Overview: Shortest Augmenting Paths

Lemma 5
The length of the shortest augmenting path never decreases.

Lemma 6
After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 7
The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

Proof.

- We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.

$\square$

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

Let L_G denote the subgraph of the residual graph G_f that contains only those edges (u,v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.

edge of G_f **edge of L_G**

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

Shortest Augmenting Path

First Lemma:
The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- Bottleneck edges on the chosen path are deleted.
- Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t.

edge of G_f edge of L_G

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let M denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k.

An s-t path in G_f that uses edges not in M has length larger than k, even when using edges added to G_f during the round.

In each augmentation an edge is deleted from M.

edge of G_f edge in M

Note that an edge cannot enter M again during the round as this would require an augmentation along a non-shortest path.

Shortest Augmenting Paths

Theorem 8
The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 9 (without proof)
There exist networks with $m = \Theta(n^2)$ that require $\Omega(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:
There always exists a set of m augmentations that gives a maximum flow (why?).

Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

Shortest Augmenting Paths

We maintain a subset M of the edges of G_f with the guarantee that a shortest s-t path using only edges from M is a shortest augmenting path.

With each augmentation some edges are deleted from M.

When M does not contain an s-t path anymore the distance between s and t strictly increases.

Note that M is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between s and t in G_f is k.

M is initialized as the level graph L_G.

Perform a DFS search to find a path from s to t using edges from M.

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from M.

Analysis

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing M for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in M and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in M for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.