

**WS 2022/23**

# **Efficient Algorithms**

Harald Räcke

Fakultät für Informatik  
TU München

<https://www.moodle.tum.de/course/view.php?id=80009>

Winter Term 2022/23

# Part I

## Organizational Matters

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- ▶ Modul: IN2003

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  - ▶ 4 SWS  
Mon 10:00–12:00 (Room Interim2)  
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“Diskrete Strukturen” (DS)
  - ▶ IN0018  
“**Discrete Probability Theory**”  
“Diskrete Wahrscheinlichkeitstheorie” (DWT)

# The Lecturer

- ▶ Harald Räche
- ▶ Email: [raecke@in.tum.de](mailto:raecke@in.tum.de)
- ▶ Room: 03.09.044
- ▶ Office hours: (by appointment)

- ▶ Omar AbdelWanis
- ▶ [omar.abdelwanis@in.tum.de](mailto:omar.abdelwanis@in.tum.de)
- ▶ Room: 03.09.042
- ▶ Office hours: (by appointment)

## **One centralized tutorial session.**

Wednesday, 14:00-16:00

<https://tum-conf.zoom.us/j/63875503821>

Meeting ID: 638 7550 3821

Passcode: 719434



# 1 Contents

- ▶ Foundations
  - ▶ Machine models
  - ▶ Efficiency measures
  - ▶ Asymptotic notation
  - ▶ Recursion

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- ▶ Cuts/Flows
- ▶ Matchings

## 2 Literatur

-  Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman:  
*The design and analysis of computer algorithms*,  
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-  Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest,  
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-  Donald E. Knuth:  
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Donald E. Knuth:

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Christos H. Papadimitriou, Kenneth Steiglitz:

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Spektrum Akademischer Verlag, 2001



Steven S. Skiena:

*The Algorithm Design Manual,*

Springer, 1998

# Part II

## Foundations



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- ▶ Learn how to design efficient algorithms.

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- ▶ Implementing and testing on representative inputs
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  - ▶ Results only hold for a specific machine and for a specific set of inputs.
  
- ▶ Theoretical analysis in a specific **model of computation**.
  - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time  $\mathcal{O}(n^2)$ ”.
  - ▶ Typically focuses on the **worst case**.
  - ▶ Can give lower bounds like “any comparison-based sorting algorithm needs at least  $\Omega(n \log n)$  comparisons in the worst case”.

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### Example 1

Suppose  $n$  numbers from the interval  $\{1, \dots, N\}$  have to be sorted. In this case we usually say that the input length is  $n$  instead of e.g.  $n \log N$ , which would be the number of bits required to encode the input.

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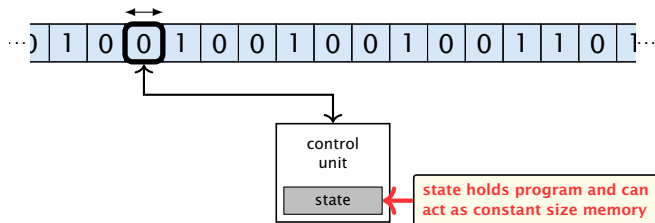
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Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

# Turing Machine

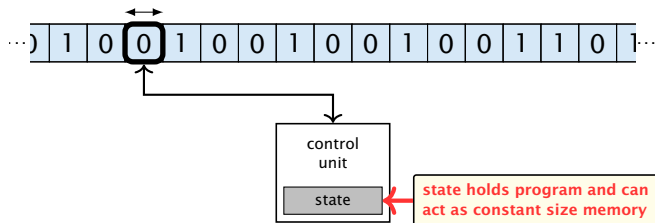
- ▶ Very simple model of computation.





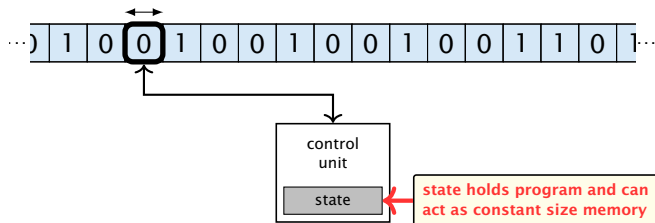
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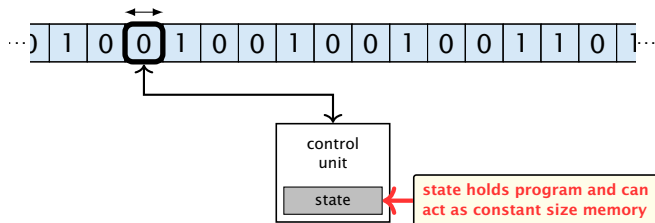
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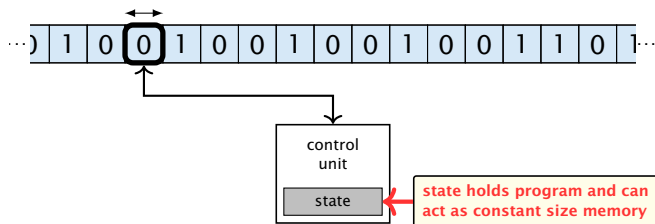
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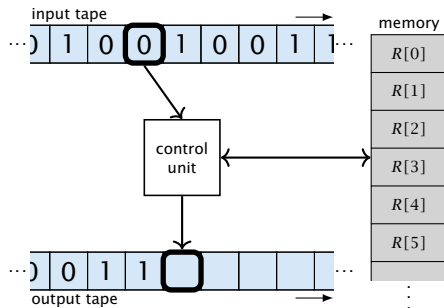
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⇒ **Not a good model for developing efficient algorithms.**



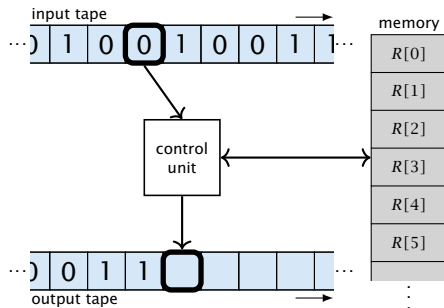
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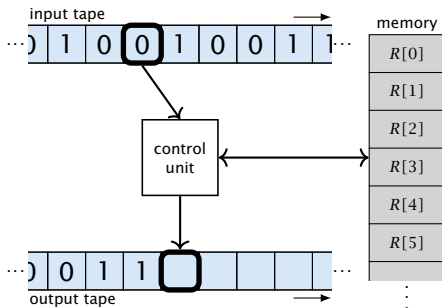
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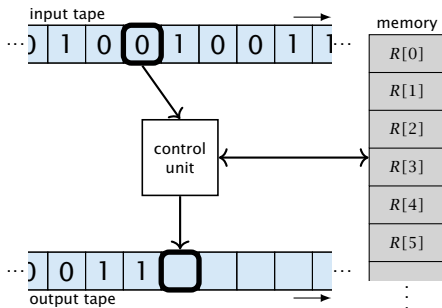
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  - ▶  $R[i] := R[j] + R[k];$
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**Bounded word RAM model:** cost is uniform but the largest value stored in a register may not exceed  $2^w$ , where usually  $w = \log_2 n$ .

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### Algorithm 1 RepeatedSquaring( $n$ )

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- ▶ space requirement:
  - ▶ uniform model:  $\mathcal{O}(1)$

# 4 Modelling Issues

## Example 2

### Algorithm 1 RepeatedSquaring( $n$ )

```
1:  $r \leftarrow 2$ ;  
2: for  $i = 1 \rightarrow n$  do  
3:    $r \leftarrow r^2$   
4: return  $r$ 
```

- ▶ running time (for Line 3):
  - ▶ uniform model:  $n$  steps
  - ▶ logarithmic model:  
 $2 + 3 + 5 + \dots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$
- ▶ space requirement:
  - ▶ uniform model:  $\mathcal{O}(1)$
  - ▶ logarithmic model:  $\mathcal{O}(2^n)$

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more general: probability measure  $\mu$

$$C_{avg}(n) := \sum_{x \in I_n} \mu(x) \cdot C(x)$$



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▶ **randomized** complexity:

The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input  $x$ . Then take the worst-case over all  $x$  with  $|x| = n$ .

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- ▶ Running time should be expressed by simple functions.

# Asymptotic Notation

## Formal Definition

Let  $f, g$  denote functions from  $\mathbb{N}$  to  $\mathbb{R}^+$ .

- ▶  $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$   
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# Asymptotic Notation

There is an equivalent definition using limes notation (**assuming that the respective limes exists**).  $f$  and  $g$  are functions from  $\mathbb{N}_0$  to  $\mathbb{R}_0^+$ .

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4. People write  $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$ , when they mean  $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$ . Again this is not an equality.

# Asymptotic Notation in Equations

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Note that  $\Theta(n)$  is on the right hand side, otw. this interpretation is wrong.



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# Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function  $f(n) \in \mathcal{O}(n)$  there is an anonymous function  $g(n) \in \Theta(n^2)$  that makes the expression true.

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**Careful!**

“It is understood” that every occurrence of an  $\Theta$ -symbol (or  $\Theta, \Omega, o, \omega$ ) on the left represents **one anonymous function**.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$$

# Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\left\{ f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n) \right. \\ \left. \text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \right\}$$

# Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

# Asymptotic Notation

## Lemma 3

Let  $f, g$  be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$  (the same for  $g$ ). Then

- ▶  $c \cdot f(n) \in \Theta(f(n))$  for any constant  $c$



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The expressions also hold for  $\Omega$ . Note that this means that  $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$ .

# Asymptotic Notation

## Comments

- ▶ Do not use asymptotic notation within induction proofs.

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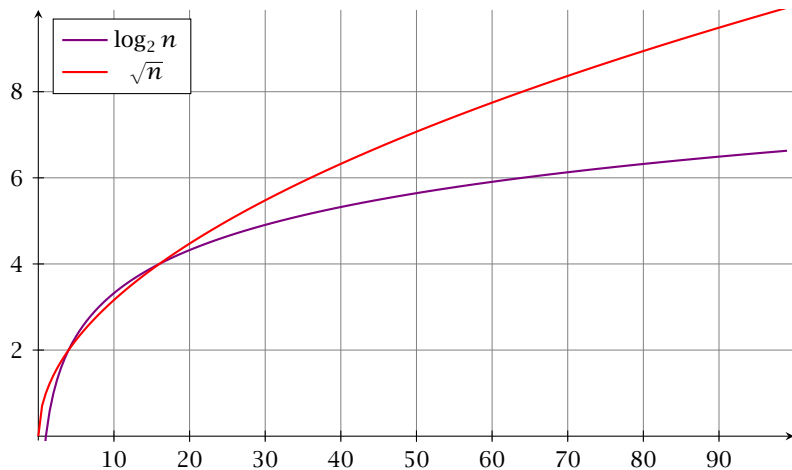
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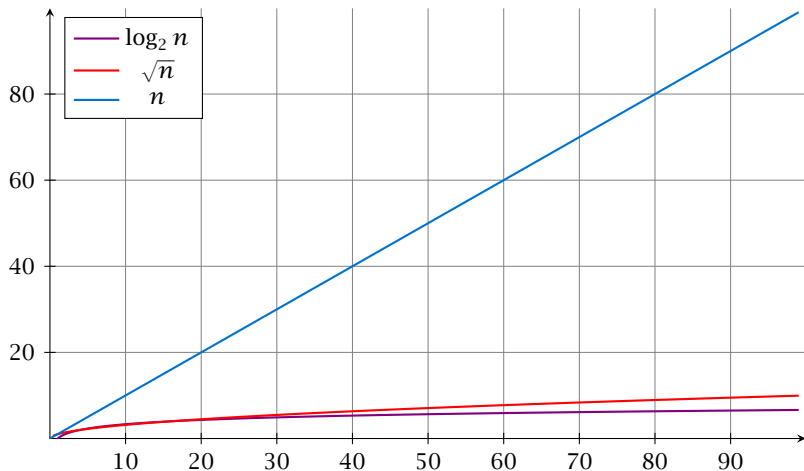
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- ▶ In general  $\log n = \log_2 n$ , i.e., we use 2 as the default base for the logarithm.

