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Definition 65

An Integer Linear Program or Integer Program is a Linear Program in which all variables are required to be integral.

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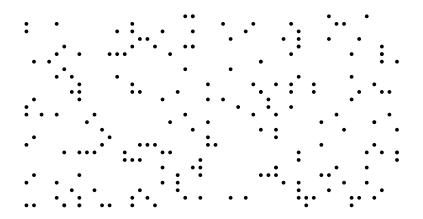
Given a ground set U, a collection of subsets $S_1, \ldots, S_k \subseteq U$, where the *i*-th subset S_i has weight/cost w_i . Find a collection $I \subseteq \{1, \ldots, k\}$ such that

 $\forall u \in U \exists i \in I : u \in S_i$ (every element is covered)

and

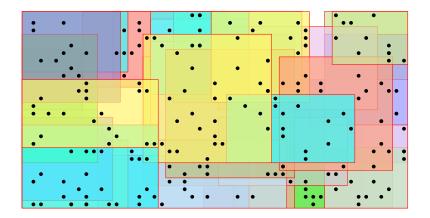
$$\sum_{i\in I} w_i$$
 is minimized.





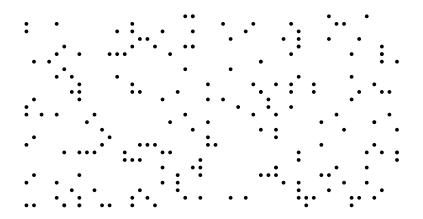


12 Integer Programs



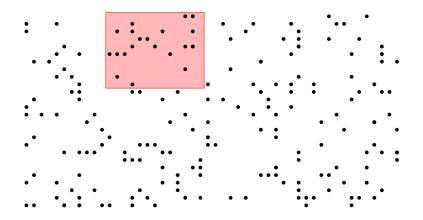


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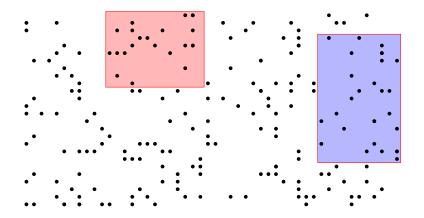


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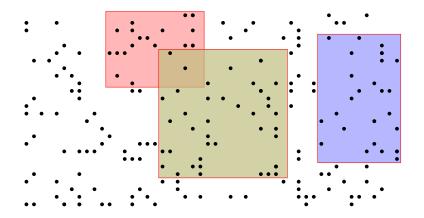


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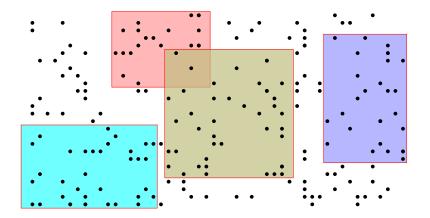


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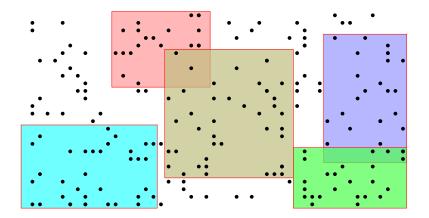


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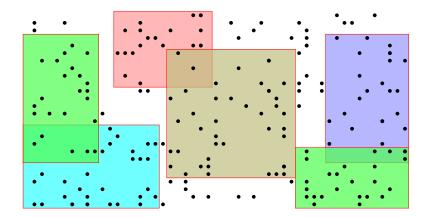


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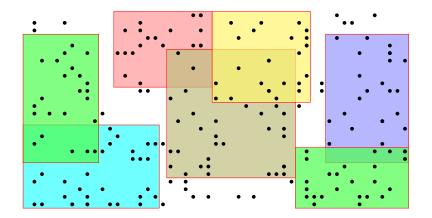


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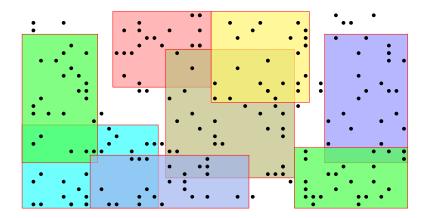


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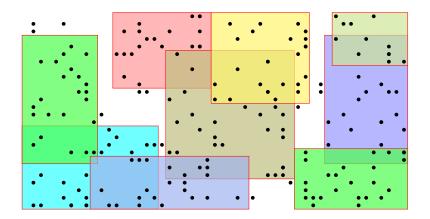


