

Complexity

LP Feasibility Problem (LP feasibility A)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$?

LP Feasibility Problem (LP feasibility B)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Find $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$!

LP Optimization A

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$. What is the maximum value of $c^T x$ for a feasible point $x \in \mathbb{R}^n$?

LP Optimization B

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$. Return feasible point $x \in \mathbb{R}^n$ with maximum value of $c^T x$?

The Bit Model

Input size

- ▶ The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- ▶ Let for an $m \times n$ matrix M , $L(M)$ denote the number of bits required to encode all the numbers in M .

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) \rceil + 1$$

- ▶ In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- ▶ Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

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- ▶ Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

- ▶ In the following we sometimes refer to $L := \langle A \rangle + \langle b \rangle$ as the input size (even though the real input size is something in $\Theta(\langle A \rangle + \langle b \rangle)$).
- ▶ Sometimes we may also refer to $L := \langle A \rangle + \langle b \rangle + n \log_2 n$ as the input size. Note that $n \log_2 n = \Theta(\langle A \rangle + \langle b \rangle)$.
- ▶ In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in L).

Suppose that $\bar{A}x = b; x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

$$x_B = \bar{A}_B^{-1}b$$

and all other entries in x are 0.

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Size of a Basic Feasible Solution

- ▶ A : original input matrix
- ▶ \bar{A} : transformation of A into standard form
- ▶ \bar{A}_B : submatrix of \bar{A} corresponding to basis B

Lemma 47

Let $\bar{A}_B \in \mathbb{Z}^{m \times m}$ and $b \in \mathbb{Z}^m$. Define $L = \langle A \rangle + \langle b \rangle + n \log_2 n$.

Then a solution to $\bar{A}_B x_B = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \leq 2^L$ and $|D| \leq 2^L$.

Proof:

Cramer's rule says that we can compute x_j as

$$x_j = \frac{\det(\bar{A}_B^j)}{\det(\bar{A}_B)}$$

where \bar{A}_B^j is the matrix obtained from \bar{A}_B by replacing the j -th column by the vector b .

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Let $X = \bar{A}_B$. Then

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Analogously for $\det(A_B^j)$.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^T x \mid Ax \leq b; x \geq 0\}$ do a **binary search** for the optimum solution

(Add constraint $c^T x \geq M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2 \left(\frac{2n2^{2L'}}{1/2^{L'}} \right) = \mathcal{O}(L') ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

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How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

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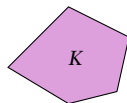
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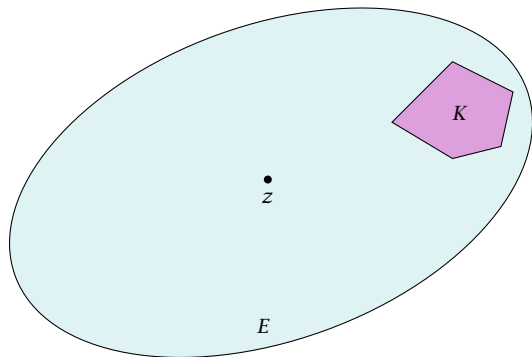
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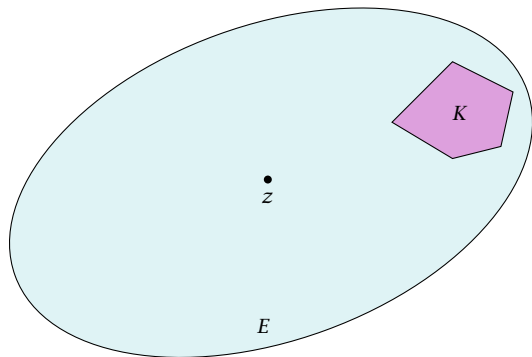
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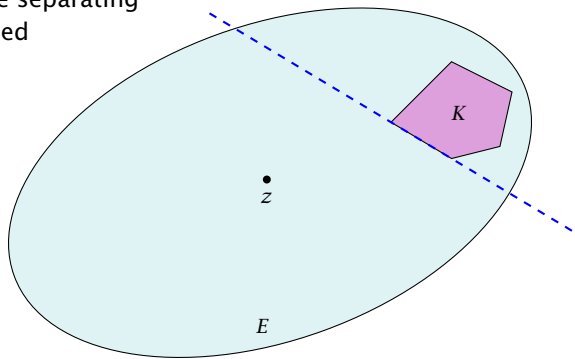
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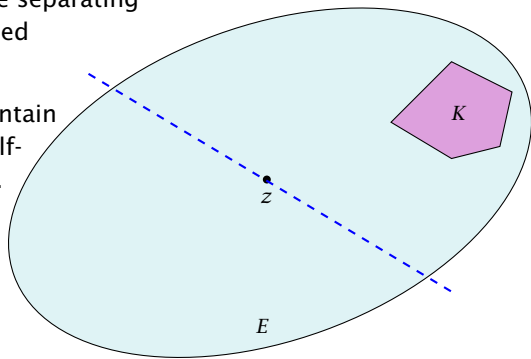
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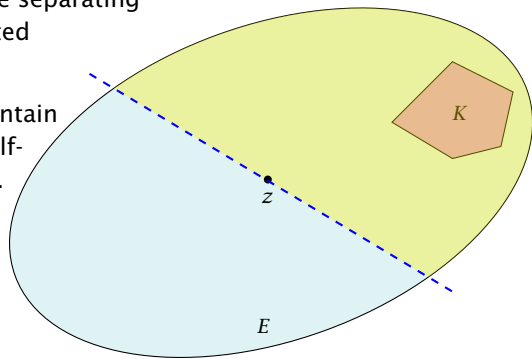
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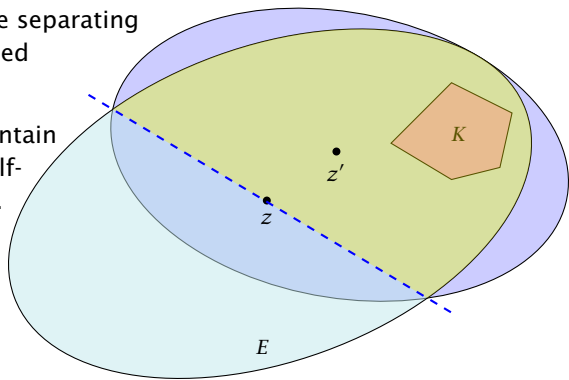
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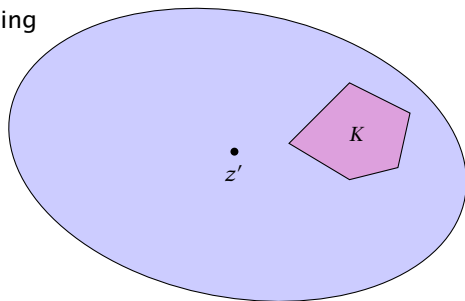
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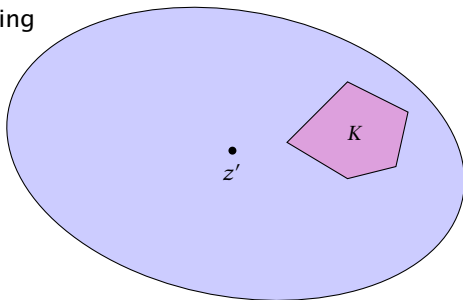
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- ▶ REPEAT



Issues/Questions:

- ▶ How do you choose the first Ellipsoid? What is its volume?
- ▶ How do you measure progress? By how much does the volume decrease in each iteration?
- ▶ When can you stop? What is the minimum volume of a non-empty polytop?

Definition 48

A mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an **affine transformation**.

Definition 49

A ball in \mathbb{R}^n with center c and radius r is given by

$$\begin{aligned} B(c, r) &= \{x \mid (x - c)^T (x - c) \leq r^2\} \\ &= \{x \mid \sum_i (x - c)_i^2 / r^2 \leq 1\} \end{aligned}$$

$B(0, 1)$ is called the **unit ball**.

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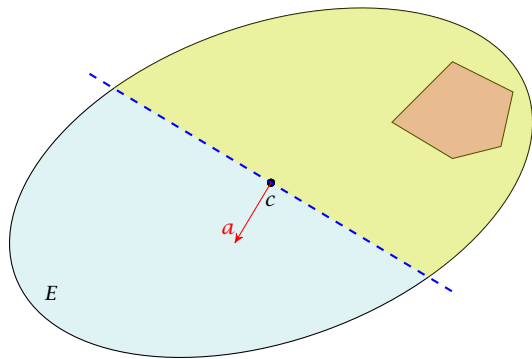
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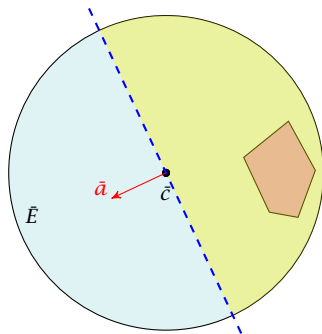
where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid



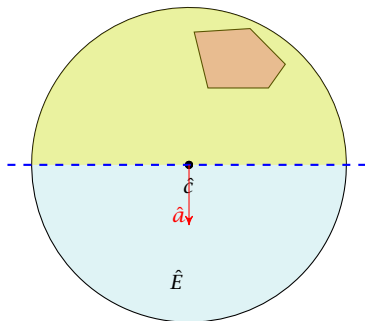
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- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



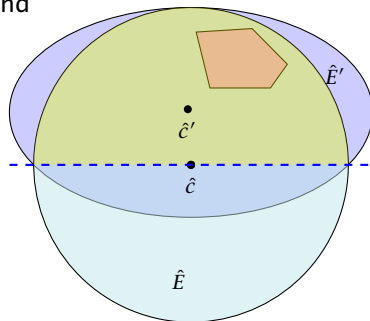
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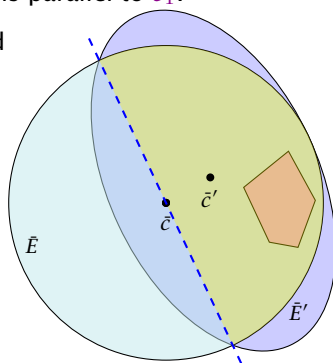
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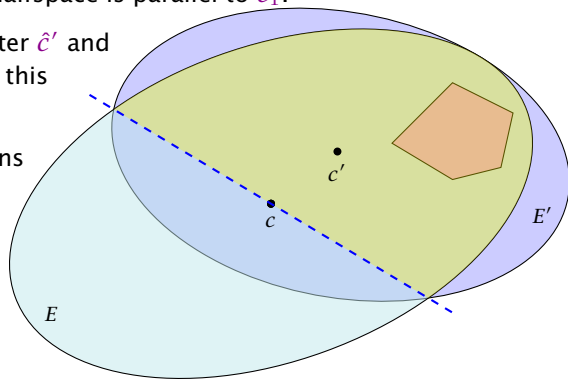
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- ▶ Compute the new center \tilde{c}' and the new matrix \tilde{Q}' for this simplified setting.
- ▶ Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E .

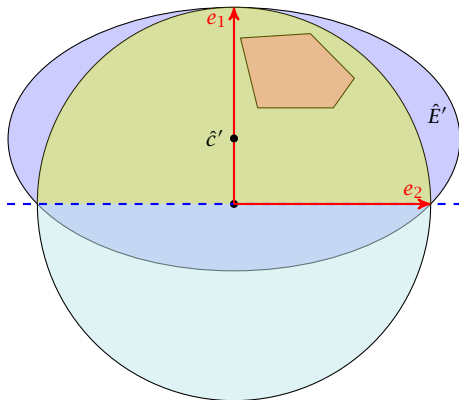


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- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
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- ▶ Compute the new center \hat{c}' and the new matrix \hat{Q}' for this simplified setting.
- ▶ Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E .

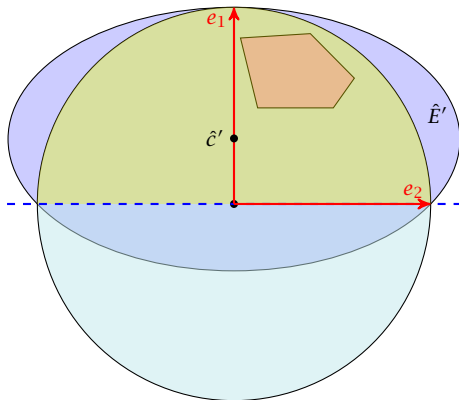


The Easy Case



- ▶ The new center lies on axis x_1 . Hence, $\hat{c}' = te_1$ for $t > 0$.
- ▶ The vectors e_1, e_2, \dots have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$.

