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$\mathcal{S}$ is the set of subsets that separate s from t.

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The Separation Problem for the Shortest Path LP is the Minimum Cut Problem.

18 Cuts & Metrics

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Suppose that ℓ_e-values are solution to Minimum Cut LP.

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How do we round the LP?

- Let $B(s,r)$ be the ball of radius r around s (w.r.t. metric d). Formally:

$$B = \{v \in V \mid d(s,v) \leq r\}$$

- For $0 \leq r < 1$, $B(s,r)$ is an s-t-cut.

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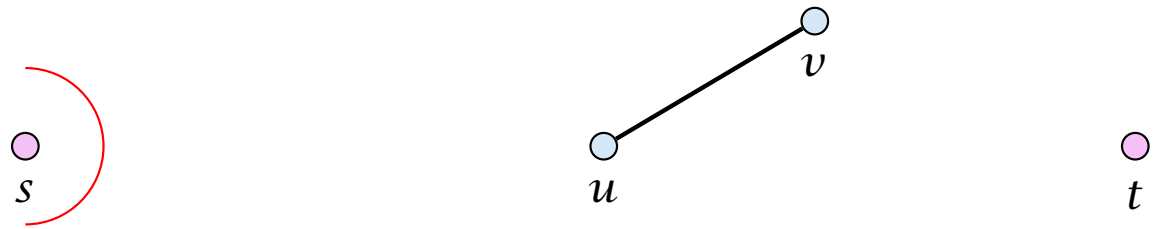

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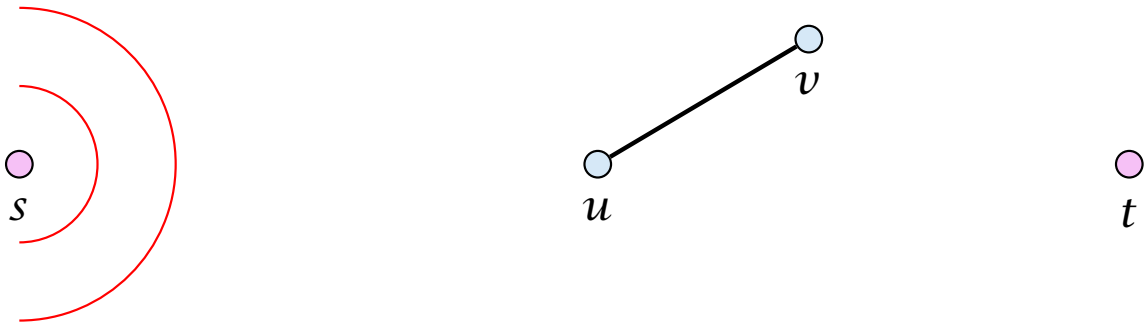

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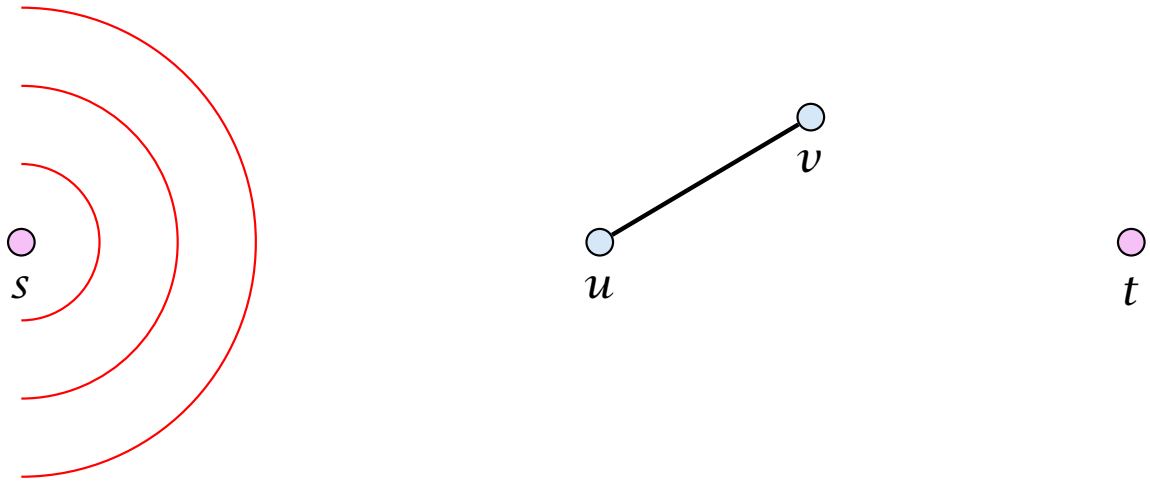