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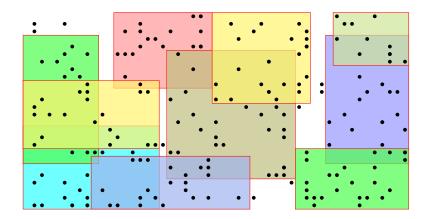


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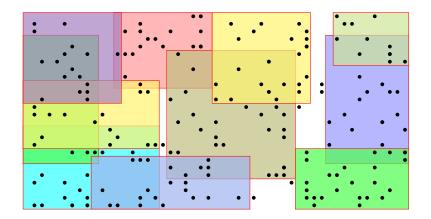


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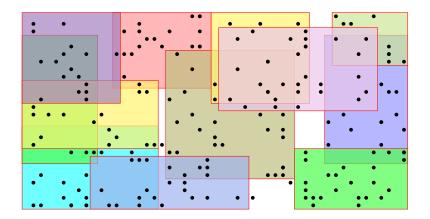


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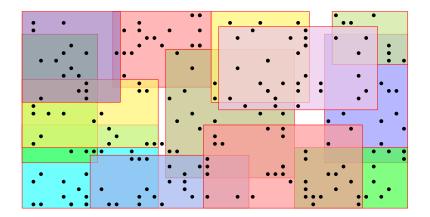


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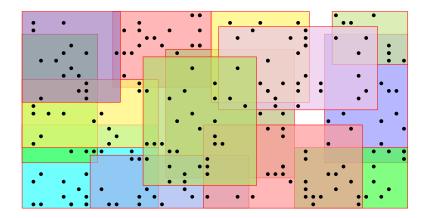


12 Integer Programs





12 Integer Programs





12 Integer Programs

IP-Formulation of Set Cover

min		$\sum_i w_i x_i$		
s.t.	$\forall u \in U$	$\sum_{i:u\in S_i} x_i$	≥	1
	$\forall i \in \{1, \ldots, k\}$	x_i	\geq	0
	$\forall i \in \{1, \ldots, k\}$	x_i	integral	



Vertex Cover

Given a graph G = (V, E) and a weight w_v for every node. Find a vertex subset $S \subseteq V$ of minimum weight such that every edge is incident to at least one vertex in S.



IP-Formulation of Vertex Cover

$$\begin{array}{c|cccc} \min & & \sum_{v \in V} w_v x_v \\ \text{s.t.} & \forall e = (i,j) \in E & & x_i + x_j & \geq & 1 \\ & \forall v \in V & & x_v & \in & \{0,1\} \end{array}$$



Maximum Weighted Matching

Given a graph G = (V, E), and a weight w_e for every edge $e \in E$. Find a subset of edges of maximum weight such that no vertex is incident to more than one edge.





12 Integer Programs

Maximum Weighted Matching

Given a graph G = (V, E), and a weight w_e for every edge $e \in E$. Find a subset of edges of maximum weight such that no vertex is incident to more than one edge.

max	$\sum_{e\in E} w_e x_e$			
s.t.	$\forall v \in V$	$\sum_{e:v \in e} x_e$	\leq	1
	$\forall e \in E$	x_e	\in	$\{0, 1\}$



Maximum Independent Set

Given a graph G = (V, E), and a weight w_v for every node $v \in V$. Find a subset $S \subseteq V$ of nodes of maximum weight such that no two vertices in S are adjacent.





12 Integer Programs

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max		$\sum_{v \in V} w_v x_v$		
s.t.	$\forall e = (i, j) \in E$	$x_i + x_j$	\leq	1
	$\forall v \in V$	x_v	\in	$\{0, 1\}$



Knapsack

Given a set of items $\{1, ..., n\}$, where the *i*-th item has weight w_i and profit p_i , and given a threshold *K*. Find a subset $I \subseteq \{1, ..., n\}$ of items of total weight at most *K* such that the profit is maximized.





12 Integer Programs

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$$\begin{array}{|c|c|c|c|c|} \max & & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} & & \sum_{i=1}^{n} w_i x_i &\leq K \\ & \forall i \in \{1, \dots, n\} & & x_i &\in \{0, 1\} \end{array}$$



12 Integer Programs

Relaxations

Definition 67

A linear program LP is a relaxation of an integer program IP if any feasible solution for IP is also feasible for LP and if the objective values of these solutions are identical in both programs.

We obtain a relaxation for all examples by writing $x_i \in [0, 1]$ instead of $x_i \in \{0, 1\}$.



12 Integer Programs

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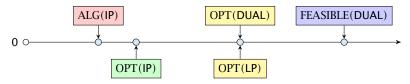


By solving a relaxation we obtain an upper bound for a maximization problem and a lower bound for a minimization problem.



Relations

Maximization Problems:



Minimization Problems:

