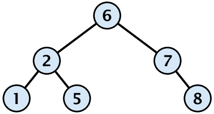
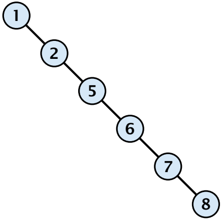
# 7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node $v$ have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:



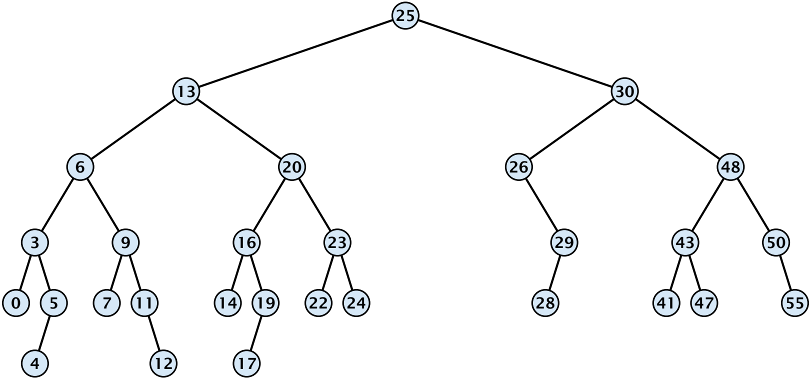



# 7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\text{insert}(x)$
- $T.\text{delete}(x)$
- $T.\text{search}(k)$
- $T.\text{successor}(x)$
- $T.\text{predecessor}(x)$
- $T.\text{minimum}()$
- $T.\text{maximum}()$

# Binary Search Trees: Searching




**Algorithm 1** TreeSearch$(x, k)$

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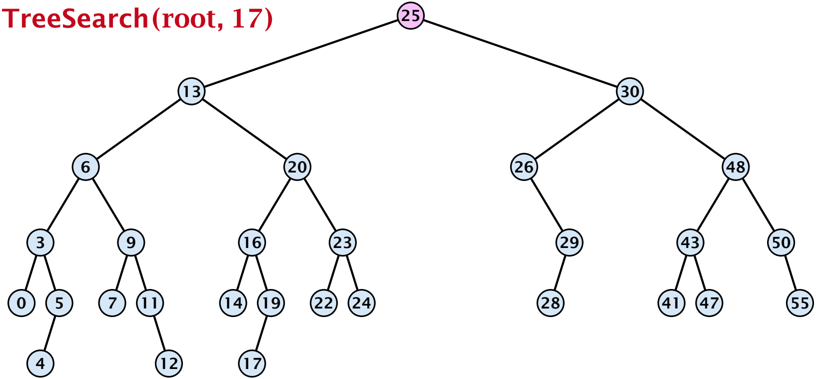
# Binary Search Trees: Searching

**TreeSearch(root, 17)**

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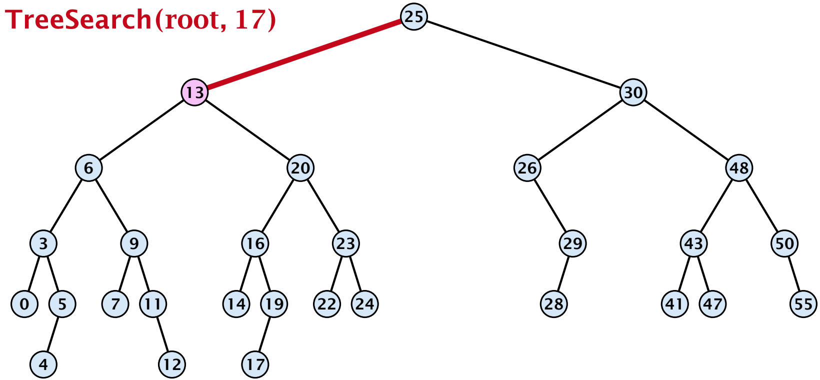


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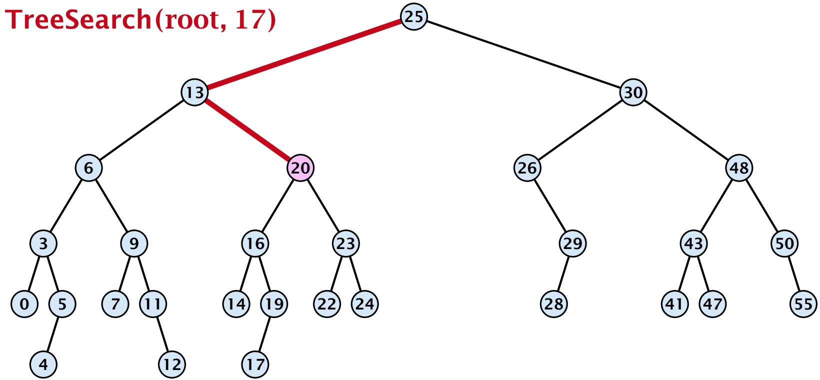


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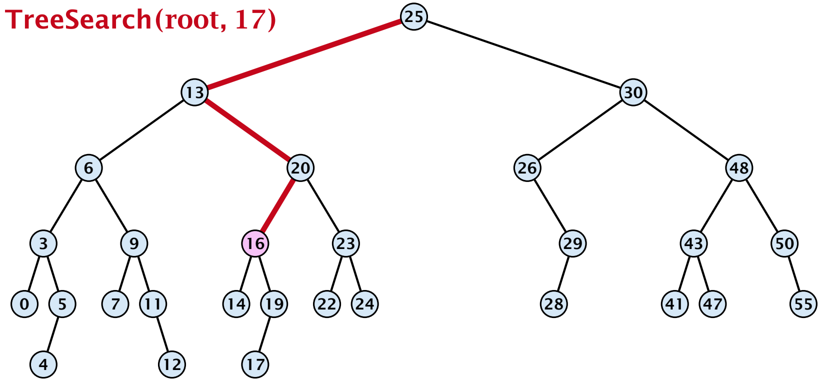


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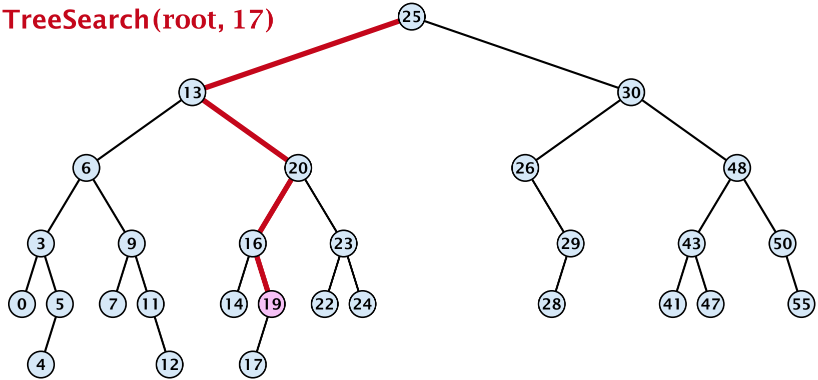


