# 4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

**Simplex Algorithm** [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

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**Simplex Algorithm** [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

# 4 Simplex Algorithm

$$\begin{array}{llllllll}
\text{max} & 13a + 23b \\
\text{s.t.} & 5a + 15b + s_c & & & = 480 \\
& 4a + \ \ 4b & + s_h & & = 160 \\
& 35a + 20b & & + s_m & = 1190 \\
& a \ , \quad b \ , s_c & , s_h & , s_m & \geq 0
\end{array}$$

$$\begin{array}{llllll}
\text{max } Z \\
& 13a + 23b & & & - Z = 0 \\
& 5a + 15b + s_c & & & = 480 \\
& 4a + \ \ 4b & + s_h & & = 160 \\
& 35a + 20b & & + s_m & = 1190 \\
& a \ , \quad b \ , s_c & , s_h & , s_m & \geq 0
\end{array}$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

# 4 Simplex Algorithm

$$
\begin{array}{llcl}
\text{max} & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 4a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a, b, s_c, s_h, s_m & \geq & 0
\end{array}
$$

$$
\begin{array}{lcl}
\text{max } Z & & \\
13a + 23b - Z & = & 0 \\
5a + 15b + s_c & = & 480 \\
4a + 4b + s_h & = & 160 \\
35a + 20b + s_m & = & 1190 \\
a, b, s_c, s_h, s_m & \geq & 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

# Pivoting Step

max $Z$

$$
\begin{array}{rcrcccccccl}
13a & + & 23b & & & & & & & -\,Z & = 0 \\
5a & + & 15b & + & s_c & & & & & & = 480 \\
4a & + & 4b & & & + & s_h & & & & = 160 \\
35a & + & 20b & & & & & + & s_m & & = 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0
\end{array}
$$

basis = $\{s_c, s_h, s_m\}$

$a = b = 0$

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$s_h = 160$

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# Pivoting Step

max $Z$

$$
\begin{array}{rcl}
13a + 23b & - Z & = 0 \\
5a + 15b + s_c & & = 480 \\
4a + 4b + s_h & & = 160 \\
35a + 20b + s_m & & = 1190 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & & \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

# Pivoting Step

$$
\begin{array}{rcrcccccccl}
\multicolumn{11}{l}{\max Z} \\
13a & + & 23b & & & & & & & -Z & = 0 \\
5a & + & 15b & + & s_c & & & & & & = 480 \\
4a & + & 4b & & & + & s_h & & & & = 160 \\
35a & + & 20b & & & & & + & s_m & & = 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
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# Pivoting Step

max $Z$

$$\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
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# Pivoting Step

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\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{aligned}
\text{basis} &= \{s_c, s_h, s_m\} \\
a &= b = 0 \\
Z &= 0 \\
s_c &= 480 \\
s_h &= 160 \\
s_m &= 1190
\end{aligned}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

# Pivoting Step

$$
\begin{array}{l}
\max Z \\
\begin{aligned}
13a + 23b \qquad\qquad\qquad\qquad - Z &= 0 \\
5a + 15b + s_c \qquad\qquad\qquad\qquad &= 480 \\
4a + 4b \qquad + s_h \qquad\qquad &= 160 \\
35a + 20b \qquad\qquad + s_m \qquad &= 1190 \\
a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m \qquad &\geq 0
\end{aligned}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- choose variable to bring into the basis
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max $Z$

$$
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

max $Z$

$$
\begin{array}{rcrcccccl}
13a & + & 23\boldsymbol{b} & & & & & -Z & = 0 \\
5a & + & 15\boldsymbol{b} & + s_c & & & & & = 480 \\
4a & + & 4\boldsymbol{b} & & + s_h & & & & = 160 \\
35a & + & 20\boldsymbol{b} & & & + s_m & & & = 1190 \\
a & , & \boldsymbol{b} & , s_c & , s_h & , s_m & & & \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- Choose variable with coefficient $> 0$ as entering variable.

max $Z$

$$\begin{array}{rcrcccl}
13a & + & 23b & & & & -Z = 0 \\
5a & + & 15b & + s_c & & & = 480 \\
4a & + & 4b & & + s_h & & = 160 \\
35a & + & 20b & & & + s_m & = 1190 \\
a & , & b & , s_c & , s_h & , s_m & \geq 0
\end{array}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- Choose variable with coefficient $> 0$ as entering variable.
- If we keep $a = 0$ and increase $b$ from $0$ to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value $Z$ will strictly increase.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcccl}
13a + 23\boldsymbol{b} & & & & & -Z & = 0 \\
5a + 15\boldsymbol{b} & + s_c & & & & & = 480 \\
4a + \phantom{1}4\boldsymbol{b} & & + s_h & & & & = 160 \\
35a + 20\boldsymbol{b} & & & + s_m & & & = 1190 \\
a \;,\; \boldsymbol{b} & ,\; s_c & ,\; s_h & ,\; s_m & & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient $> 0$ as entering variable.
- If we keep $a = 0$ and increase $b$ from $0$ to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value $Z$ will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcccccl} 13a & + & 23b & & & & & -Z & = 0 \\ 5a & + & 15\mathbf{b} & + s_c & & & & & = 480 \\ 4a & + & 4b & & + s_h & & & & = 160 \\ 35a & + & 20b & & & + s_m & & & = 1190 \\ a & , & b & , s_c & , s_h & , s_m & & & \geq 0 \end{array} \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- Choose variable with coefficient $> 0$ as entering variable.
- If we keep $a = 0$ and increase $b$ from $0$ to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value $Z$ will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes $0$, and no variable goes negative.

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\mathbf{b} + \mathbf{s_c} \qquad\qquad = 480 \\
\quad 4a + 4b \qquad + s_h \qquad\quad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m \quad = 1190 \\
\quad a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient $> 0$ as entering variable.
- If we keep $a = 0$ and increase $b$ from $0$ to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value $Z$ will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes $0$, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

max $Z$

$$
\begin{array}{rrrrrrr}
13a + 23b & & & & -\ Z & = 0 \\
5a + 15b + s_c & & & & & = 480 \\
4a + \ \ 4b & + s_h & & & & = 160 \\
35a + 20b & & + s_m & & & = 1190 \\
a\ ,\ \ b\ ,\ s_c & ,\ s_h & ,\ s_m & & & \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

$$
\begin{array}{l}
\max Z \\
\begin{aligned}
13a + 23\boldsymbol{b} \quad\quad\quad\quad\quad\quad\quad - Z &= 0 \\
5a + 15\boldsymbol{b} + s_c \quad\quad\quad\quad\quad\quad &= 480 \\
4a + 4\boldsymbol{b} \quad\quad + s_h \quad\quad\quad &= 160 \\
35a + 20\boldsymbol{b} \quad\quad\quad\quad + s_m \quad &= 1190 \\
a,\ \boldsymbol{b},\ s_c,\ s_h,\ s_m &\geq 0
\end{aligned}
\end{array}
$$

$$
\begin{aligned}
&\text{basis} = \{s_c, s_h, s_m\} \\
&a = b = 0 \\
&Z = 0 \\
&s_c = 480 \\
&s_h = 160 \\
&s_m = 1190
\end{aligned}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23\boldsymbol{b} & & & & & & & - & Z & = & 0 \\
5a & + & 15\boldsymbol{b} & + & s_c & & & & & & & = & 480 \\
4a & + & 4\boldsymbol{b} & & & + & s_h & & & & & = & 160 \\
35a & + & 20\boldsymbol{b} & & & & & + & s_m & & & = & 1190 \\
a & , & \boldsymbol{b} & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\text{basis} & = \{s_c, s_h, s_m\} \\
a & = b = 0 \\
Z & = 0 \\
s_c & = 480 \\
s_h & = 160 \\
s_m & = 1190
\end{array}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\text{basis} & = \{b, s_h, s_m\} \\
a & = s_c = 0 \\
Z & = 736 \\
b & = 32 \\
s_h & = 32 \\
s_m & = 550
\end{array}
$$

max $Z$

$$
\begin{array}{rcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - Z = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

max $Z$

$$
\begin{aligned}
\tfrac{16}{3}\boldsymbol{a} \qquad - \tfrac{23}{15}s_c \qquad\qquad\qquad - Z &= -736 \\
\tfrac{1}{3}\boldsymbol{a} + b + \tfrac{1}{15}s_c \qquad\qquad\qquad &= 32 \\
\tfrac{8}{3}\boldsymbol{a} \qquad - \tfrac{4}{15}s_c + s_h \qquad\qquad &= 32 \\
\tfrac{85}{3}\boldsymbol{a} \qquad - \tfrac{4}{3}s_c \qquad + s_m \qquad &= 550 \\
\boldsymbol{a}\,,\ b\,,\ s_c\,,\ s_h\,,\ s_m &\geq 0
\end{aligned}
$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

Choose variable $a$ to bring into basis.

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
\boldsymbol{a} & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

basis = $\{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable $a$ to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

max $Z$

$$
\begin{array}{rrrrrrcl}
\frac{16}{3}a & & -\ \frac{23}{15}s_c & & & -\ Z & = & -736 \\
\frac{1}{3}a & +\ b & +\ \frac{1}{15}s_c & & & & = & 32 \\
\frac{8}{3}a & & -\ \frac{4}{15}s_c & +\ s_h & & & = & 32 \\
\frac{85}{3}a & & -\ \frac{4}{3}s_c & & +\ s_m & & = & 550 \\
a\ , & b\ , & s_c\ , & s_h\ , & s_m & & \geq & 0
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable $a$ to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \max Z \\ \frac{16}{3}a - \frac{23}{15}s_c - Z = -736 \\ \frac{1}{3}a + b + \frac{1}{15}s_c = 32 \\ \frac{8}{3}\mathbf{a} - \frac{4}{15}s_c + s_h = 32 \\ \frac{85}{3}a - \frac{4}{3}s_c + s_m = 550 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{b, s_h, s_m\} \\ a = s_c = 0 \\ Z = 736 \\ b = 32 \\ s_h = 32 \\ s_m = 550 \end{array}$$

Choose variable $a$ to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \max Z \\ - s_c - 2s_h - Z = -800 \\ b + \frac{1}{10}s_c - \frac{1}{8}s_h = 28 \\ a - \frac{1}{10}s_c + \frac{3}{8}s_h = 12 \\ \frac{3}{2}s_c - \frac{85}{8}s_h + s_m = 210 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{a, b, s_m\} \\ s_c = s_h = 0 \\ Z = 800 \\ b = 28 \\ a = 12 \\ s_m = 210 \end{array}$$

# 4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

**Solution is optimal:**

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

# 4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

**Solution is optimal:**

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**Solution is optimal:**

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most $800$
- the current solution has value $800$

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Pivoting stops when all coefficients in the objective function are non-positive.

**Solution is optimal:**

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most $800$
- the current solution has value $800$

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Pivoting stops when all coefficients in the objective function are non-positive.

**Solution is optimal:**

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most $800$
- the current solution has value $800$

# 4 Simplex Algorithm

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**Solution is optimal:**

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most $800$
- the current solution has value $800$

# Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis $B$ is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

# Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis $B$ is

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# Matrix View

Let our linear program be

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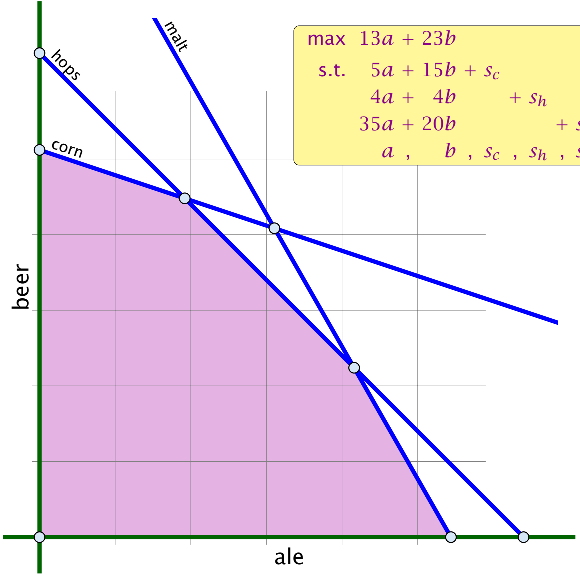


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\begin{array}{llllllll}
\text{max} & 13a + 23b & & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & & = 480 \\
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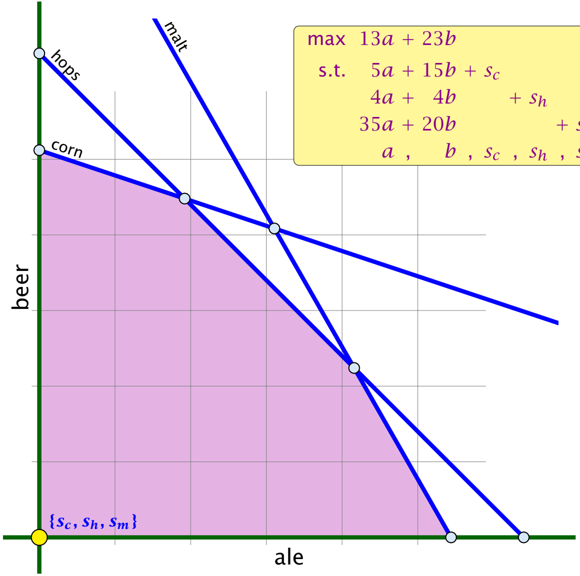


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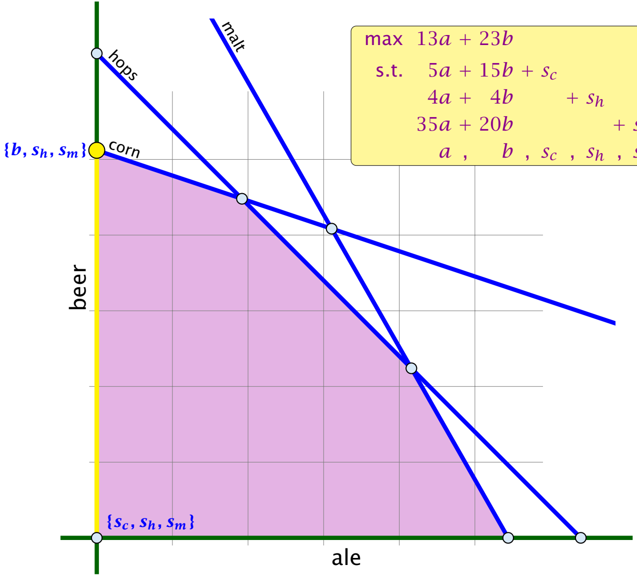


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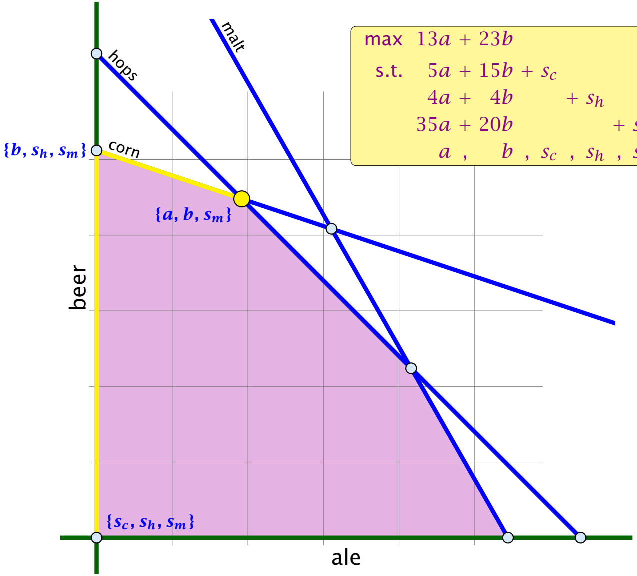


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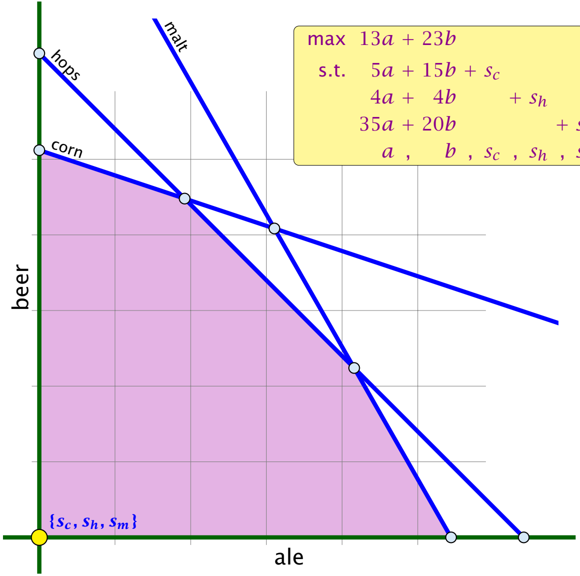


