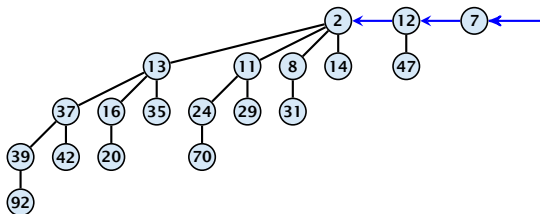
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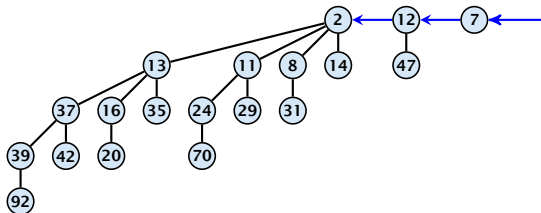
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- ▶ The minimum must be contained in one of the roots.
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- ▶ The trees are stored in a single-linked list; ordered by dimension/size.



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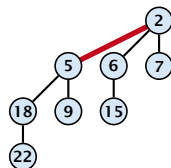
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Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.



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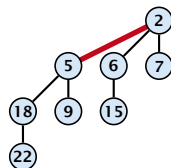
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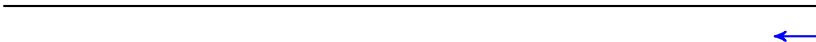
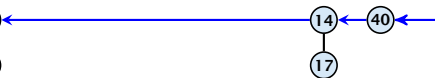
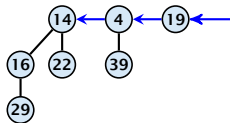
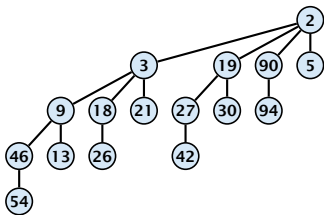
A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

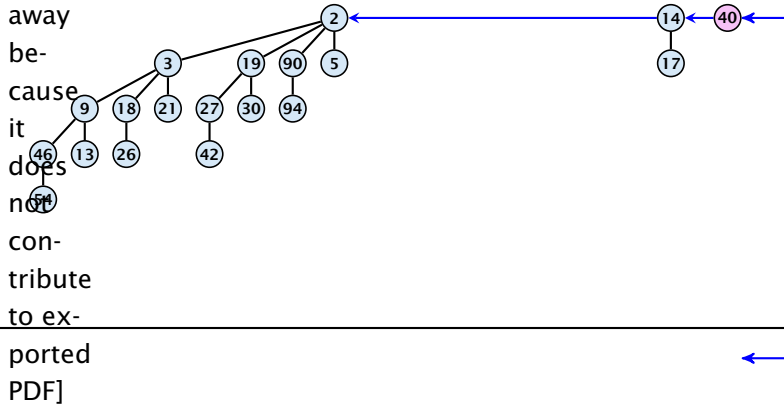
For more trees the technique is analogous to binary addition.