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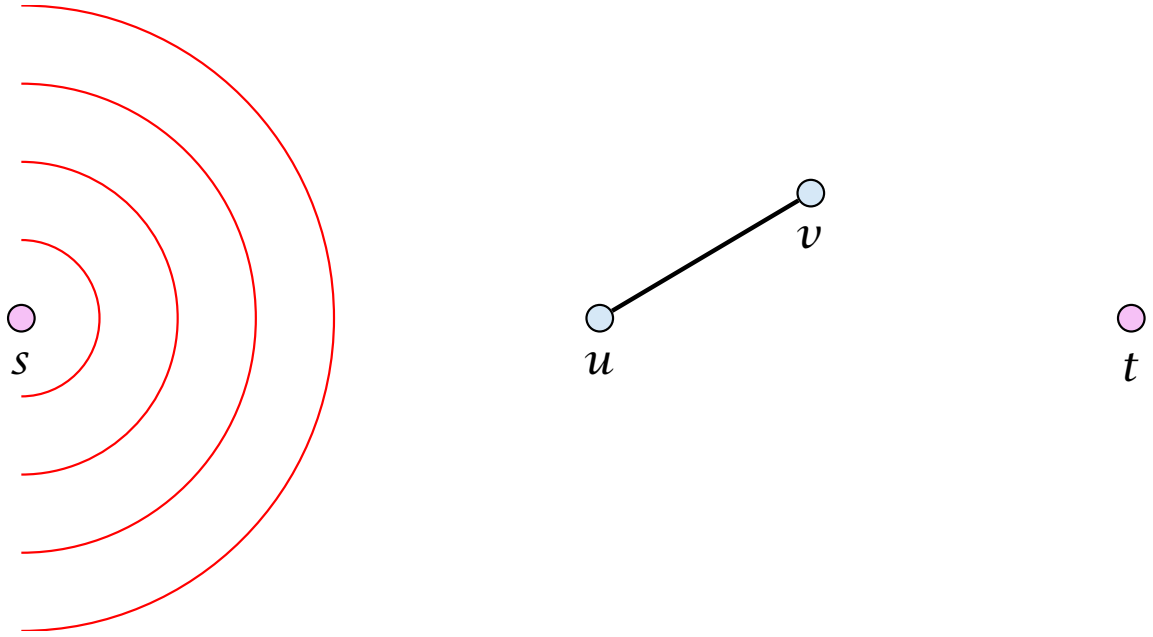

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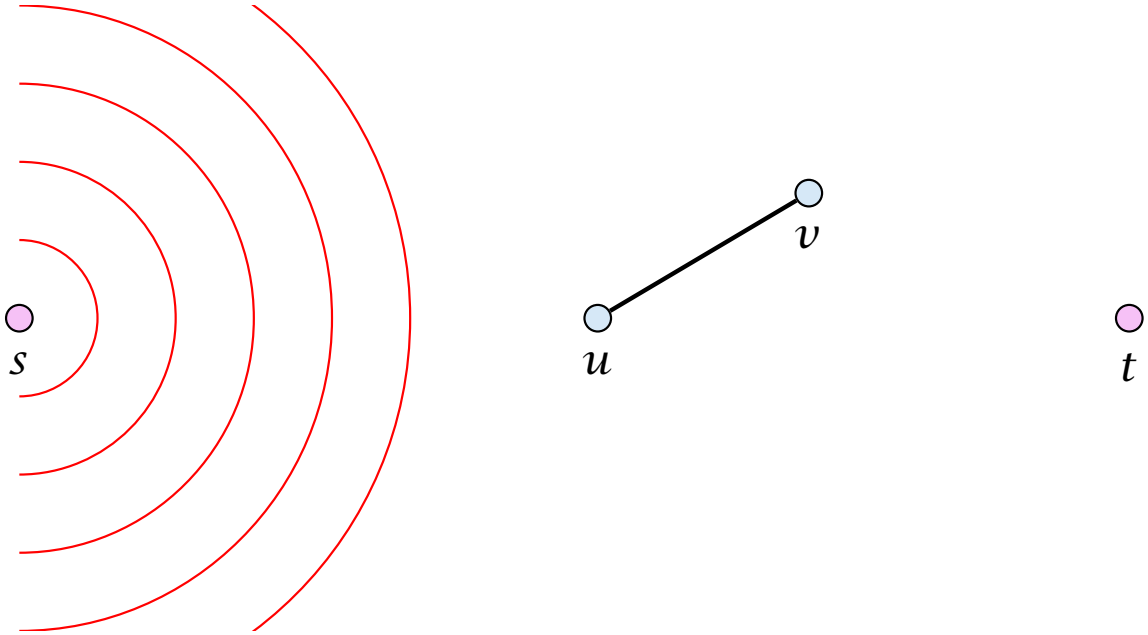

