

## 5.3 Strong Duality

$$P = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

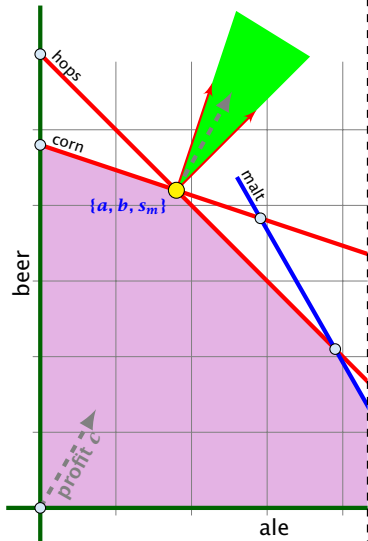
$n_A$ : number of variables,  $m_A$ : number of constraints

We can put the non-negativity constraints into  $A$  (which gives us unrestricted variables):  $\bar{P} = \max\{c^T x \mid \bar{A}x \leq \bar{b}\}$

$$n_{\bar{A}} = n_A, m_{\bar{A}} = m_A + n_A$$

Dual  $D = \min\{\bar{b}^T y \mid \bar{A}^T y = c, y \geq 0\}$ .

## 5.3 Strong Duality



If we have a conic combination  $y$  of  $c$  then  $b^T y$  is an upper bound of the profit we can obtain (**weak duality**):

$$c^T x = (\bar{A}^T y)^T x = y^T \bar{A} x \leq y^T \bar{b}$$

If  $x$  and  $y$  are optimal then the **duality gap** is 0 (**strong duality**). This means

$$\begin{aligned} 0 &= c^T x - y^T \bar{b} \\ &= (\bar{A}^T y)^T x - y^T \bar{b} \\ &= y^T (\bar{A} x - \bar{b}) \end{aligned}$$

The last term can only be 0 if  $y_i$  is 0 whenever the  $i$ -th constraint is not tight. This means we have a conic combination of  $c$  by normals (columns of  $\bar{A}^T$ ) of *tight* constraints.

Conversely, if we have  $x$  such that the normals of tight constraint (at  $x$ ) give rise to a conic combination of  $c$ , we know that  $x$  is optimal.

The profit vector  $c$  lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

# Strong Duality

## Theorem 33 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z^*$  and  $w^*$  denote the optimal solution to  $P$  and  $D$ , respectively.

Then

$$z^* = w^*$$

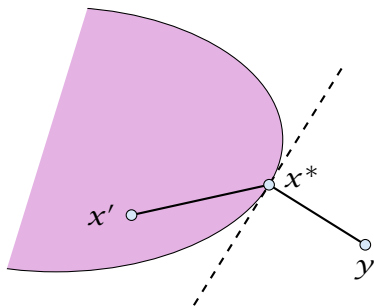
### Lemma 34 (Weierstrass)

Let  $X$  be a compact set and let  $f(x)$  be a continuous function on  $X$ . Then  $\min\{f(x) : x \in X\}$  exists.

**(without proof)**

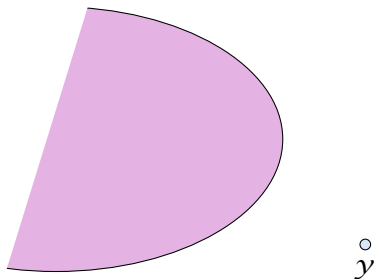
### Lemma 35 (Projection Lemma)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty convex set, and let  $y \notin X$ . Then there exist  $x^* \in X$  with minimum distance from  $y$ . Moreover for all  $x \in X$  we have  $(y - x^*)^T(x - x^*) \leq 0$ .



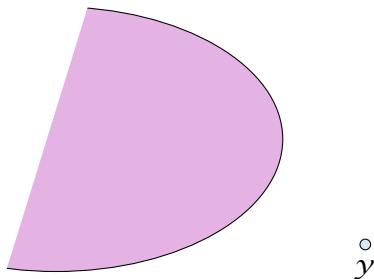
# Proof of the Projection Lemma

- ▶ Define  $f(x) = \|y - x\|$ .
- ▶ We want to apply Weierstrass but  $X$  may not be bounded.
- ▶  $X \neq \emptyset$ . Hence, there exists  $x' \in X$ .
- ▶ Define  $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$ . This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



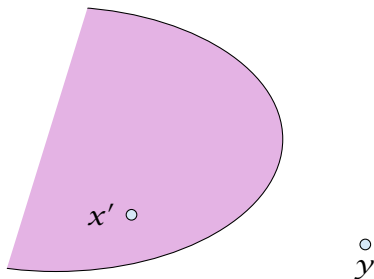
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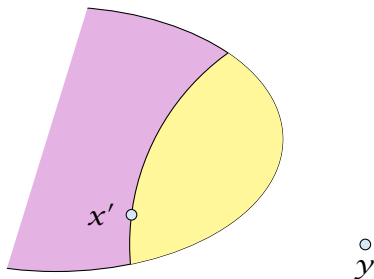
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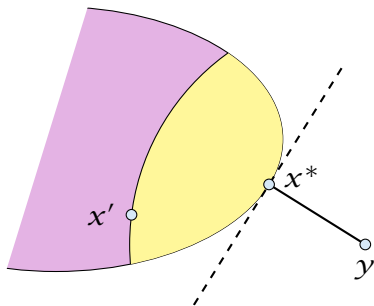
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# Proof of the Projection Lemma (continued)

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$$\|y - x^*\|^2 \leq \|y - x^* - \epsilon(x - x^*)\|^2$$

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$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$



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Hence,  $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon \|x - x^*\|^2$ .