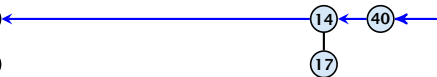
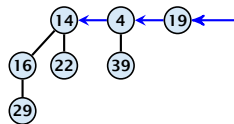
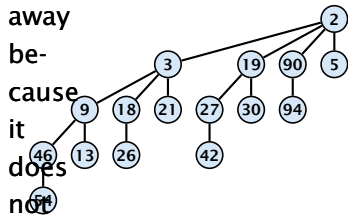
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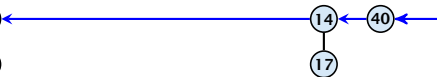
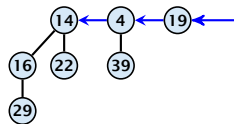
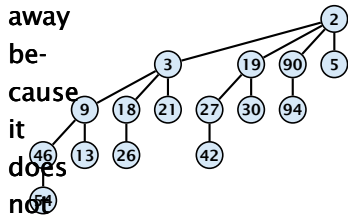
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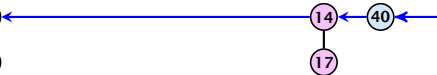
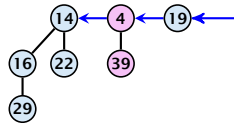
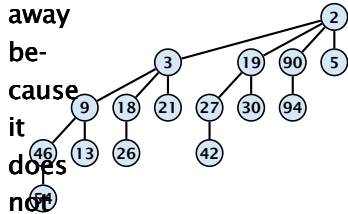
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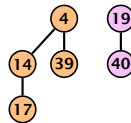
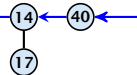
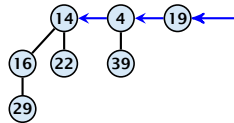
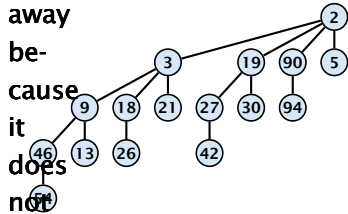
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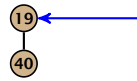
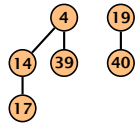
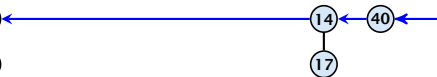
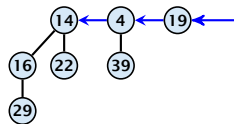
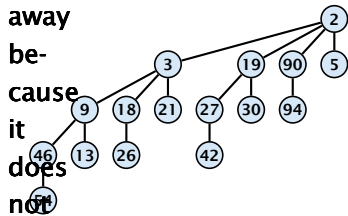
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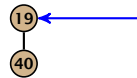
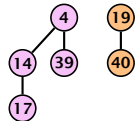
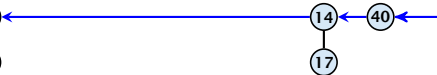
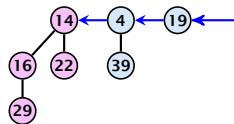
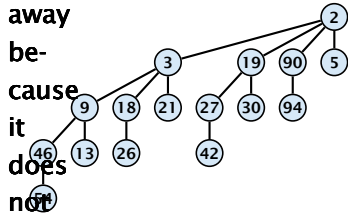
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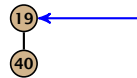
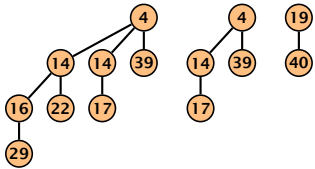
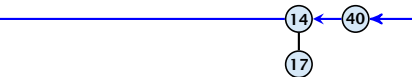
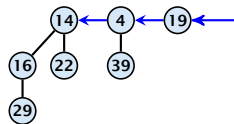
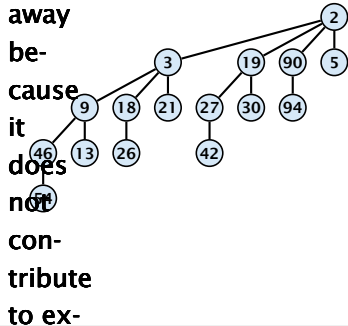
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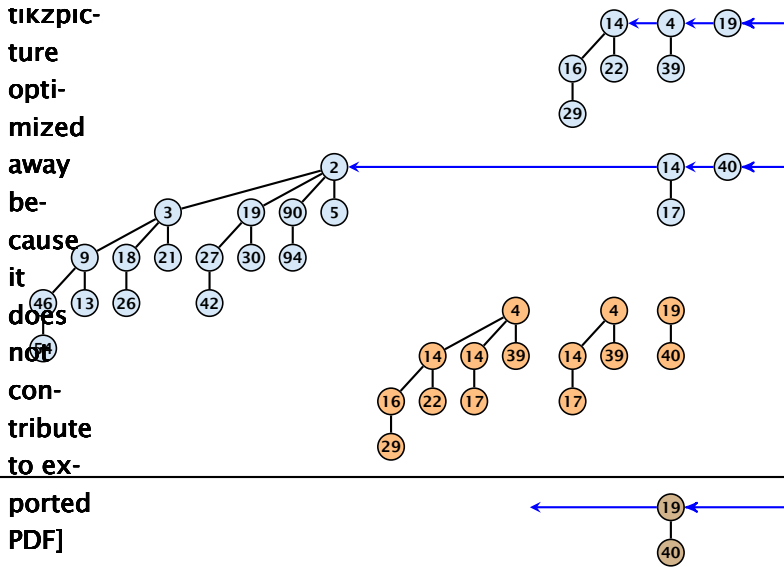
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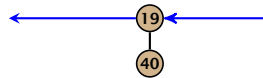
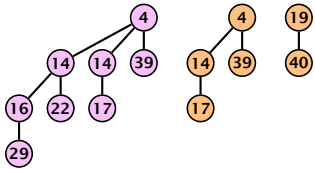
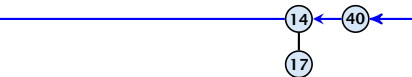
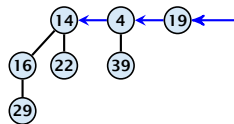
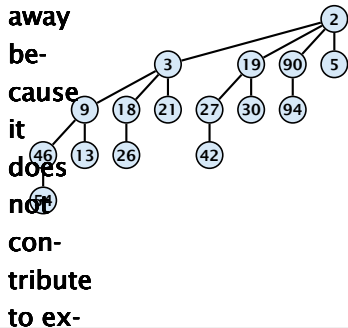
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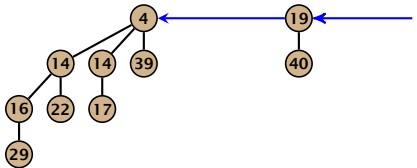
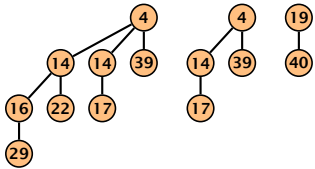
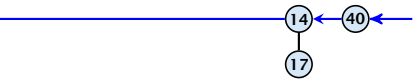
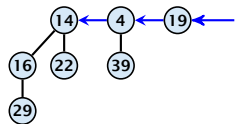
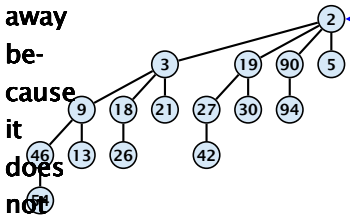
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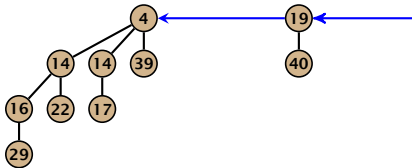
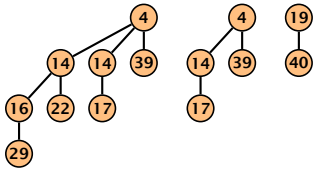
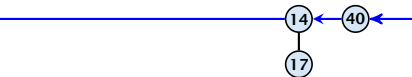
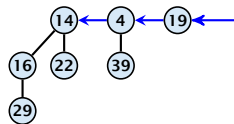
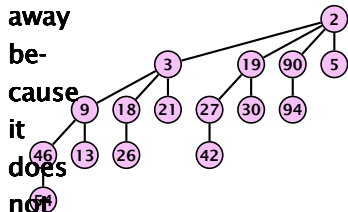
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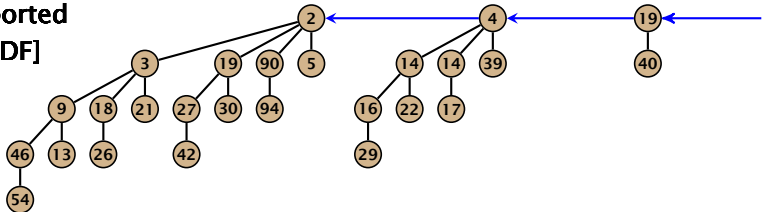
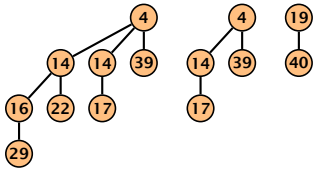
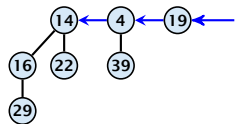
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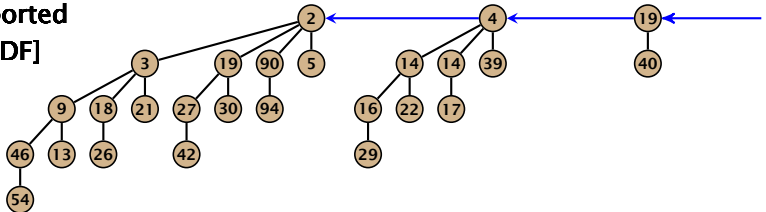
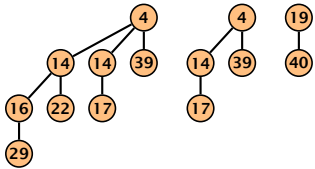
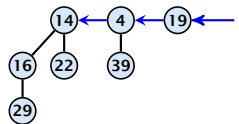
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## 8.2 Binomial Heaps