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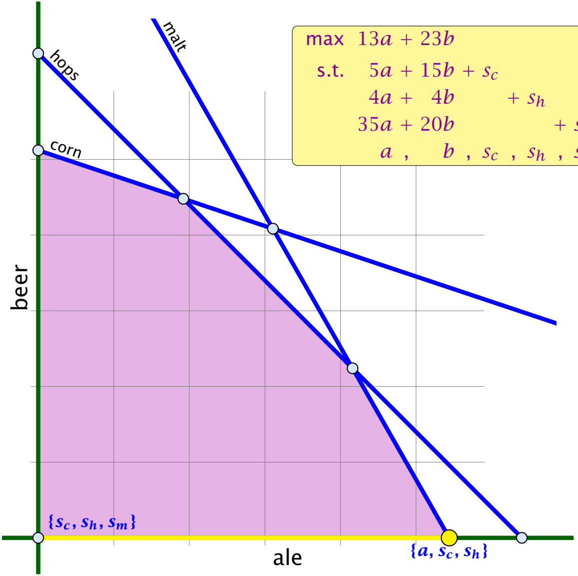


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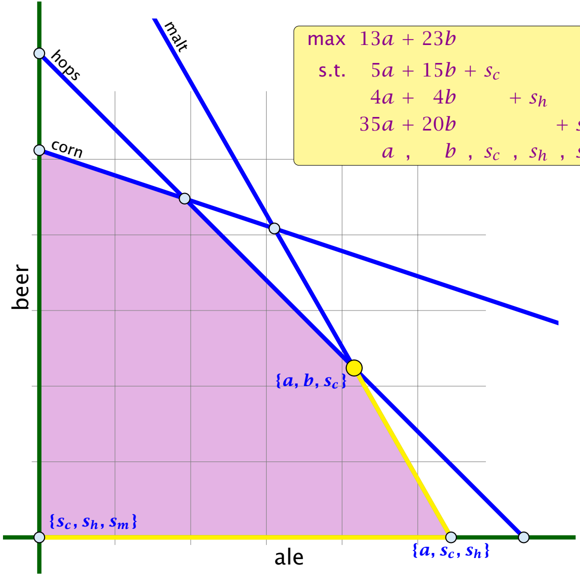

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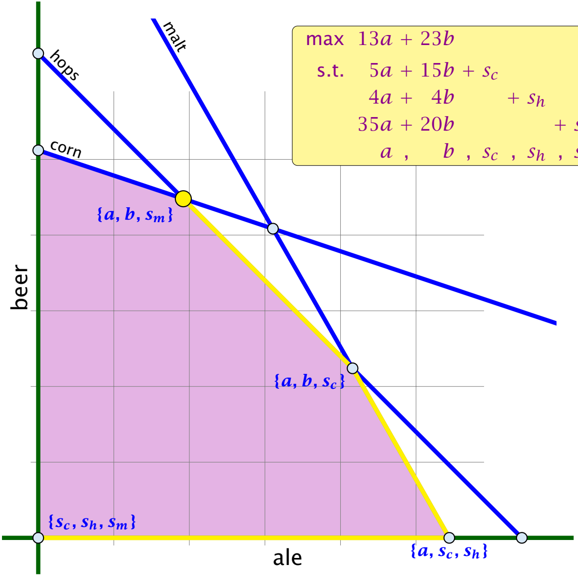


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# Algebraic Definition of Pivoting

- Given basis $B$ with BFS $x^*$.
- Choose index $j \notin B$ in order to increase $x_j^*$ from $0$ to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from $x^*$ to $x^* + \theta \cdot d$.

**Requirements for $d$:**

- $d_j = 1$ (normalization)
- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

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# Algebraic Definition of Pivoting

**Definition 26 ($j$-th basis direction)**

Let $B$ be a basis, and let $j \notin B$. The vector $d$ with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the $j$-th basis direction for $B$.