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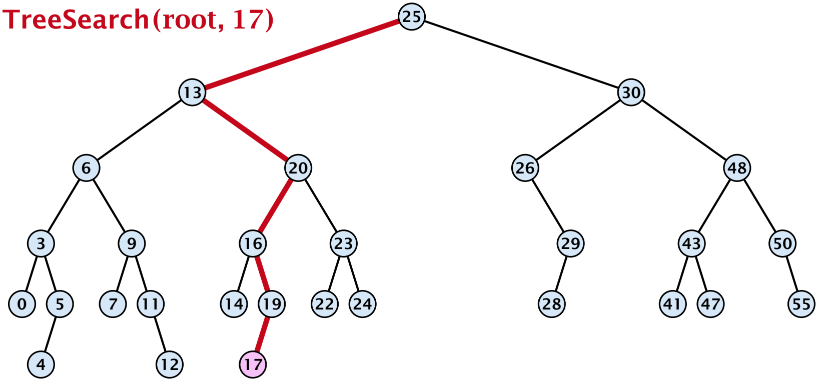


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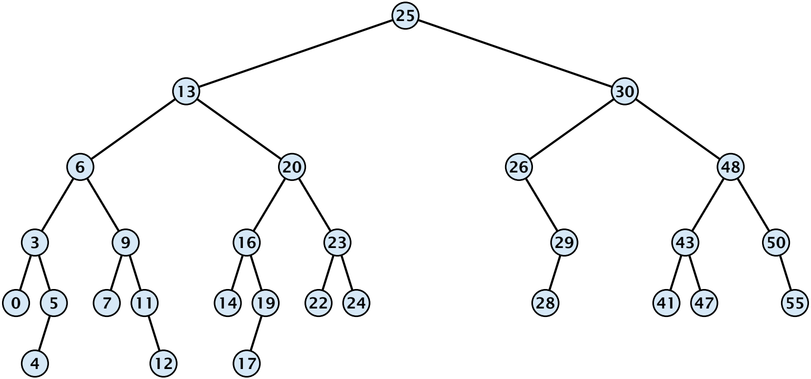


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# Binary Search Trees: Searching

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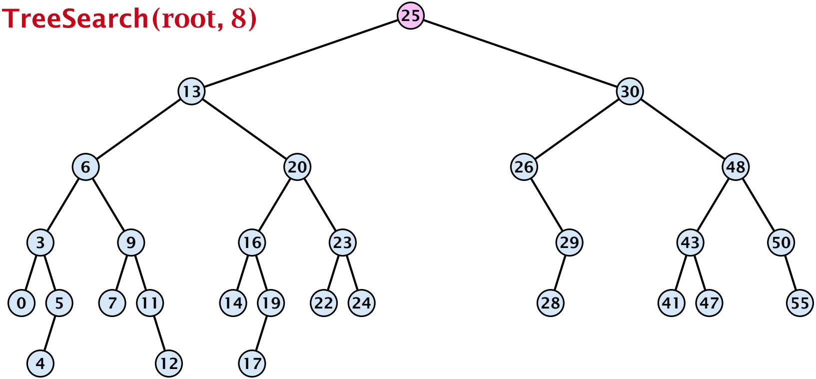


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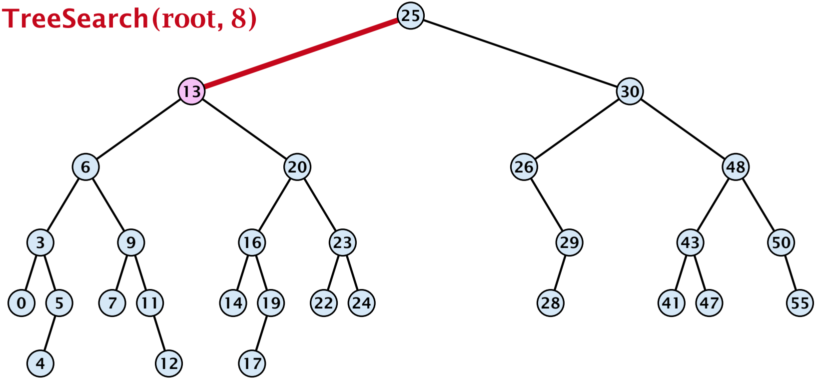


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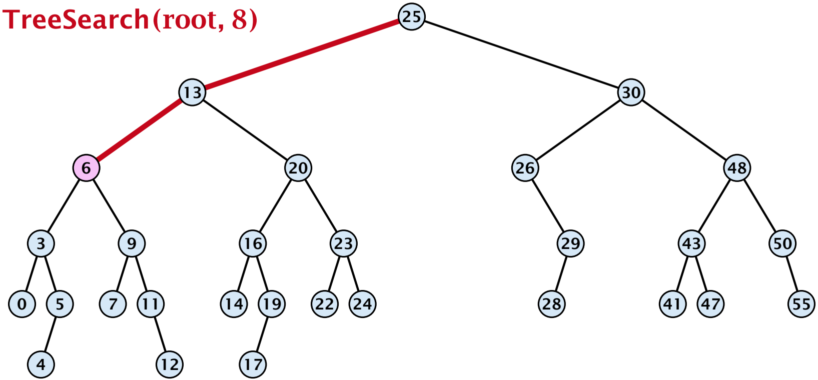


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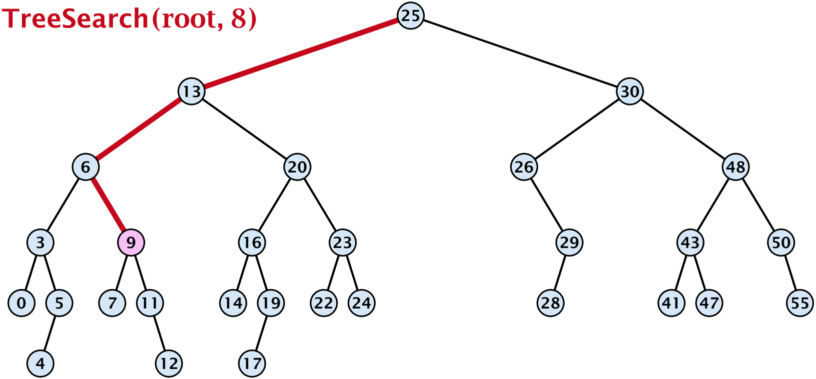