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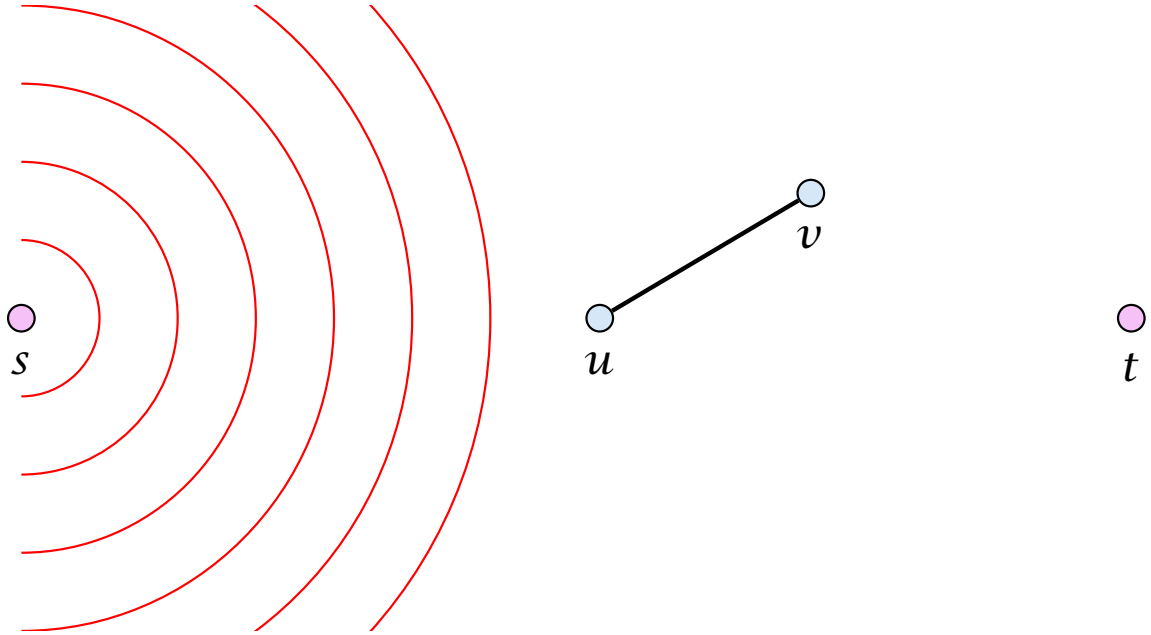

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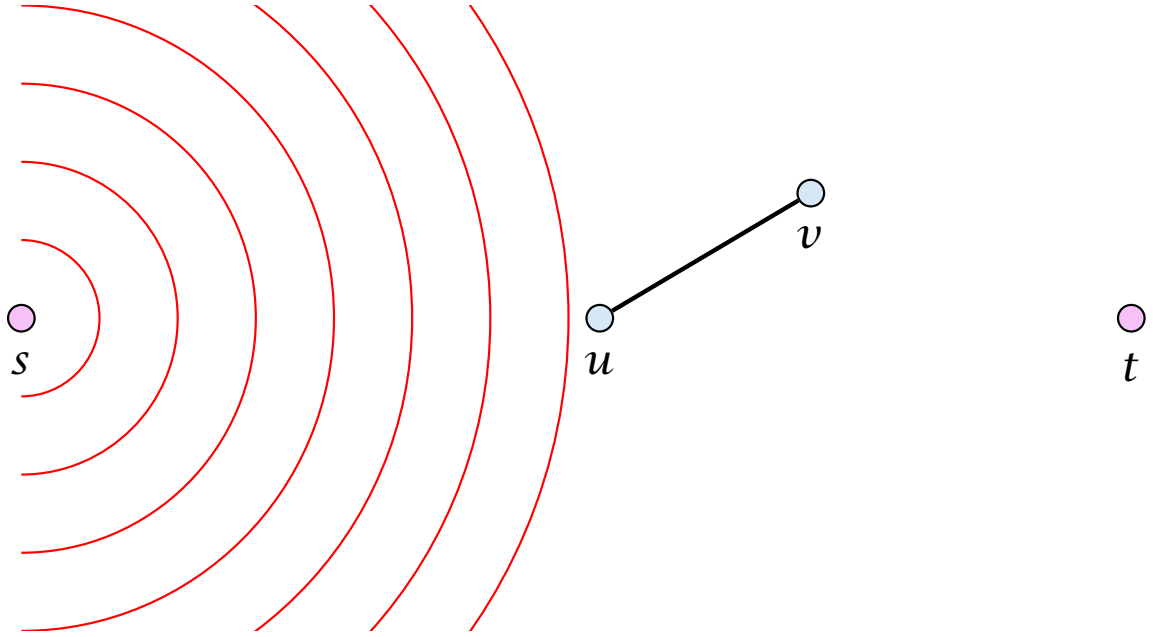

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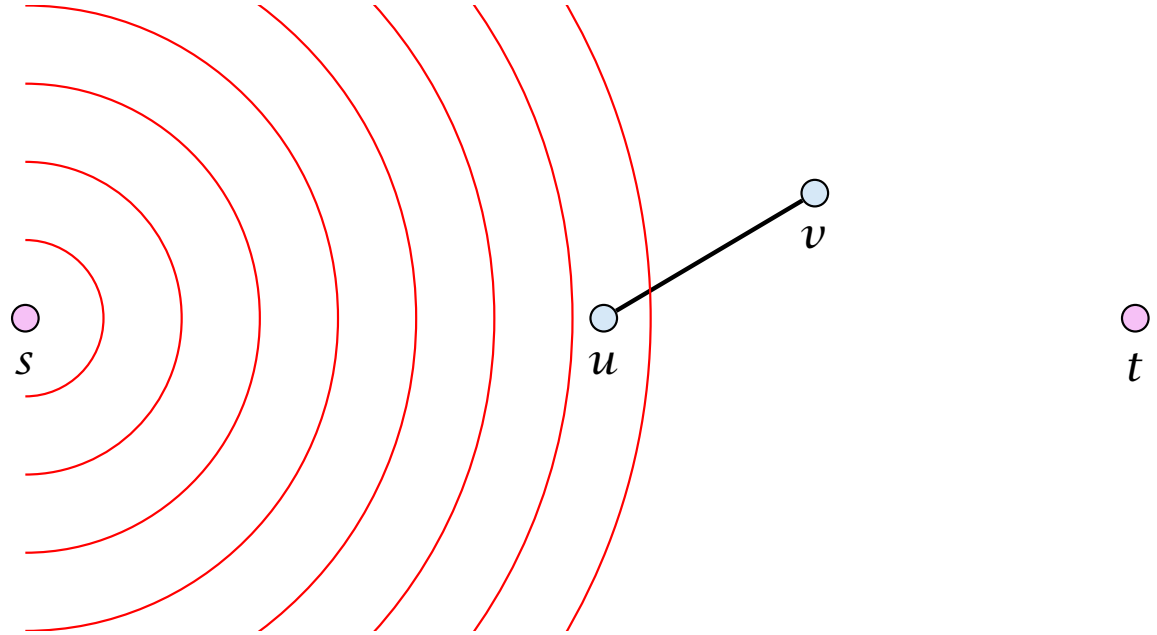