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The Easy Case

- ▶ To obtain the matrix \hat{Q}'^{-1} for our ellipsoid \hat{E}' note that \hat{E}' is **axis-parallel**.
- ▶ Let a denote the radius along the x_1 -axis and let b denote the (common) radius for the other axes.
- ▶ The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

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The Easy Case

- ▶ As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix \hat{Q}'^{-1} is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- ▶ $(e_1 - \hat{c}')^T \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $(1-t)^2 = a^2$.

The Easy Case

- ▶ For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ looks like (here $i = 2$)

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2}$$

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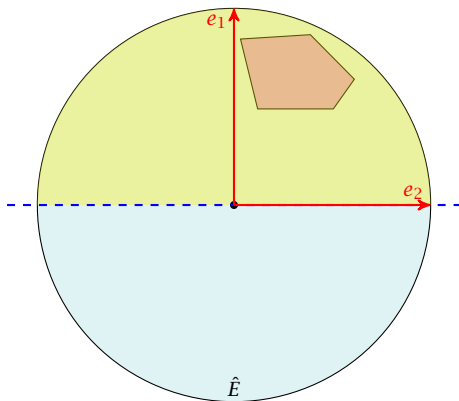
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1 - t}{\sqrt{1 - 2t}}$$

The Easy Case

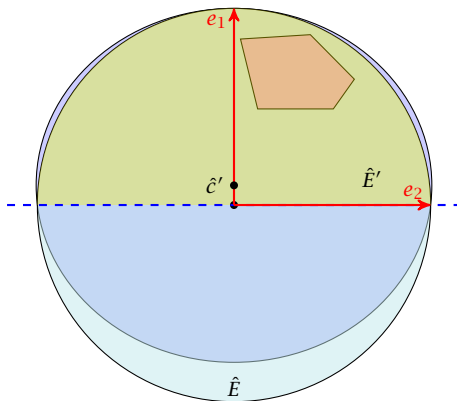
We still have many choices for t :



Choose t such that the volume of \hat{E}' is minimal!!!

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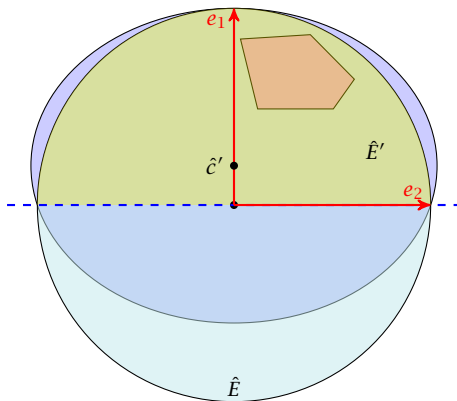
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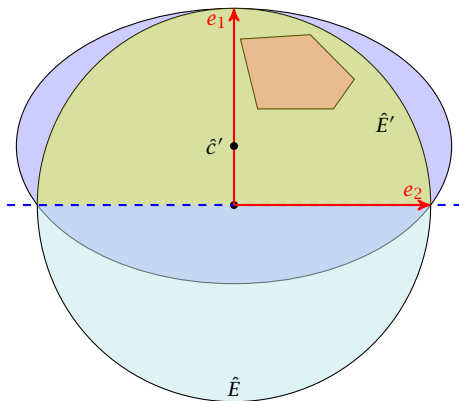
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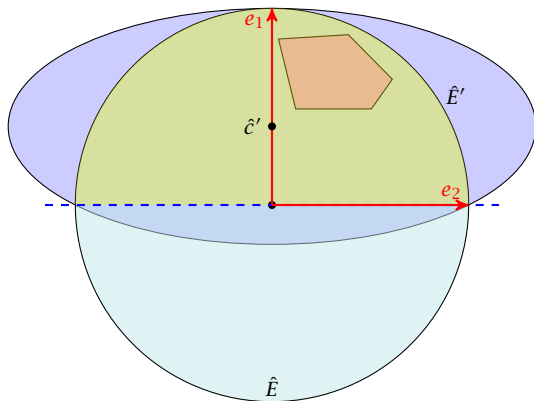
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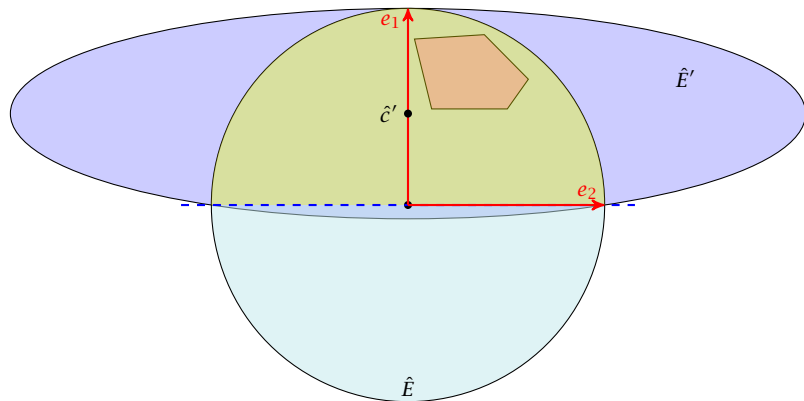
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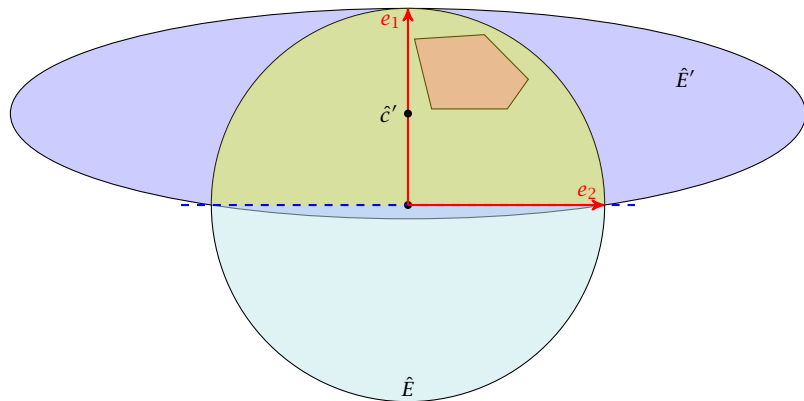
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We want to choose t such that the volume of \hat{E}' is minimal.

Lemma 51

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\text{vol}(L(K)) = |\det(L)| \cdot \text{vol}(K) .$$

The Easy Case

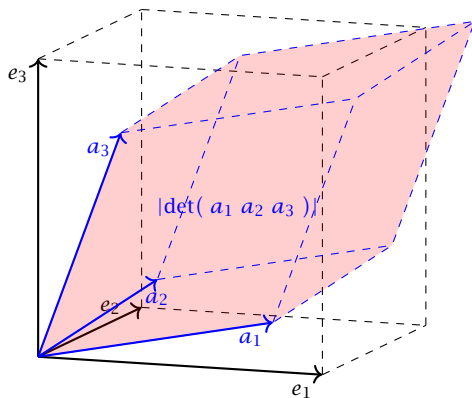
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Lemma 51

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n-dimensional volume



The Easy Case

- ▶ We want to choose t such that the volume of \hat{E}' is minimal.

$$\text{vol}(\hat{E}') = \text{vol}(B(0, 1)) \cdot |\det(\hat{L}')| ,$$

- ▶ Recall that

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

- ▶ Note that a and b in the above equations depend on t , by the previous equations.

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$$\begin{aligned}\text{vol}(\hat{E}') &= \text{vol}(B(0, 1)) \cdot |\det(\hat{L}')| \\ &= \text{vol}(B(0, 1)) \cdot ab^{n-1}\end{aligned}$$

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We use the shortcut $\Phi := \text{vol}(B(0, 1))$.

The Easy Case

$$\frac{d \operatorname{vol}(\hat{E}')}{dt}$$

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$N = \text{denominator}$

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$$\begin{aligned}\frac{d \operatorname{vol}(\hat{E}')}{dt} &= \frac{d}{dt} \left(\Phi \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}} \right) \\ &= \frac{\Phi}{N^2} \cdot \left(\underbrace{(-1) \cdot n(1-t)^{n-1}}_{\text{derivative of numerator}} \right)\end{aligned}$$

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outer derivative

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inner derivative

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numerator

The Easy Case

$$\begin{aligned}\frac{d \operatorname{vol}(\hat{E}')}{d t} &= \frac{d}{d t} \left(\Phi \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}} \right) \\ &= \frac{\Phi}{N^2} \cdot \left((-1) \cdot n(1-t)^{n-1} \cdot (\sqrt{1-2t})^{n-1} \right. \\ &\quad \left. - (n-1)(\sqrt{1-2t})^{n-2} \cdot \frac{1}{2\sqrt{1-2t}} \cdot (-2) \cdot (1-t)^n \right) \\ &= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1}\end{aligned}$$

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 &= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \\
 &\quad \cdot \left((n-1)(1-t) - n(1-2t) \right)
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 &\quad \left. - (n-1)(\sqrt{1-2t})^{n-2} \cdot \frac{1}{2\sqrt{1-2t}} \cdot (-2) \cdot (1-t)^n \right) \\
 &= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \\
 &\quad \cdot \left((n-1)(1-t) - n(1-2t) \right) \\
 &= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \left((n+1)t - 1 \right)
 \end{aligned}$$

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- ▶ We obtain the minimum for $t = \frac{1}{n+1}$.
- ▶ For this value we obtain

a

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To see the equation for b , observe that

$$b^2$$

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The Easy Case

Let $\gamma_n = \frac{\text{vol}(\hat{E}')} {\text{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

$$\gamma_n^2$$

The Easy Case

Let $\gamma_n = \frac{\text{vol}(\hat{E}')} {\text{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

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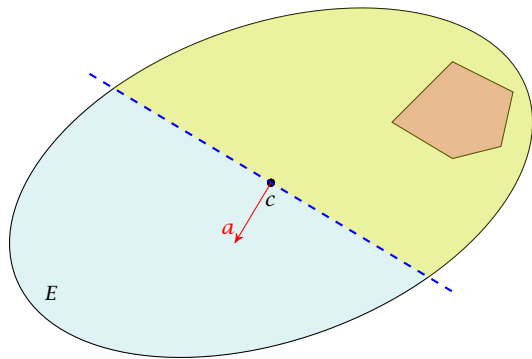
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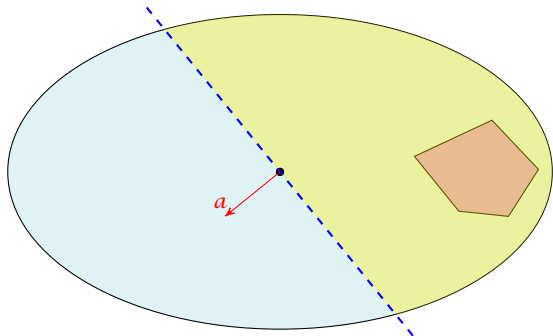
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How to Compute the New Ellipsoid



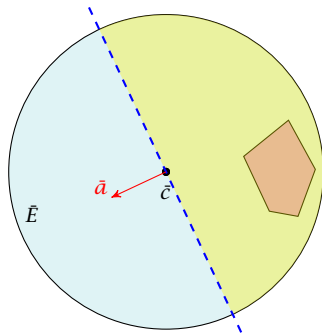
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- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to translate/distort the ellipsoid (back) into the unit ball.