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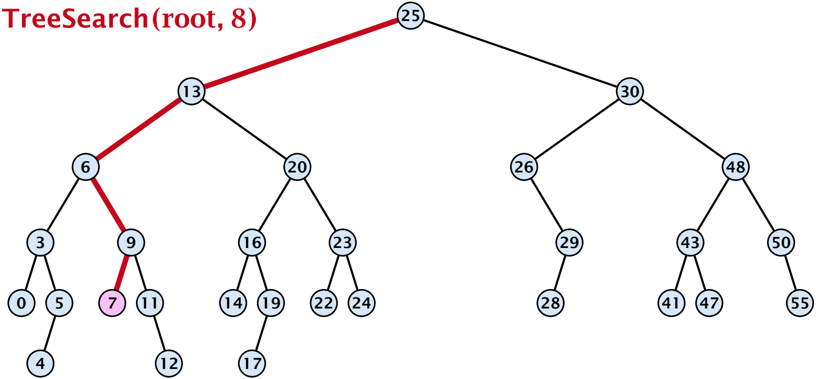


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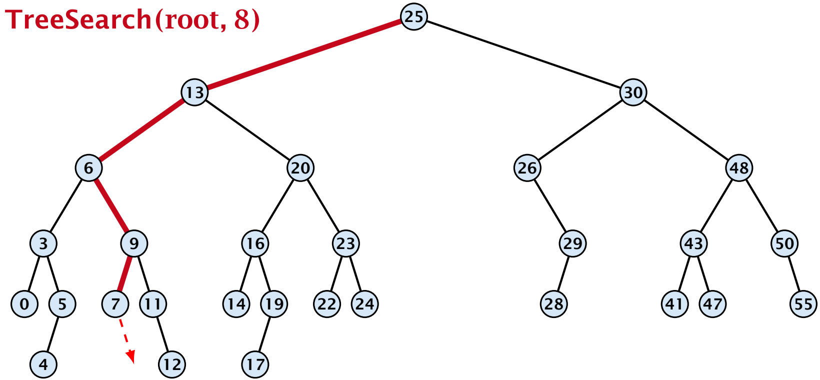


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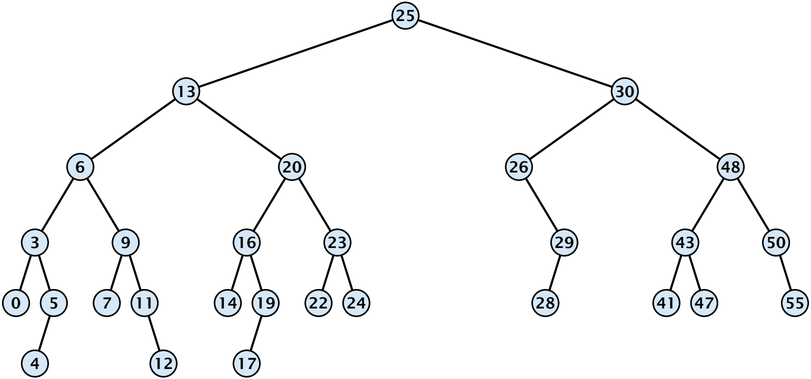


**Algorithm 2** TreeMin($x$)

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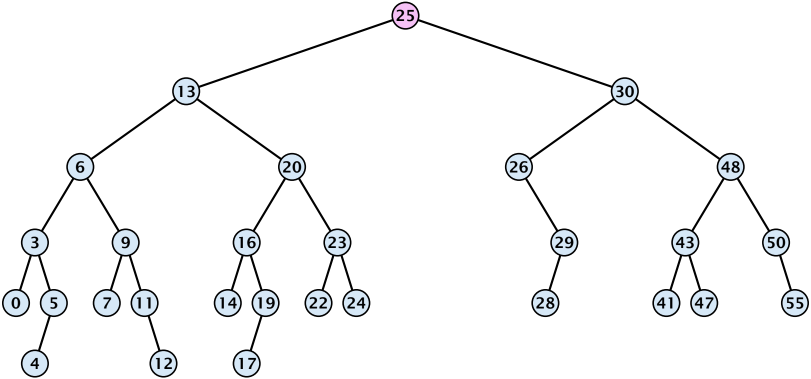


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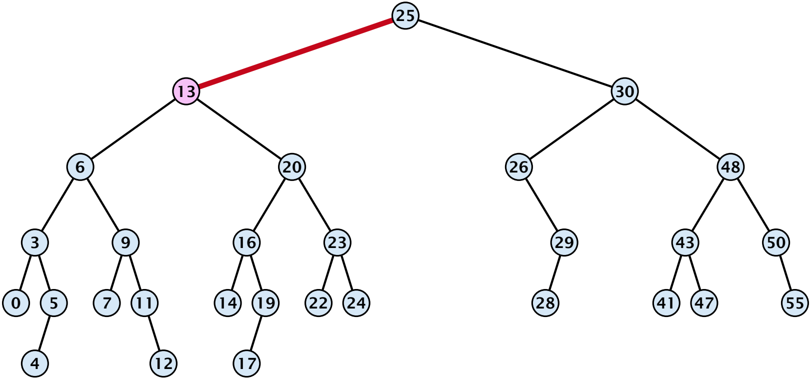


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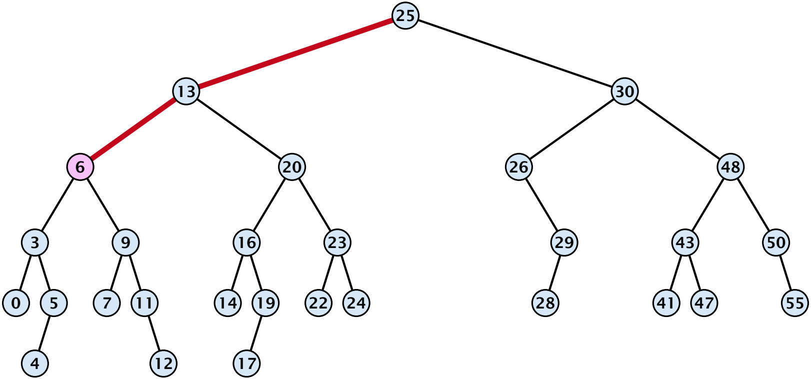


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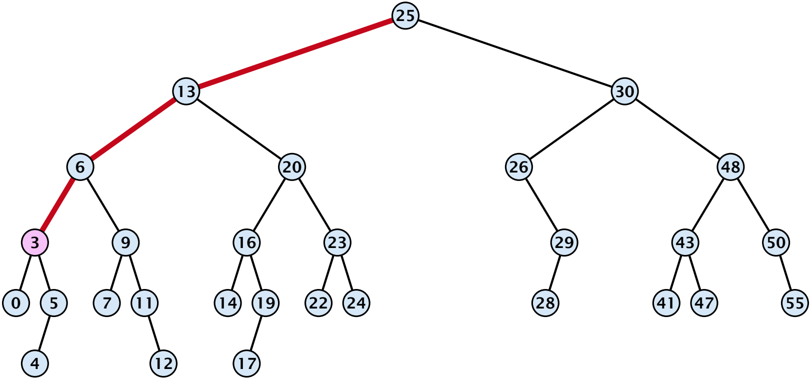


