

## Brewery Problem

### Brewery brews ale and beer.

- ▶ Production limited by supply of corn, hops and barley malt
- ▶ Recipes for ale and beer require different amounts of resources

	Corn (kg)	Hops (kg)	Malt (kg)	Profit (€)
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

## Brewery Problem

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### How can brewer maximize profits?

- ▶ only brew ale: 34 barrels of ale  $\Rightarrow$  442 €
- ▶ only brew beer: 32 barrels of beer  $\Rightarrow$  736 €
- ▶ 7.5 barrels ale, 29.5 barrels beer  $\Rightarrow$  776 €
- ▶ 12 barrels ale, 28 barrels beer  $\Rightarrow$  800 €

## Brewery Problem

### Linear Program

- ▶ Introduce variables  $a$  and  $b$  that define how much ale and beer to produce.
- ▶ Choose the variables in such a way that the objective function (profit) is maximized.
- ▶ Make sure that no constraints (due to limited supply) are violated.

$$\begin{aligned}
 \max \quad & 13a + 23b \\
 \text{s.t.} \quad & 5a + 15b \leq 480 \\
 & 4a + 4b \leq 160 \\
 & 35a + 20b \leq 1190 \\
 & a, b \geq 0
 \end{aligned}$$

## Standard Form LPs

### LP in standard form:

- ▶ input: numbers  $a_{ij}, c_j, b_i$
- ▶ output: numbers  $x_j$
- ▶  $n = \#$ decision variables,  $m = \#$ constraints
- ▶ maximize linear objective function subject to linear (in)equalities

$$\begin{aligned}
 \max \quad & \sum_{j=1}^n c_j x_j \\
 \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad 1 \leq i \leq m \\
 & x_j \geq 0 \quad 1 \leq j \leq n
 \end{aligned}$$

$$\begin{aligned}
 \max \quad & c^T x \\
 \text{s.t.} \quad & Ax = b \\
 & x \geq 0
 \end{aligned}$$

## Standard Form LPs

### Original LP

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

### Standard Form

Add a **slack variable** to every constraint.

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

## Standard Form LPs

There are different standard forms:

standard form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

standard maximization form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

standard minimization form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

## Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

► less or equal to equality:

$$a - 3b + 5c \leq 12 \Rightarrow \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\geq 0 \end{aligned}$$

► greater or equal to equality:

$$a - 3b + 5c \geq 12 \Rightarrow \begin{aligned} a - 3b + 5c - s &= 12 \\ s &\geq 0 \end{aligned}$$

► min to max:

$$\min a - 3b + 5c \Rightarrow \max -a + 3b - 5c$$

## Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

► equality to less or equal:

$$a - 3b + 5c = 12 \Rightarrow \begin{aligned} a - 3b + 5c &\leq 12 \\ -a + 3b - 5c &\leq -12 \end{aligned}$$

► equality to greater or equal:

$$a - 3b + 5c = 12 \Rightarrow \begin{aligned} a - 3b + 5c &\geq 12 \\ -a + 3b - 5c &\geq -12 \end{aligned}$$

► unrestricted to nonnegative:

$$x \text{ unrestricted} \Rightarrow x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$$

## Standard Form LPs

### Observations:

- ▶ a linear program does not contain  $x^2$ ,  $\cos(x)$ , etc.
- ▶ transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- ▶ for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

## Fundamental Questions

### Definition 1 (Linear Programming Problem (LP))

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^T x \geq \alpha$ ?

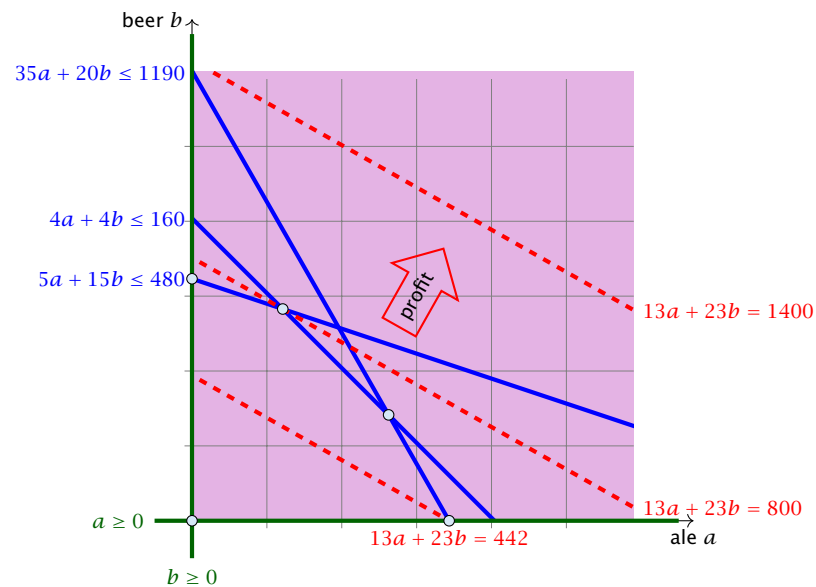
### Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