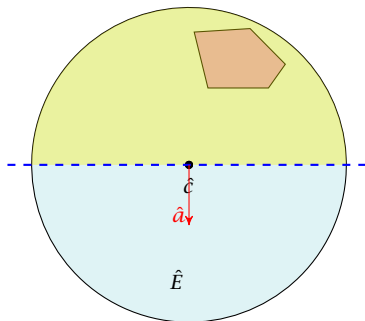
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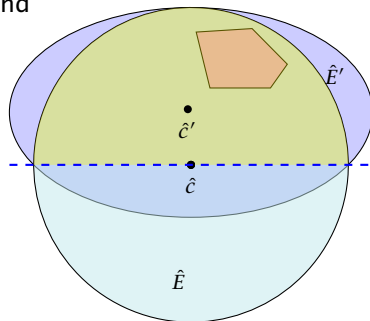
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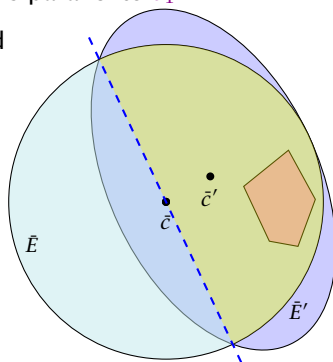
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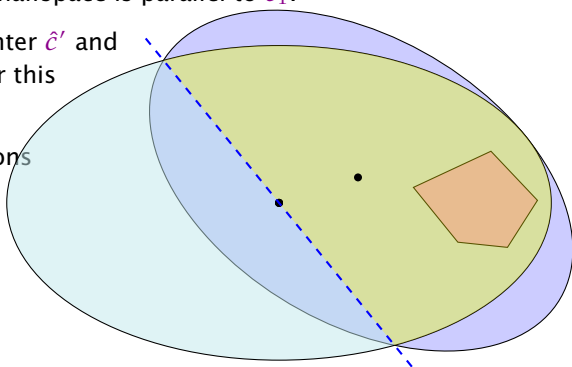
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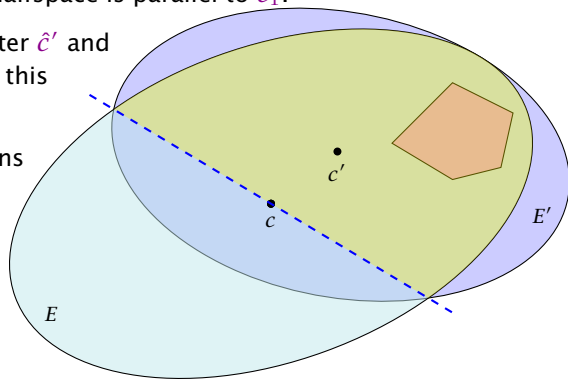
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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

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This means $\bar{a} = L^T a$.

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After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1 -direction. Hence,

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For computing the matrix Q' of the new ellipsoid we assume in the following that \hat{E}' , \bar{E}' and E' refer to the ellipsoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1} \left(I - \frac{2}{n + 1} e_1 e_1^T \right)$$

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9 The Ellipsoid Algorithm

\bar{E}'

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Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

- 1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
- 2: **output:** point $x \in K$ or “ K is empty”
- 3: $Q \leftarrow ???$
- 4: **repeat**
- 5: **if** $c \in K$ **then return** c
- 6: **else**
- 7: choose a violated hyperplane a
- 8:
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
- 9:
$$Q \leftarrow \frac{n^2}{n^2-1} \left(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$
- 10: **endif**
- 11: **until** $???$
- 12: **return** “ K is empty”

Repeat: Size of basic solutions

Lemma 52

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a bounded polyhedron. Let $L := 2\langle A \rangle + \langle b \rangle + 2n(1 + \log_2 n)$. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \leq 2^L$.

In the following we use $\delta := 2^L$.

Proof:

We can replace P by $P' := \{x \mid A'x \leq b; x \geq 0\}$ where $A' = \begin{bmatrix} A & -A \end{bmatrix}$. The lemma follows by applying Lemma 47, and observing that $\langle A' \rangle = 2\langle A \rangle$ and $n' = 2n$.

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How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \leq \delta$.

Hence, P is contained in the cube $-\delta \leq x_i \leq \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \text{vol}(B(0, 1)) \leq (n\delta)^n \text{vol}(B(0, 1))$.

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When can we terminate?

Let $P := \{x \mid Ax \leq b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \leq b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

where $\lambda = \delta^2 + 1$.

Note that the volume of P_λ cannot be 0

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P_λ is feasible if and only if P is feasible.

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Consider the polyhedrons

$$\bar{P} = \{x \mid [A \ -A \ I_m]x = b; x \geq 0\}$$

and

$$\bar{P}_\lambda = \left\{x \mid [A \ -A \ I_m]x = b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0\right\}.$$

P is feasible if and only if \bar{P} is feasible, and P_λ feasible if and only if \bar{P}_λ feasible.

\bar{P}_λ is bounded since P_λ and P are bounded.

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Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$.

\bar{P}_λ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1}b + \frac{1}{\lambda}\bar{A}_B^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x -values are zero)

The only reason that this basic feasible solution is not feasible for \bar{P} is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1}b)_i < 0 \leq (\bar{A}_B^{-1}b)_i + \frac{1}{\lambda}(\bar{A}_B^{-1}\bar{1})_i$$

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$$(\bar{A}_B^{-1}b)_i < 0 \quad \Rightarrow \quad (\bar{A}_B^{-1}b)_i \leq -\frac{1}{\det(\bar{A}_B)} \leq -1/\delta$$

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where \bar{A}_B^j is obtained by replacing the j -th column of \bar{A}_B by $\vec{1}$.

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Lemma 54

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \text{vol}(B(0, 1)) = \frac{1}{\delta^{3n}} \text{vol}(B(0, 1))$.

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Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

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$$e^{-\frac{i}{2(n+1)}} \cdot \text{vol}(B(0, R)) < \text{vol}(B(0, r))$$

Hence,

$$\begin{aligned} i &> 2(n+1) \ln\left(\frac{\text{vol}(B(0, R))}{\text{vol}(B(0, r))}\right) \\ &= 2(n+1) \ln\left(n^n \delta^n \cdot \delta^{3n}\right) \\ &= 8n(n+1) \ln(\delta) + 2(n+1)n \ln(n) \\ &= \mathcal{O}(\text{poly}(n) \cdot L) \end{aligned}$$

Algorithm 1 ellipsoid-algorithm

- 1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
- 2: with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
- 3: **output:** point $x \in K$ or “ K is empty”
- 4: $Q \leftarrow \text{diag}(R^2, \dots, R^2)$ // i.e., $L = \text{diag}(R, \dots, R)$
- 5: **repeat**
- 6: **if** $c \in K$ **then return** c
- 7: **else**
- 8: choose a violated hyperplane a
- 9:
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
- 10:
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$
- 11: **endif**
- 12: **until** $\det(Q) \leq r^{2n}$ // i.e., $\det(L) \leq r^n$
- 13: **return** “ K is empty”

Separation Oracle

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- ▶ certifies that $x \in K$,
- ▶ or finds a hyperplane separating x from K .

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- ▶ a guarantee that a ball of radius r is contained in K ,
- ▶ an initial ball B with radius R that contains K ,
- ▶ a separation oracle for K .

The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

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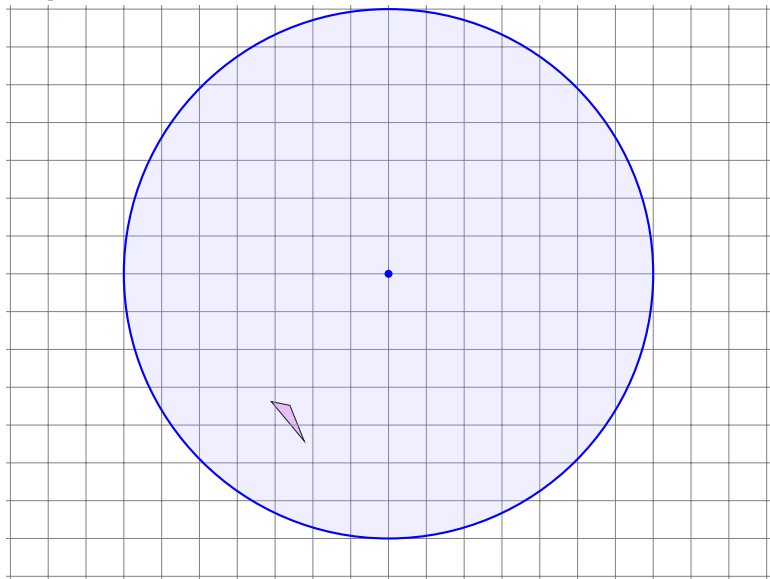
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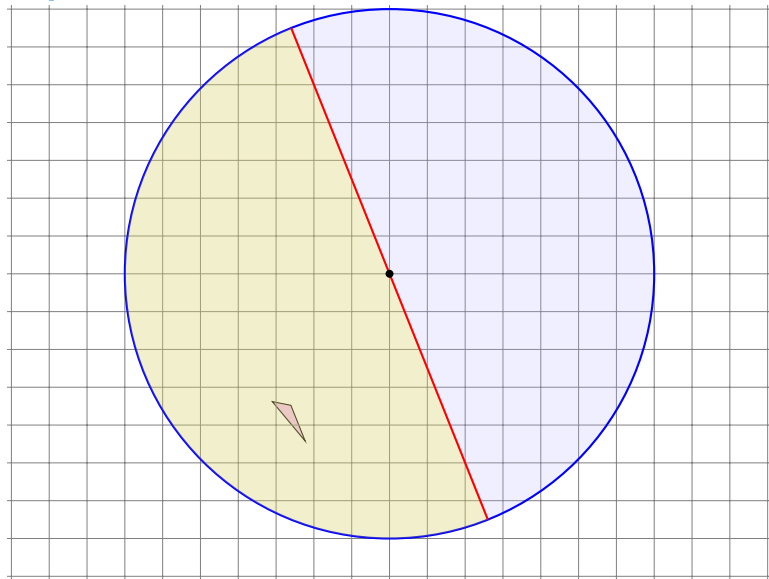
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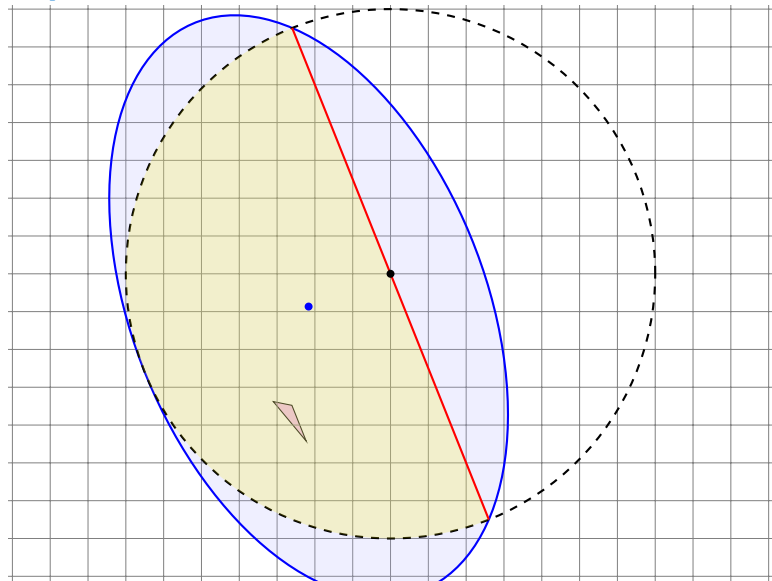
Example



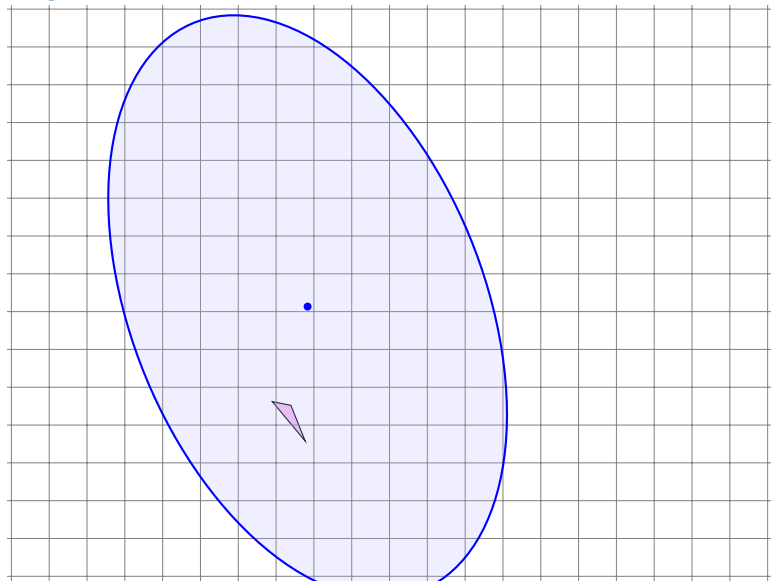
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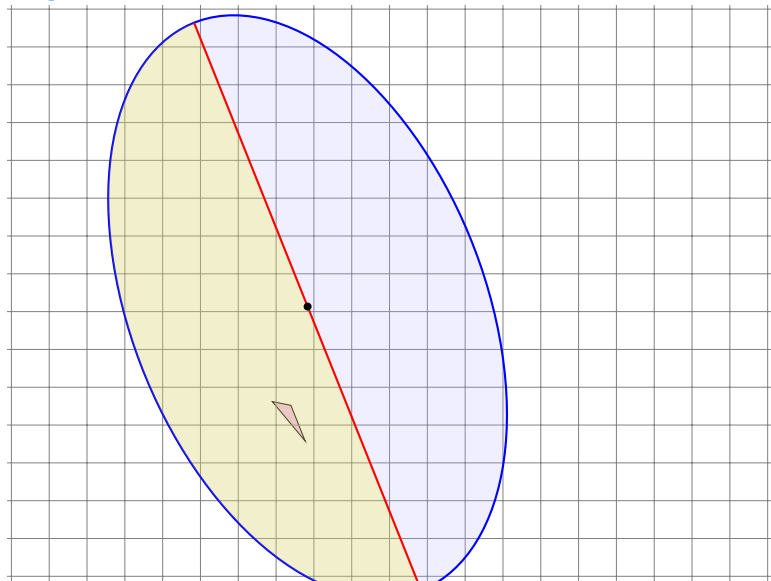
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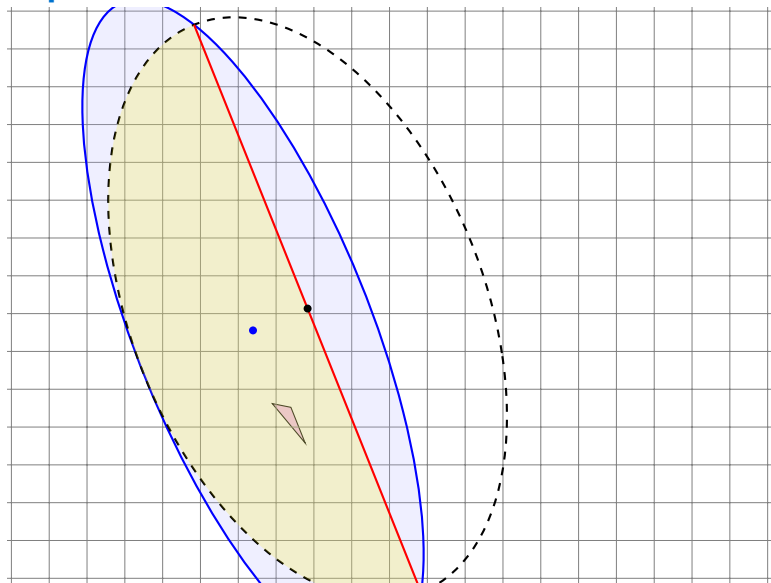
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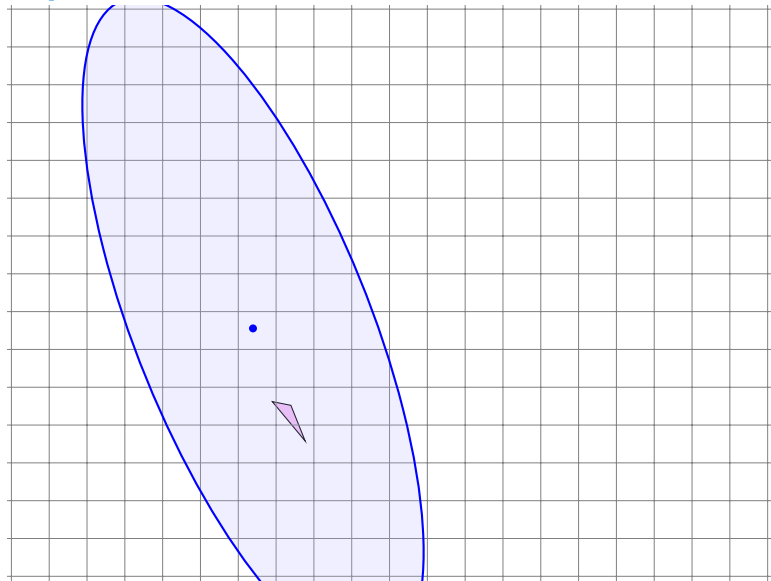
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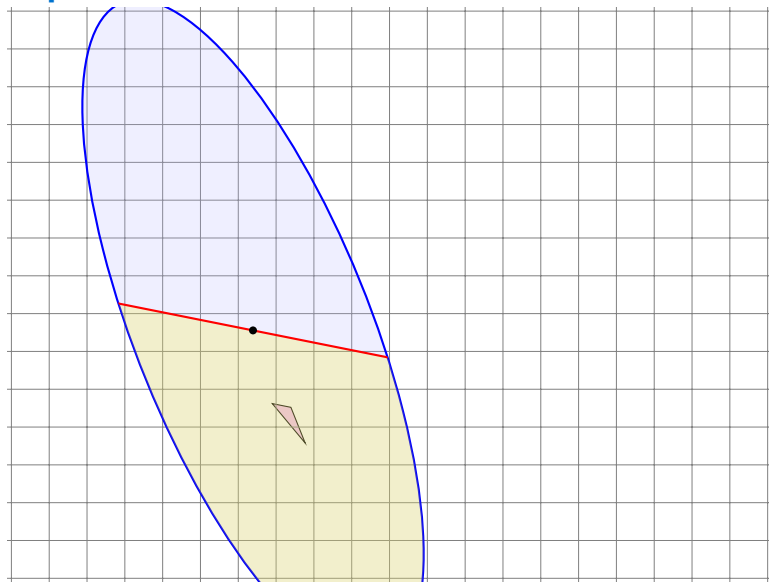
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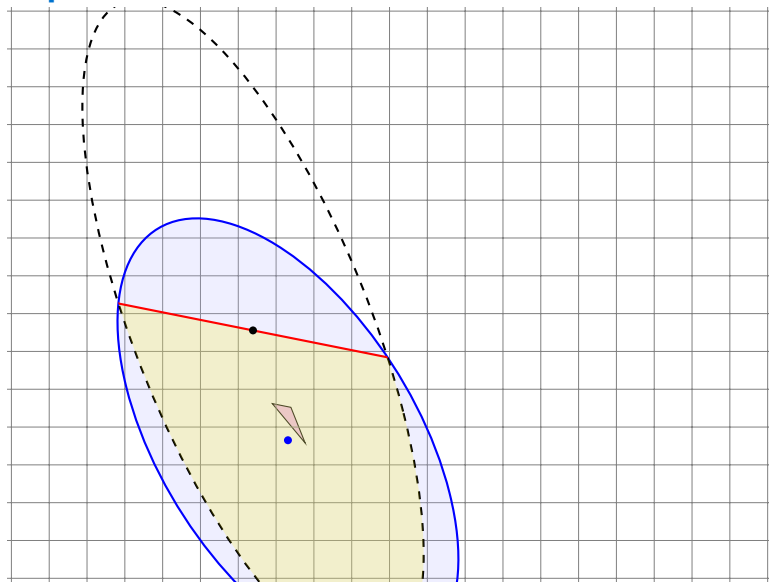
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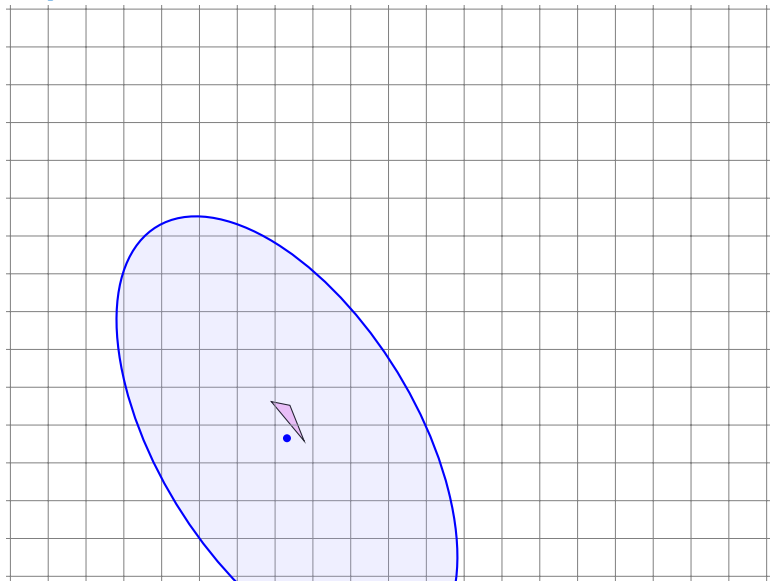
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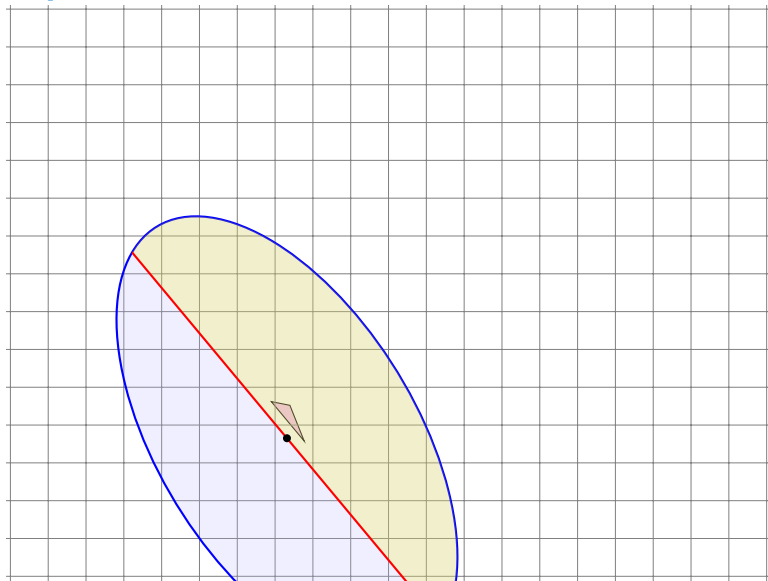
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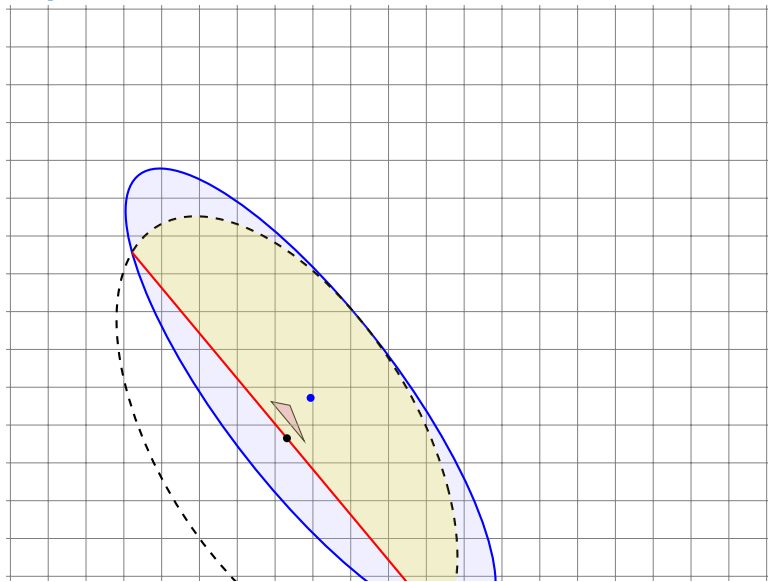
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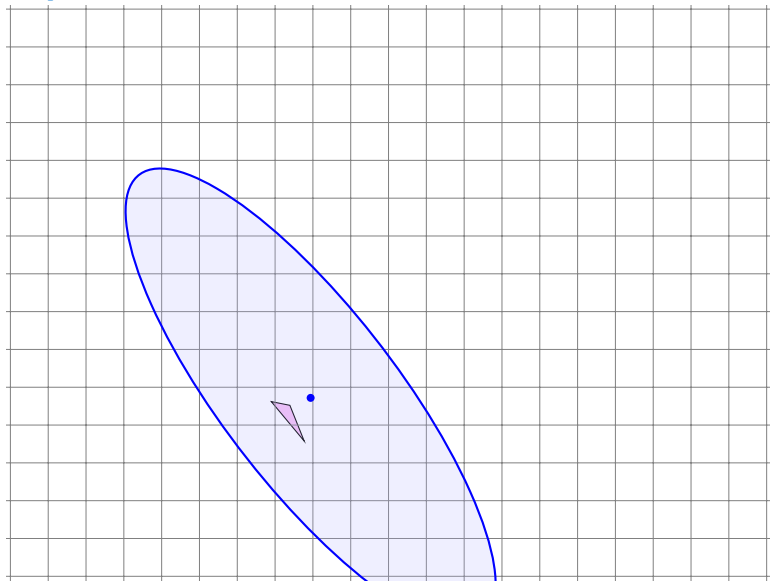