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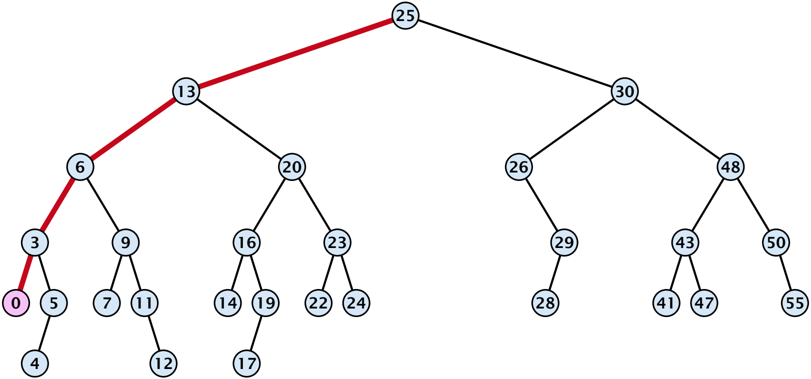


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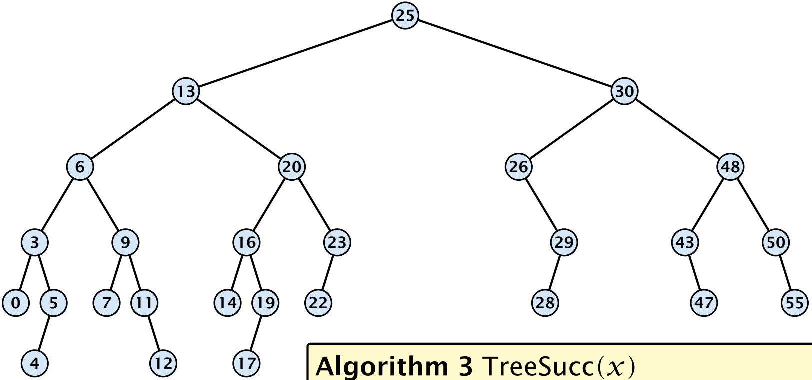


**Algorithm 3** TreeSucc($x$)

**if** right[$x$] $\neq$ null **return** TreeMin(right[$x$])
$y \leftarrow$ parent[$x$]
**while** $y \neq$ null **and** $x =$ right[$y$] **do**
    $x \leftarrow y$; $y \leftarrow$ parent[$x$]
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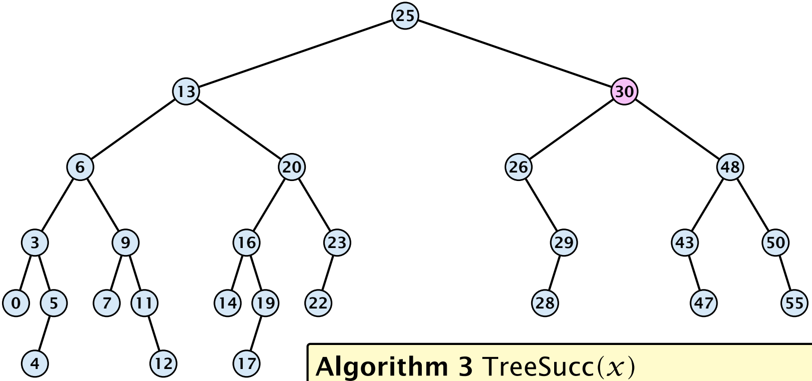


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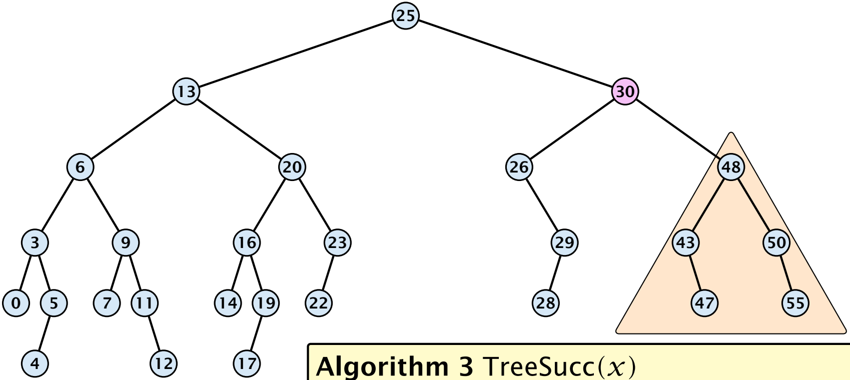


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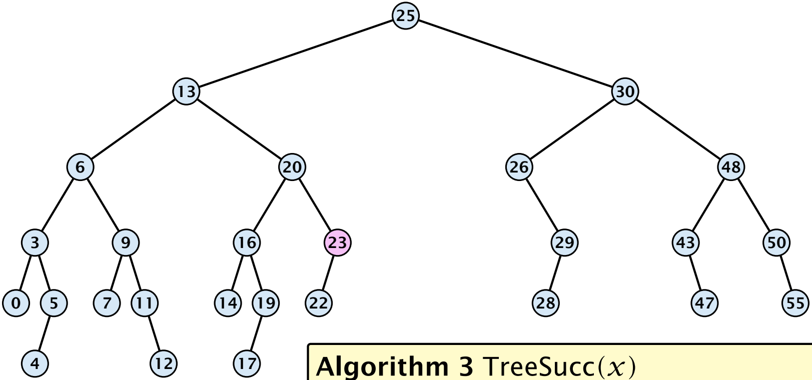


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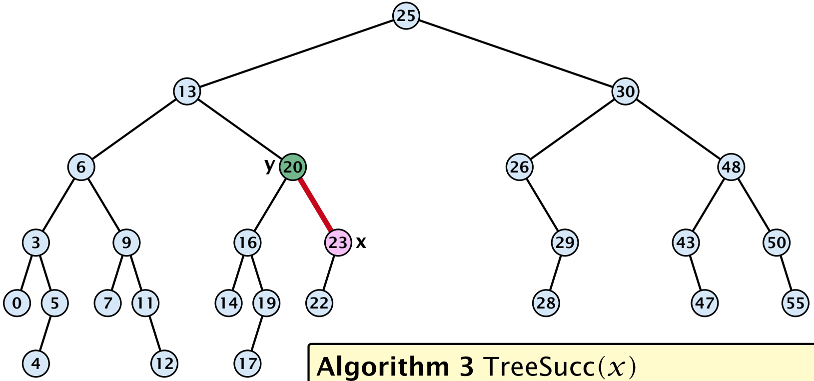


