## Degeneracy Revisited

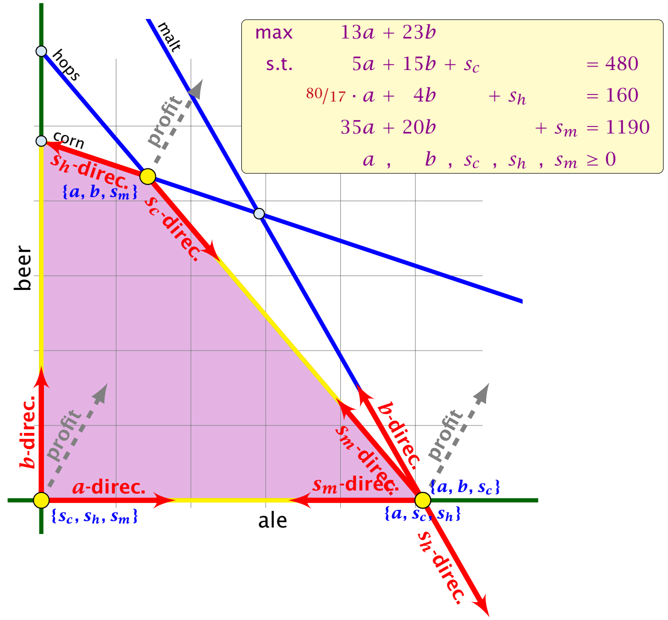
If a basis variable is $0$ in the basic feasible solution then we may not make progress during an iteration of simplex.

**Idea:**
Change $\text{LP} := \max\{c^Tx, Ax = b; x \geq 0\}$ into
$\text{LP}' := \max\{c^Tx, Ax = b', x \geq 0\}$ such that

- **I.** LP is feasible
- **II.** If a set $B$ of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \ngeq 0$) then $B$ corresponds to an infeasible basis in $\text{LP}'$ (note that columns in $A_B$ are linearly independent).
- **III.** LP has no degenerate basic solutions



## Degenerate Example




## Degeneracy Revisited

If a basis variable is $0$ in the basic feasible solution then we may not make progress during an iteration of simplex.

**Idea:**
Given feasible $\text{LP} := \max\{c^Tx, Ax = b; x \geq 0\}$. Change it into
$\text{LP}' := \max\{c^Tx, Ax = b', x \geq 0\}$ such that

- **I.** $\text{LP}'$ is feasible
- **II.** If a set $B$ of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \ngeq 0$) then $B$ corresponds to an infeasible basis in $\text{LP}'$ (note that columns in $A_B$ are linearly independent).
- **III.** $\text{LP}'$ has no degenerate basic solutions



## Perturbation

Let $B$ be index set of some basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad \text{(i.e. } B \text{ is feasible)}$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \quad \text{for } \varepsilon > 0 \ .$$

This is the perturbation that we are using.

## Property I

The new LP is feasible because the set $B$ of basis variables provides a feasible basis:

$$A_B^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right) = x_B^* + \begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix} \geq 0 \ .$$



## Property II

Let $\tilde{B}$ be a non-feasible basis. This means $(A_{\tilde{B}}^{-1}b)_i < 0$ for some row $i$.

Then for small enough $\epsilon > 0$

$$\left(A_{\tilde{B}}^{-1}\left(b + A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)\right)_i = (A_{\tilde{B}}^{-1}b)_i + \left(A_{\tilde{B}}^{-1}A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}\right)_i < 0$$

Hence, $\tilde{B}$ is not feasible.



## Property III

Let $\tilde{B}$ be a basis. It has an associated solution

$$x_{\tilde{B}}^* = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B\begin{pmatrix}\varepsilon\\ \vdots\\ \varepsilon^m\end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynom with variable $\varepsilon$ of degree at most $m$.

$A_{\tilde{B}}^{-1}A_B$ has rank $m$. Therefore no polynom is $0$.

A polynom of degree at most $m$ has at most $m$ roots (Nullstellen).

Hence, $\epsilon > 0$ small enough gives that no component of the above vector is $0$. Hence, no degeneracies.



Since, there are no degeneracies Simplex will terminate when run on $\mathrm{LP}'$.

- If it terminates because the reduced cost vector fulfills

  $$\tilde{c} = (c^T - c_B^T A_B^{-1}A) \leq 0$$

  then we have found an optimal basis. Note that this basis is also optimal for LP, as the above constraint does not depend on $b$.
- If it terminates because it finds a variable $x_j$ with $\tilde{c}_j > 0$ for which the $j$-th basis direction $d$, fulfills $d \geq 0$ we know that $\mathrm{LP}'$ is unbounded. The basis direction does not depend on $b$. Hence, we also know that LP is unbounded.

## Lexicographic Pivoting

Doing calculations with perturbed instances may be costly. Also the right choice of $\varepsilon$ is difficult.

**Idea:**
Simulate behaviour of $\mathrm{LP}'$ without explicitly doing a perturbation.



## Lexicographic Pivoting

We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then $\mathrm{LP}'$ and $\mathrm{LP}$ do the same (i.e., choose the same variable).

Otherwise we have to be careful.



## Lexicographic Pivoting

In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where $B$ is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

$$b' = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$



## Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis $B$ is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

## Lexicographic Pivoting

LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_\ell = \frac{\hat{b}_\ell}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_\ell}{(A_B^{-1}A_{*e})_\ell} \ .$$

$\ell$ is the index of a leaving variable within $B$. This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is $3$ then $\ell = 2$.



## Lexicographic Pivoting

**Definition 44**
$u \leq_{\text{lex}} v$ if and only if the first component in which $u$ and $v$ differ fulfills $u_i \leq v_i$.



## Lexicographic Pivoting

LP′ chooses an index that minimizes

$$\theta_\ell = \frac{\left(A_B^{-1}\left(b + \begin{pmatrix}\varepsilon\\ \vdots \\ \varepsilon^m\end{pmatrix}\right)\right)_\ell}{(A_B^{-1}A_{*e})_\ell} = \frac{\left(A_B^{-1}(b \mid I)\begin{pmatrix}1\\ \varepsilon\\ \vdots \\ \varepsilon^m\end{pmatrix}\right)_\ell}{(A_B^{-1}A_{*e})_\ell}$$

$$= \frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}\begin{pmatrix}1\\ \varepsilon\\ \vdots \\ \varepsilon^m\end{pmatrix}$$



## Lexicographic Pivoting

This means you can choose the variable/row $\ell$ for which the vector

$$\frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}$$

is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.