Going from $x^*$ to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

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**Definition 27 (Reduced Cost)**

For a basis $B$ the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the reduced cost for variable $x_j$.

Note that this is defined for every $j$. If $j \in B$ then the above term is $0$.

# Algebraic Definition of Pivoting

Let our linear program be

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\begin{array}{rcrcl}
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# 4 Simplex Algorithm

**Questions:**

- What happens if the min ratio test fails to give us a value by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
- Is there always a basis $B$ such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \le 0 \ ?$$

  Then we can terminate because we know that the solution is optimal.
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# Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to $\theta$ the leaving variable becomes $0$ and all other variables stay non-negative.

For this, one computes $b_i/A_{ie}$ for all constraints $i$ and calculates the minimum positive value.

What does it mean that the ratio $b_i/A_{ie}$ (and hence $A_{ie}$) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase $b$. Hence, there is no danger of this basic variable becoming negative

What happens if **all** $b_i/A_{ie}$ are negative? Then we do not have a leaving variable. Then the LP is unbounded!

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The objective function may not increase!

Because a variable $x_\ell$ with $\ell \in B$ is already $0$.

The set of inequalities is degenerate (also the basis is degenerate).

**Definition 28 (Degeneracy)**

A BFS $x^*$ is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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The objective function may not increase!

Because a variable $x_\ell$ with $\ell \in B$ is already $0$.

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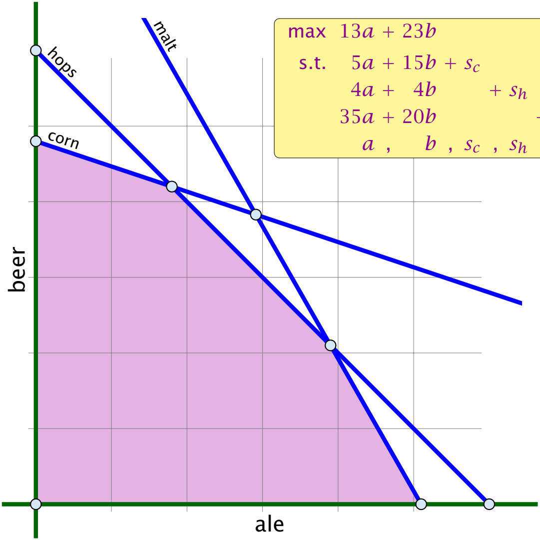


$$
\begin{array}{lrcrcrcrcrcl}
\text{max} & 13a & + & 23b & & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
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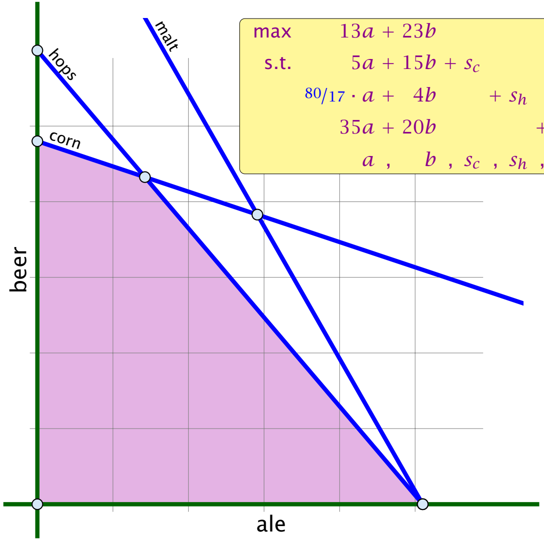


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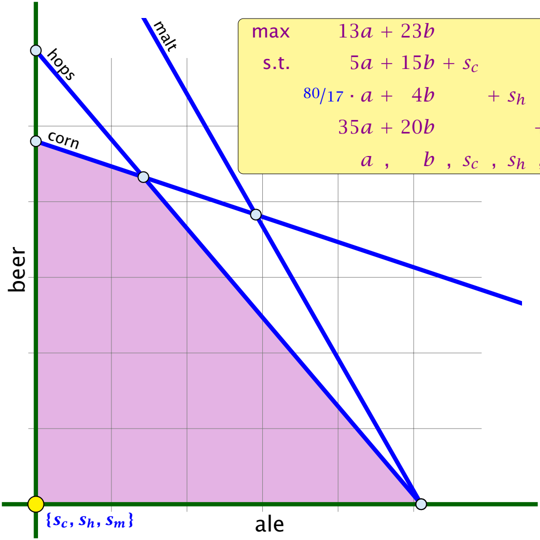