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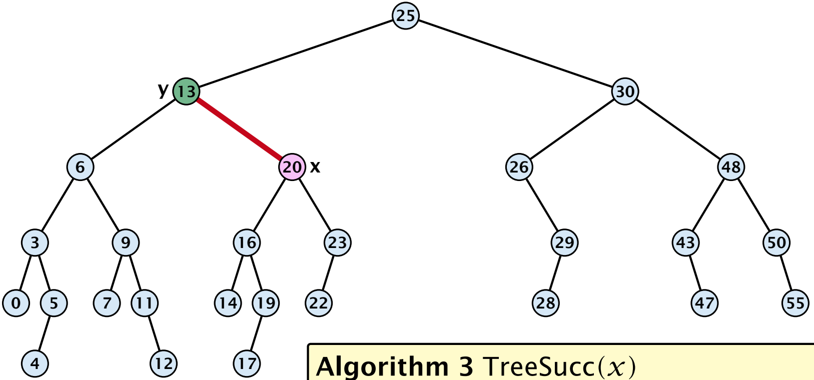


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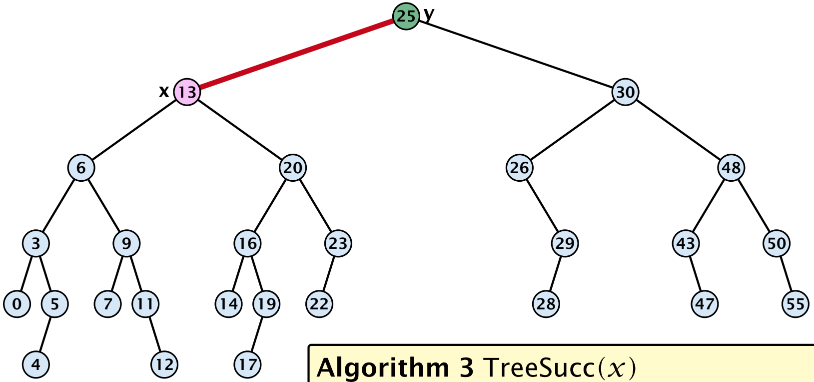


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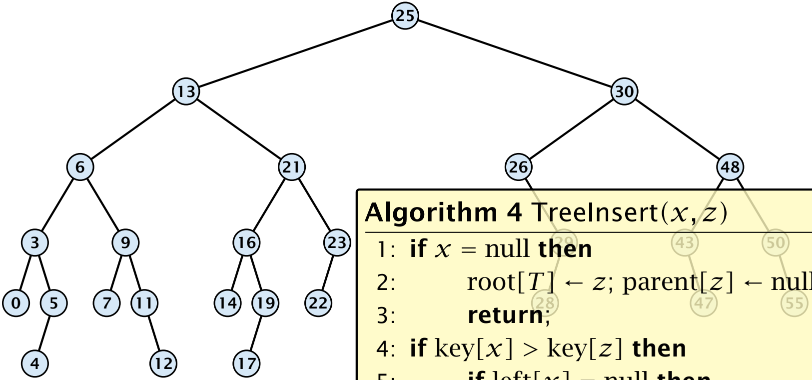


**Algorithm 4** TreeInsert$(x, z)$

**if** $x$ = null **then**
  root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
  **return**;
**if** key$[x] >$ key$[z]$ **then**
  **if** left$[x]$ = null **then**
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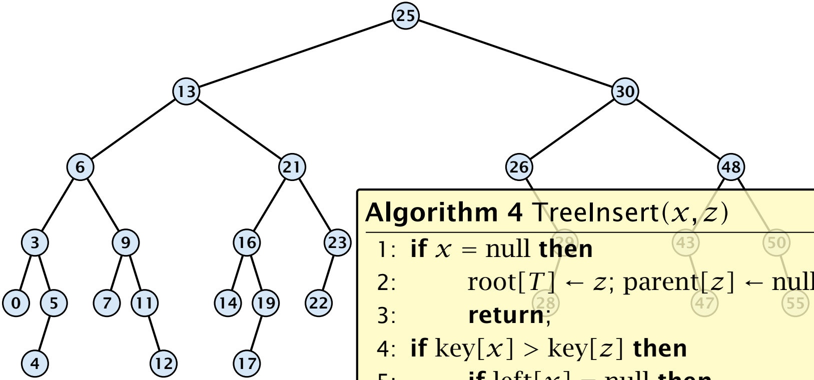
# Binary Search Trees: Insert

Insert element **not** in the tree.

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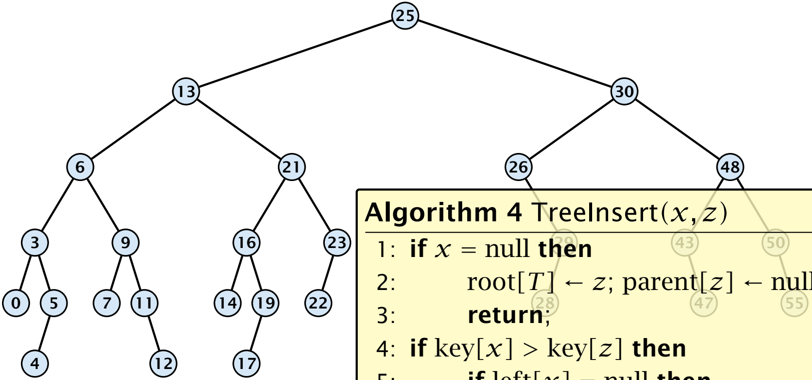


Search for $z$. At some point the search stops at a null-pointer. This is the place to insert $z$.

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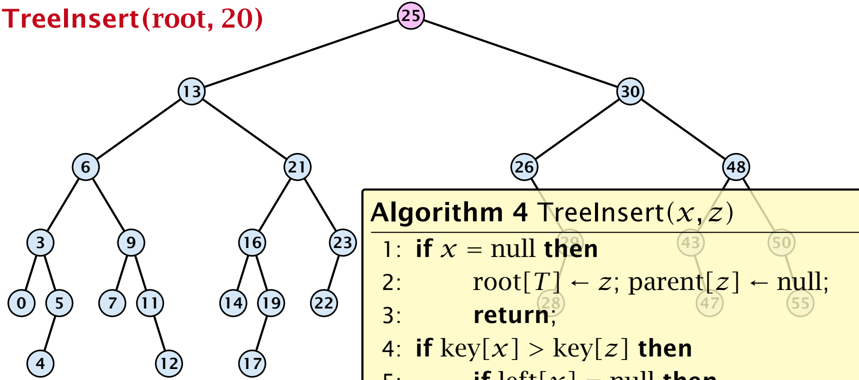