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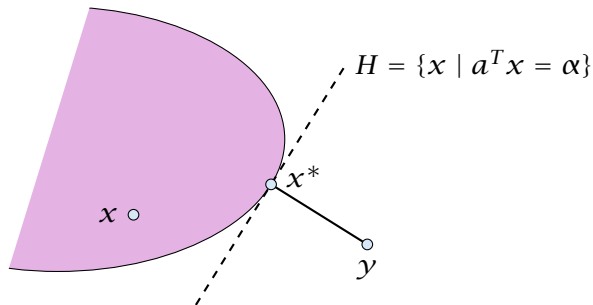
Letting  $\epsilon \rightarrow 0$  gives the result.

### Theorem 36 (Separating Hyperplane)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty closed convex set, and let  $y \notin X$ . Then there exists a *separating hyperplane*  $\{x \in \mathbb{R}^m : a^T x = \alpha\}$  where  $a \in \mathbb{R}^m$ ,  $\alpha \in \mathbb{R}$  that *separates*  $y$  from  $X$ . ( $a^T y < \alpha$ ;  $a^T x \geq \alpha$  for all  $x \in X$ )

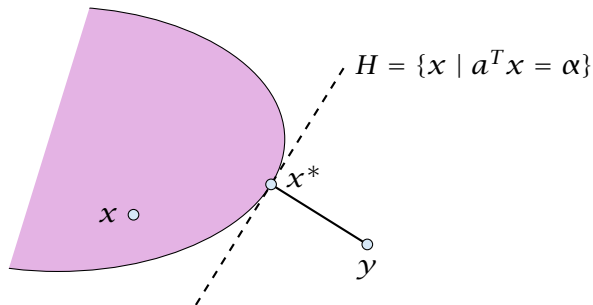
# Proof of the Hyperplane Lemma

- ▶ Let  $x^* \in X$  be closest point to  $y$  in  $X$ .
- ▶ By previous lemma  $(y - x^*)^T(x - x^*) \leq 0$  for all  $x \in X$ .
- ▶ Choose  $a = (x^* - y)$  and  $\alpha = a^T x^*$ .
- ▶ For  $x \in X$ :  $a^T(x - x^*) \geq 0$ , and, hence,  $a^T x \geq \alpha$ .
- ▶ Also,  $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$



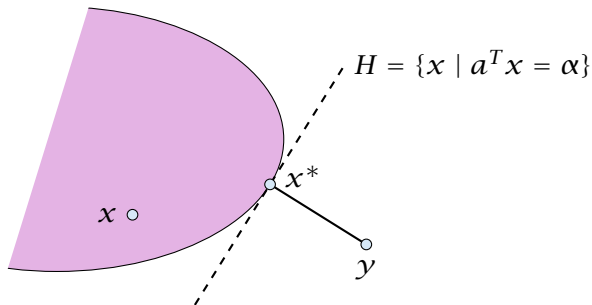
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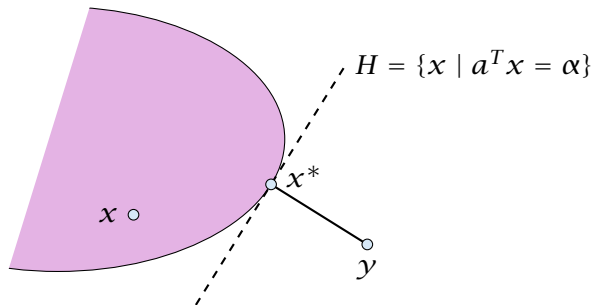
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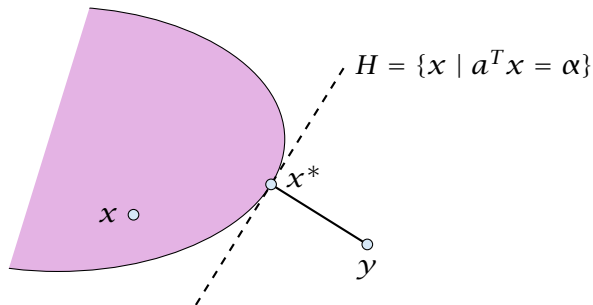
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### Lemma 37 (Farkas Lemma)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then *exactly one* of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax = b$ ,  $x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^T y \geq 0$ ,  $b^T y < 0$

Assume  $\hat{x}$  satisfies 1. and  $\hat{y}$  satisfies 2. Then

$$0 > \hat{y}^T b = \hat{y}^T A \hat{x} \geq 0$$

Hence, at most one of the statements can hold.