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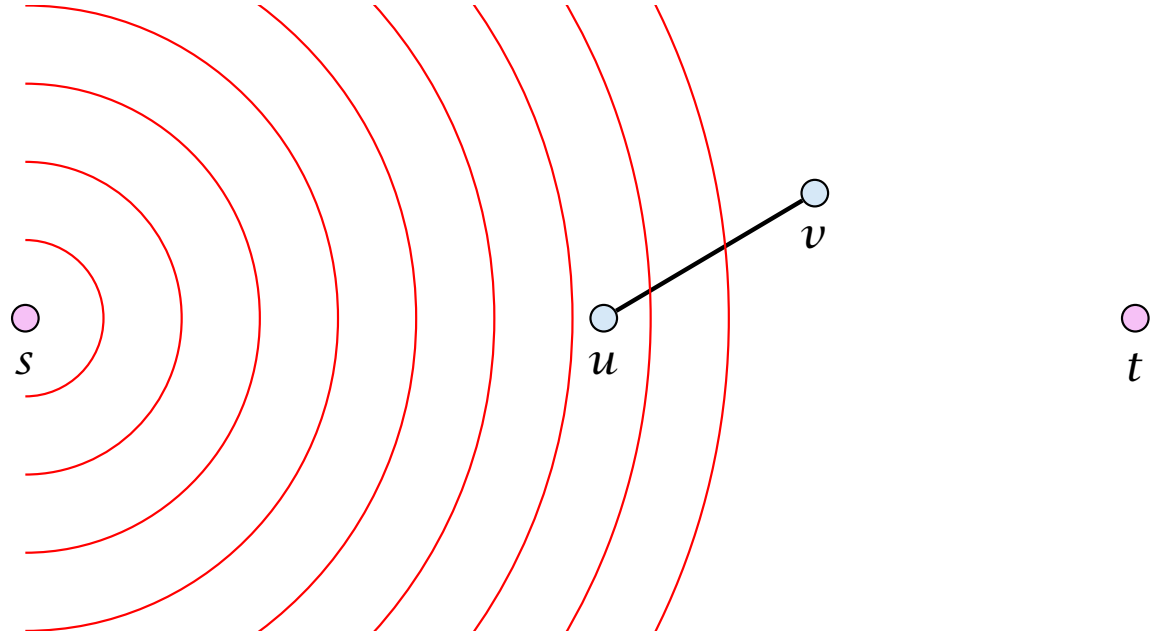

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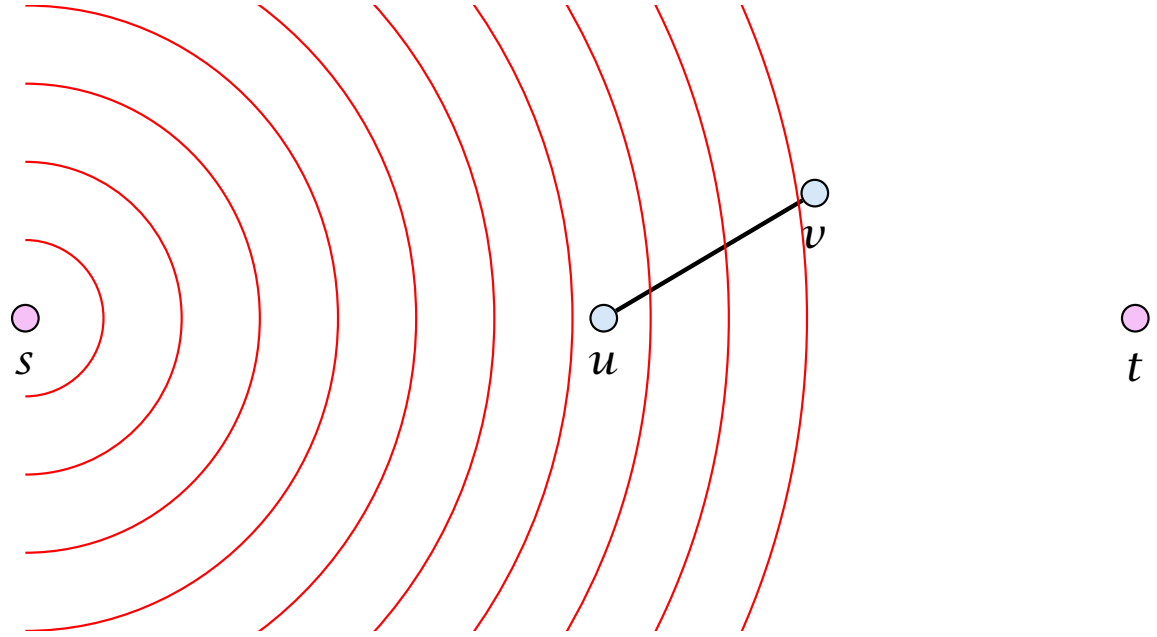