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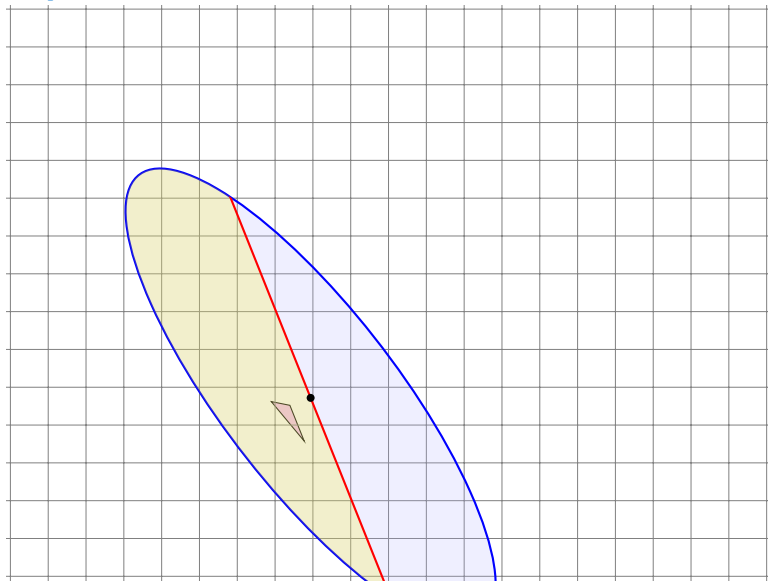
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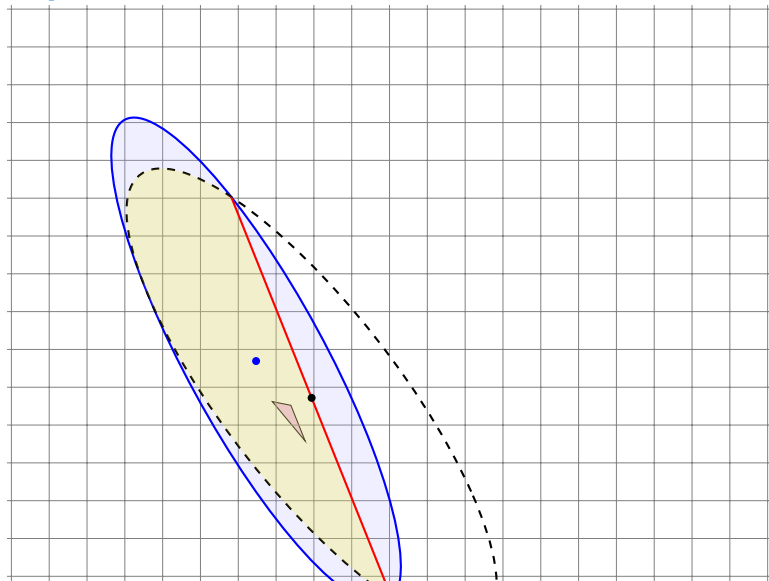
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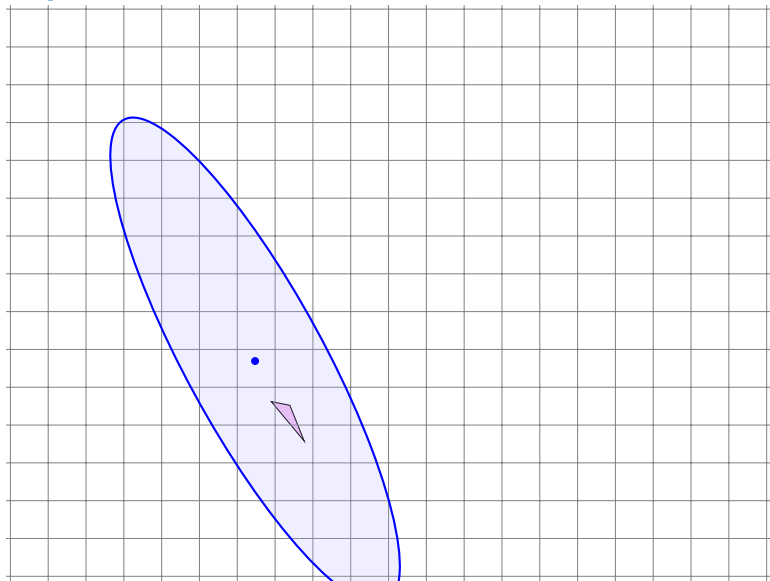
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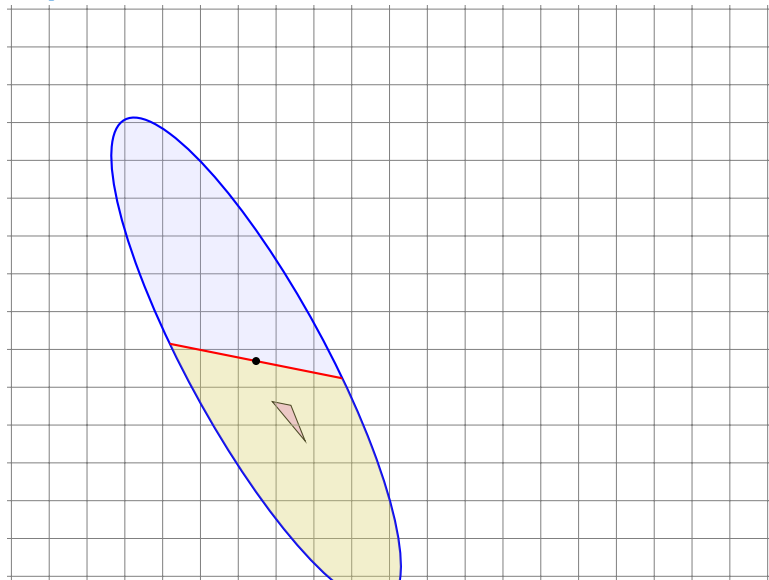
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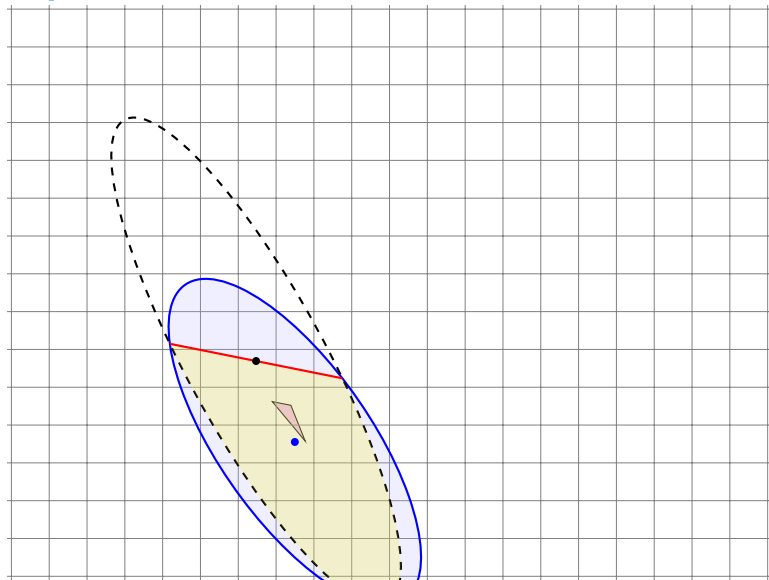
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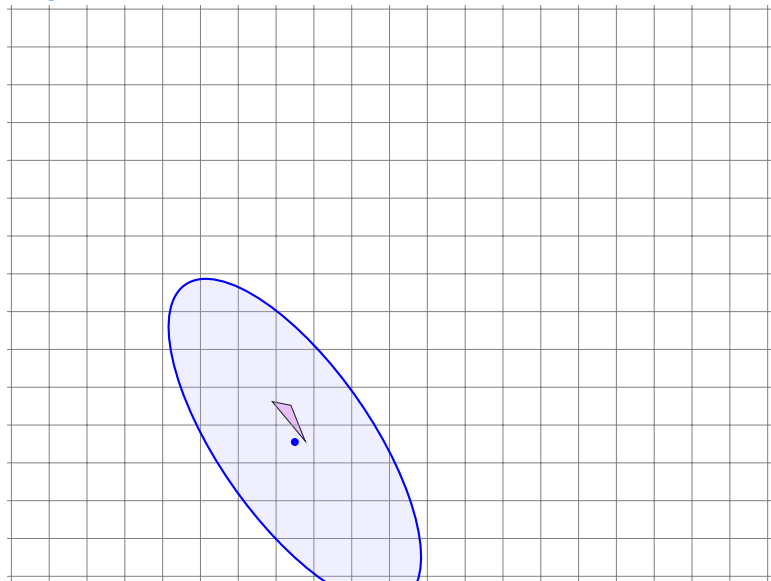
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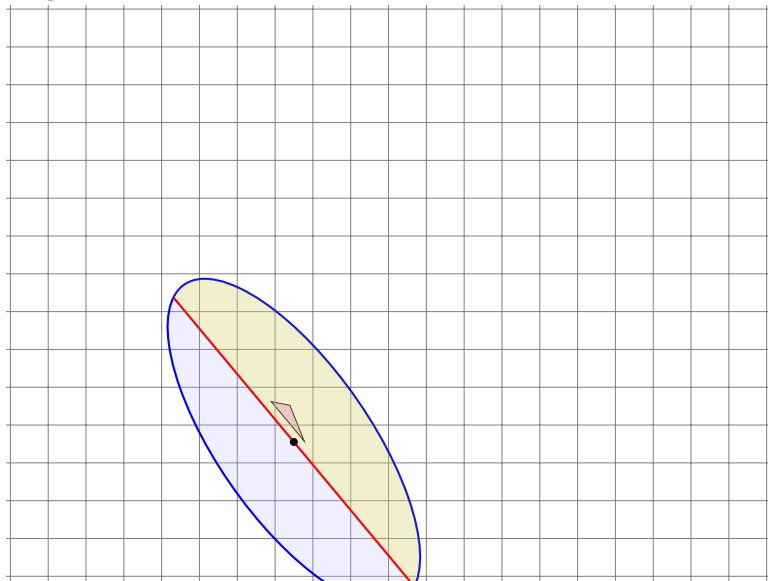
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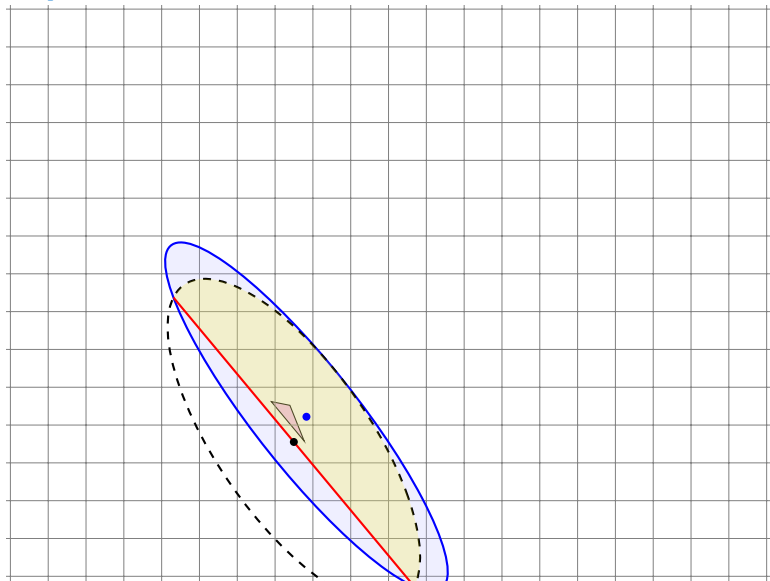
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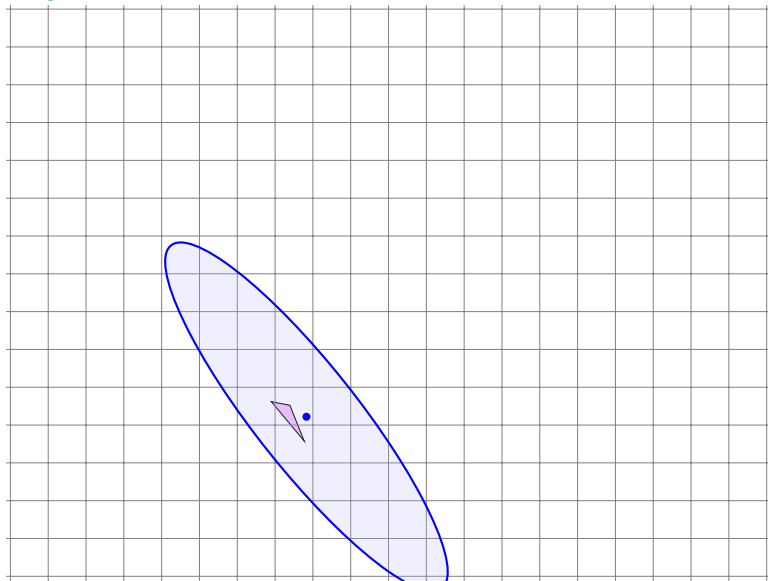