# Funktionen

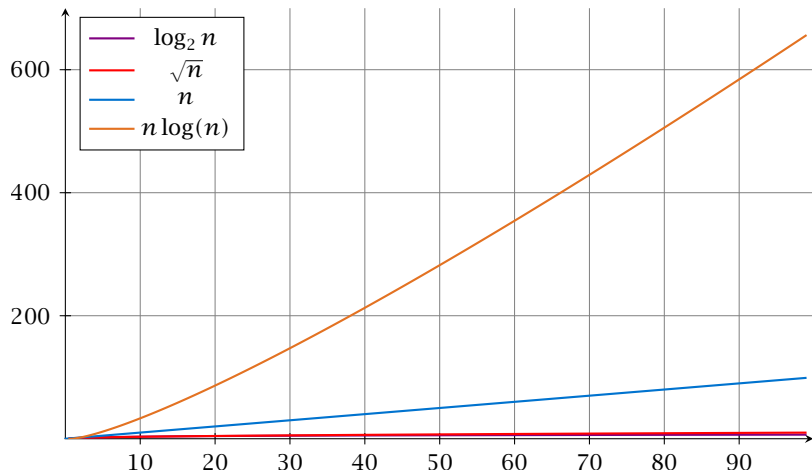




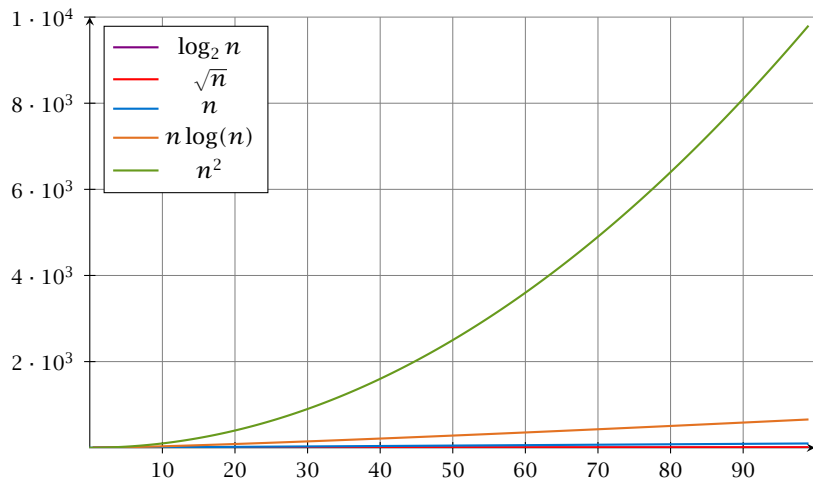
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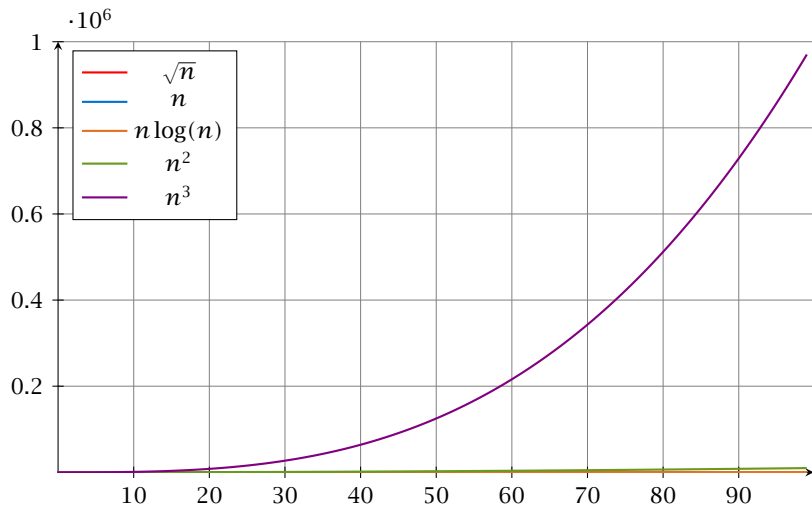
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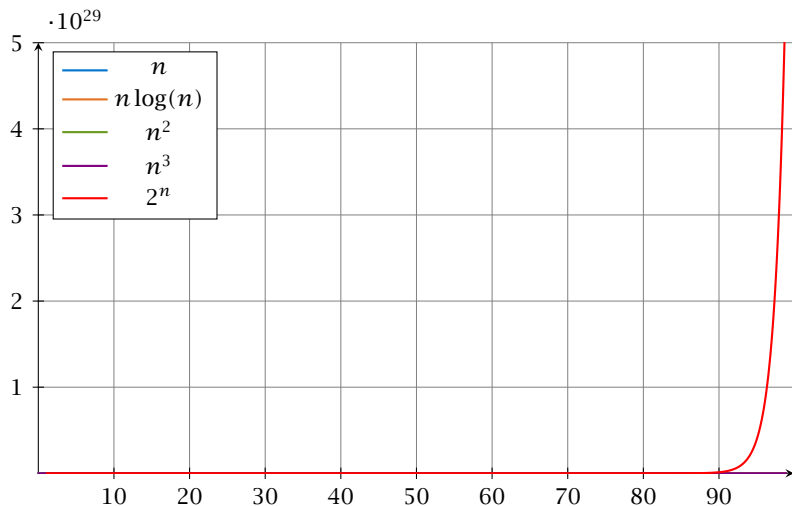
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# Laufzeiten

Funktion	Eingabelänge $n$							
	10	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$	$10^7$	$10^8$
$\log n$	33ns	66ns	0.1 $\mu$ s	0.1 $\mu$ s	0.2 $\mu$ s	0.2 $\mu$ s	0.2 $\mu$ s	0.3 $\mu$ s
$\sqrt{n}$	32ns	0.1 $\mu$ s	0.3 $\mu$ s	1 $\mu$ s	3.1 $\mu$ s	10 $\mu$ s	31 $\mu$ s	0.1ms
$n$	100ns	1 $\mu$ s	10 $\mu$ s	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3 $\mu$ s	6.6 $\mu$ s	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3 $\mu$ s	10 $\mu$ s	0.3ms	10ms	0.3s	10s	5.2min	2.7h
$n^2$	1 $\mu$ s	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
$n^3$	10 $\mu$ s	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$ y	
$1.1^n$	26ns	0.1ms	$7.8 \cdot 10^{25}$ y					
$2^n$	10 $\mu$ s	$4 \cdot 10^{14}$ y						
$n!$	36ms	$3 \cdot 10^{142}$ y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca.  $13.8 \cdot 10^9$ y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

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Clearly  $f = o(g)$ . However, as long as  $\log n \leq 1000$  Algorithm B will be more efficient.

# Multiple Variables in Asymptotic Notation

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## Formal Definition

Let  $f, g$  denote functions from  $\mathbb{N}^d$  to  $\mathbb{R}_0^+$ .

$$\blacktriangleright \mathcal{O}(f) = \{g \mid \exists c > 0 \exists N \in \mathbb{N}_0 \forall \vec{n} \text{ with } n_i \geq N \text{ for some } i : [g(\vec{n}) \leq c \cdot f(\vec{n})]\}$$

(set of functions that asymptotically grow **not faster** than  $f$ )

# Multiple Variables in Asymptotic Notation

## Example 4

►  $f : \mathbb{N} \rightarrow \mathbb{R}_0^+$ ,  $f(n, m) = 1$  und  $g : \mathbb{N} \rightarrow \mathbb{R}_0^+$ ,  $g(n, m) = n - 1$