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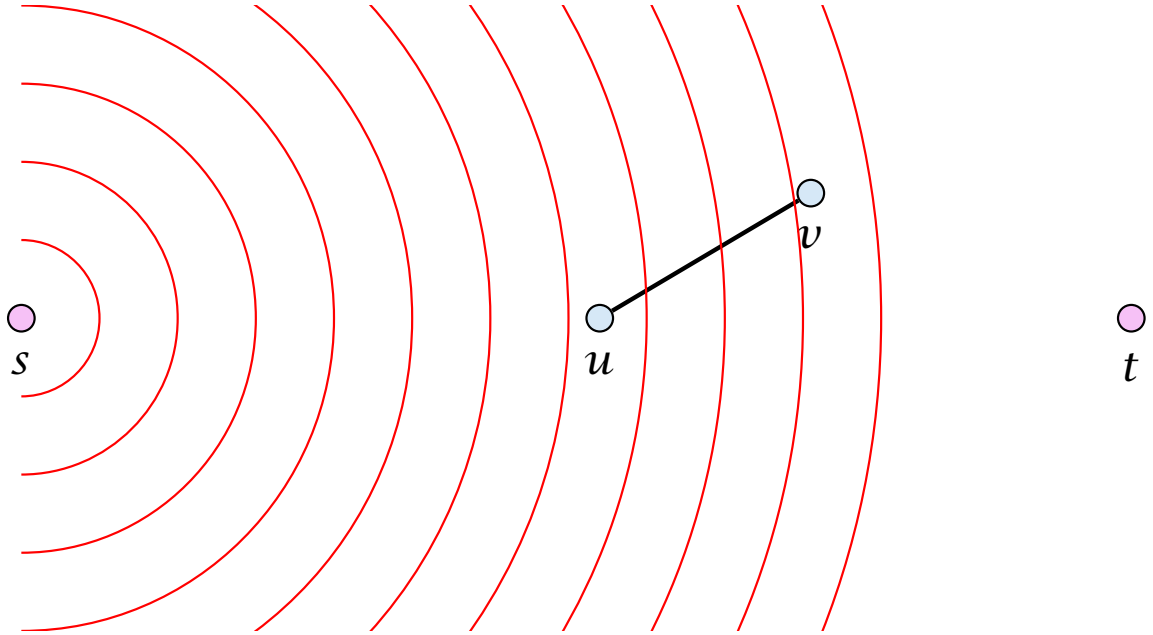

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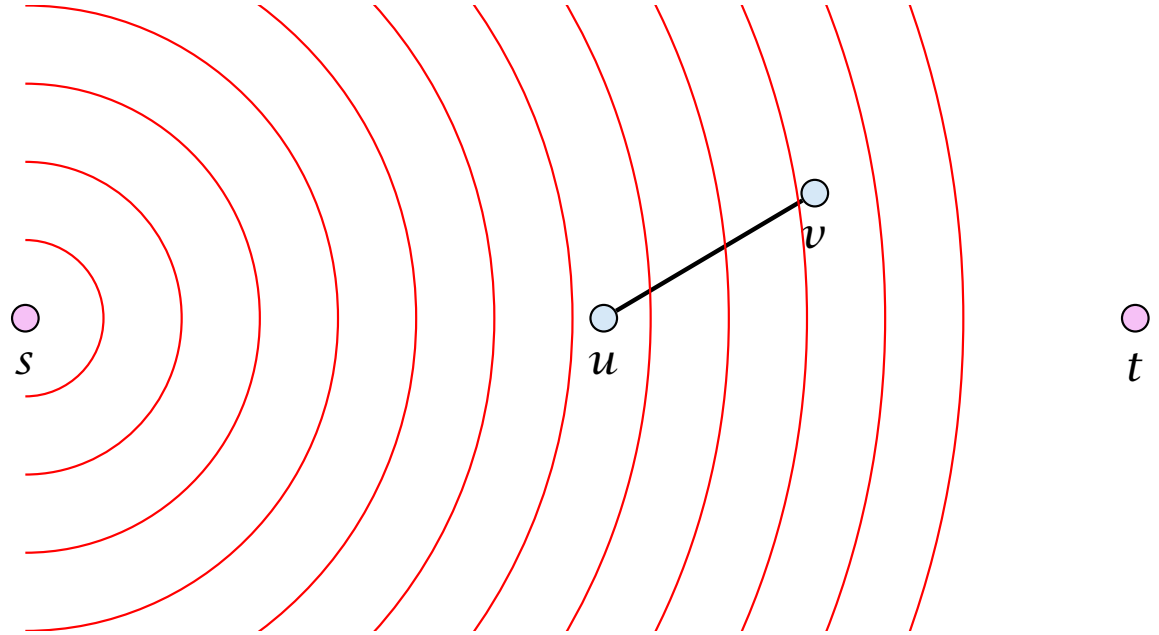