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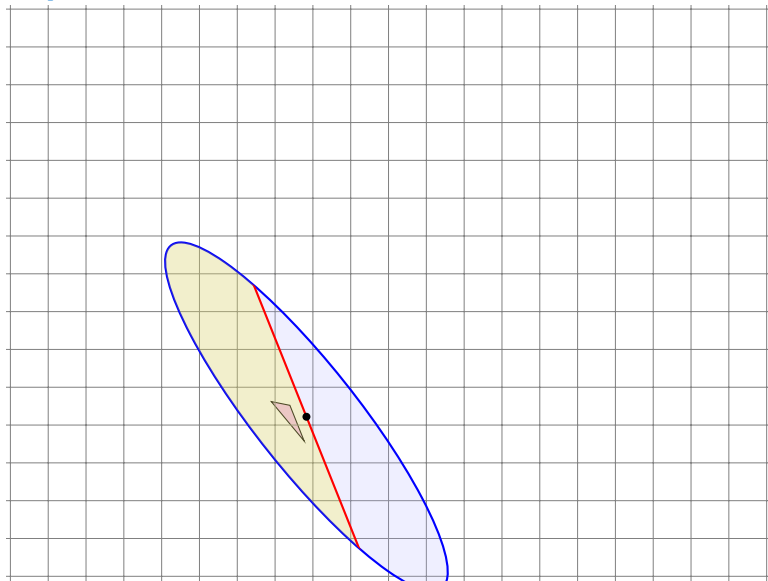
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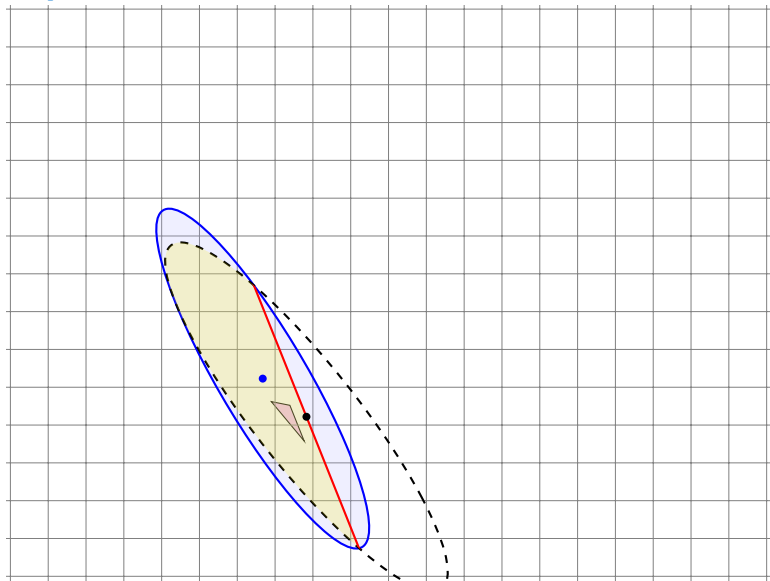
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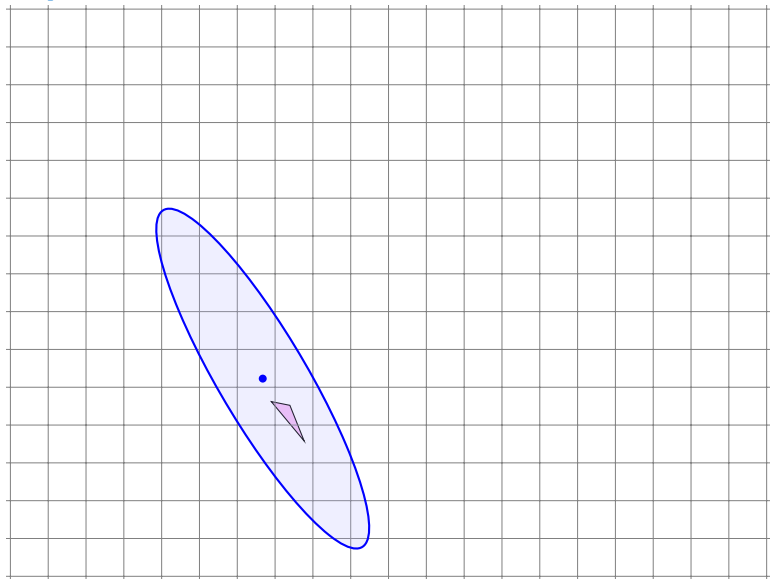
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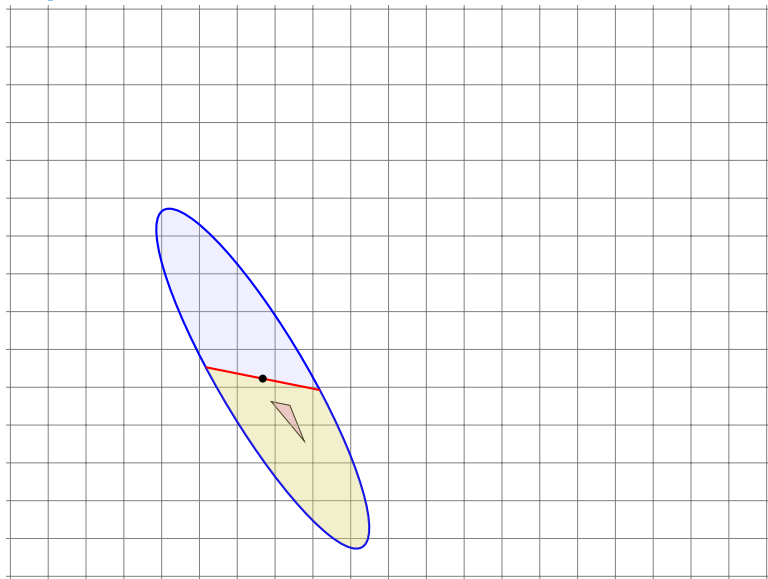
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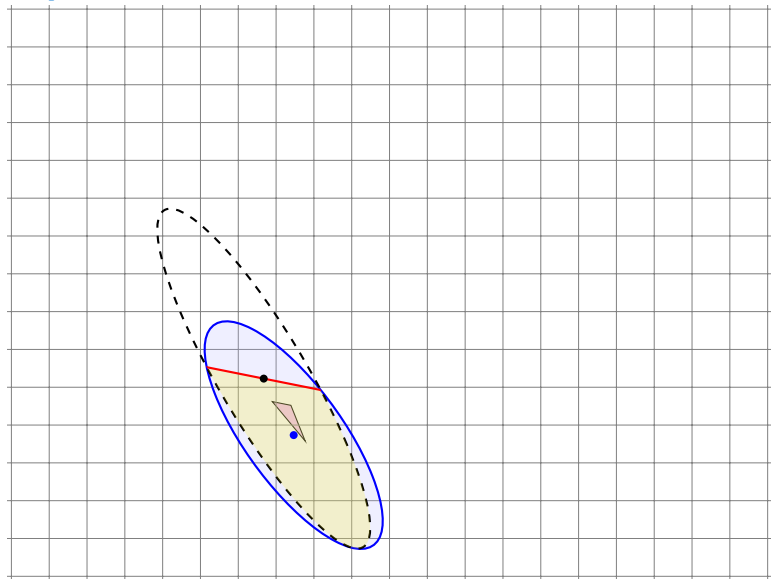
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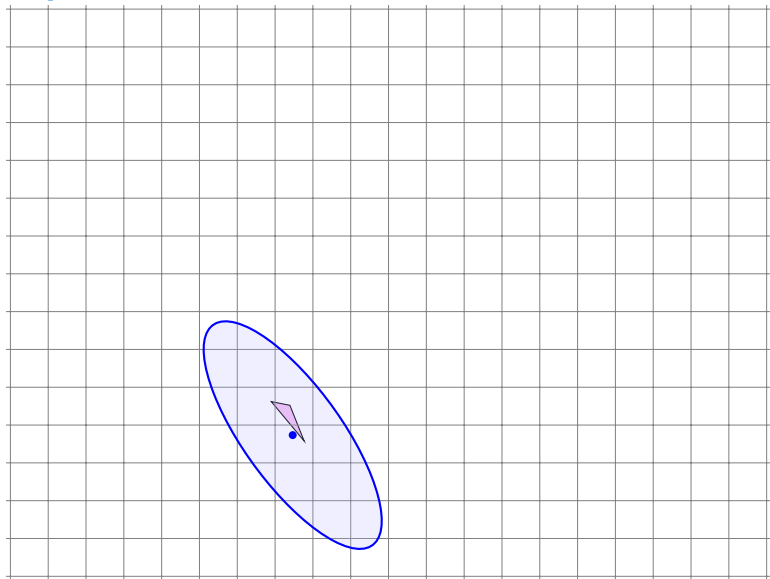
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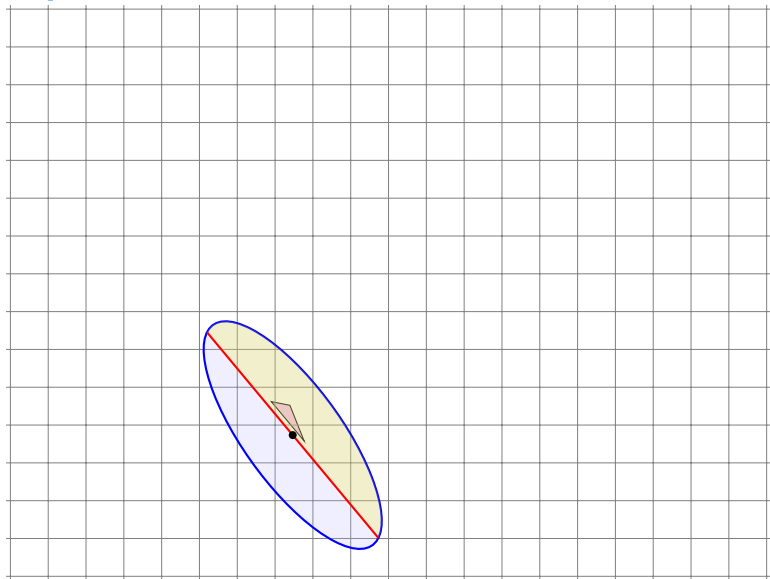
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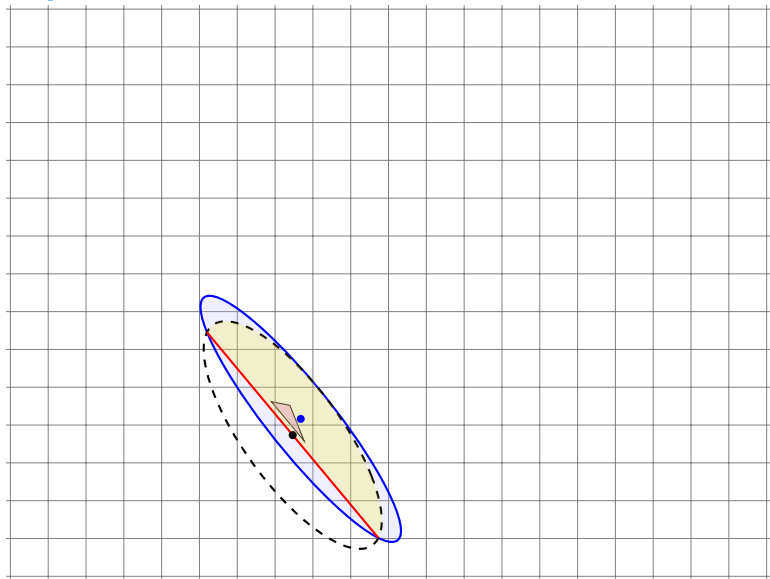
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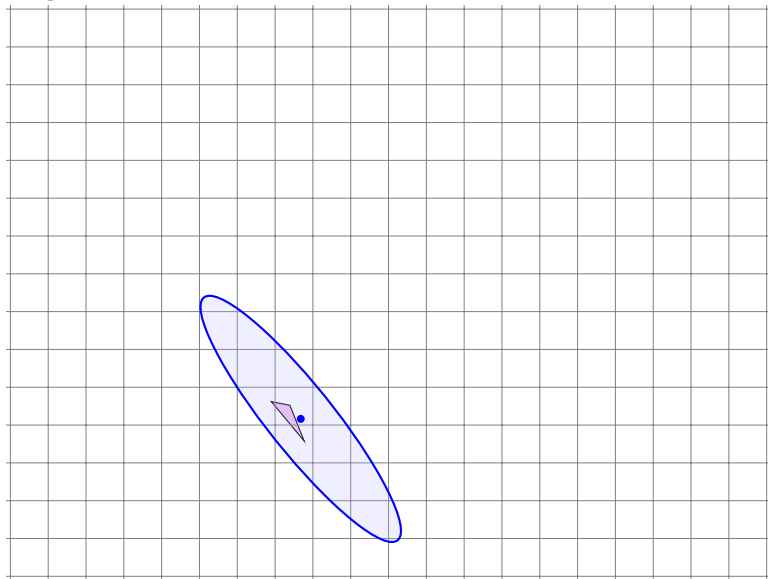