$S_1$ . merge( $S_2$ ):

- ▶ Analogous to binary addition.

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- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

All other operations can be reduced to `merge()`.

**S.insert( $x$ ):**

- ▶ Create a new heap  $S'$  that contains just the element  $x$ .

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**`S.insert(x)`:**

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- ▶ Execute `S.merge(S')`.



## 8.2 Binomial Heaps

All other operations can be reduced to `merge()`.

**`S.insert(x)`:**

- ▶ Create a new heap  $S'$  that contains just the element  $x$ .
- ▶ Execute  $S.merge(S')$ .
- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

### **S. minimum():**

- ▶ Find the minimum key-value among all roots.
- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

**S. delete-min():**

## 8.2 Binomial Heaps

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### S. delete-min():

- ▶ Find the minimum key-value among all roots.
- ▶ Remove the corresponding tree  $T_{\min}$  from the heap.
- ▶ Create a new heap  $S'$  that contains the trees obtained from  $T_{\min}$  after deleting the root (note that these are just  $\mathcal{O}(\log n)$  trees).

## 8.2 Binomial Heaps

### **S.delete-min():**

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## 8.2 Binomial Heaps

### **S.** delete-min():

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- ▶ Compute  $S.\text{merge}(S')$ .
- ▶ Time:  $\mathcal{O}(\log n)$ .



## 8.2 Binomial Heaps

**S. decrease-key(handle  $h$ ):**

## 8.2 Binomial Heaps

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## 8.2 Binomial Heaps

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## 8.2 Binomial Heaps

### S. decrease-key(handle $h$ ):

- ▶ Decrease the key of the element pointed to by  $h$ .
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time:  $\mathcal{O}(\log n)$  since the trees have height  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

***S.* delete(handle *h*):**

## 8.2 Binomial Heaps

**$S$ . delete(handle  $h$ ):**

- ▶ Execute  $S$ . decrease-key( $h, -\infty$ ).

## 8.2 Binomial Heaps

**$S$ . delete(handle  $h$ ):**

- ▶ Execute  $S$ . decrease-key( $h, -\infty$ ).
- ▶ Execute  $S$ . delete-min().

## 8.2 Binomial Heaps

**$S$ . delete(handle  $h$ ):**

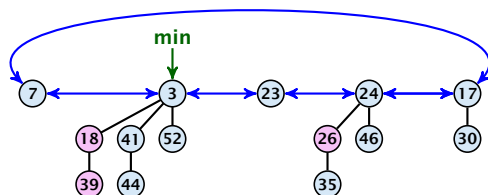
- ▶ Execute  $S$ . decrease-key( $h, -\infty$ ).
- ▶ Execute  $S$ . delete-min().
- ▶ Time:  $\mathcal{O}(\log n)$ .



## 8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



## 8.3 Fibonacci Heaps

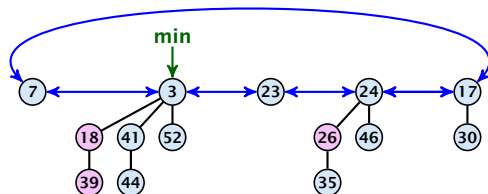
### Additional implementation details:

- ▶ Every node  $x$  stores its degree in a field  $x.degree$ . Note that this can be updated in constant time when adding a child to  $x$ .
- ▶ Every node stores a boolean value  $x.marked$  that specifies whether  $x$  is **marked** or not.

## 8.3 Fibonacci Heaps

### The potential function:

- ▶  $t(S)$  denotes the number of trees in the heap.
- ▶  $m(S)$  denotes the number of marked nodes.
- ▶ We use the potential function  $\Phi(S) = t(S) + 2m(S)$ .



The potential is  $\Phi(S) = 5 + 2 \cdot 3 = 11$ .

## 8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use  $c$  to denote the amount of work that a unit of potential can pay for.