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## 6 Recurrences

### Algorithm 2 mergesort(list $L$ )

```
1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
8: return  $L$ 
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This algorithm requires

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n)$$

comparisons when  $n > 1$  and 0 comparisons when  $n \leq 1$ .

# Recurrences

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For this we need to **solve** the recurrence.



# Methods for Solving Recurrences

## 1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

## 2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

## 3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

## 4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

## 5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

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First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

**Informal way:**

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One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

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Formally, this is not correct if  $n$  is not a power of 2. Also even in this case one would need to do an induction proof.

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

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Hence, statement is **true** if we choose  $d \geq c$ .

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Note that we can do this as for constant-sized inputs the running time is always some constant ( $b$  in the above case).

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$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

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$$\leq dn \log n - 0.33dn + cn$$



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$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

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$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

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$$\log n \leq \frac{n}{4}$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of  $d$ .

## 6.2 Master Theorem

### Lemma 5

Let  $a \geq 1$ ,  $b \geq 1$  and  $\epsilon > 0$  denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

#### Case 1.

If  $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$  then  $T(n) = \Theta(n^{\log_b a})$ .

#### Case 2.

If  $f(n) = \Theta(n^{\log_b(a)} \log^k n)$  then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ ,  
 $k \geq 0$ .

#### Case 3.

If  $f(n) = \Omega(n^{\log_b(a)+\epsilon})$  and for sufficiently large  $n$   
 $af\left(\frac{n}{b}\right) \leq cf(n)$  for some constant  $c < 1$  then  $T(n) = \Theta(f(n))$ .

## 6.2 Master Theorem

We prove the Master Theorem for the case that  $n$  is of the form  $b^\ell$ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

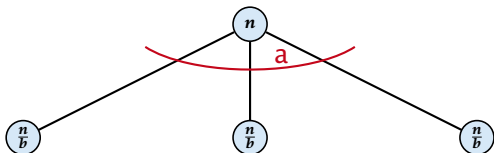
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The running time of a recursive algorithm can be visualized by a recursion tree:



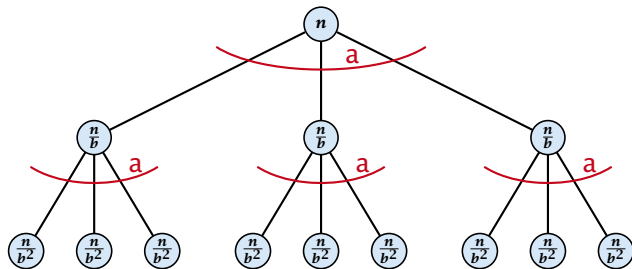
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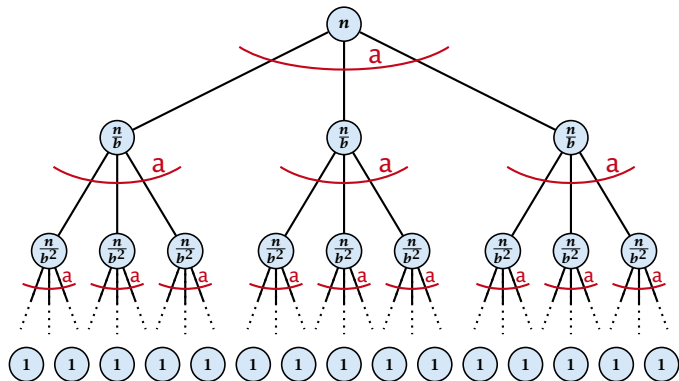
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# The Recursion Tree

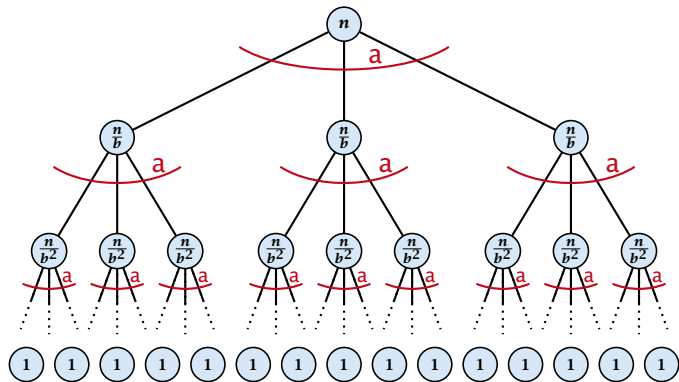
The running time of a recursive algorithm can be visualized by a recursion tree:





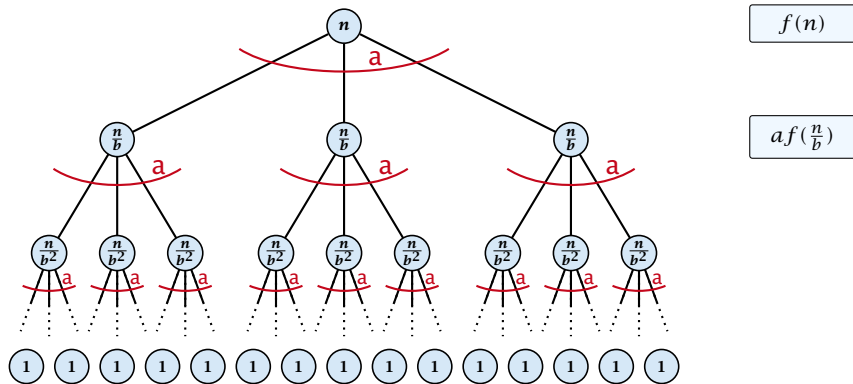
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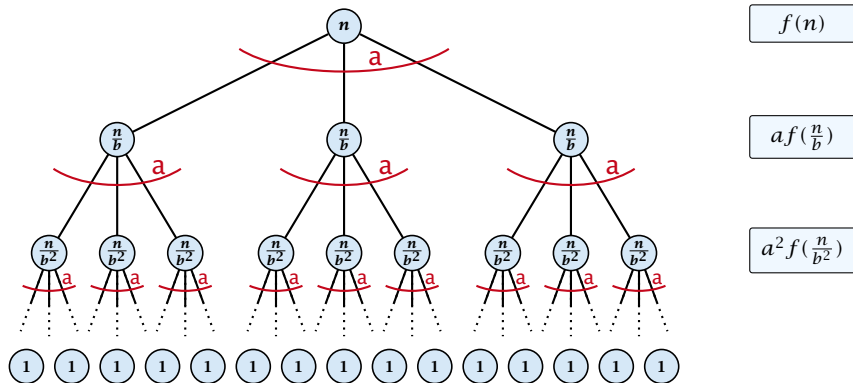
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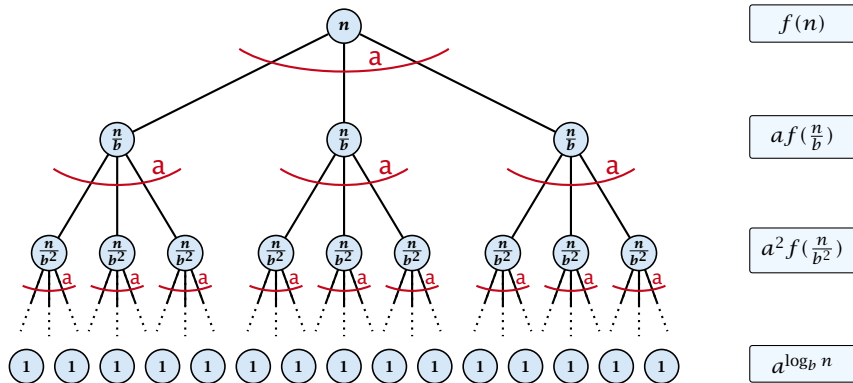
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:





## 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

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$$T(n) = n^{\log_b a}$$



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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

$$\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

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$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$



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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$

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Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

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$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \end{aligned}$$

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$



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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

The diagram shows two 9-bit integers,  $A$  and  $B$ , aligned for addition. Integer  $A$  is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer  $B$  is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of  $B$ . A vertical light blue box highlights the rightmost bit of  $A$  (the least significant bit) and the bit of  $B$  directly below it, indicating the first step in the addition process.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \phantom{1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1} 1 \\ \phantom{1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1} 0 \end{array}$$

The diagram shows the addition of two 9-bit integers, A and B. Integer A is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer B is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. A vertical box on the right contains the result of the addition, which is 0. A small '1' is written below the line under the 8th bit position, indicating a carry.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>								1	
									0