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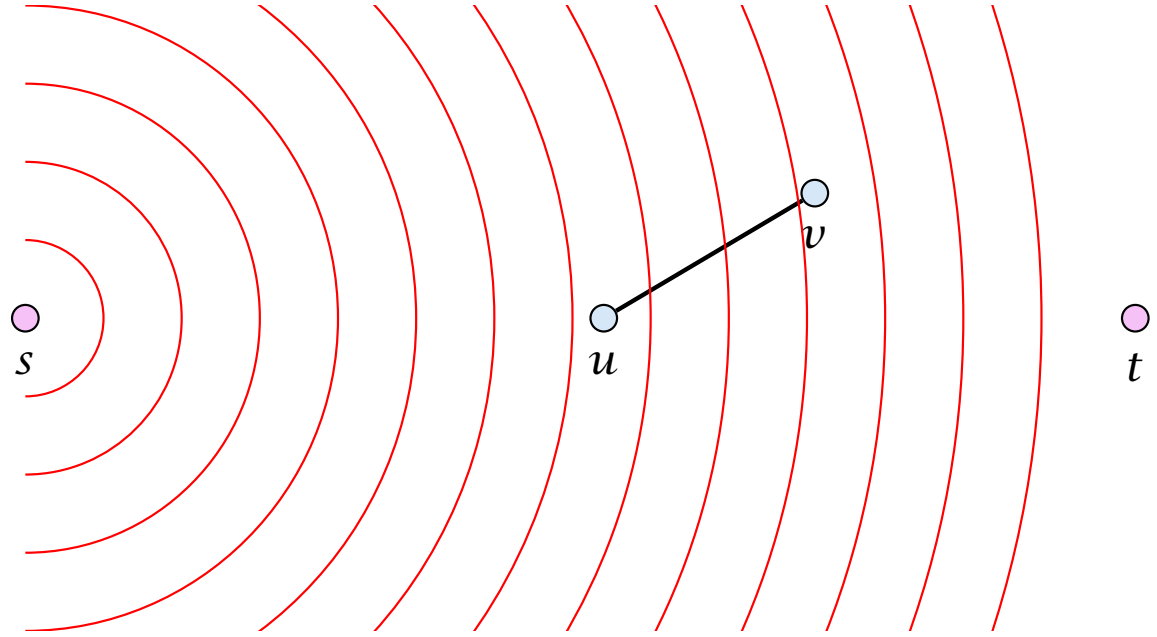

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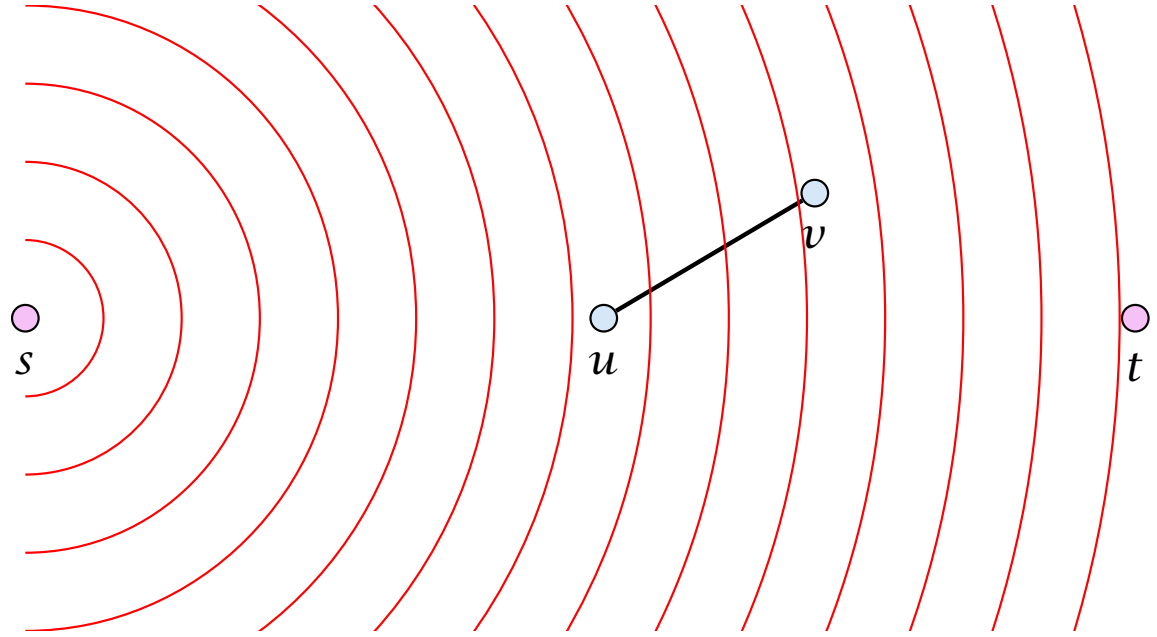