## 8.3 Fibonacci Heaps

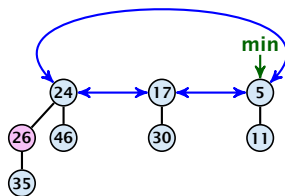
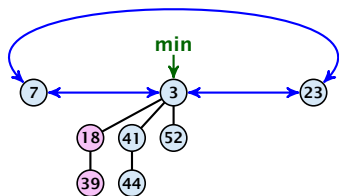
### S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Amortized cost  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

### $S$ . merge( $S'$ )

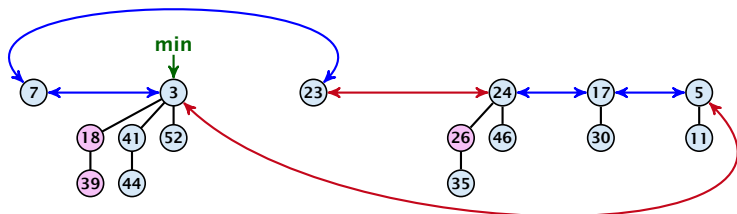
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



## 8.3 Fibonacci Heaps

### S. merge( $S'$ )

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



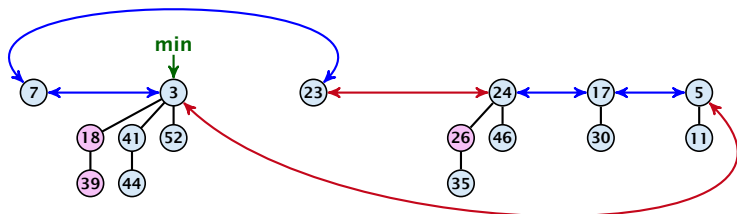
### Running time:

- ▶ Actual cost  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

### S. merge( $S'$ )

- ▶ Merge the root lists.
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### Running time:

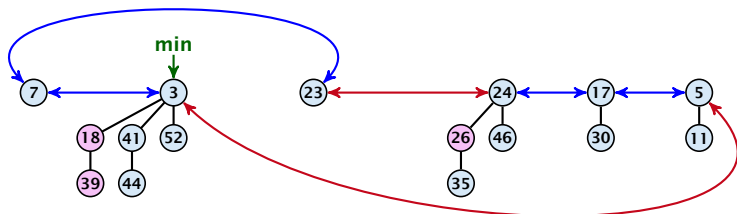
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## 8.3 Fibonacci Heaps

### S. merge( $S'$ )

- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



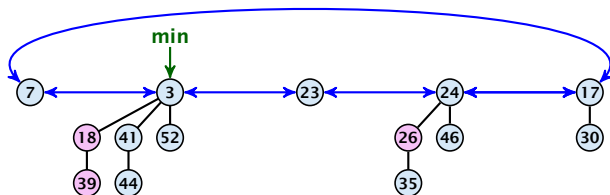
### Running time:

- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Hence, amortized cost is  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

### S. insert( $x$ )

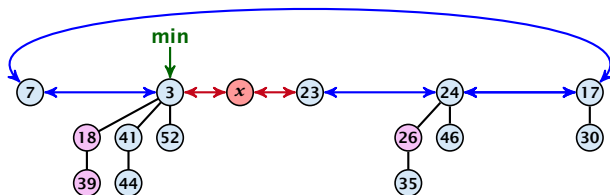
- ▶ Create a new tree containing  $x$ .
- ▶ Insert  $x$  into the root-list.
- ▶ Update min-pointer, if necessary.



## 8.3 Fibonacci Heaps

### S. insert( $x$ )

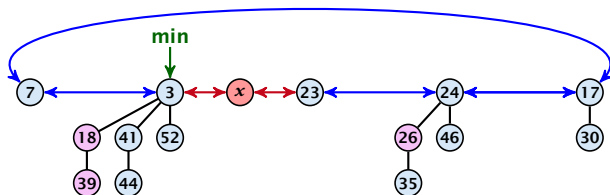
- ▶ Create a new tree containing  $x$ .
- ▶ Insert  $x$  into the root-list.
- ▶ Update min-pointer, if necessary.



## 8.3 Fibonacci Heaps

### S. insert( $x$ )

- ▶ Create a new tree containing  $x$ .
- ▶ Insert  $x$  into the root-list.
- ▶ Update min-pointer, if necessary.

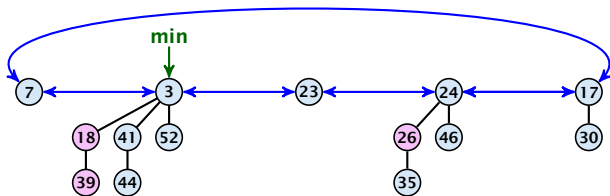


### Running time:

- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ Change in potential is  $+1$ .
- ▶ Amortized cost is  $c + \mathcal{O}(1) = \mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

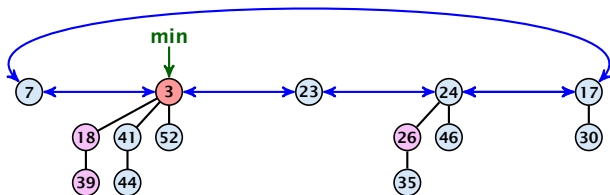
S. delete-min( $x$ )



## 8.3 Fibonacci Heaps

### S. delete-min( $x$ )

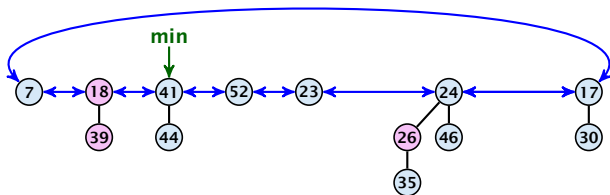
- ▶ Delete minimum; add child-trees to heap;  
time:  $D(\min) \cdot \mathcal{O}(1)$ .



