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- **$\mathcal{P}.\mathbf{find}(x)$:** Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.
- **$\mathcal{P}.\mathbf{union}(x, y)$:** Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

9 Union Find

Applications:

- Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

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- Kruskals Minimum Spanning Tree Algorithm

9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

```
A ← ∅;
for all v ∈ V do
        v.set ← P.makeset(v.label)
sort edges in non-decreasing order of weight w
for all (u, v) ∈ E in non-decreasing order do
        if P.find(u.set) ≠ P.find(v.set) then
                A ← A ∪ {(u, v)}
                P.union(u.set, v.set)
```

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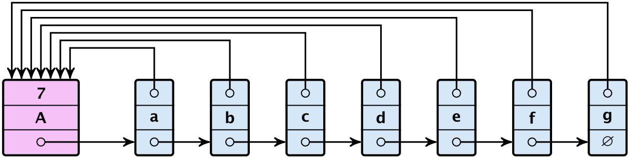

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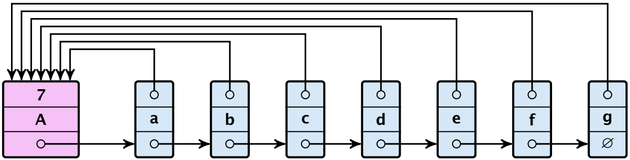

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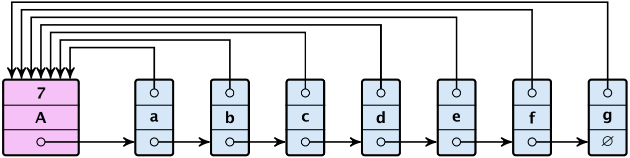

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- Time: $\min\{|S_x|, |S_y|\}$.

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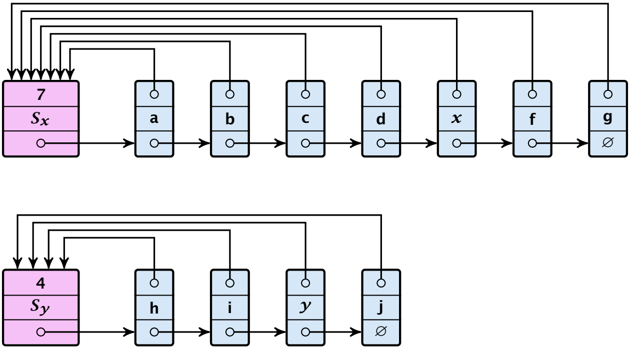

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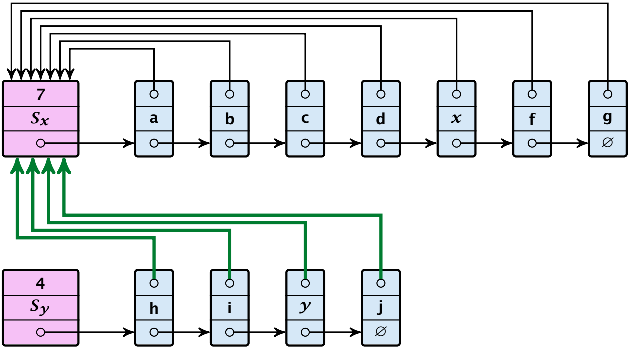

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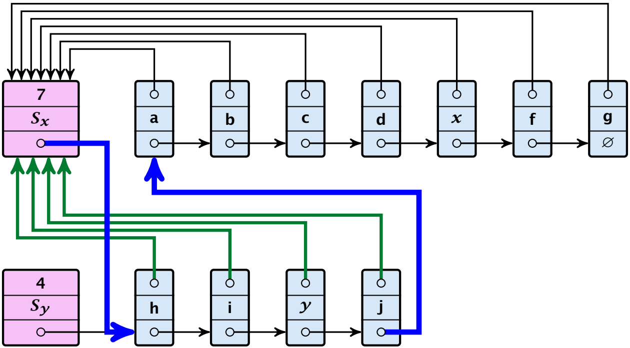

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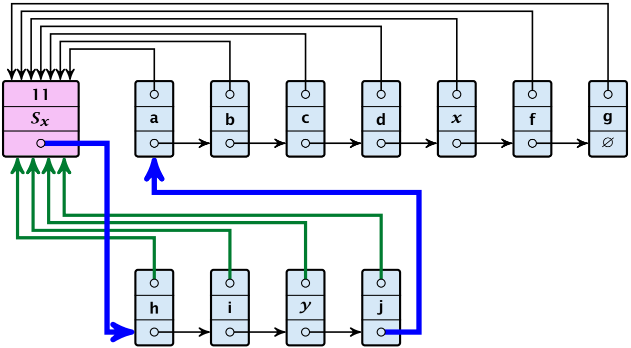

List Implementation

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 1
The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\text{find}(x)$: $\mathcal{O}(1)$.
- $\text{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\text{union}(x, y)$: $\mathcal{O}(1)$.

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- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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- Later operations charge the account but the balance never drops below zero.

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- Charge c to every element in set S_x.

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Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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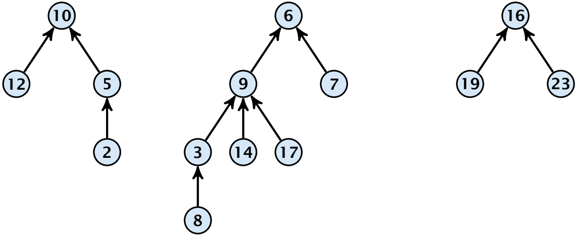

Set system $\{2,5,10,12\}$, $\{3,6,7,8,9,14,17\}$, $\{16,19,23\}$.