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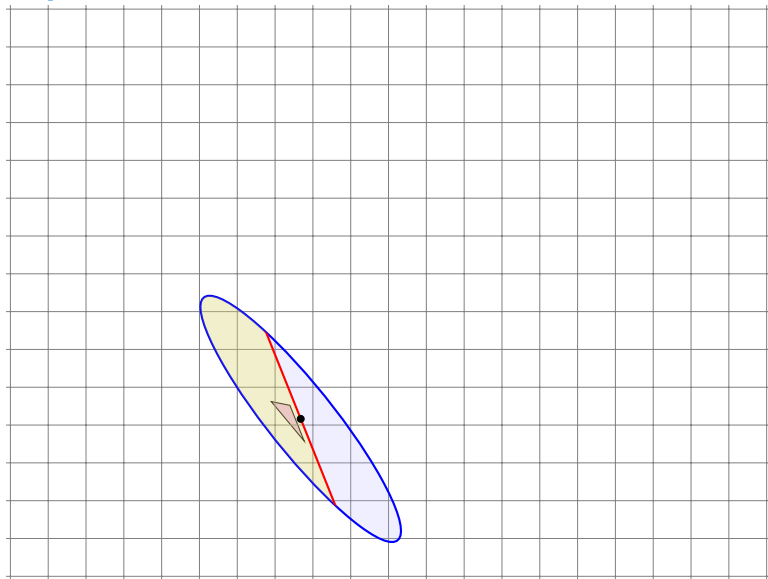
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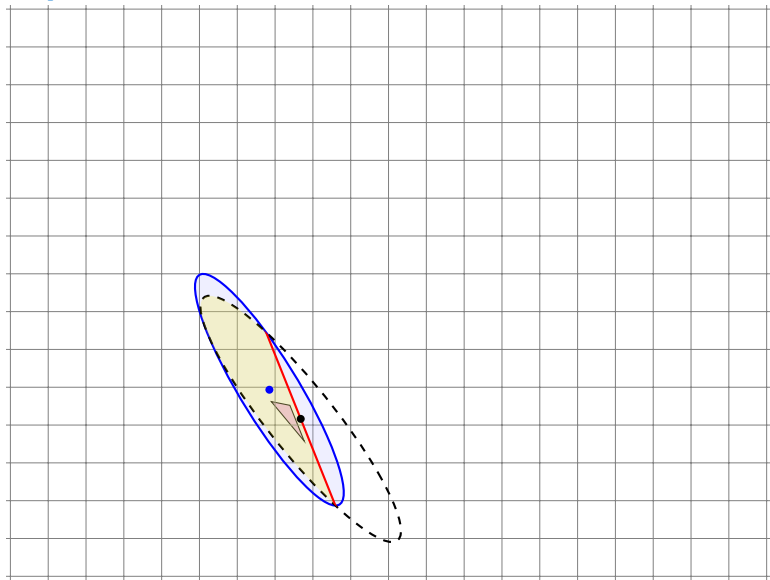
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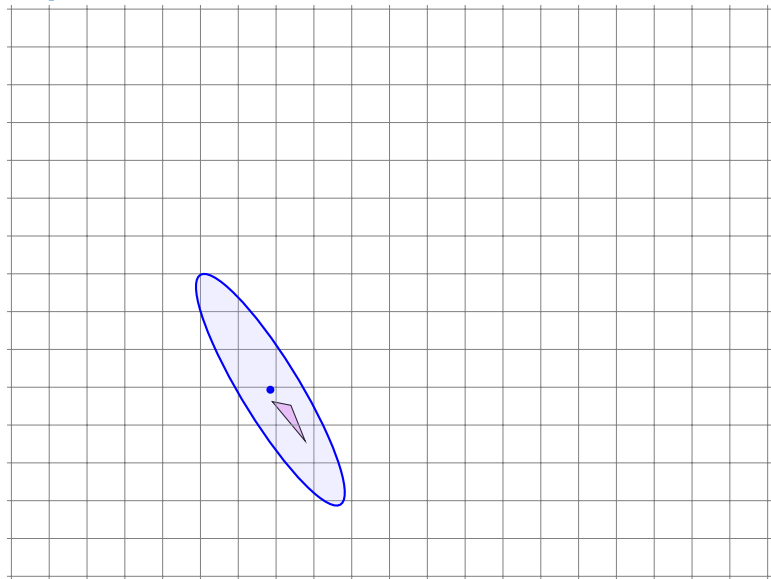
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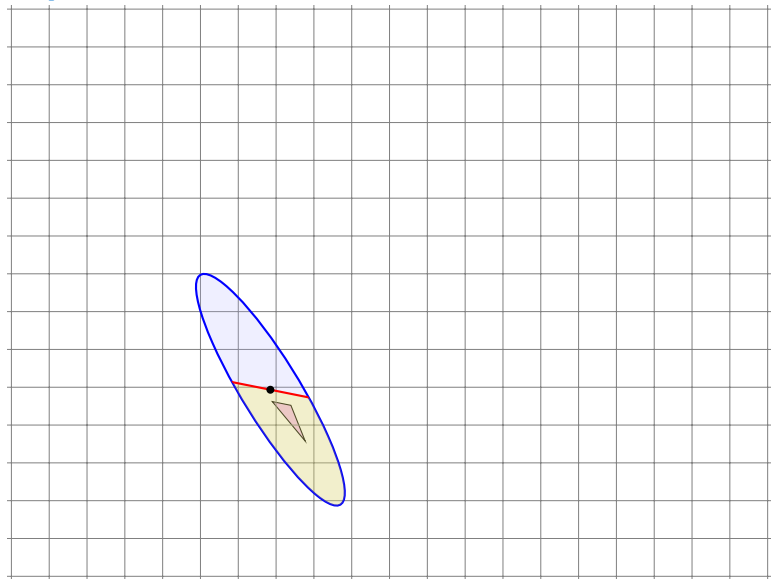
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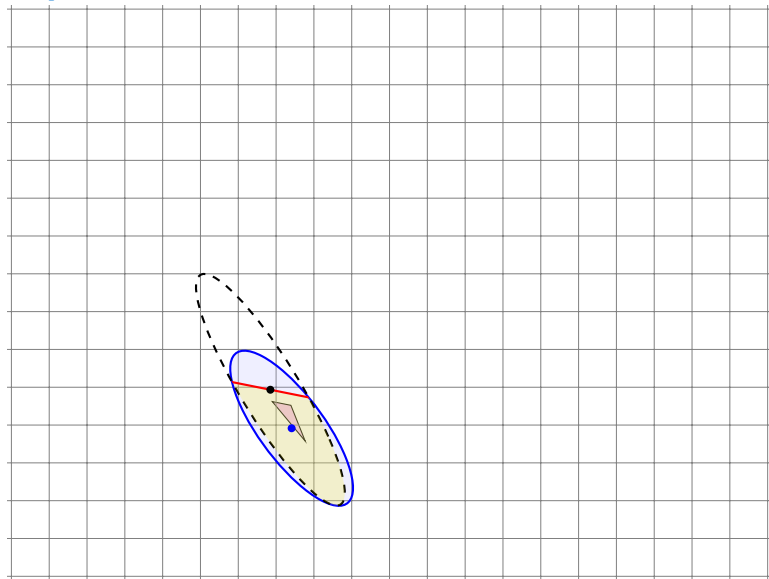
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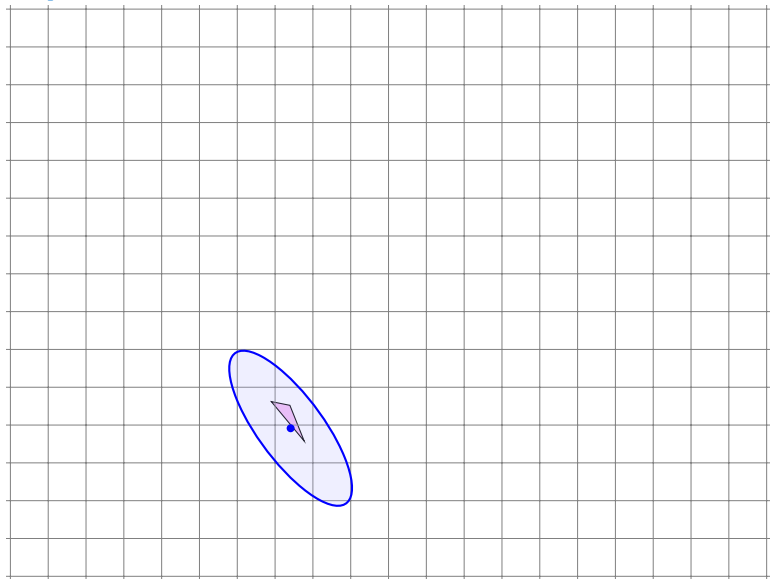
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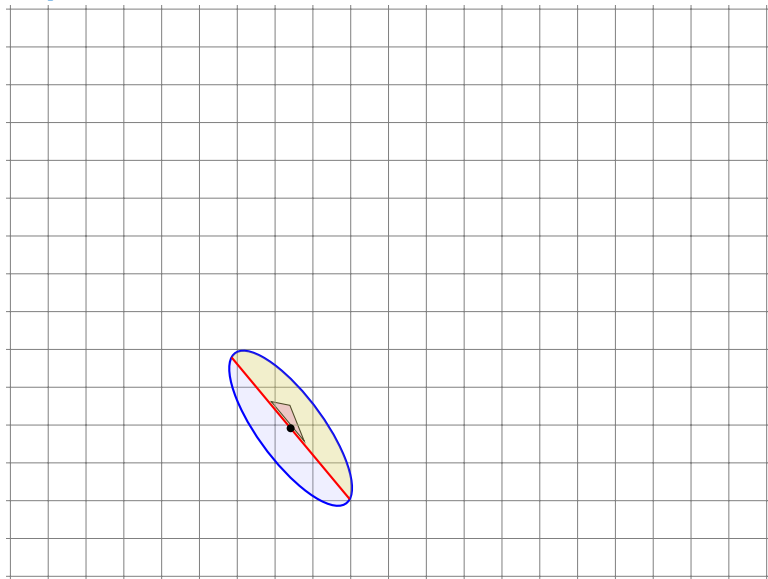
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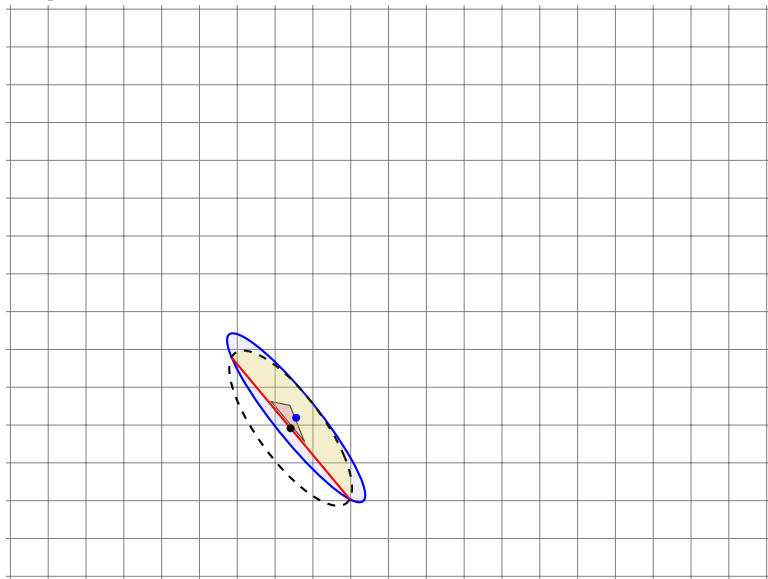
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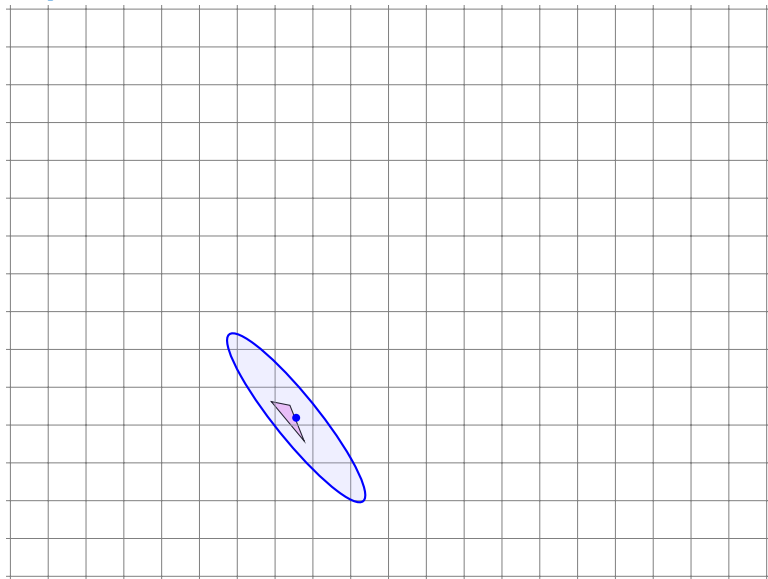