## 8.3 Fibonacci Heaps

### S. delete-min( $x$ )

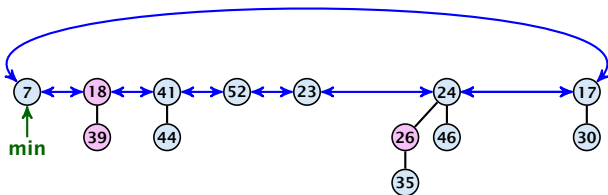
- ▶ Delete minimum; add child-trees to heap; time:  $D(\min) \cdot \mathcal{O}(1)$ .
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## 8.3 Fibonacci Heaps

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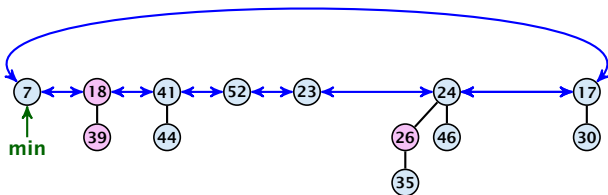




## 8.3 Fibonacci Heaps

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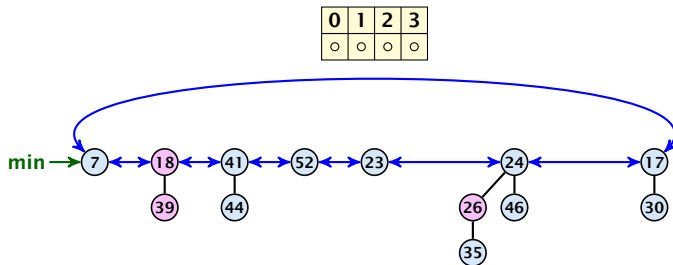
- ▶ Delete minimum; add child-trees to heap; time:  $D(\min) \cdot \mathcal{O}(1)$ .
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- ▶ Consolidate root-list so that no roots have the same degree. Time  $t \cdot \mathcal{O}(1)$  (see next slide).

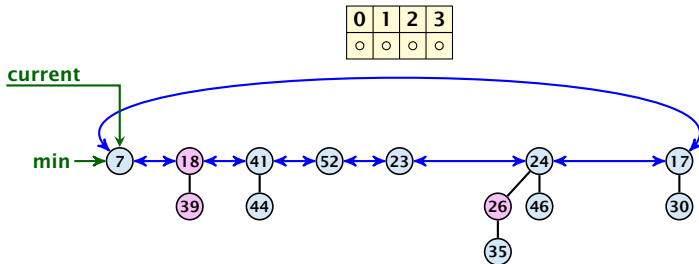
## 8.3 Fibonacci Heaps

Consolidate:



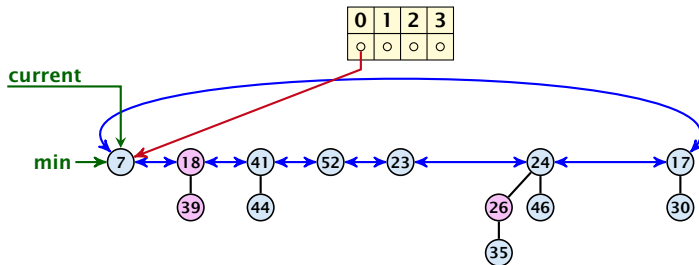
# 8.3 Fibonacci Heaps

Consolidate:



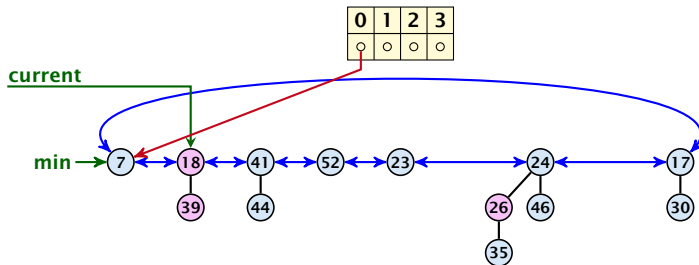
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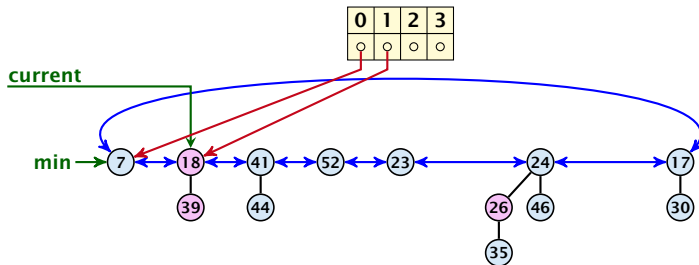
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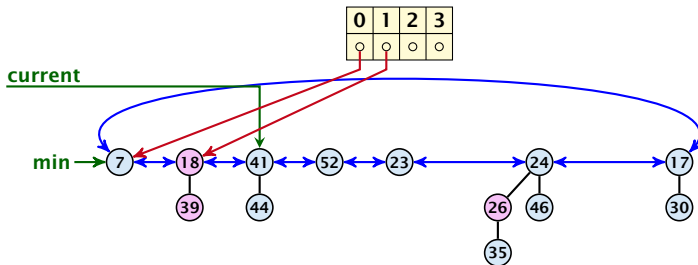
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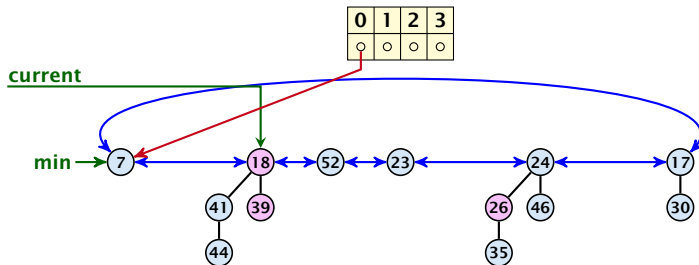
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Consolidate:



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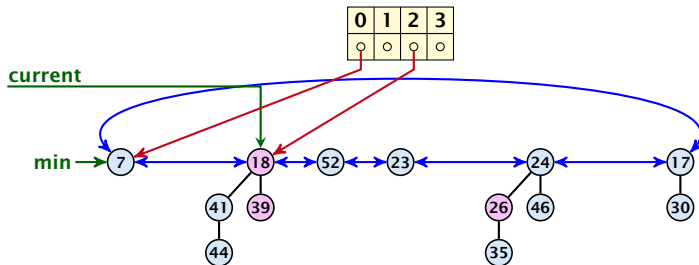
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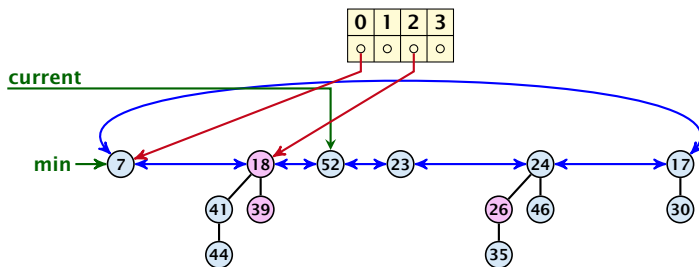
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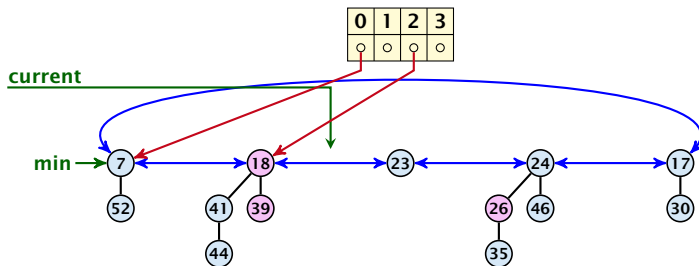
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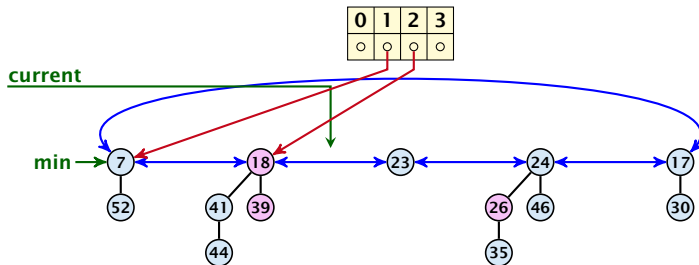
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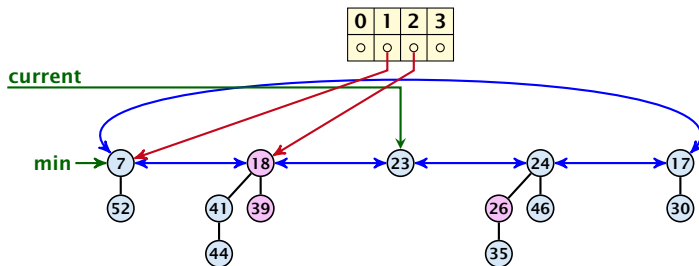
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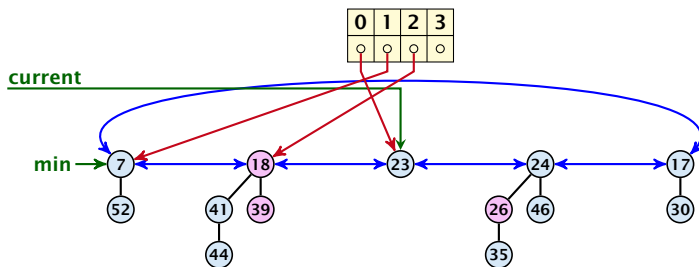
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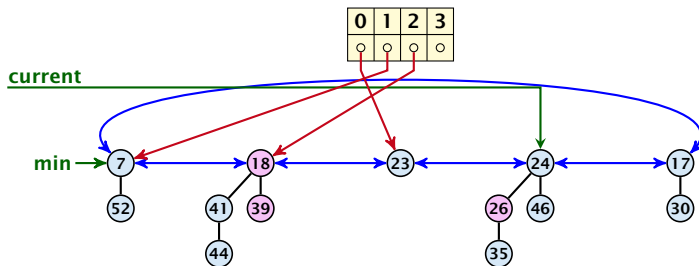
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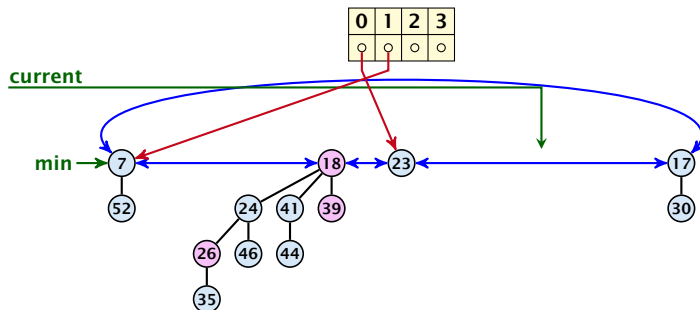
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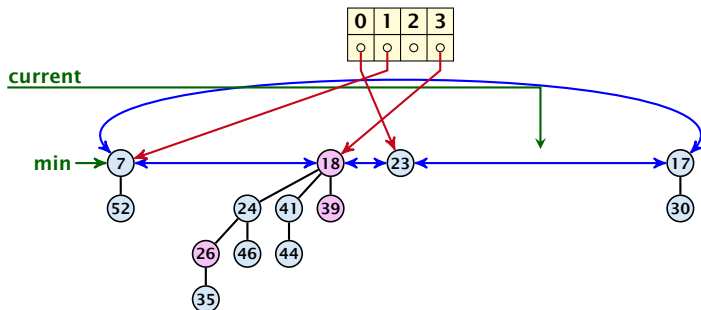
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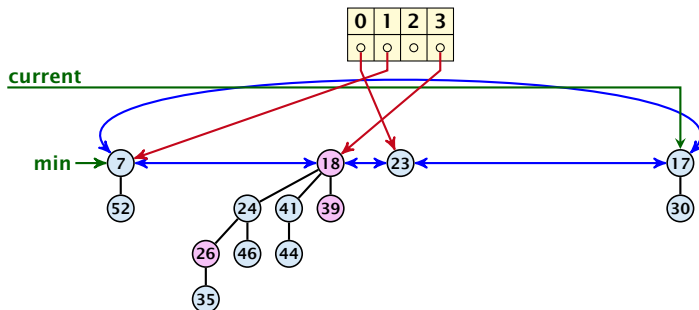
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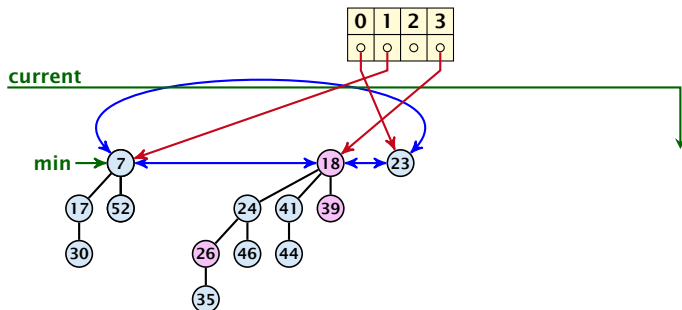
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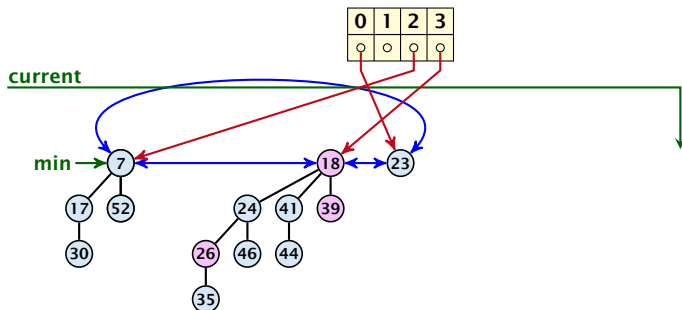
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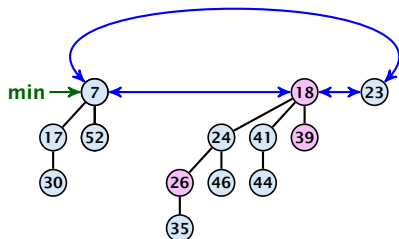
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for  $c \geq c_1$  .