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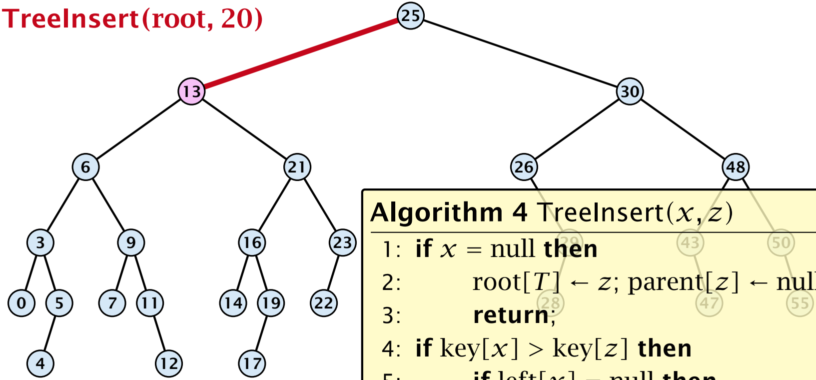


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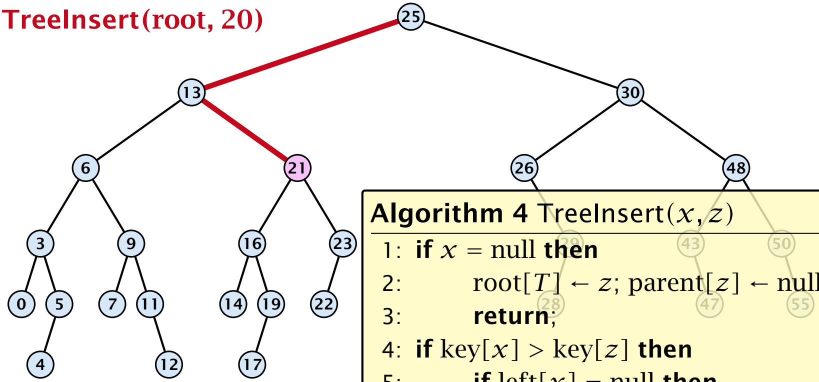


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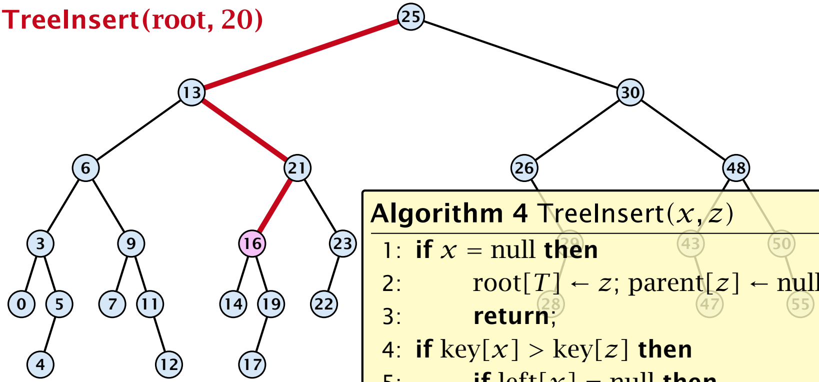


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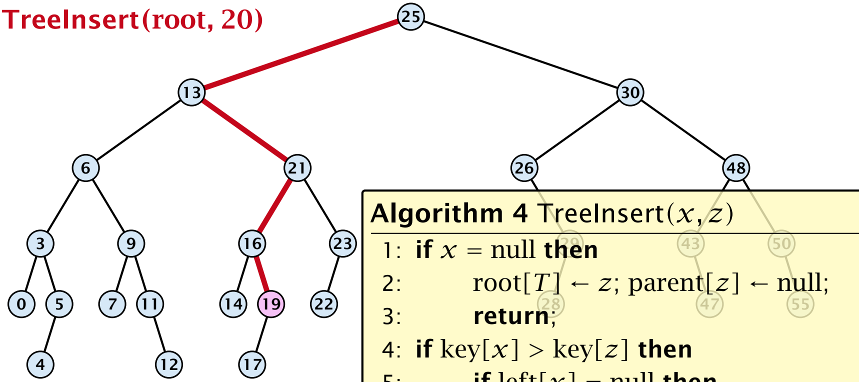


Search for $z$. At some point the search stops at a null-pointer. This is the place to insert $z$.

**Algorithm 4** TreeInsert$(x, z)$

```
if x = null then
    root[T] ← z; parent[z] ← null;
    return;
if key[x] > key[z] then
    if left[x] = null then
        left[x] ← z; parent[z] ← x;
    else TreeInsert(left[x], z);
else
    if right[x] = null then
        right[x] ← z; parent[z] ← x;
    else TreeInsert(right[x], z);
```

# Binary Search Trees: Insert

Insert element **not** in the tree.

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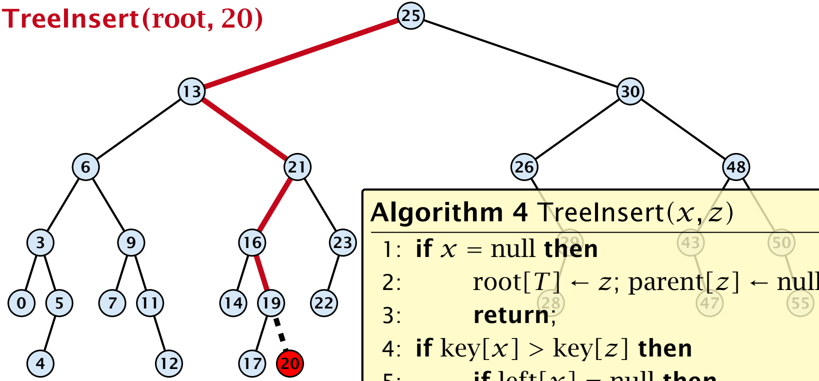


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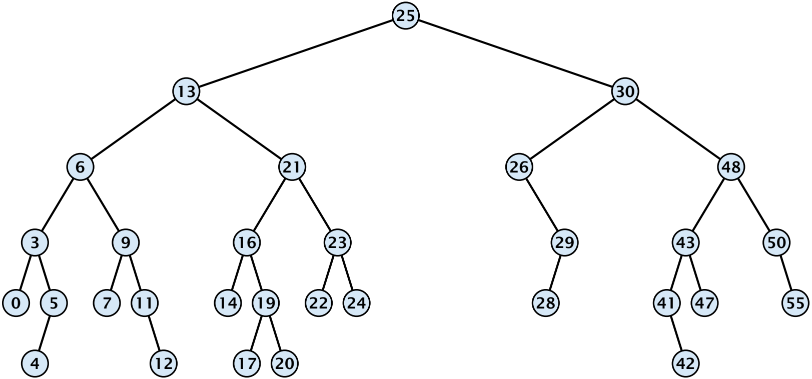