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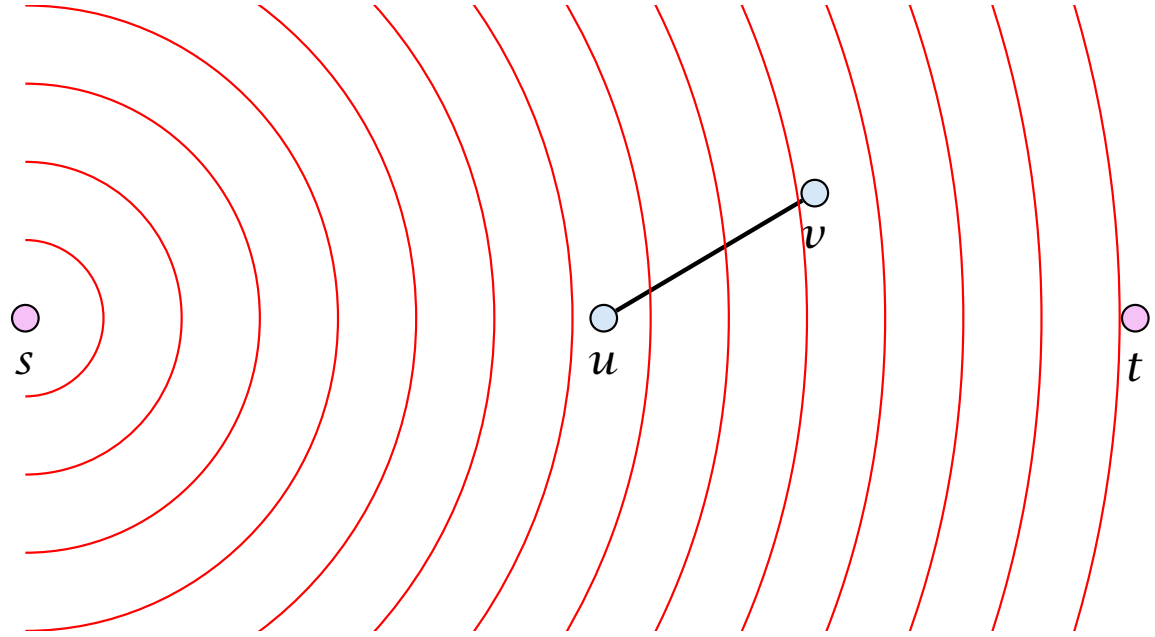

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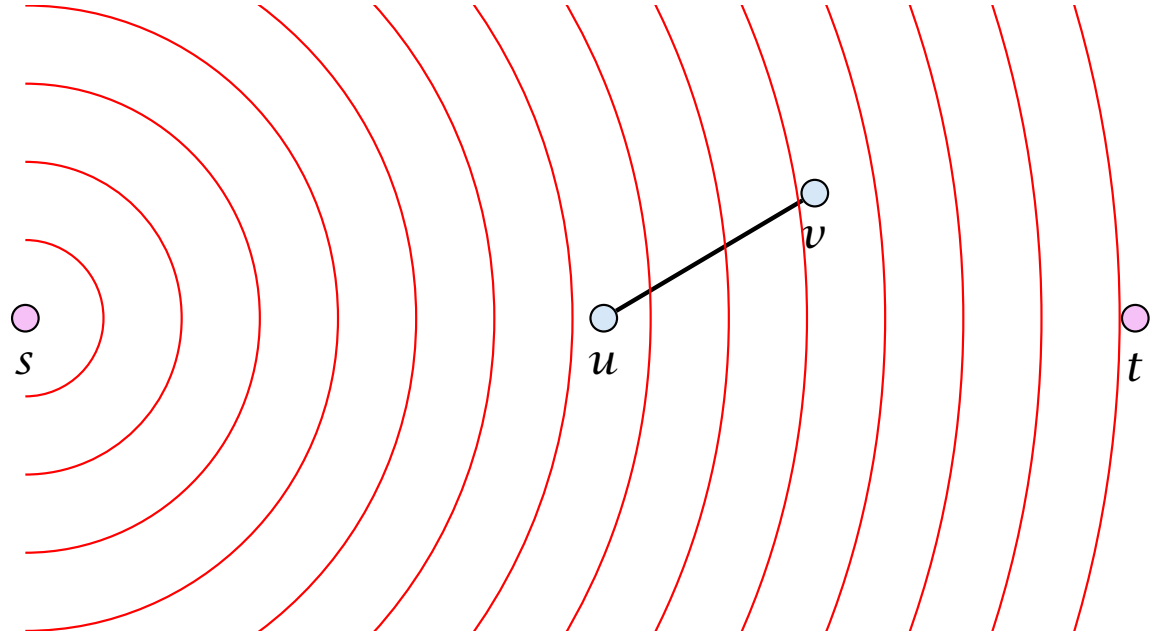