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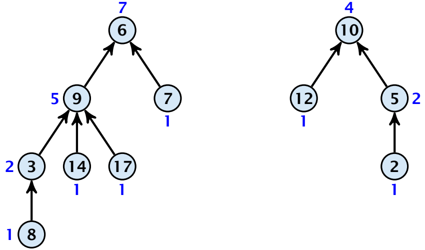

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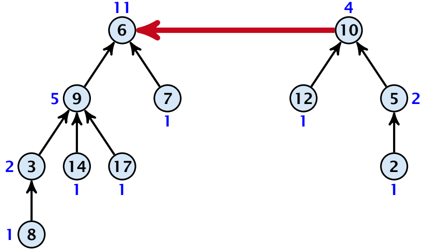

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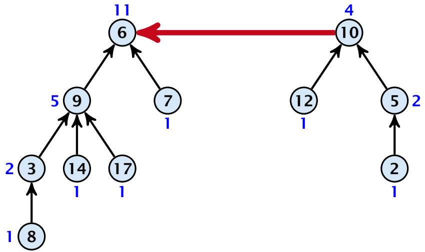

- Time: constant for $\text{link}(a, b)$ plus two find-operations.

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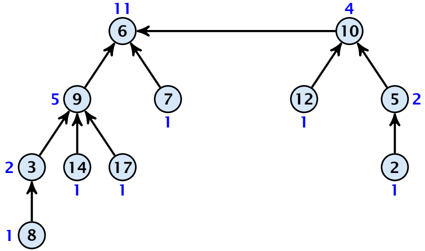

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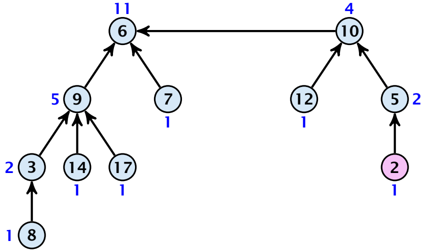

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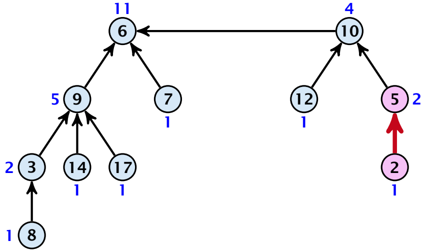

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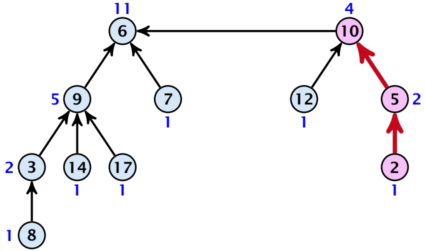

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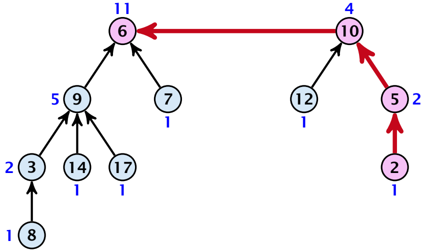

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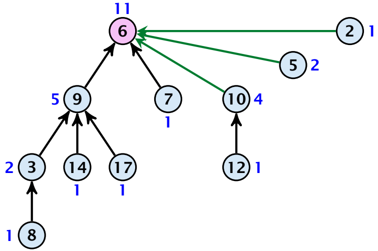

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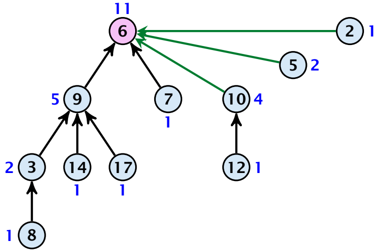

- Note that the size-fields now only give an upper bound on the size of a sub-tree.

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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

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Lemma 4

The rank of a parent must be strictly larger than the rank of a child.

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Lemma 5

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- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node *sees* at most one rank s node, but every rank s node is seen by at least 2^s different nodes. $\square$

Amortized Analysis

We define

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Theorem 6
Union find with path compression fulfills the following amortized running times:

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
- $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- Hence, the total number of rank-groups is at most $\log^* n$.

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- If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of v.
- Otherwise we charge the cost to the find-account.

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- After a node v is charged its parent-edge is re-assigned. The rank of the parent strictly increases.
- After some charges to v the parent will be in a larger rank-group. $\Rightarrow v$ will never be charged again.
- The total charge made to a node in rank-group g is at most $\mathrm{tow}(g) - \mathrm{tow}(g-1) - 1 \leq \mathrm{tow}(g)$.

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- The total charge is at most

$$\sum_g n(g) \cdot \mathrm{tow}(g) \enspace ,$$

where $n(g)$ is the number of nodes in group g.

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Hence,

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Hence,

$$\sum_g n(g)\,\mathrm{tow}(g) \leq n(0)\,\mathrm{tow}(0) + \sum_{g\geq 1} n(g)\,\mathrm{tow}(g) \leq n\log^*(n)$$

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This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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There is also a lower bound of $\Omega(\alpha(m,n))$.

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$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1, A(x,y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

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- $A(0,y) = y + 1$
- $A(1,y) = y + 2$
- $A(2,y) = 2y + 3$
- $A(3,y) = 2^{y+3} - 3$
- $A(4,y) = \underbrace{2^{2^{2^{2}}}}_{y+3 \text{ times}} - 3$