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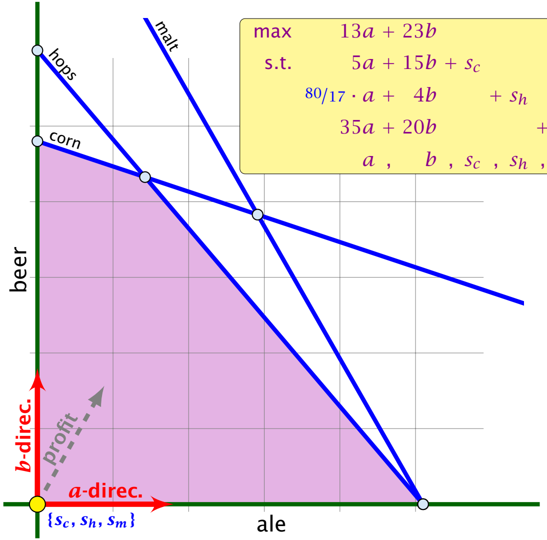

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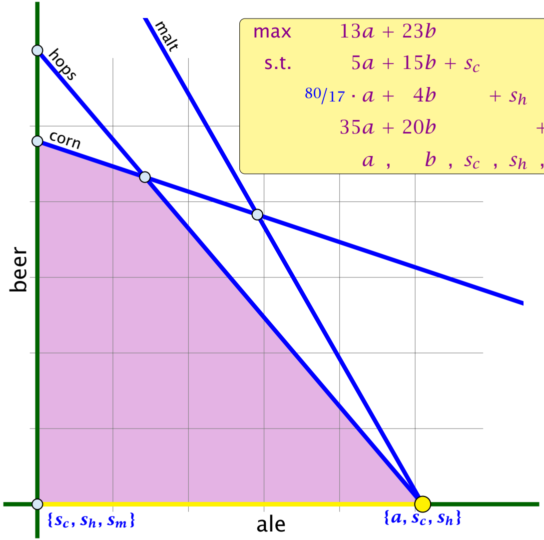

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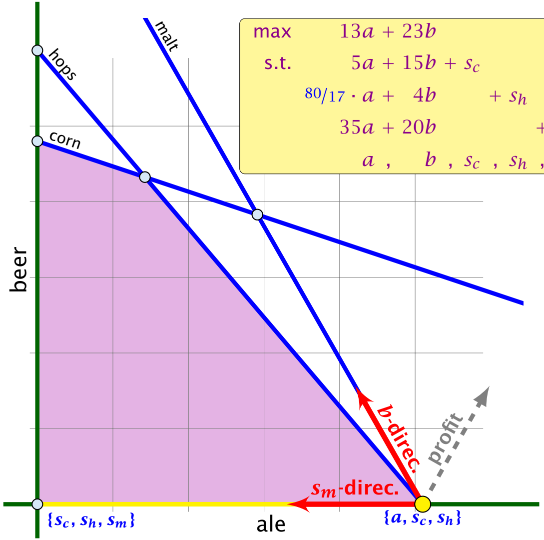

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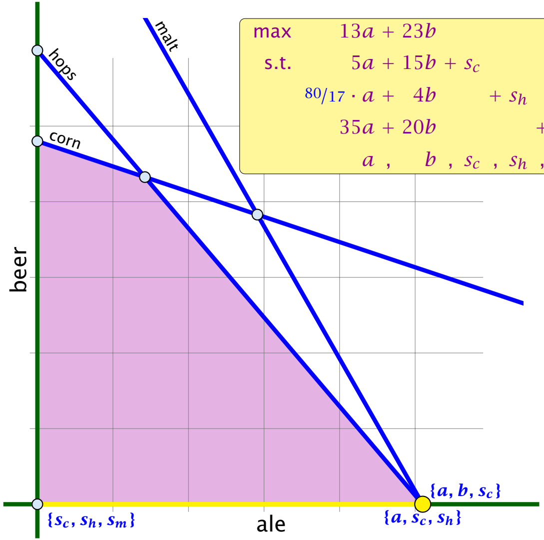