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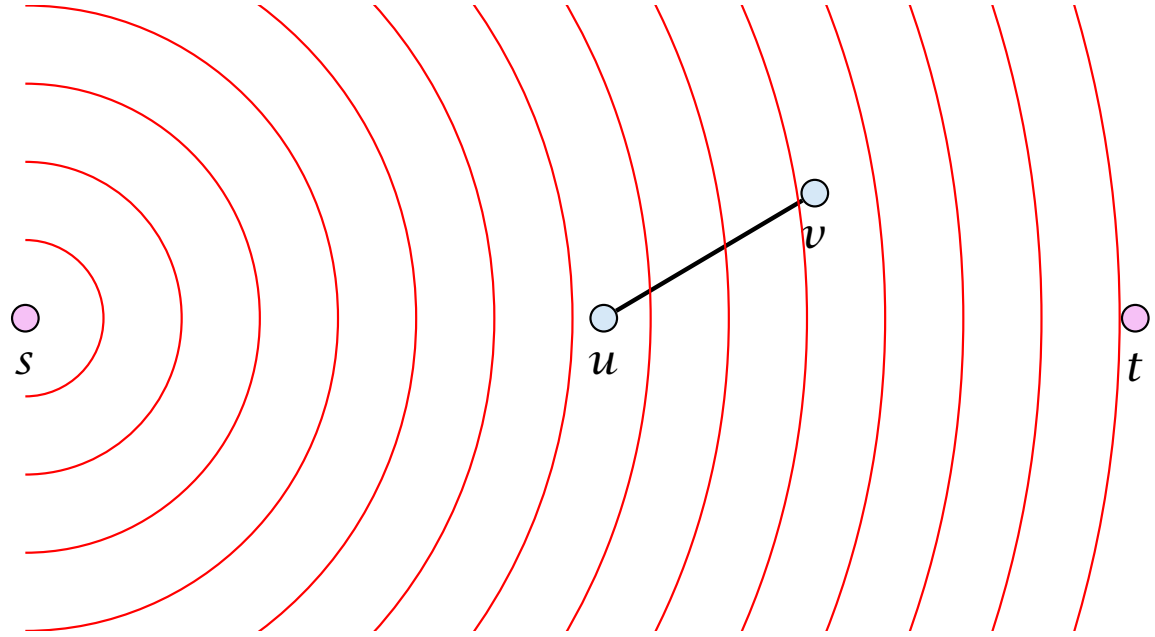

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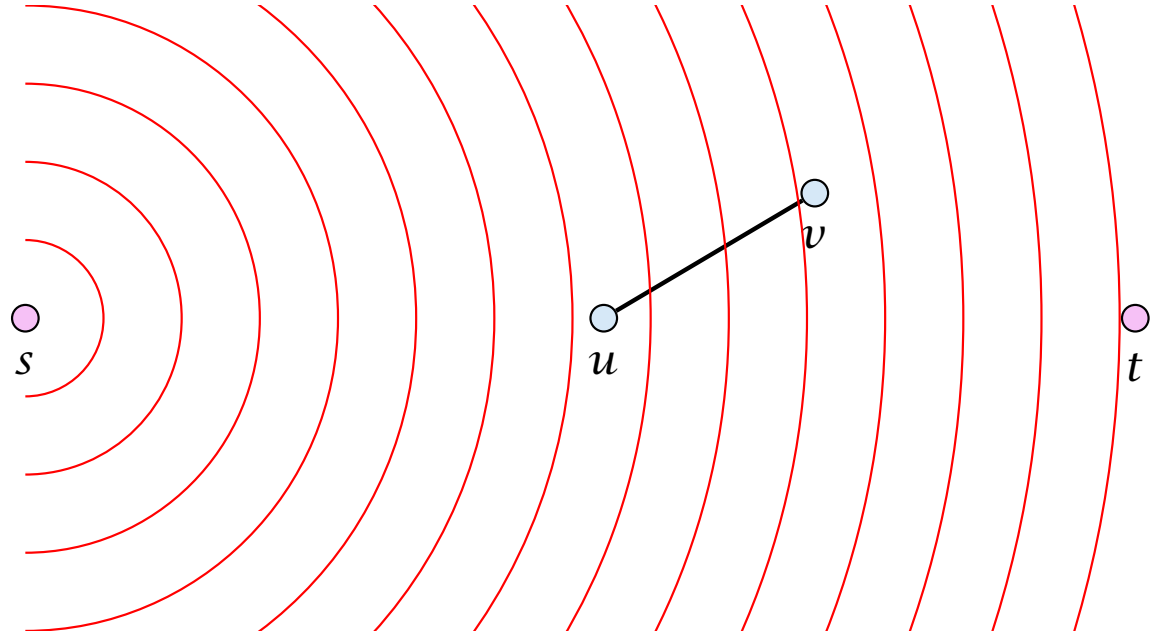

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On the other hand:

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2: repeat
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Algorithm 1 Multicut(G')

1: **while** $\exists s_i$-t_i pair in G' **do**
2: $\quad C \leftarrow$ RegionGrowing(s_i, p)
3: $\quad G' = G' \setminus C$ // cuts edges leaving C
4: **return** $B(s_i, z)$

- probability of cutting an edge is only p
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- choose $p = 6 \ln k \cdot \delta$
- we make $\frac{1}{2\delta}$ trials before reaching radius $1/2$.
- we say a Region Growing is not successful if it does not terminate before reaching radius $1/2$.

$$\Pr[\text{not successful}] \leq (1-p)^{\frac{1}{2\delta}} = \left((1-p)^{1/p}\right)^{\frac{p}{2\delta}} \leq e^{-\frac{p}{2\delta}} \leq \frac{1}{k^3}$$

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If we are not successful we simply perform a trivial k-approximation.

This only increases the expected cost by at most $\frac{1}{k^2} \cdot k\mathrm{OPT} \leq \mathrm{OPT}/k$.

Hence, our final cost is $\mathcal{O}(\ln k) \cdot \mathrm{OPT}$ in expectation.