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 0\ 0 \end{array}$$

The diagram shows the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 and 0 highlighted in a light blue box. The carry bits are indicated by small '1's below the 7th and 8th bits of B.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 1 & 1 & & \\ & & & & & & & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. A horizontal line is drawn under the numbers. A vertical light blue box highlights the 7th bit position (from the right). In this position, the bits from A and B are 1 and 0, respectively, and their sum is 1. The 8th bit position also shows a carry of 1 from the 7th position, with the sum of the bits being 1 + 1 = 0. The final result of the addition is 00 in the 8th and 9th positions.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, using a ripple carry adder. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the numbers. The result of the addition is shown below the line as 0 0 0. A vertical box highlights the carry propagation from the 6th bit to the 7th bit, and from the 7th bit to the 8th bit, illustrating the ripple carry effect.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
					1	1	1		
						0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 0\ 0 \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. The sum is 1, 0, 0, 0. The carry propagation is shown by the 0 and 1 bits below the horizontal line. The bit 0 is under the 5th bit from the left, and the bit 1 is under the 6th bit from the left. The bit 1 is also under the 7th bit from the left, and the bit 0 is under the 8th bit from the left. The bit 1 is also under the 9th bit from the left, and the bit 0 is under the 10th bit from the left.



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 9-bit result. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. The result is 0, 1, 0, 0, 0. A vertical box highlights the 5th bit of A (1) and the 5th bit of B (1), which are added to produce the 5th bit of the result (0). Small subscripts are placed below the 4th, 5th, 6th, 7th, and 8th bits of the result row, indicating carry bits from the previous step.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
				1	0	1	1	1	
				0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			0	0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A vertical box highlights the 4th bit (1) of A and the 4th bit (0) of B, which are being added together. Below the horizontal line, the result of the addition is shown as 0, 0, 1, 0, 0, 0. Small subscripts are present under the 4th and 5th bits of B: 1, 0, 1, 1, 1.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & 1 & 1 & 0 & 1 & 1 & 1 & & \\ & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A vertical blue box highlights the third bit position (index 2 from the right), where a carry of 1 is generated from the addition of the two 0s. This carry is then added to the next bit position (index 3), where the sum of 0 and 0 plus the carry 1 results in 1. This process repeats for the next bit position (index 4), where the sum of 1 and 1 plus the carry 1 results in 1 and a carry of 1. The final result of the addition is 0, 0, 1, 0, 0, 0, 0.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

The diagram shows the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under B. Below the line, the sum is shown as 1, 0, 0, 1, 0, 0, 0. A vertical box highlights the third bit position (index 2 from the right) of the input numbers and the resulting carry bit '1' below the line. Small subscripts '0' and '1' are placed below the second and third bits of B, respectively, indicating carry propagation.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
		1	0	0	1	0	0	0	

*Note: In the original image, a blue box highlights the first column of bits (1, 1, 0) and the carry bit 0 below the horizontal line. Small subscripts 0, 1, 1, 0, 1, 1, 1 are placed below the bits of B from the second column onwards.*

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	<hr/>									
	1	1	0	0	1	0	0	0		

The diagram illustrates the addition of two 8-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 and the bits of B are 1 0 0 0 1 0 0 1. The result of the addition is 1 1 0 0 1 0 0 0. A vertical box highlights the first two bits of the result, 1 1, which correspond to the carry bits from the addition of the first two bits of A and B.



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
1	0	0	1	1	0	1	0	1	1	$B$
	0	1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	1	$B$
<hr/>										
1	0	1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two  $n$ -bit integers can be added in time  $\mathcal{O}(n)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$



## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0 \\ \phantom{1\ 0\ 0\ 0\ 1} 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1 \\ \phantom{1\ 0\ 0\ 0\ 1} 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0 \end{array}$$

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

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- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .
- ▶ Adding  $m$  numbers of length  $\leq 2n$ :  $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$ .

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**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .



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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$



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**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