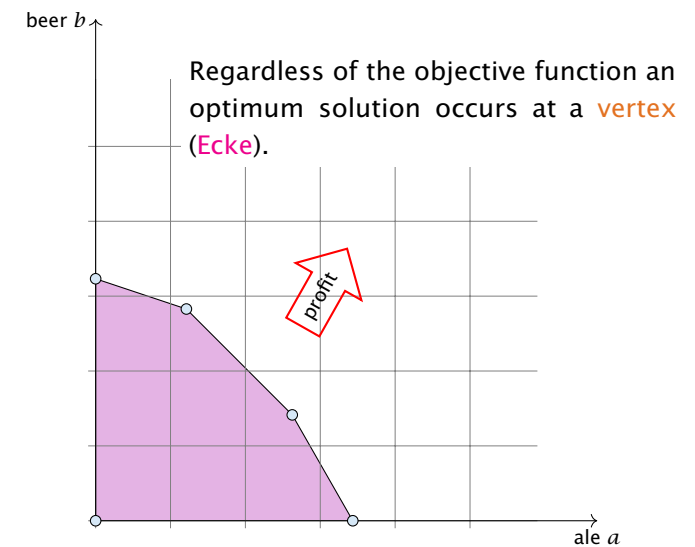
### Input size:

- ▶  $n$  number of variables,  $m$  constraints,  $L$  number of bits to encode the input

## Geometry of Linear Programming



## Geometry of Linear Programming



## Definitions

Let for a Linear Program in standard form

$$P = \{x \mid Ax = b, x \geq 0\}.$$

- ▶  $P$  is called the **feasible region** (Lösungsraum) of the LP.
- ▶ A point  $x \in P$  is called a **feasible point** (gültige Lösung).
- ▶ If  $P \neq \emptyset$  then the LP is called **feasible** (erfüllbar). Otherwise, it is called **infeasible** (unerfüllbar).
- ▶ An LP is **bounded** (beschränkt) if it is feasible and
  - ▶  $c^T x < \infty$  for all  $x \in P$  (for maximization problems)
  - ▶  $c^T x > -\infty$  for all  $x \in P$  (for minimization problems)

## Definition 2

Given vectors/points  $x_1, \dots, x_k \in \mathbb{R}^n$ ,  $\sum \lambda_i x_i$  is called

- ▶ **linear combination** if  $\lambda_i \in \mathbb{R}$ .
- ▶ **affine combination** if  $\lambda_i \in \mathbb{R}$  and  $\sum_i \lambda_i = 1$ .
- ▶ **convex combination** if  $\lambda_i \in \mathbb{R}$  and  $\sum_i \lambda_i = 1$  and  $\lambda_i \geq 0$ .
- ▶ **conic combination** if  $\lambda_i \in \mathbb{R}$  and  $\lambda_i \geq 0$ .

Note that a combination involves only finitely many vectors.

## Definition 3

A set  $X \subseteq \mathbb{R}^n$  is called

- ▶ a **linear subspace** if it is closed under linear combinations.
- ▶ an **affine subspace** if it is closed under affine combinations.
- ▶ **convex** if it is closed under convex combinations.
- ▶ a **convex cone** if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

## Definition 4

Given a set  $X \subseteq \mathbb{R}^n$ .

- ▶ **span**( $X$ ) is the set of all linear combinations of  $X$  (**linear hull**, **span**)
- ▶ **aff**( $X$ ) is the set of all affine combinations of  $X$  (**affine hull**)
- ▶ **conv**( $X$ ) is the set of all convex combinations of  $X$  (**convex hull**)
- ▶ **cone**( $X$ ) is the set of all conic combinations of  $X$  (**conic hull**)

### Definition 5

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is **convex** if for  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$  we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

### Lemma 6

If  $P \subseteq \mathbb{R}^n$ , and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  convex then also

$$Q = \{x \in P \mid f(x) \leq t\}$$

## Dimensions

### Definition 7

The **dimension**  $\dim(A)$  of an affine subspace  $A \subseteq \mathbb{R}^n$  is the dimension of the vector space  $\{x - a \mid x \in A\}$ , where  $a \in A$ .