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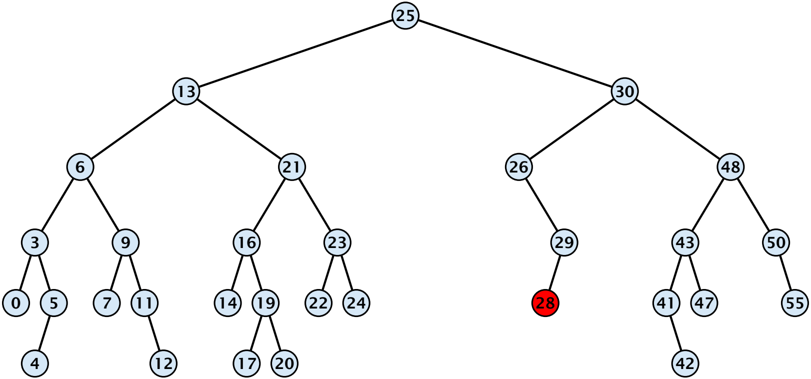
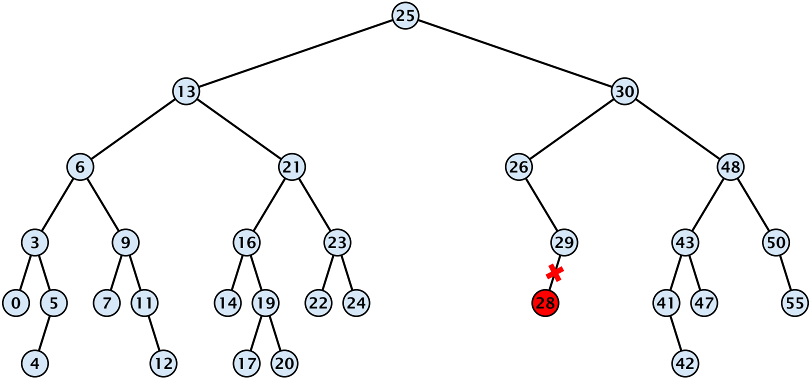



Case 1:

Element does not have any children

- Simply go to the parent and set the corresponding pointer to null.

# Binary Search Trees: Delete

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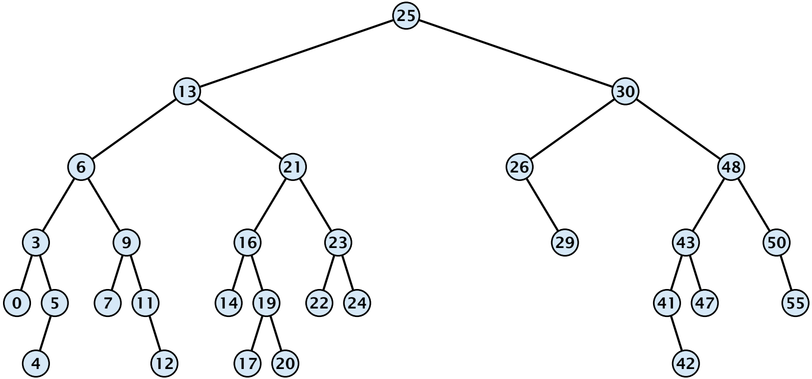


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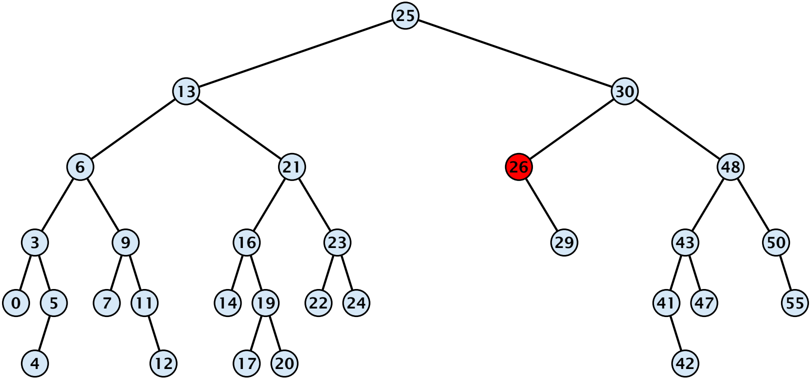
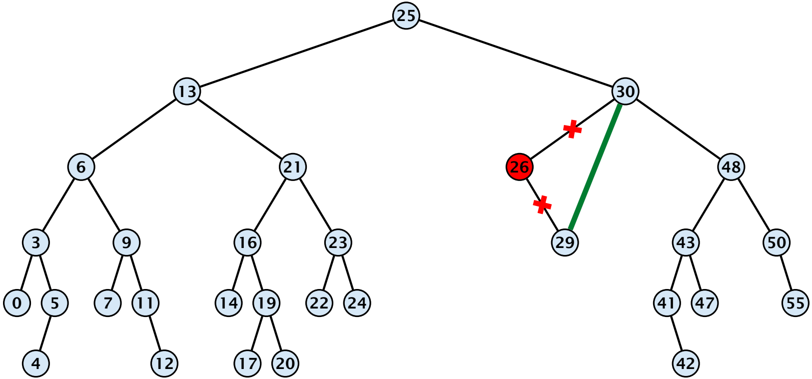



Case 2:

Element has exactly one child

- Splice the element out of the tree by connecting its parent to its successor.

# Binary Search Trees: Delete

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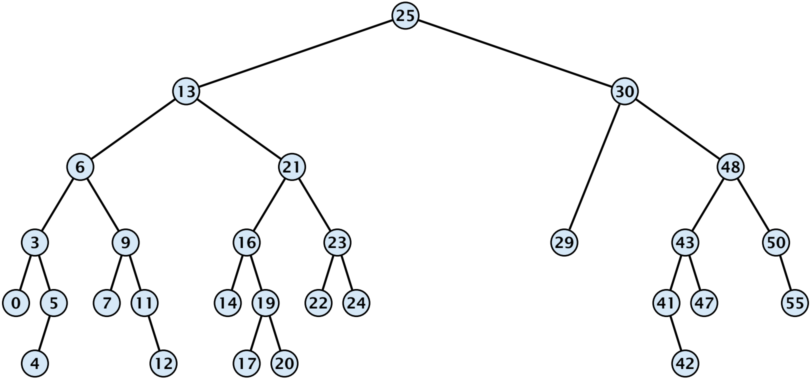


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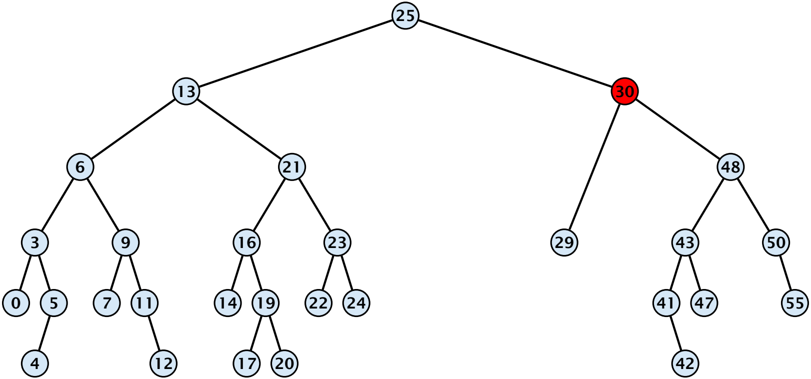


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Element has two children

- Find the successor of the element
- Splice successor out of the tree
- Replace content of element by content of successor

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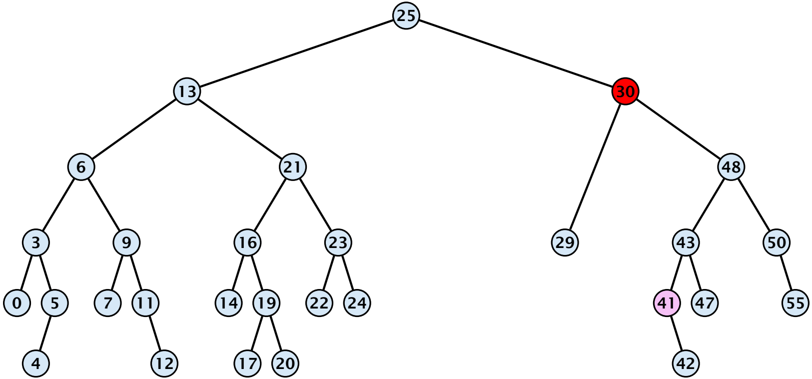


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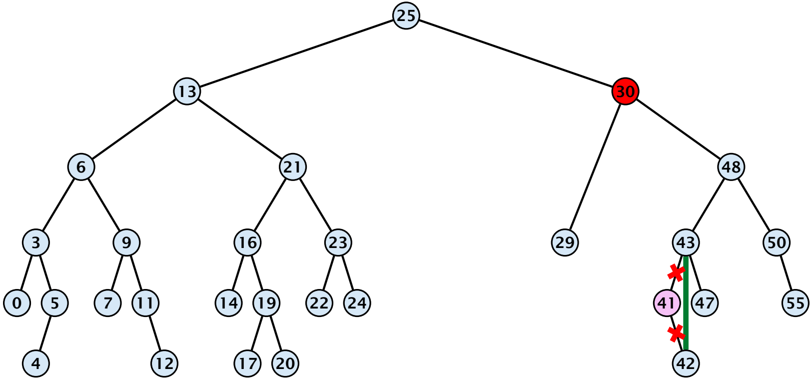


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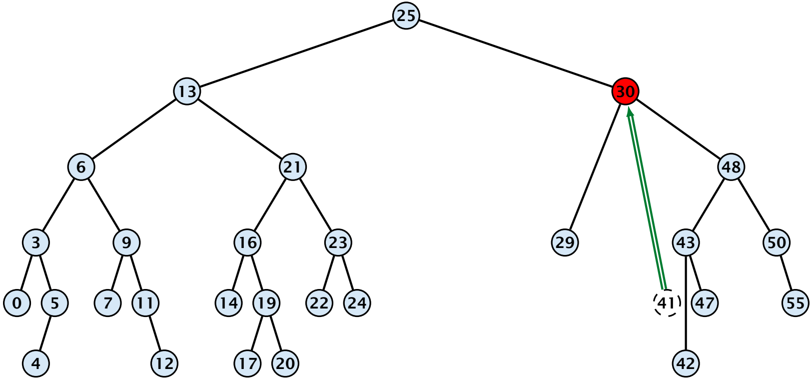


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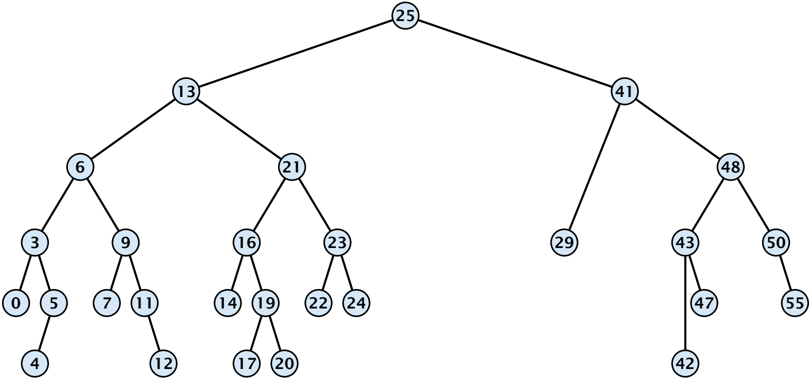


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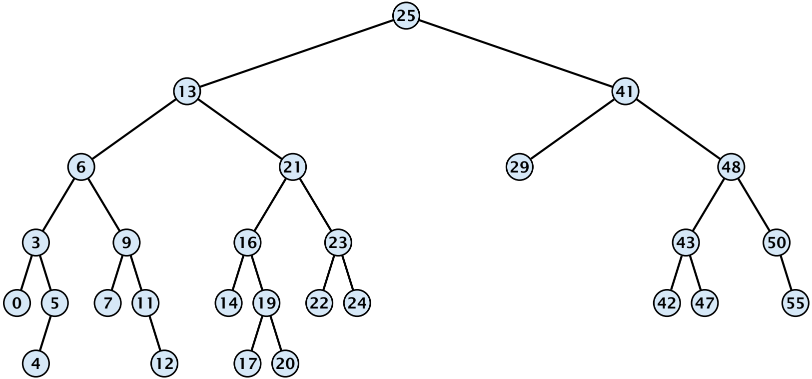


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# Binary Search Trees: Delete

**Algorithm 9** TreeDelete($z$)

```
if left[z] = null or right[z] = null
        then y ← z else y ← TreeSucc(z);                select y to splice out
if left[y] ≠ null
        then x ← left[y] else x ← right[y];     x is child of y (or null)
if x ≠ null then parent[x] ← parent[y];             parent[x] is correct
if parent[y] = null then                                ⎫
        root[T] ← x                                     ⎪
else                                                    ⎪
        if y = left[parent[y]] then                     ⎬ fix pointer to x
                left[parent[y]] ← x                     ⎪
        else                                            ⎪
                right[parent[y]] ← x                    ⎭
if y ≠ z then copy y-data to z
```

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With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.