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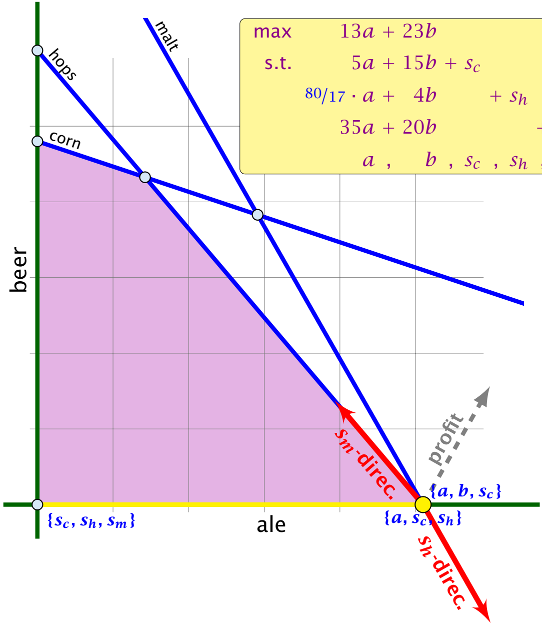

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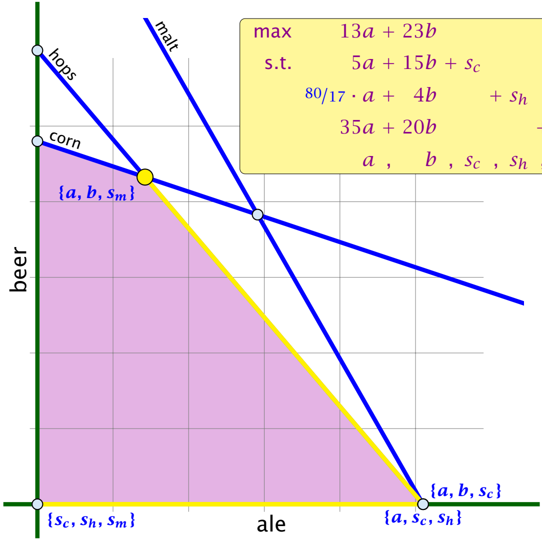

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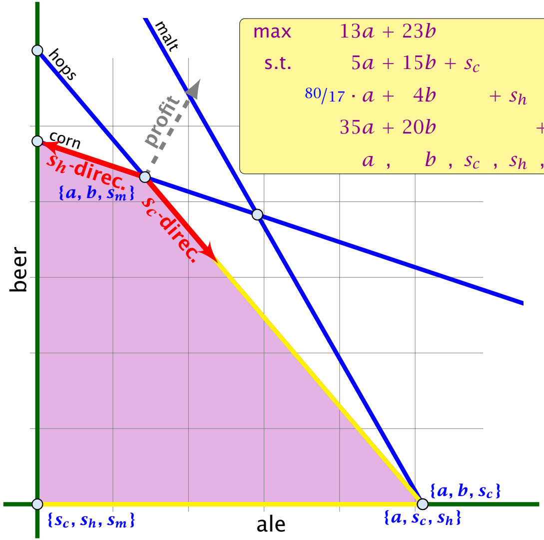


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# Summary: How to choose pivot-elements

- We can choose a column $e$ as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for $x_e$).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \ldots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable $\ell$ such that $b_\ell / A_{\ell e}$ is minimal among all variables $i$ with $A_{ie} > 0$.
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
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# Termination

**What do we have so far?**

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

**How do we come up with an initial solution?**

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b} \geq \mathbf{0}$.
- The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where $s$ denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

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# Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by $-1$.
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b$, $x \geq 0$, $v \geq 0$ using Simplex. $x = 0$, $v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

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# Optimality

**Lemma 29**
*Let $B$ be a basis and $x^*$ a BFS corresponding to basis $B$. $\tilde{c} \leq 0$ implies that $x^*$ is an optimum solution to the LP.*