### Definition 8

The **dimension**  $\dim(X)$  of a convex set  $X \subseteq \mathbb{R}^n$  is the dimension of its affine hull  $\text{aff}(X)$ .

### Definition 9

A set  $H \subseteq \mathbb{R}^n$  is a **hyperplane** if  $H = \{x \mid a^T x = b\}$ , for  $a \neq 0$ .

### Definition 10

A set  $H' \subseteq \mathbb{R}^n$  is a (closed) **halfspace** if  $H' = \{x \mid a^T x \leq b\}$ , for  $a \neq 0$ .

## Definitions

### Definition 11

A **polytop** is a set  $P \subseteq \mathbb{R}^n$  that is the convex hull of a **finite** set of points, i.e.,  $P = \text{conv}(X)$  where  $|X| = c$ .

## Definitions

### Definition 12

A **polyhedron** is a set  $P \subseteq \mathbb{R}^n$  that can be represented as the intersection of **finitely** many half-spaces

$\{H(a_1, b_1), \dots, H(a_m, b_m)\}$ , where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} .$$

### Definition 13

A polyhedron  $P$  is **bounded** if there exists  $B$  s.t.  $\|x\|_2 \leq B$  for all  $x \in P$ .

## Definitions

### Theorem 14

$P$  is a bounded polyhedron iff  $P$  is a polytop.

### Definition 15

Let  $P \subseteq \mathbb{R}^n$ ,  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ . The hyperplane

$$H(a, b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a **supporting hyperplane** of  $P$  if  $\max\{a^T x \mid x \in P\} = b$ .

### Definition 16

Let  $P \subseteq \mathbb{R}^n$ .  $F$  is a **face** of  $P$  if  $F = P$  or  $F = P \cap H$  for some supporting hyperplane  $H$ .

### Definition 17

Let  $P \subseteq \mathbb{R}^n$ .

- ▶ a face  $v$  is a **vertex** of  $P$  if  $\{v\}$  is a face of  $P$ .
- ▶ a face  $e$  is an **edge** of  $P$  if  $e$  is a face and  $\dim(e) = 1$ .
- ▶ a face  $F$  is a **facet** of  $P$  if  $F$  is a face and  $\dim(F) = \dim(P) - 1$ .

### Equivalent definition for vertex:

### Definition 18

Given polyhedron  $P$ . A point  $x \in P$  is a **vertex** if  $\exists c \in \mathbb{R}^n$  such that  $c^T y < c^T x$ , for all  $y \in P$ ,  $y \neq x$ .

### Definition 19

Given polyhedron  $P$ . A point  $x \in P$  is an **extreme point** if  $\nexists a, b \neq x$ ,  $a, b \in P$ , with  $\lambda a + (1 - \lambda)b = x$  for  $\lambda \in [0, 1]$ .

### Lemma 20

A vertex is also an extreme point.

## Observation

The feasible region of an LP is a Polyhedron.

## Convex Sets

### Theorem 21

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

### Proof

- ▶ suppose  $x$  is optimal solution that is not extreme point
- ▶ there exists direction  $d \neq 0$  such that  $x \pm d \in P$
- ▶  $Ad = 0$  because  $A(x \pm d) = b$
- ▶ Wlog. assume  $c^T d \geq 0$  (by taking either  $d$  or  $-d$ )
- ▶ Consider  $x + \lambda d, \lambda > 0$

## Convex Sets

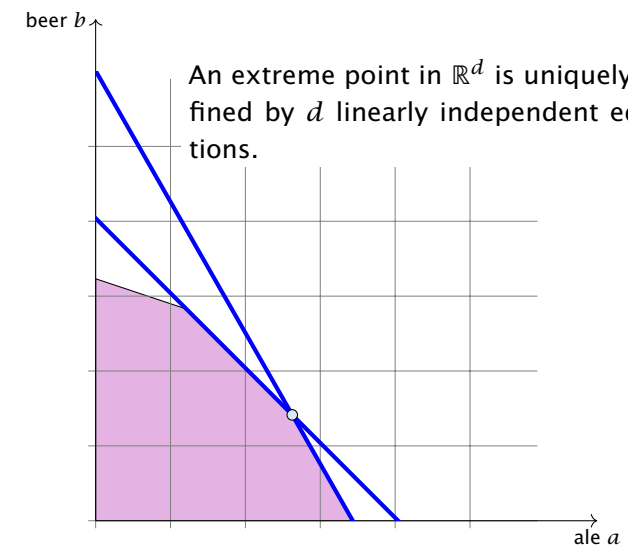
### Case 1. [ $\exists j$ s.t. $d_j < 0$ ]

