5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \ge 0\}$$
$$w = \min\{b^T y \mid A^T y \ge c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned}\max\{c^T x \mid Ax = b, x \ge 0\} &= \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\}\\ &= \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\}\end{aligned}$$

Dual:

$$\begin{aligned}\min\{[b^T\ -b^T] y \mid [A^T\ -A^T] y \ge c, y \ge 0\} &= \min\left\{[b^T\ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T\ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\}\\ &= \min\left\{b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0\right\}\\ &= \min\left\{b^T y' \,\middle|\, A^T y' \ge c\right\}\end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned}b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^*\\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B\\ &= c^T x^*\end{aligned}$$

Hence, the solution is optimal.