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In our case  $a = 4$ ,  $b = 2$ , and  $f(n) = \Theta(n)$ . Hence, we are in Case 1, since  $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$ .

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A huge improvement over the “school method”.

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

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Note that we ignore **boundary conditions** for the moment.

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- ▶ First consider the homogenous case.

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The solution space

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**How do we find a non-trivial solution?**

We guess that the solution is of the form  $\lambda^n$ ,  $\lambda \neq 0$ , and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all  $n \geq k$ .

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Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

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Let  $\lambda_1, \dots, \lambda_k$  be the  $k$  (complex) roots of  $P[\lambda]$ . Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values  $\alpha_i$ .

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## Lemma 6

Assume that the characteristic polynomial has  $k$  *distinct* roots  $\lambda_1, \dots, \lambda_k$ . Then *all* solutions to the recurrence relation are of the form

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We show that the above set of solutions contains one solution for every choice of boundary conditions.



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## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$



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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$



# Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all  $\lambda_i$ 's are different, then the determinant is non-zero.

# The Homogeneous Case

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Suppose we have a root  $\lambda_i$  with multiplicity (**Vielfachheit**) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

Since  $\lambda_i$  is a root we can write this as  $Q[\lambda] \cdot (\lambda - \lambda_i)^2$ . Calculating the derivative gives a polynomial that still has root  $\lambda_i$ .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

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Hence,

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Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$



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We can continue  $j-1$  times.

Hence,  $n^\ell \lambda_i^n$  is a solution for  $\ell \in 0, \dots, j-1$ .

# The Homogeneous Case

## Lemma 7

Let  $P[\lambda]$  denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let  $\lambda_i, i = 1, \dots, m$  be the (complex) roots of  $P[\lambda]$  with multiplicities  $\ell_i$ . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of  $\alpha_{ij}$ 's is a solution to the recurrence.

## Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

## Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

## Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

# The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

with  $f(n) \neq 0$ .

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where  $T_h$  is **any** solution to the homogeneous equation, and  $T_p$  is **one** particular solution to the inhomogeneous equation.



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There is no general method to find a particular solution.

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I get a completely determined recurrence if I add  $T[0] = 1$  and  $T[1] = 2$ .

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$T[0] = 1$  gives  $\alpha = 1$ .

$T[1] = 2$  gives  $1 + \beta = 2 \Rightarrow \beta = 1$ .

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3 \end{aligned}$$

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$$T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2$$

and so on...

## 6.4 Generating Functions

### Definition 8 (Generating Function)

Let  $(a_n)_{n \geq 0}$  be a sequence. The corresponding

- ▶ **generating function** (**Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} a_n z^n ;$$



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$$F(z) := \sum_{n \geq 0} a_n z^n ;$$

- ▶ **exponential generating function** (**exponentielle Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n .$$

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### Example 9

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- ▶ **Multiplication:**  $f \cdot g := \sum_{n \geq 0} c_n z^n$  with  $c_n = \sum_{p=0}^n a_p b_{n-p}$ .

There are no convergence issues here.

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Then, it is important to think about convergence/convergence radius etc.

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This is well-defined.

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Hence, the generating function of the sequence  $a_n = n + 1$  is  $1/(1-z)^2$ .

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$\frac{1}{n!}$	$e^z$

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6. The coefficients of the resulting power series are the  $a_n$ .

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means  $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$ .

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### Example 10

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$



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Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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# Part III