- ▶ increase  $\lambda$  to  $\lambda'$  until first component of  $x + \lambda d$  hits 0
- ▶  $x + \lambda' d$  is feasible. Since  $A(x + \lambda' d) = b$  and  $x + \lambda' d \geq 0$
- ▶  $x + \lambda' d$  has one more zero-component ( $d_k = 0$  for  $x_k = 0$  as  $x \pm d \in P$ )
- ▶  $c^T x' = c^T(x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

### Case 2. [ $d_j \geq 0$ for all $j$ and $c^T d > 0$ ]

- ▶  $x + \lambda d$  is feasible for all  $\lambda \geq 0$  since  $A(x + \lambda d) = b$  and  $x + \lambda d \geq x \geq 0$
- ▶ as  $\lambda \rightarrow \infty, c^T(x + \lambda d) \rightarrow \infty$  as  $c^T d > 0$

## Algebraic View



### Notation

Suppose  $B \subseteq \{1 \dots n\}$  is a set of column-indices. Define  $A_B$  as the subset of columns of  $A$  indexed by  $B$ .

### Theorem 22

Let  $P = \{x \mid Ax = b, x \geq 0\}$ . For  $x \in P$ , define  $B = \{j \mid x_j > 0\}$ . Then  $x$  is extreme point iff  $A_B$  has linearly independent columns.

### Theorem 22

Let  $P = \{x \mid Ax = b, x \geq 0\}$ . For  $x \in P$ , define  $B = \{j \mid x_j > 0\}$ . Then  $x$  is extreme point iff  $A_B$  has linearly independent columns.

### Proof ( $\Leftarrow$ )

- ▶ assume  $x$  is not extreme point
- ▶ there exists direction  $d$  s.t.  $x \pm d \in P$
- ▶  $Ad = 0$  because  $A(x \pm d) = b$
- ▶ define  $B' = \{j \mid d_j \neq 0\}$
- ▶  $A_{B'}$  has linearly dependent columns as  $Ad = 0$
- ▶  $d_j = 0$  for all  $j$  with  $x_j = 0$  as  $x \pm d \geq 0$
- ▶ Hence,  $B' \subseteq B$ ,  $A_{B'}$  is sub-matrix of  $A_B$

### Theorem 22

Let  $P = \{x \mid Ax = b, x \geq 0\}$ . For  $x \in P$ , define  $B = \{j \mid x_j > 0\}$ . Then  $x$  is extreme point iff  $A_B$  has linearly independent columns.

### Proof ( $\Rightarrow$ )

- ▶ assume  $A_B$  has linearly dependent columns
- ▶ there exists  $d \neq 0$  such that  $A_B d = 0$
- ▶ extend  $d$  to  $\mathbb{R}^n$  by adding 0-components
- ▶ now,  $Ad = 0$  and  $d_j = 0$  whenever  $x_j = 0$
- ▶ for sufficiently small  $\lambda$  we have  $x \pm \lambda d \in P$
- ▶ hence,  $x$  is not extreme point

### Theorem 23

Let  $P = \{x \mid Ax = b, x \geq 0\}$ . For  $x \in P$ , define  $B = \{j \mid x_j > 0\}$ . If  $A_B$  has linearly independent columns then  $x$  is a vertex of  $P$ .

- ▶ define  $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- ▶ then  $c^T x = 0$  and  $c^T y \leq 0$  for  $y \in P$
- ▶ assume  $c^T y = 0$ ; then  $y_j = 0$  for all  $j \notin B$
- ▶  $b = Ay = A_B y_B = Ax = A_B x_B$  gives that  $A_B(x_B - y_B) = 0$ ;
- ▶ this means that  $x_B = y_B$  since  $A_B$  has linearly independent columns
- ▶ we get  $y = x$
- ▶ hence,  $x$  is a vertex of  $P$



## Observation

For an LP we can assume wlog. that the matrix  $A$  has full row-rank. This means  $\text{rank}(A) = m$ .