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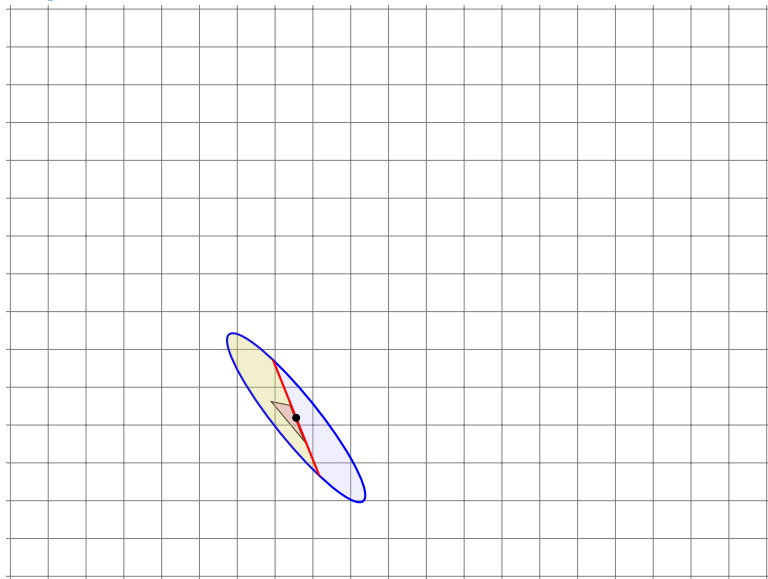
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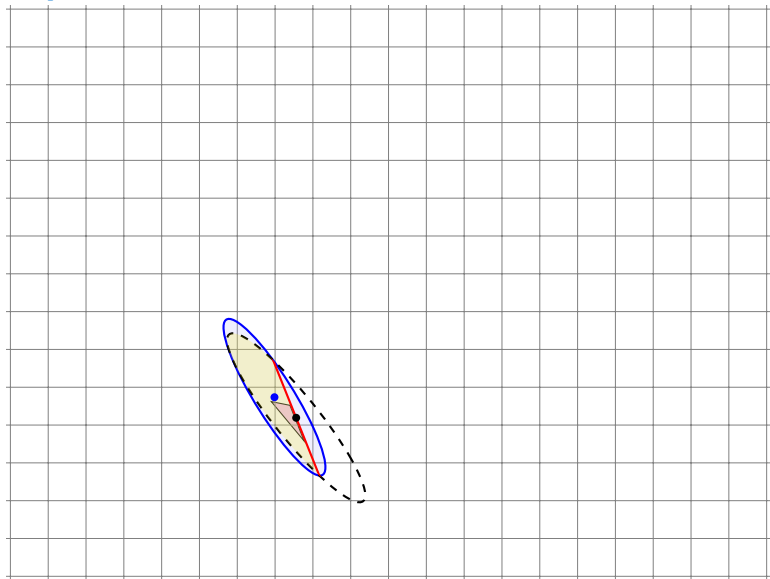
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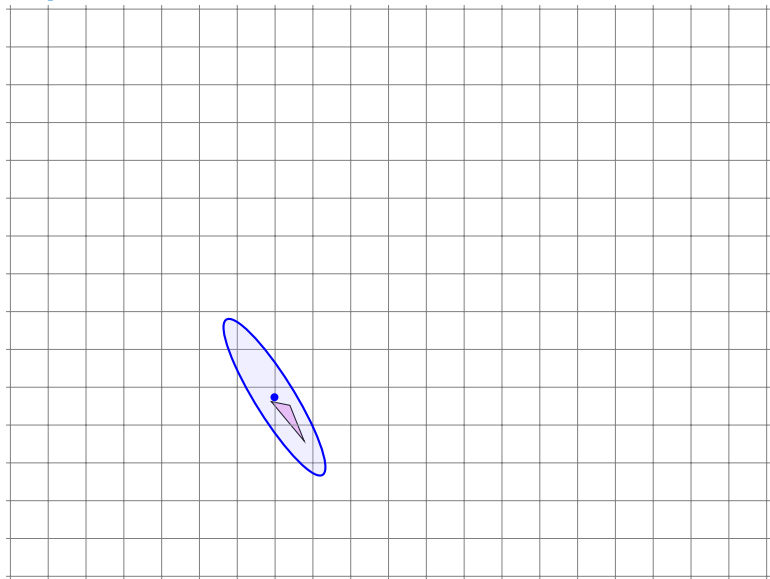
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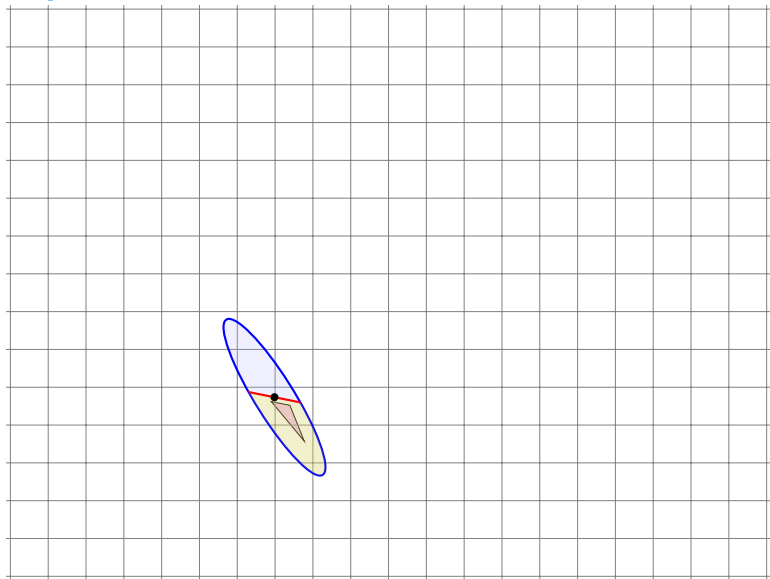
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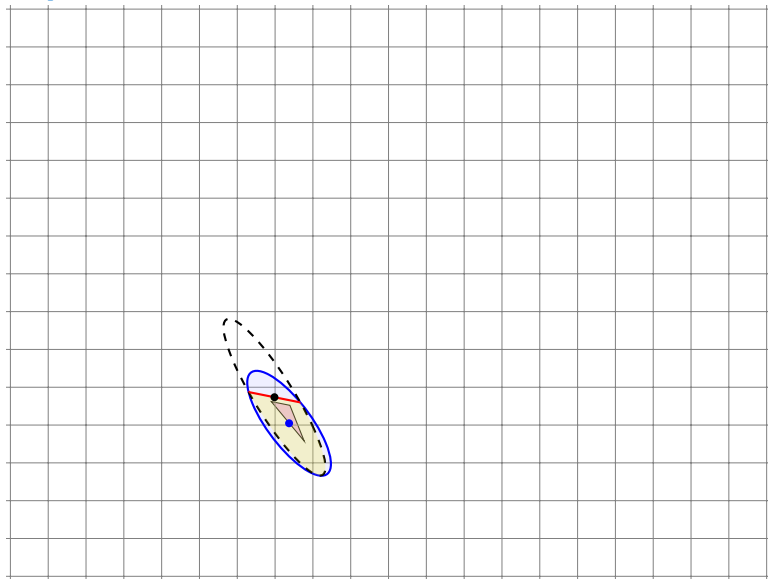
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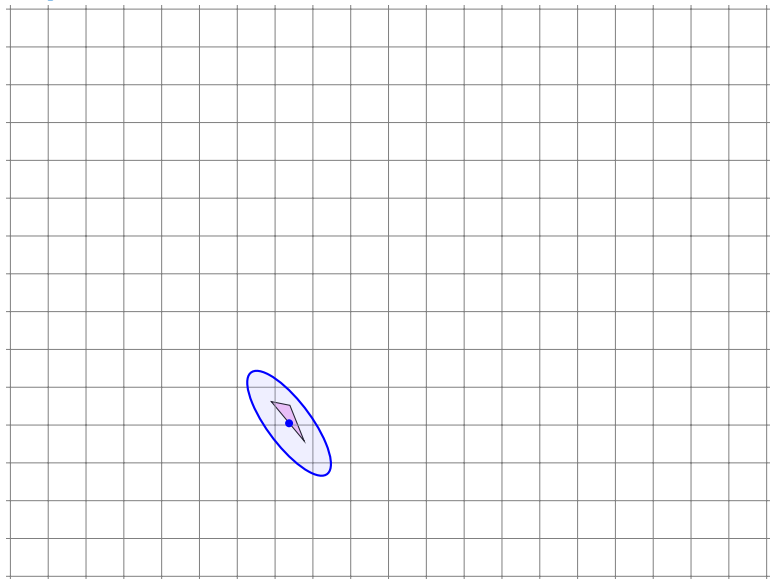
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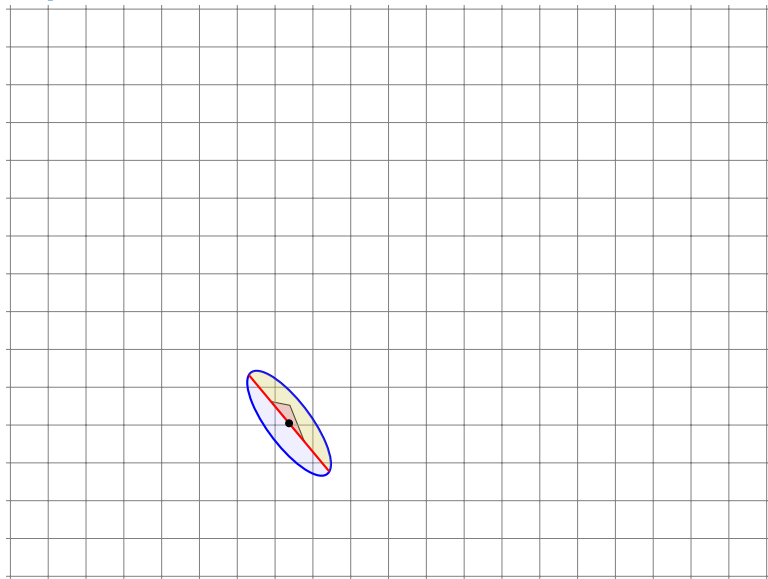
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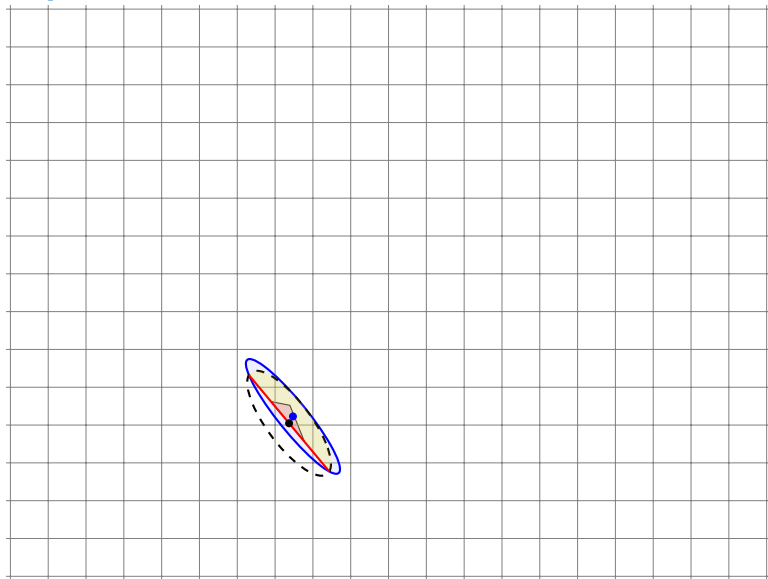
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