## Data Structures

# Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.

The data types in this lecture all operate on objects that are represented by a `[key, value]` pair.

- ▶ The `key` comes from a totally ordered set, and we assume that there is an efficient comparison function.
- ▶ The `value` can be anything; it usually carries satellite information important for the application that uses the ADT.

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- ▶  **$S$ .search( $k$ )**: Returns pointer to object  $x$  from  $S$  with  $\text{key}[x] = k$  or **null**.



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- ▶  **$S$ . predecessor( $x$ )**: Return pointer to the next smaller element in  $S$  or **null** if  $x$  is minimum.

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- ▶ **S. decrease-key( $x, k$ ):** Replace  $\text{key}[x]$  by  $k \leq \text{key}[x]$ .

## Examples of ADTs

### Stack:

- ▶  $S.$  **push**( $x$ ): Insert an element.
- ▶  $S.$  **pop**(): Return the element from  $S$  that was inserted most recently; delete it from  $S$ .
- ▶  $S.$  **empty**(): Tell if  $S$  contains any object.

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## Examples of ADTs

### Stack:

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### Priority-Queue:

- ▶ ***S.insert(x)***: Insert an element.
- ▶ ***S.delete-min()***: Return the element with lowest key-value; delete it from *S*.

## 7 Dictionary

### Dictionary:

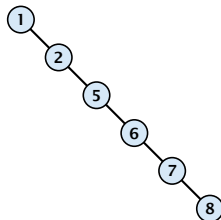
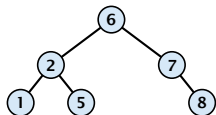
- ▶  **$S.$  insert( $x$ )**: Insert an element  $x$ .
- ▶  **$S.$  delete( $x$ )**: Delete the element pointed to by  $x$ .
- ▶  **$S.$  search( $k$ )**: Return a pointer to an element  $e$  with  $\text{key}[e] = k$  in  $S$  if it exists; otherwise return **null**.

## 7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node  $v$  have a smaller key-value than  $\text{key}[v]$  and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:



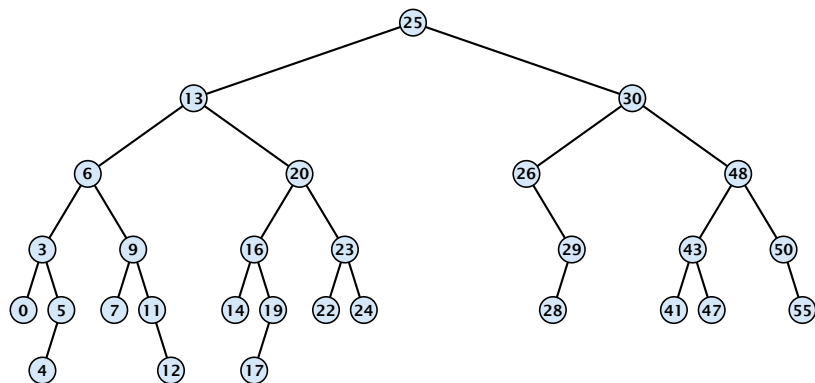


## 7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶  $T.\text{insert}(x)$
- ▶  $T.\text{delete}(x)$
- ▶  $T.\text{search}(k)$
- ▶  $T.\text{successor}(x)$
- ▶  $T.\text{predecessor}(x)$
- ▶  $T.\text{minimum}()$
- ▶  $T.\text{maximum}()$

# Binary Search Trees: Searching

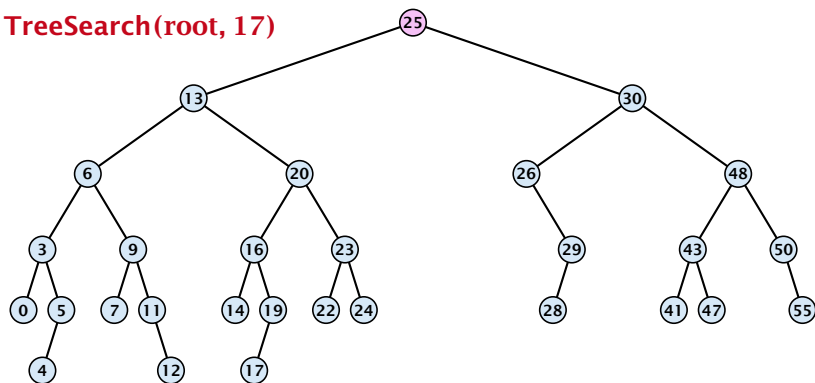


## Algorithm 1 TreeSearch( $x, k$ )

- 1: **if**  $x = \text{null}$  **or**  $k = \text{key}[x]$  **return**  $x$
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# Binary Search Trees: Searching

TreeSearch(root, 17)

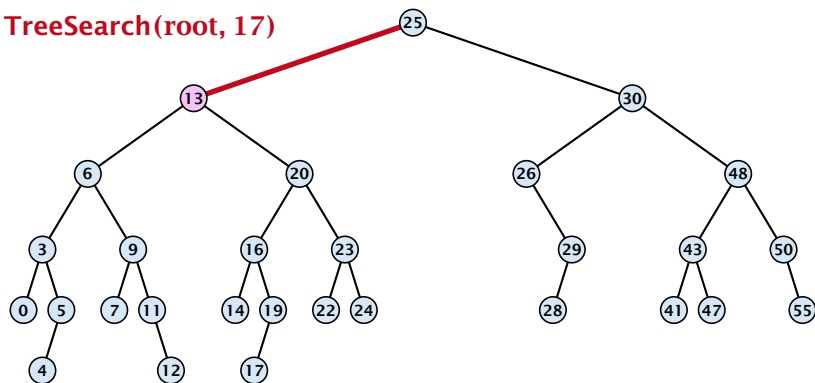


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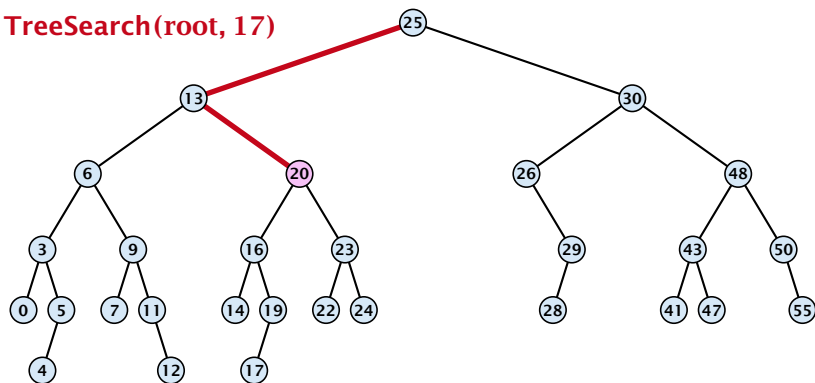


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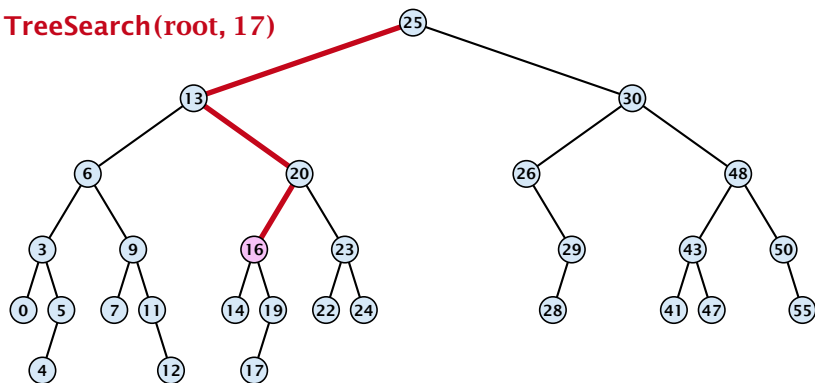


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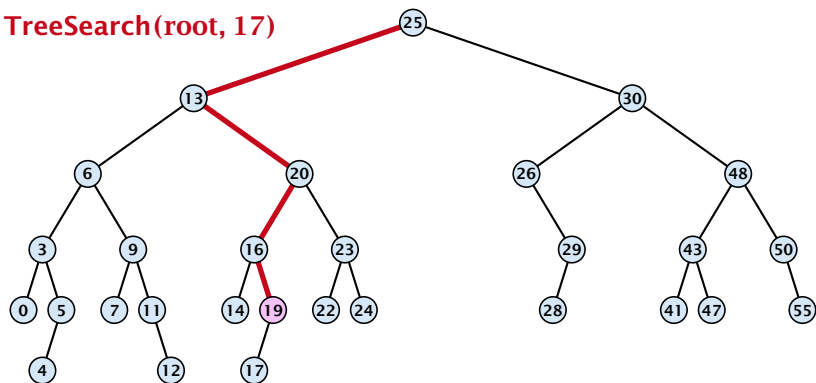


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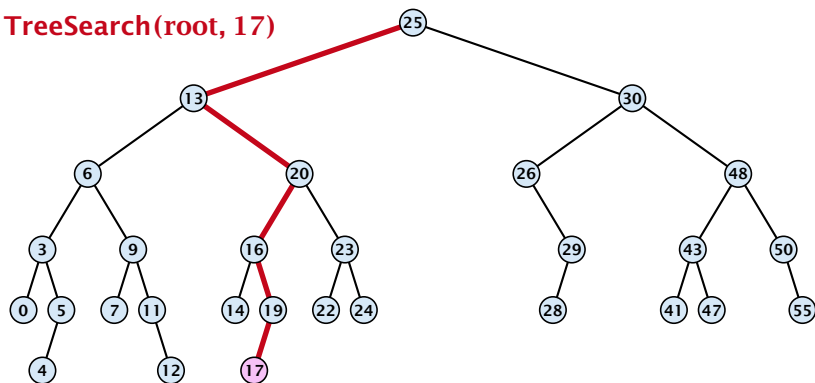


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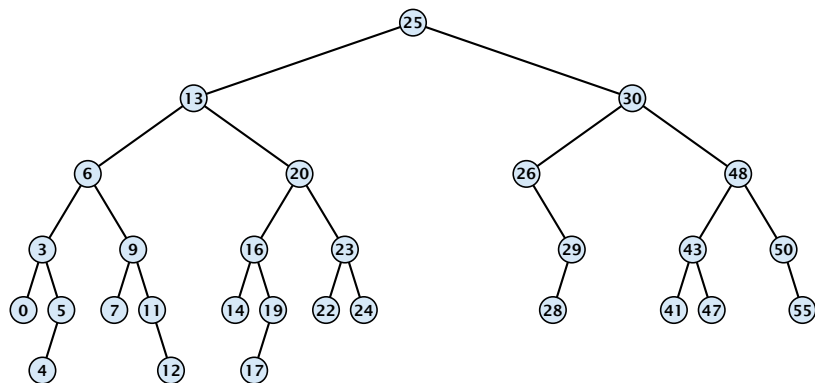


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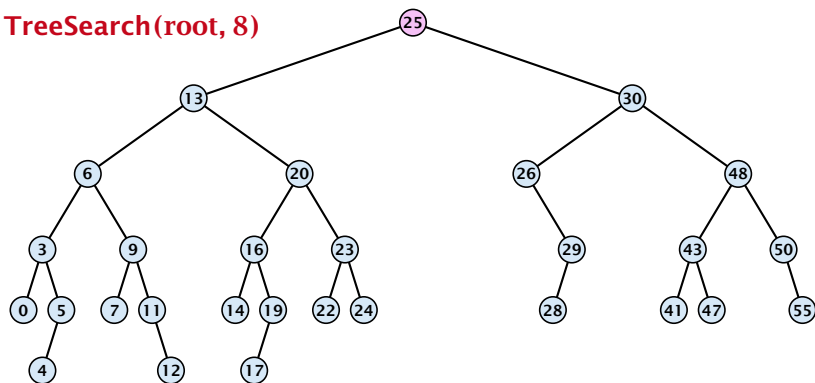


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# Binary Search Trees: Searching

TreeSearch(root, 8)

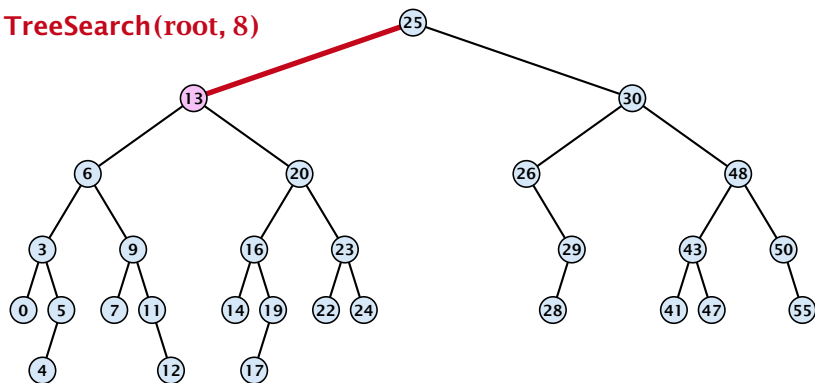


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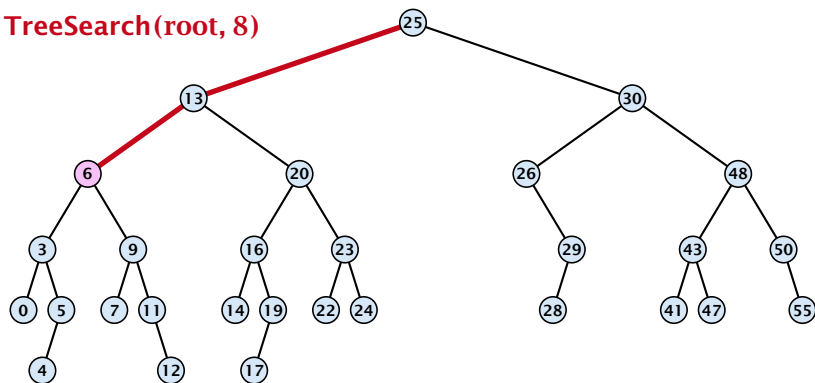


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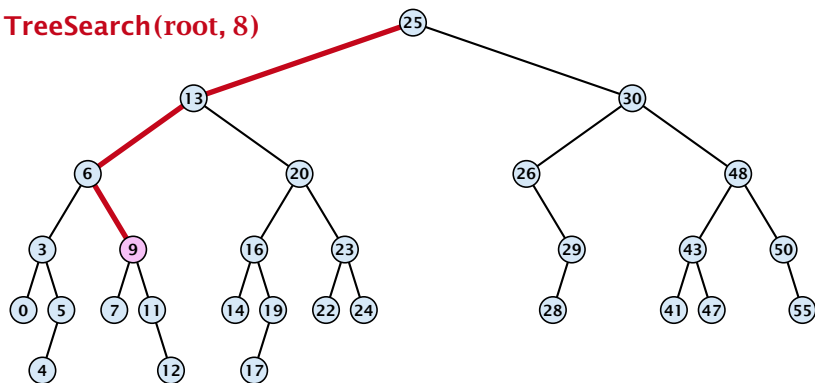


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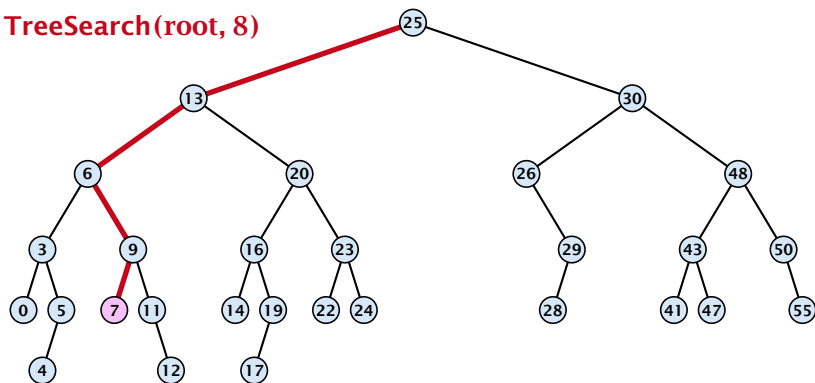


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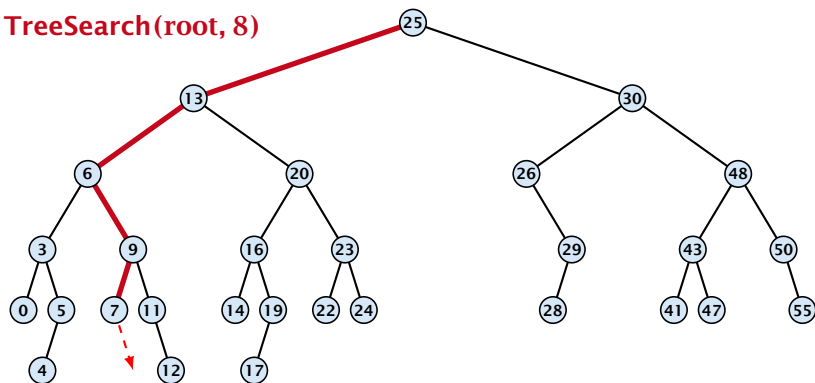


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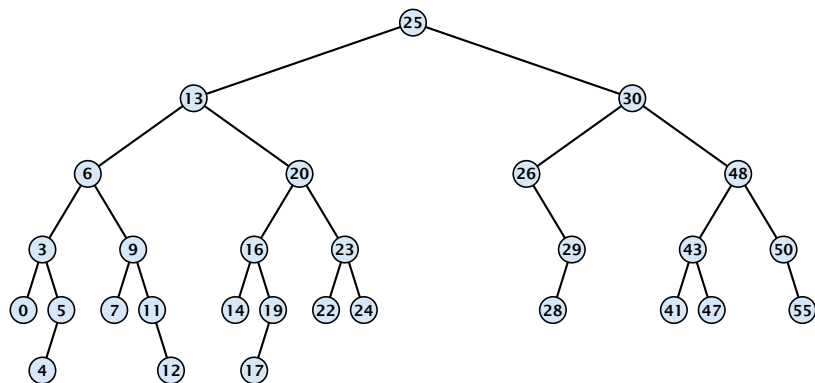
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# Binary Search Trees: Minimum

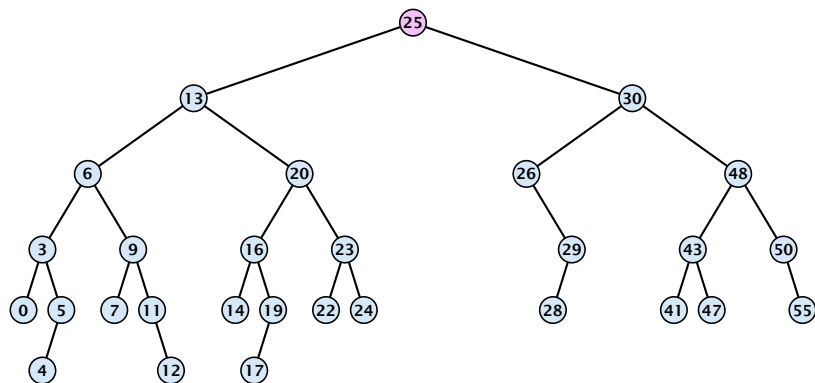


## Algorithm 2 TreeMin( $x$ )

- 1: **if**  $x = \text{null}$  **or**  $\text{left}[x] = \text{null}$  **return**  $x$
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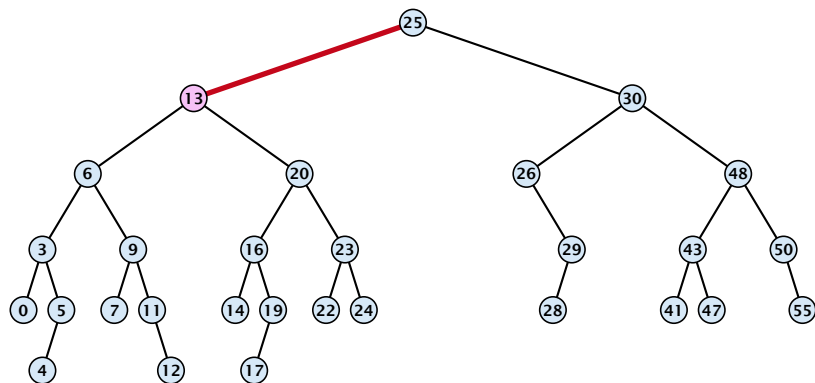
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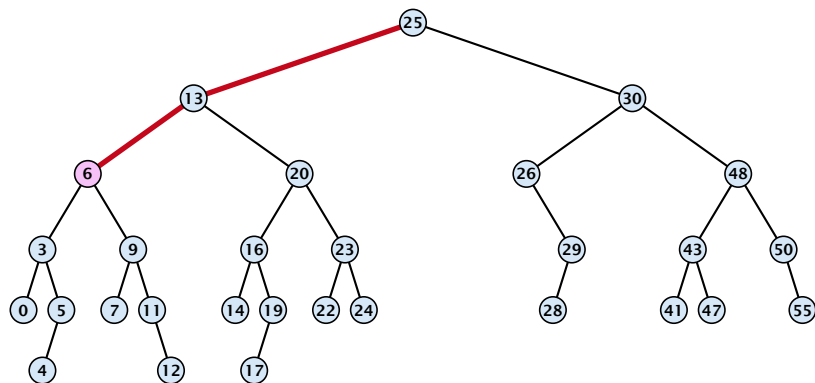
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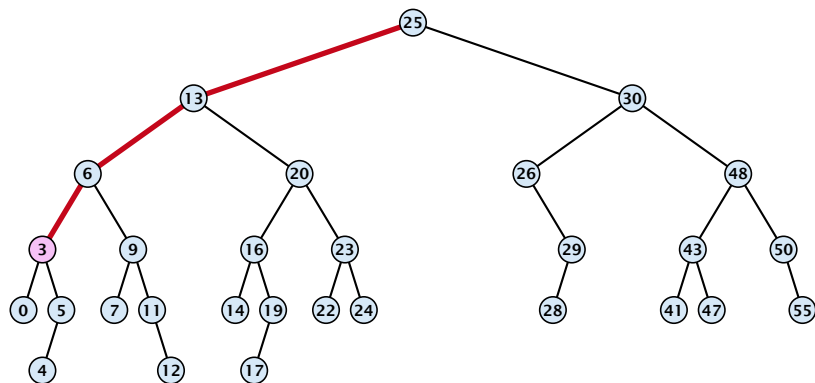
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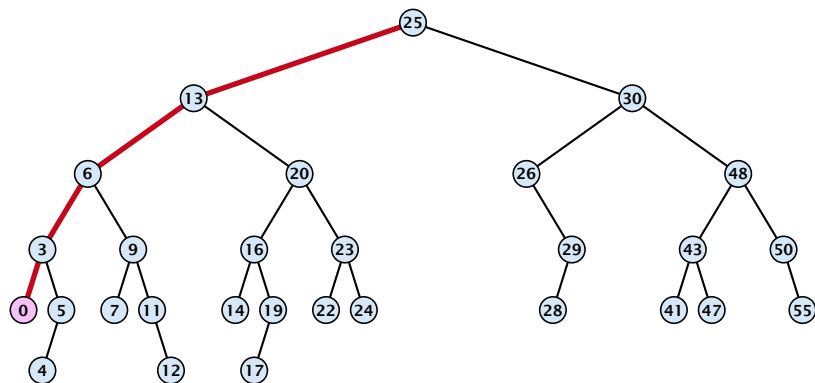
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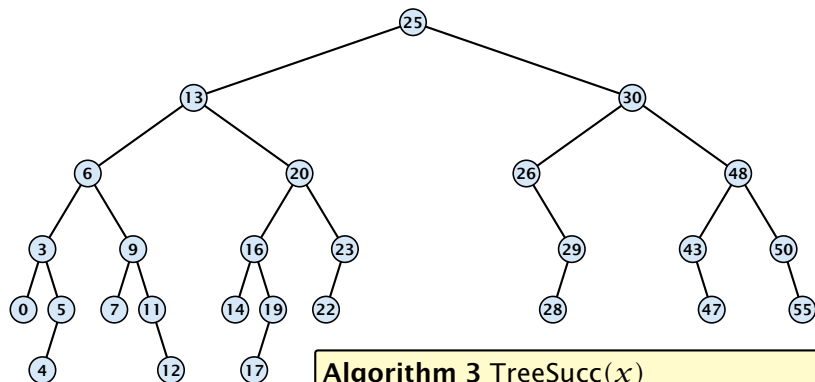
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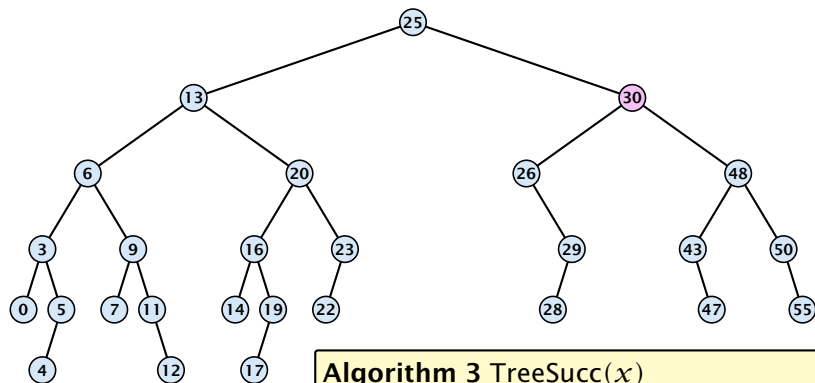
# Binary Search Trees: Successor



## Algorithm 3 TreeSucc( $x$ )

- 1: **if**  $\text{right}[x] \neq \text{null}$  **return**  $\text{TreeMin}(\text{right}[x])$
- 2:  $y \leftarrow \text{parent}[x]$
- 3: **while**  $y \neq \text{null}$  **and**  $x = \text{right}[y]$  **do**
- 4:      $x \leftarrow y; y \leftarrow \text{parent}[x]$
- 5: **return**  $y$ ;

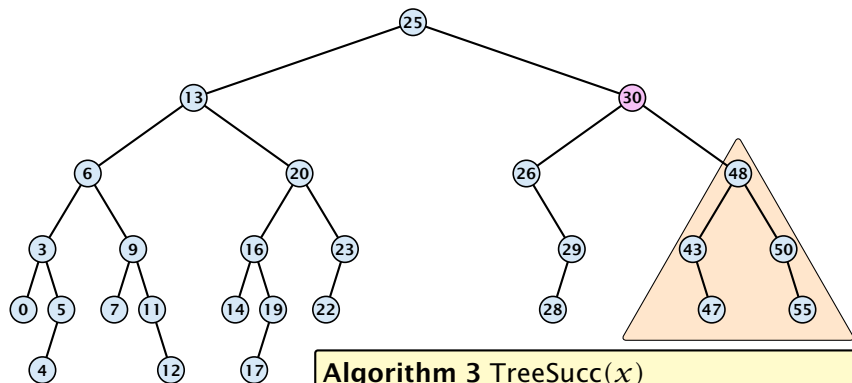
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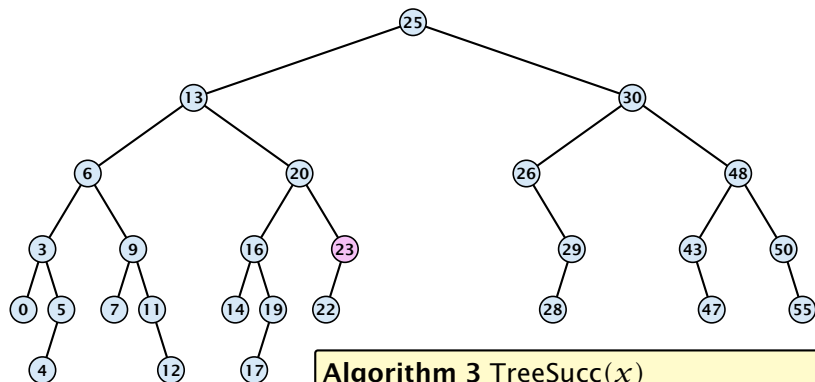


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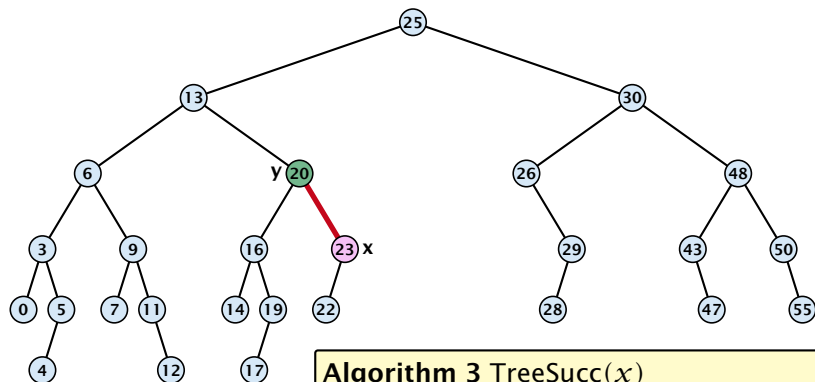
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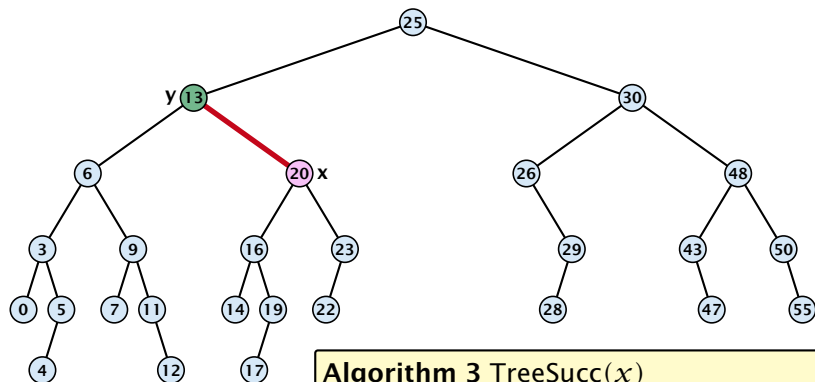
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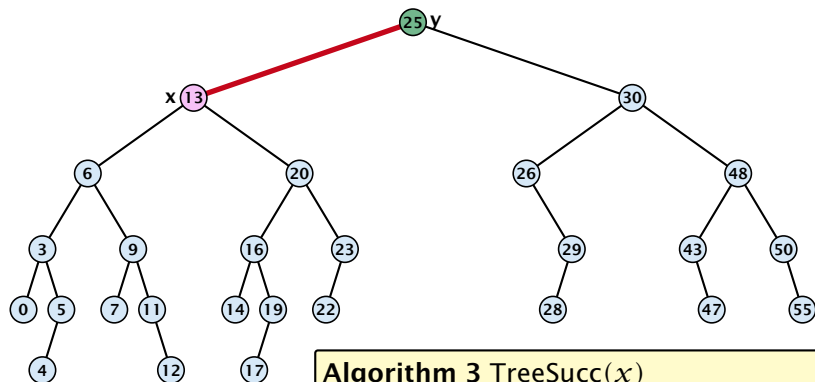
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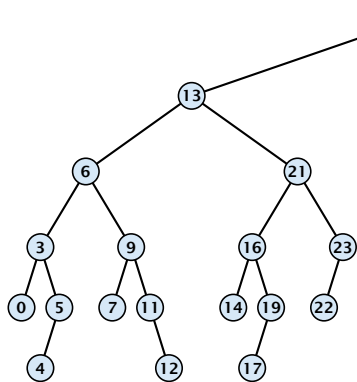
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## Binary Search Trees: Insert

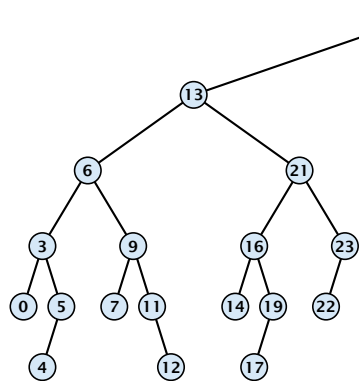


### Algorithm 4 TreeInsert( $x, z$ )