- ▶ assume that  $\text{rank}(A) < m$
- ▶ assume wlog. that the first row  $A_1$  lies in the span of the other rows  $A_2, \dots, A_m$ ; this means

$$A_1 = \sum_{i=2}^m \lambda_i \cdot A_i, \text{ for suitable } \lambda_i$$

**C1** if now  $b_1 = \sum_{i=2}^m \lambda_i \cdot b_i$  then for all  $x$  with  $A_i x = b_i$  we also have  $A_1 x = b_1$ ; hence the first constraint is superfluous

**C2** if  $b_1 \neq \sum_{i=2}^m \lambda_i \cdot b_i$  then the LP is infeasible, since for all  $x$  that fulfill constraints  $A_2, \dots, A_m$  we have

$$A_1 x = \sum_{i=2}^m \lambda_i \cdot A_i x = \sum_{i=2}^m \lambda_i \cdot b_i \neq b_1$$

From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

## Theorem 24

Given  $P = \{x \mid Ax = b, x \geq 0\}$ .  $x$  is extreme point iff there exists  $B \subseteq \{1, \dots, n\}$  with  $|B| = m$  and

- ▶  $A_B$  is non-singular
- ▶  $x_B = A_B^{-1} b \geq 0$
- ▶  $x_N = 0$

where  $N = \{1, \dots, n\} \setminus B$ .

## Proof

Take  $B = \{j \mid x_j > 0\}$  and augment with linearly independent columns until  $|B| = m$ ; always possible since  $\text{rank}(A) = m$ .

## Basic Feasible Solutions

$x \in \mathbb{R}^n$  is called **basic solution (Basislosung)** if  $Ax = b$  and  $\text{rank}(A_J) = |J|$  where  $J = \{j \mid x_j \neq 0\}$ ;

$x$  is a **basic feasible solution (gultige Basislosung)** if in addition  $x \geq 0$ .

A **basis (Basis)** is an index set  $B \subseteq \{1, \dots, n\}$  with  $\text{rank}(A_B) = m$  and  $|B| = m$ .

$x \in \mathbb{R}^n$  with  $A_B x_B = b$  and  $x_j = 0$  for all  $j \notin B$  is the **basic solution associated to basis B (die zu B assoziierte Basislosung)**

## Basic Feasible Solutions

A BFS fulfills the  $m$  equality constraints.

In addition, at least  $n - m$  of the  $x_i$ 's are zero. The corresponding non-negativity constraint is fulfilled with equality.

### Fact:

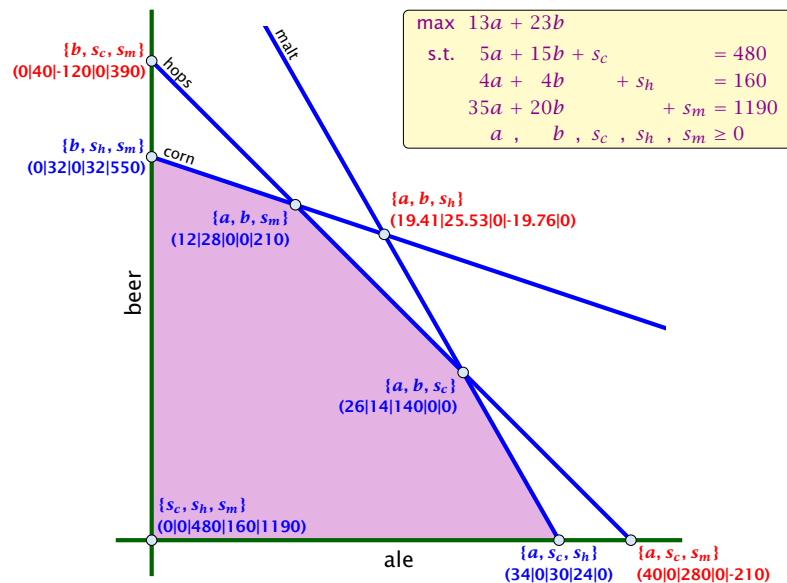
In a BFS at least  $n$  constraints are fulfilled with equality.

## Basic Feasible Solutions

### Definition 25

For a general LP ( $\max\{c^T x \mid Ax \leq b\}$ ) with  $n$  variables a point  $x$  is a **basic feasible solution** if  $x$  is feasible and there exist  $n$  (linearly independent) constraints that are tight.

## Algebraic View



## Fundamental Questions

### Linear Programming Problem (LP)

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^T x \geq \alpha$ ?

### Questions:

- ▶ Is LP in NP? yes!
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

### Proof:

- ▶ Given a basis  $B$  we can compute the associated basis solution by calculating  $A_B^{-1}b$  in polynomial time; then we can also compute the profit.

### Observation

We can compute an optimal solution to a linear program in time  $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n, m)\right)$ .

- ▶ there are only  $\binom{n}{m}$  different bases.
- ▶ compute the profit of each of them and take the maximum

What happens if LP is unbounded?