## 8.3 Fibonacci Heaps

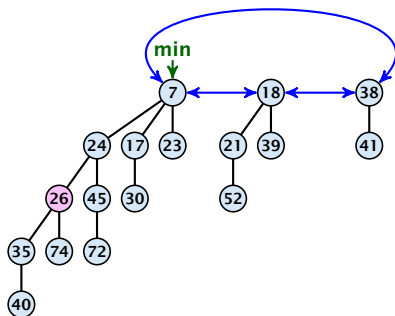
If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

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If we do not have delete or decrease-key operations then  
 $D_n \leq \log n$ .

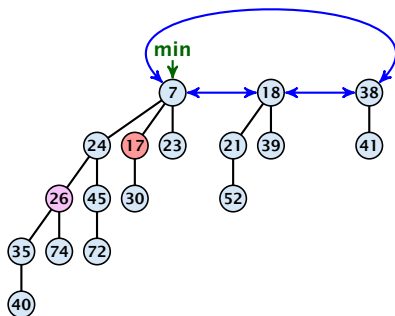
## Fibonacci Heaps: decrease-key(handle $h, v$ )



### Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by  $h$ . Nothing else to do.

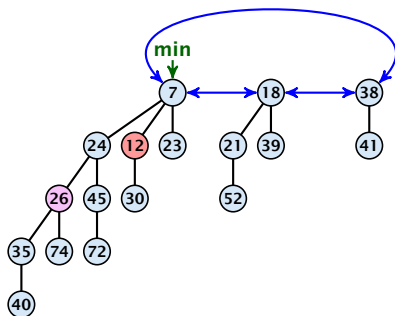
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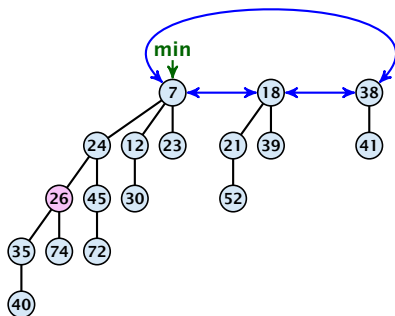
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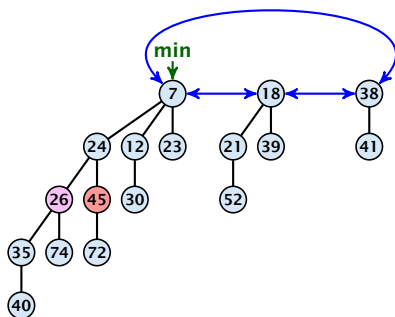
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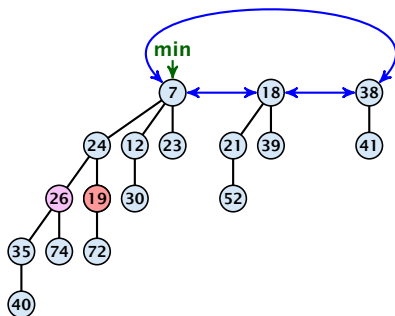
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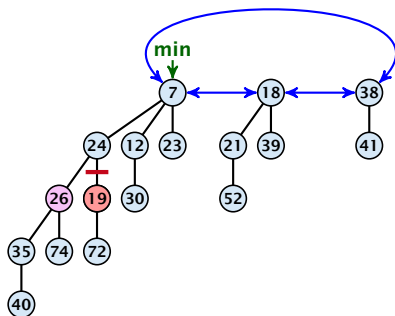