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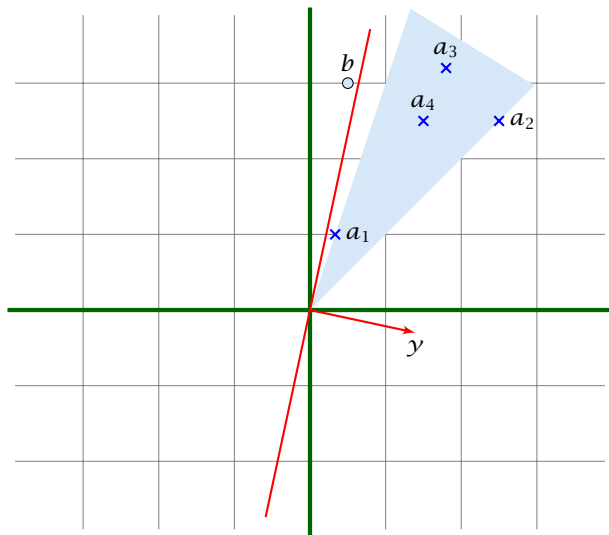
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## Farkas Lemma



If  $b$  is not in the cone generated by the columns of  $A$ , there exists a hyperplane  $y$  that separates  $b$  from the cone.

## Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider  $S = \{Ax : x \geq 0\}$  so that  $S$  closed, convex,  $b \notin S$ .

We want to show that there is  $y$  with  $A^T y \geq 0$ ,  $b^T y < 0$ .

Let  $y$  be a hyperplane that separates  $b$  from  $S$ . Hence,  $y^T b < \alpha$  and  $y^T s \geq \alpha$  for all  $s \in S$ .

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$$

$y^T Ax \geq \alpha$  for all  $x \geq 0$ . Hence,  $y^T A \geq 0$  as we can choose  $x$  arbitrarily large.

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### Lemma 38 (Farkas Lemma; different version)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then exactly one of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax \leq b, x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^T y \geq 0, b^T y < 0, y \geq 0$

Rewrite the conditions:

1.  $\exists x \in \mathbb{R}^n$  with  $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b, x \geq 0, s \geq 0$
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# Proof of Strong Duality

$$P: z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

## Theorem 39 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z$  and  $w$  denote the optimal solution to  $P$  and  $D$ , respectively (i.e.,  $P$  and  $D$  are non-empty). Then

$$z = w .$$

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$$\begin{array}{l} \exists x \in \mathbb{R}^n \\ \text{s.t.} \quad Ax \leq b \\ \quad \quad -c^T x \leq -\alpha \\ \quad \quad x \geq 0 \end{array}$$

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$z \leq w$ : follows from weak duality

$z \geq w$ :

We show  $z < \alpha$  implies  $w < \alpha$ .

$$\exists x \in \mathbb{R}^n$$

$$\begin{aligned} \text{s.t.} \quad Ax &\leq b \\ -c^T x &\leq -\alpha \\ x &\geq 0 \end{aligned}$$

$$\exists y \in \mathbb{R}^m; v \in \mathbb{R}$$

$$\begin{aligned} \text{s.t.} \quad A^T y - cv &\geq 0 \\ b^T y - \alpha v &< 0 \\ y, v &\geq 0 \end{aligned}$$

# Proof of Strong Duality

$z \leq w$ : follows from weak duality

$z \geq w$ :

We show  $z < \alpha$  implies  $w < \alpha$ .

$$\exists x \in \mathbb{R}^n$$

$$\begin{aligned} \text{s.t.} \quad Ax &\leq b \\ -c^T x &\leq -\alpha \\ x &\geq 0 \end{aligned}$$

$$\exists y \in \mathbb{R}^m; v \in \mathbb{R}$$

$$\begin{aligned} \text{s.t.} \quad A^T y - cv &\geq 0 \\ b^T y - \alpha v &< 0 \\ y, v &\geq 0 \end{aligned}$$

From the definition of  $\alpha$  we know that the first system is infeasible; hence the second must be feasible.

# Proof of Strong Duality

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If the solution  $\mathbf{y}, \mathbf{v}$  has  $\mathbf{v} = 0$  we have that

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is feasible. By Farkas lemma this gives that LP  $P$  is infeasible.  
Contradiction to the assumption of the lemma.

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Hence, there exists a solution  $y, v$  with  $v > 0$ .

We can rescale this solution (scaling both  $y$  and  $v$ ) s.t.  $v = 1$ .

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# Fundamental Questions

## Definition 40 (Linear Programming Problem (LP))

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^T x \geq \alpha$ ?

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- ▶ Is LP in NP?
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