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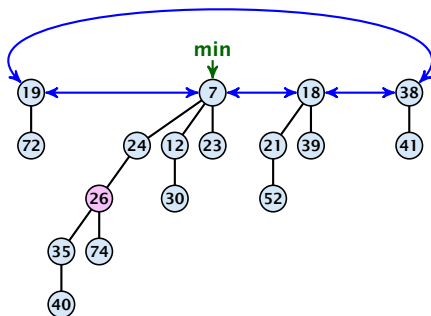
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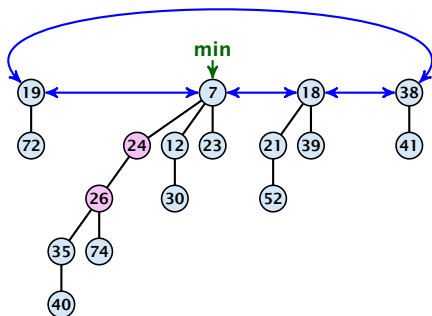
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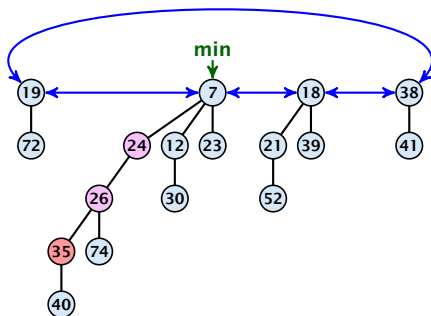
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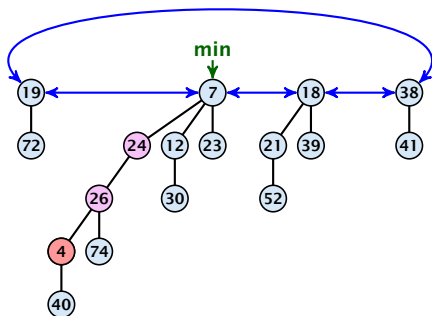
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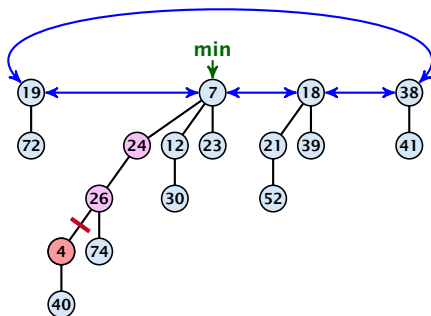
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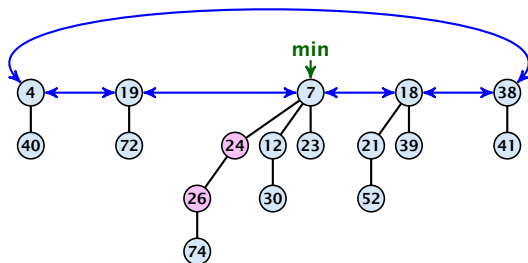
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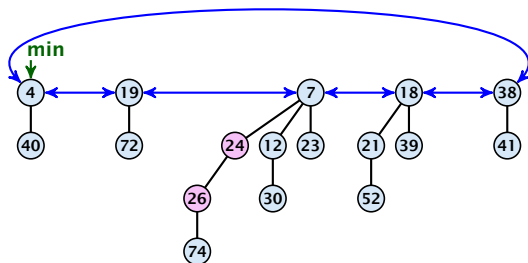
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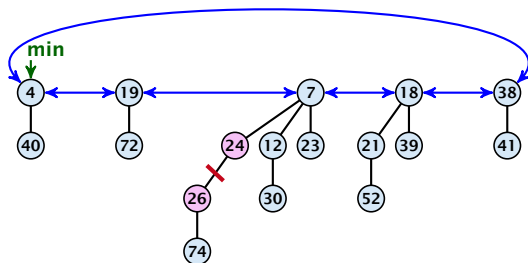


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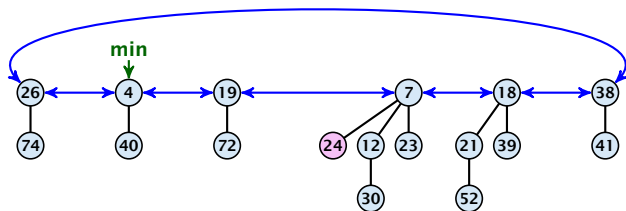
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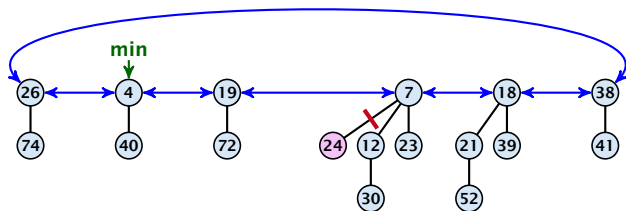
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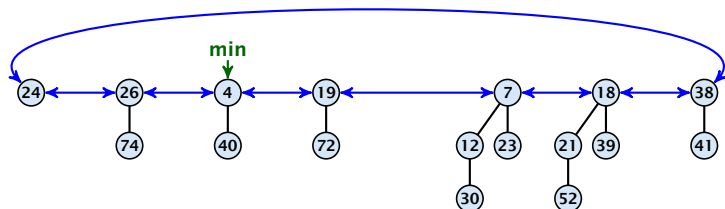
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- ▶ Adjust min-pointers, if necessary.
- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

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$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$
if  $c \geq c_2$ .



# Delete node

***H. delete( $x$ ):***

- ▶ decrease value of  $x$  to  $-\infty$ .
- ▶ delete-min.

**Amortized cost:  $\mathcal{O}(D_n)$**

- ▶  $\mathcal{O}(1)$  for decrease-key.
- ▶  $\mathcal{O}(D_n)$  for delete-min.

## 8.3 Fibonacci Heaps

### Lemma 1

Let  $x$  be a node with degree  $k$  and let  $y_1, \dots, y_k$  denote the children of  $x$  in the order that they were linked to  $x$ . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

## 8.3 Fibonacci Heaps

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## 8.3 Fibonacci Heaps

$\phi = \frac{1}{2}(1 + \sqrt{5})$  denotes the *golden ratio*.  
Note that  $\phi^2 = 1 + \phi$ .

### Definition 2

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

### Facts:

1.  $F_k \geq \phi^k$ .
2. For  $k \geq 2$ :  $F_k = 2 + \sum_{i=0}^{k-2} F_i$ .

The above facts can be easily proved by induction. From this it follows that  $s_k \geq F_k \geq \phi^k$ , which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$