```
1: if  $x = \text{null}$  then
2:      $\text{root}[T] \leftarrow z$ ;  $\text{parent}[z] \leftarrow \text{null}$ ;
3:     return;
4: if  $\text{key}[x] > \text{key}[z]$  then
5:     if  $\text{left}[x] = \text{null}$  then
6:          $\text{left}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
7:     else TreeInsert( $\text{left}[x], z$ );
8: else
9:     if  $\text{right}[x] = \text{null}$  then
10:         $\text{right}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
11:    else TreeInsert( $\text{right}[x], z$ );
```

# Binary Search Trees: Insert

Insert element **not** in the tree.

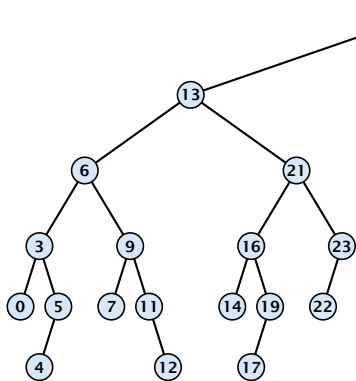


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- 3:     **return**;
- 4: **if**  $\text{key}[x] > \text{key}[z]$  **then**
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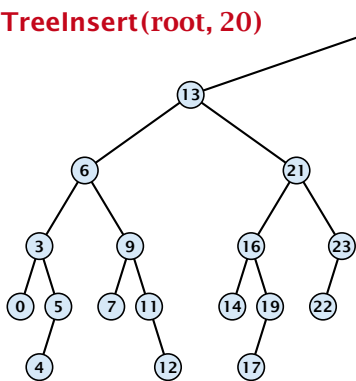
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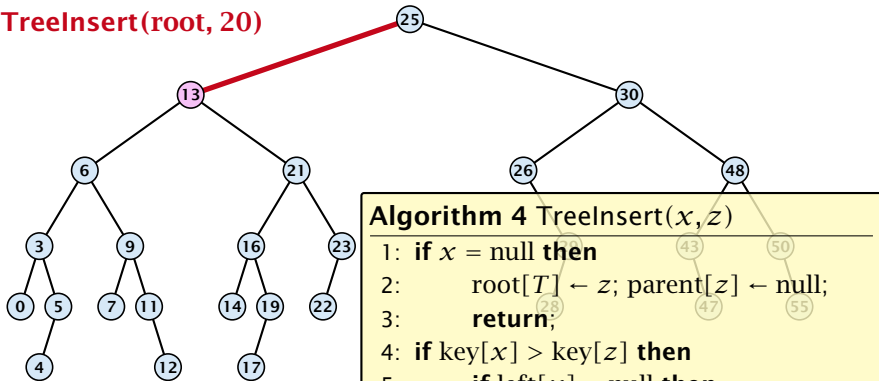
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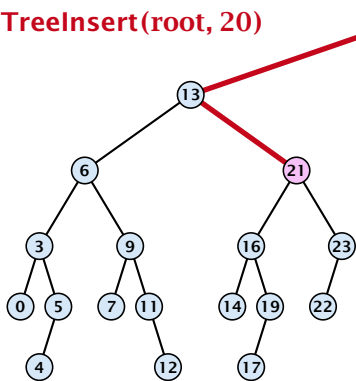
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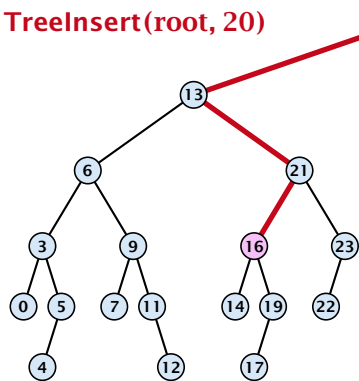
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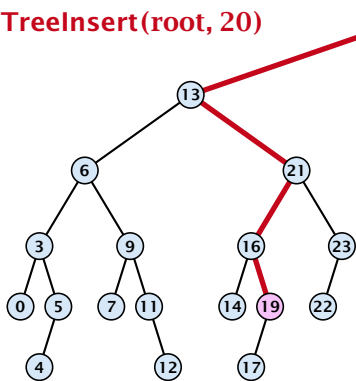
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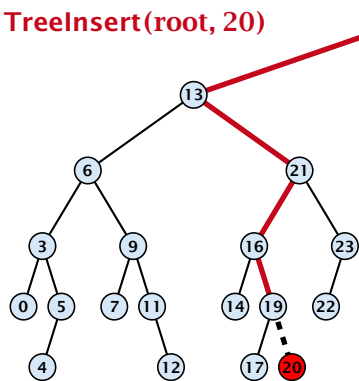
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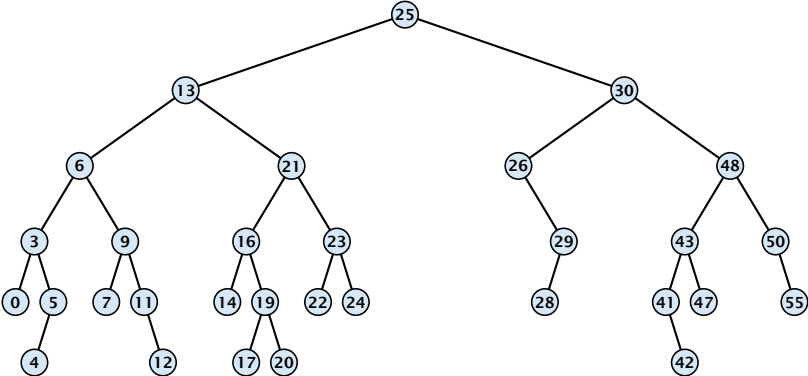


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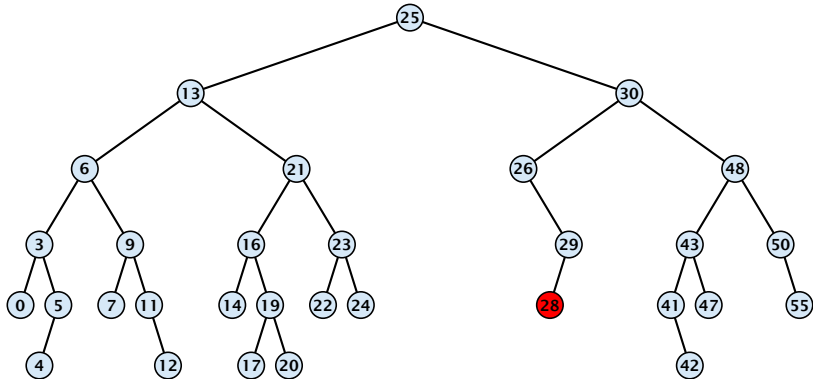
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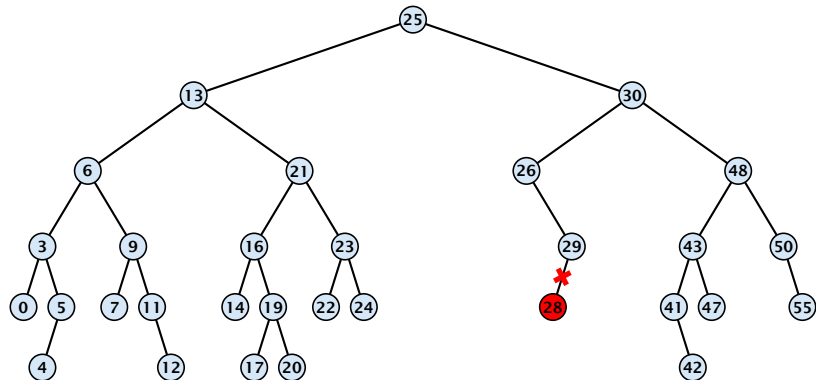


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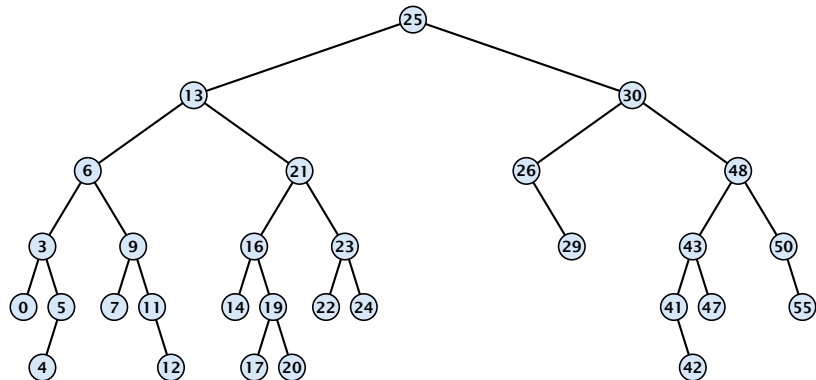
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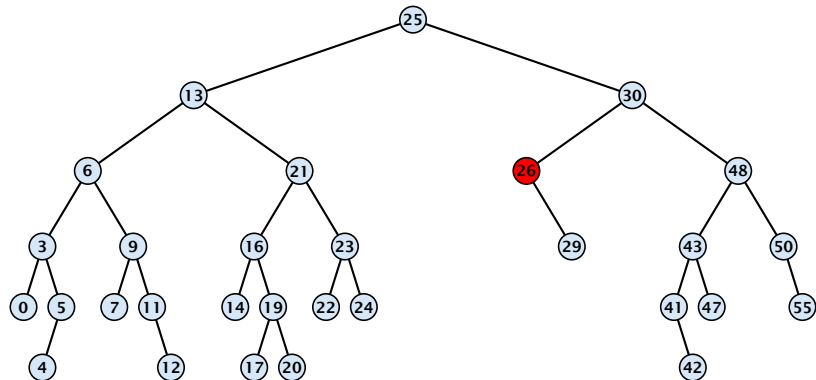


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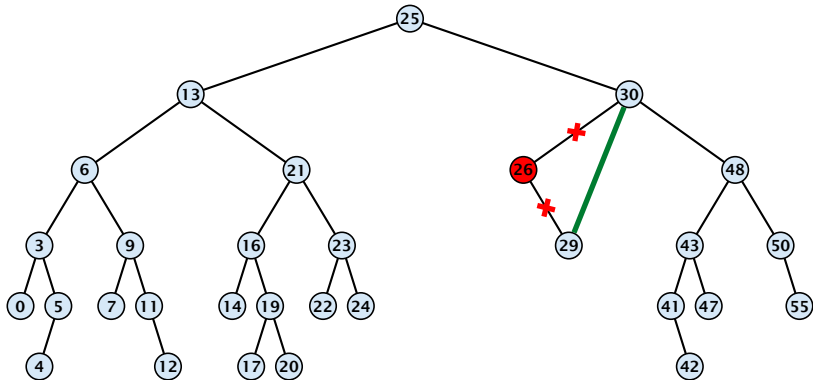


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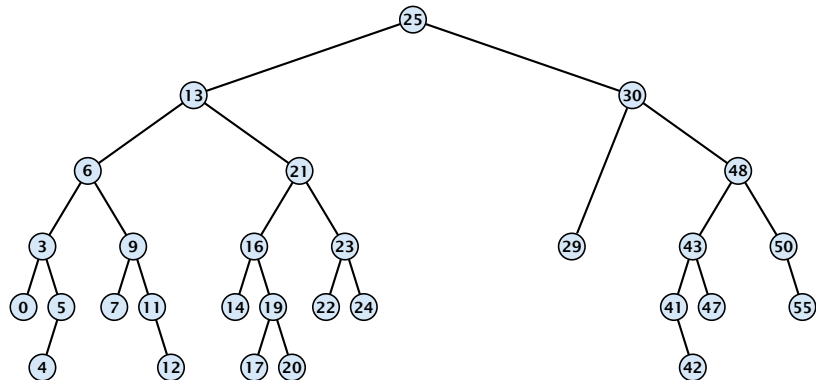


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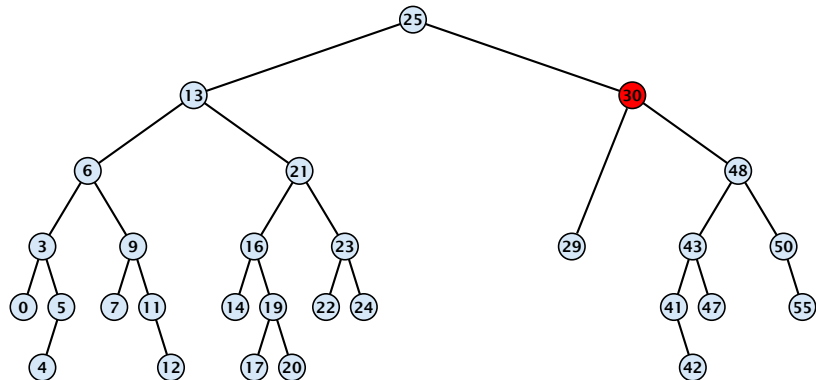


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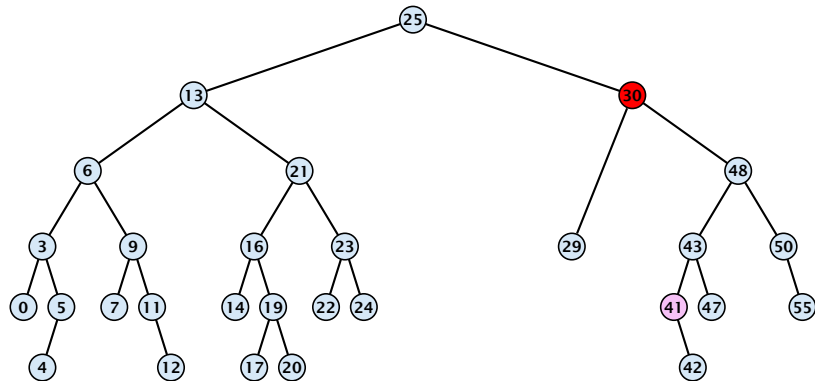


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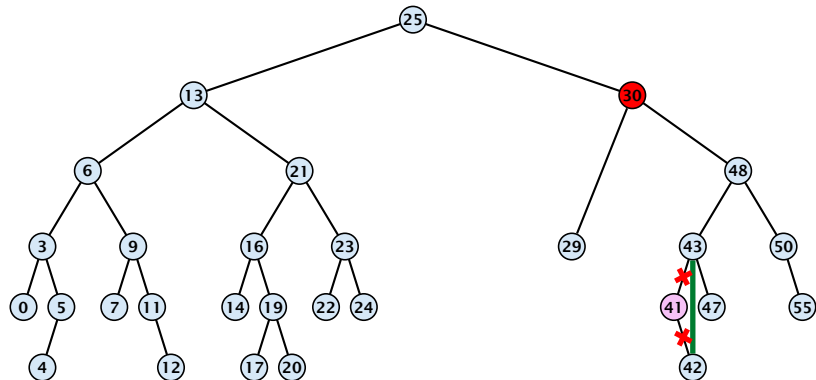


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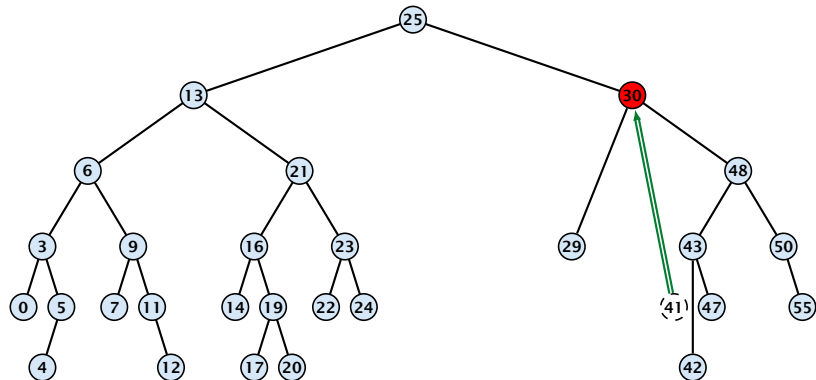


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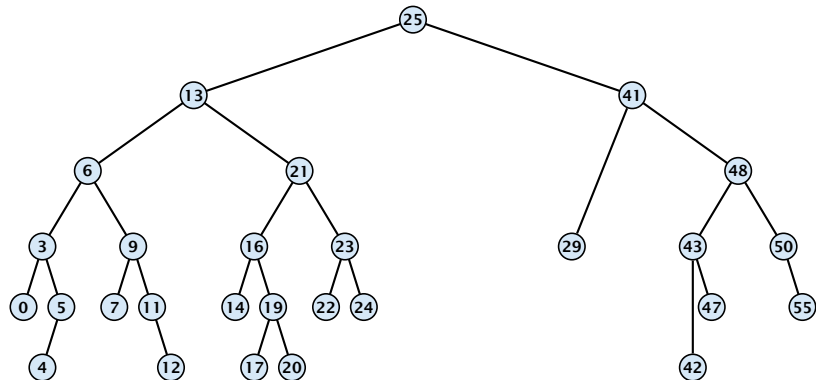
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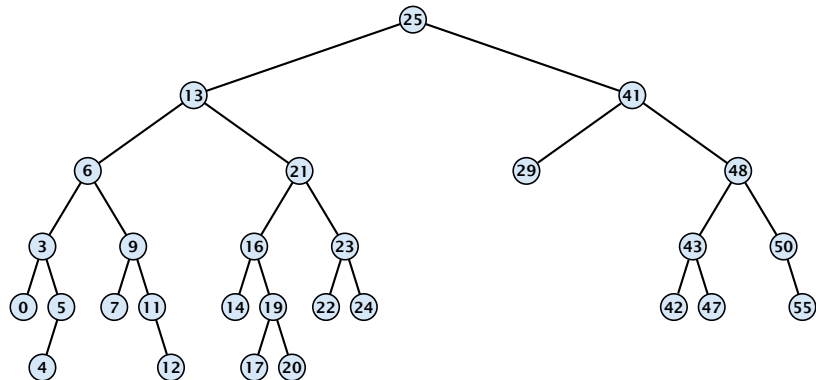


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# Binary Search Trees: Delete

## Algorithm 9 TreeDelete( $z$ )

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:     else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to  $x$

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AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.



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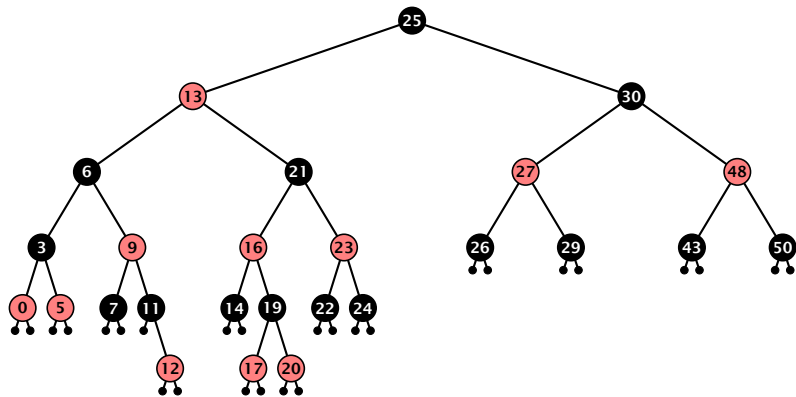
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# Red Black Trees: Example



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*A red-black tree with  $n$  internal nodes has height at most  $\mathcal{O}(\log n)$ .*



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We first show:

### Lemma 15

A sub-tree of black height  $\text{bh}(v)$  in a red black tree contains at least  $2^{\text{bh}(v)} - 1$  internal vertices.

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- ▶ Then  $T_v$  contains at least  $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$  vertices.



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Hence,  $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$ . □

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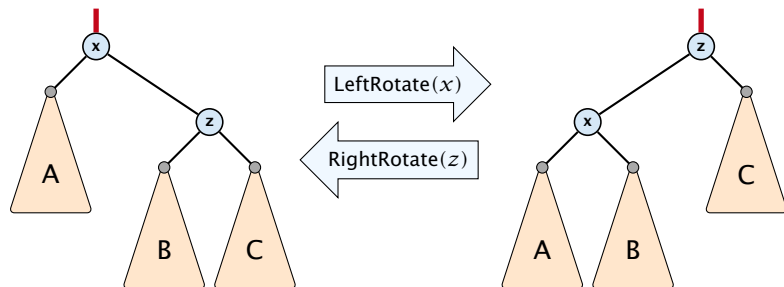
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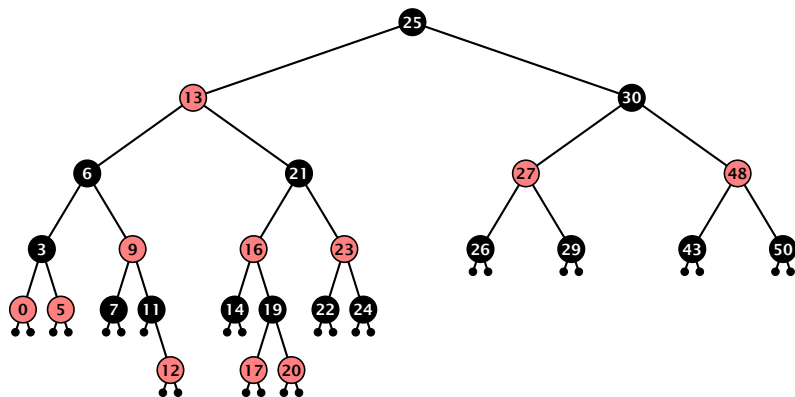
We need to adapt the insert and delete operations so that the red black properties are maintained.

# Rotations

The properties will be maintained through rotations:



# Red Black Trees: Insert

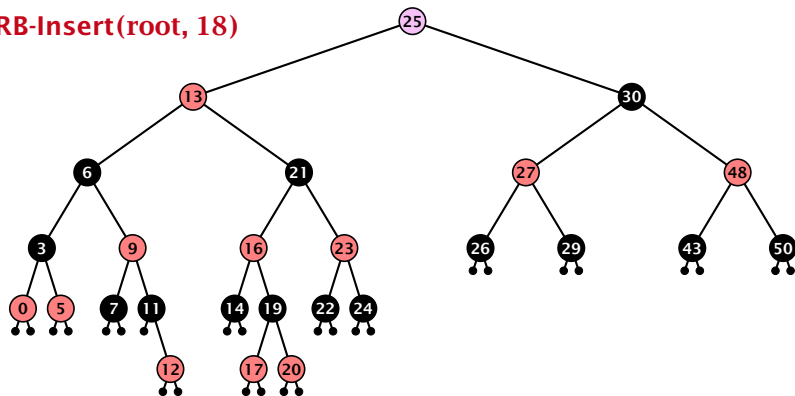


## Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

# Red Black Trees: Insert

RB-Insert(root, 18)



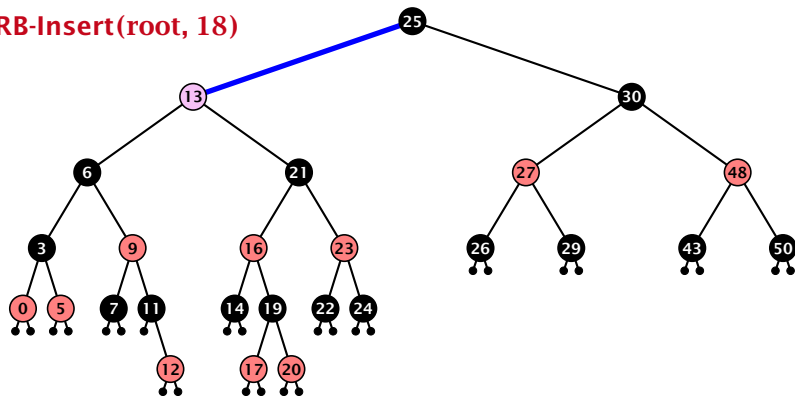
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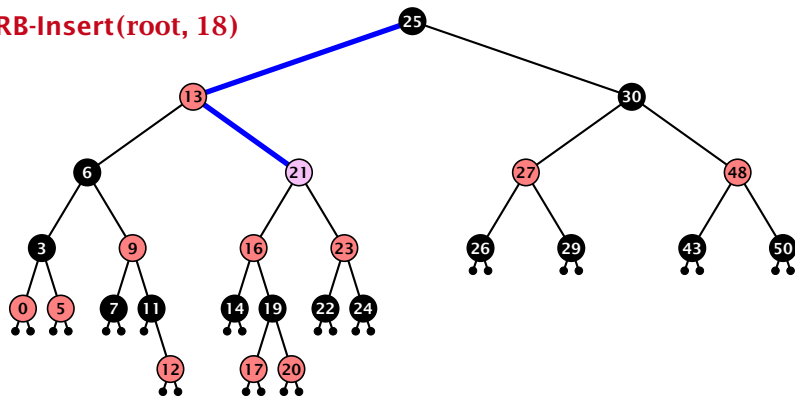


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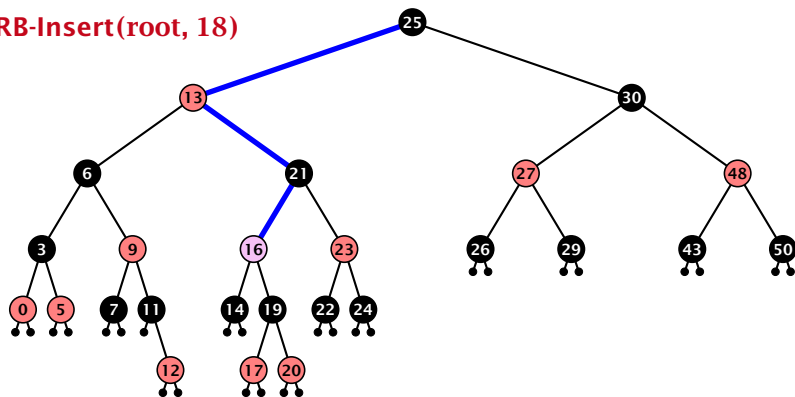


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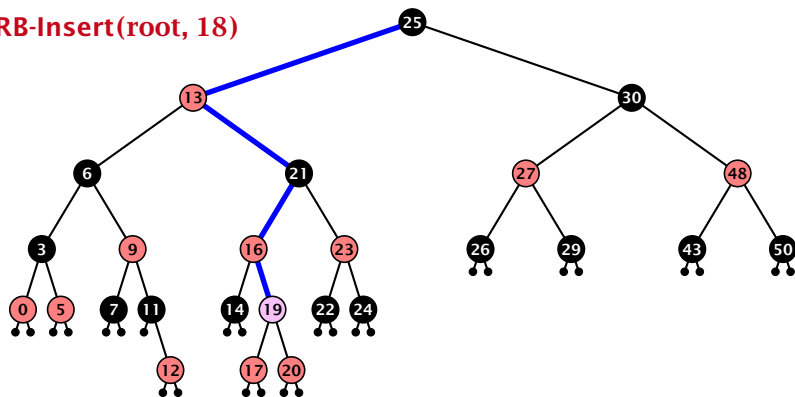


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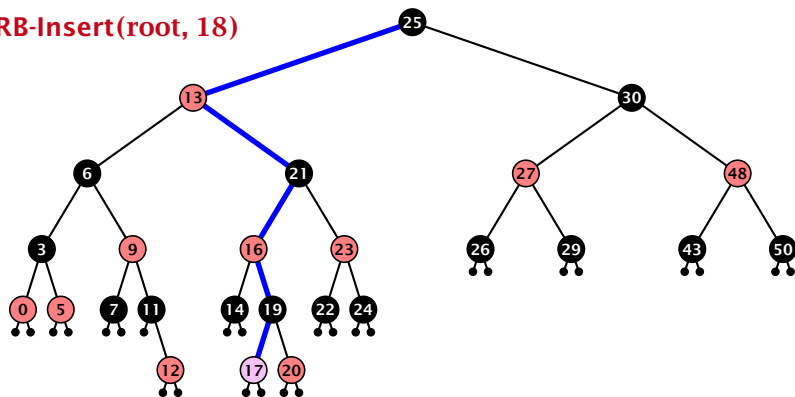


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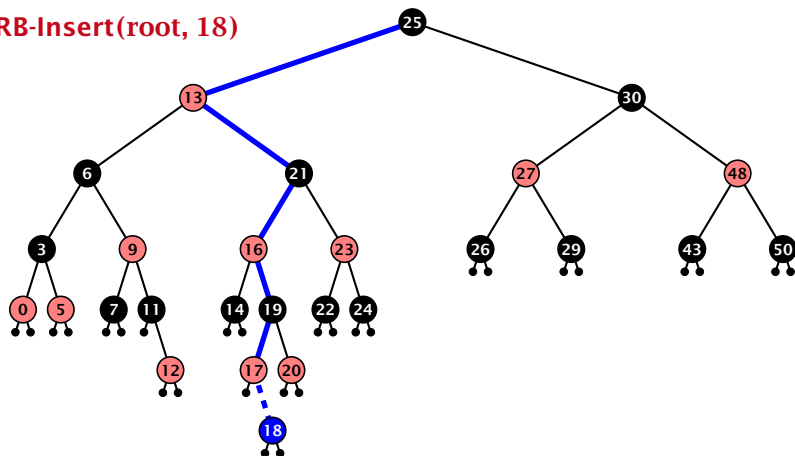


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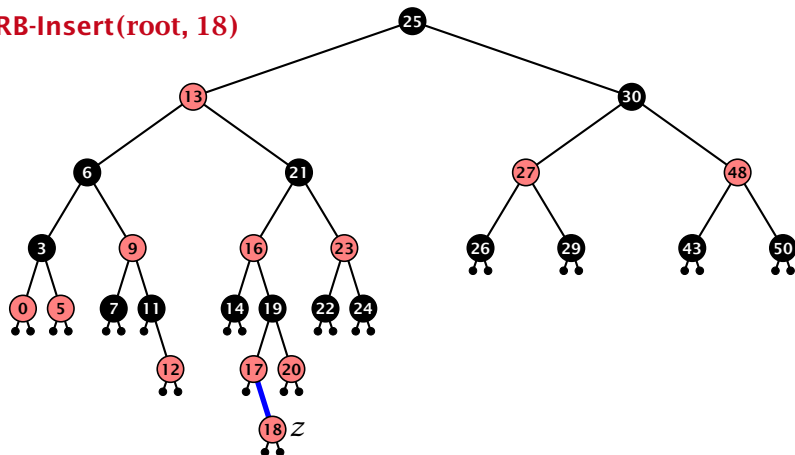


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**Invariant of the fix-up algorithm:**

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  - ▶ or the parent does not exist  
(violation since root must be black)

If  $z$  has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
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12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
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```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
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3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then Case 1: uncle red
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
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8:     if  $z$  = right[parent[ $z$ ]] then
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# Red Black Trees: Insert

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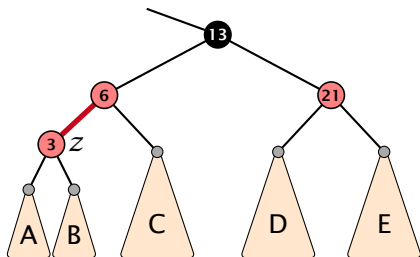
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1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
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11:        RightRotate(gp[ $z$ ]);
12:       else same as then-clause but right and left exchanged
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# Red Black Trees: Insert

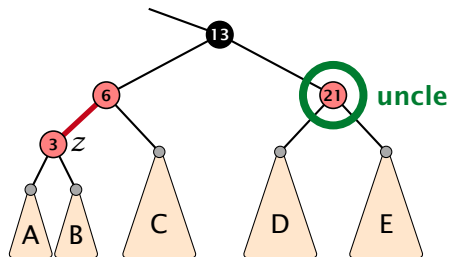
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10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red; 2b:  $z$  left child
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

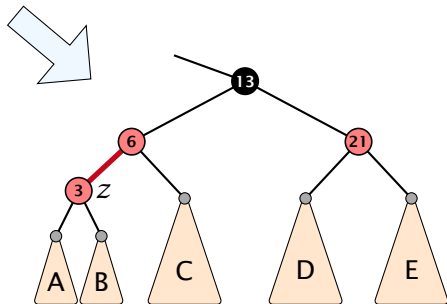
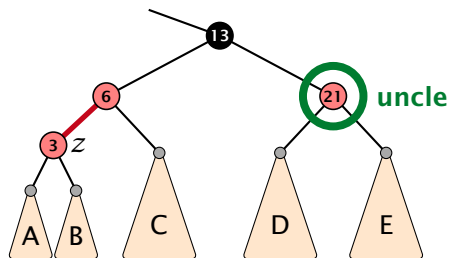
## Case 1: Red Uncle



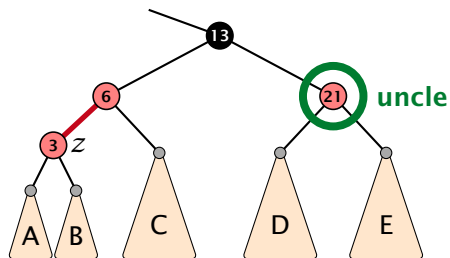
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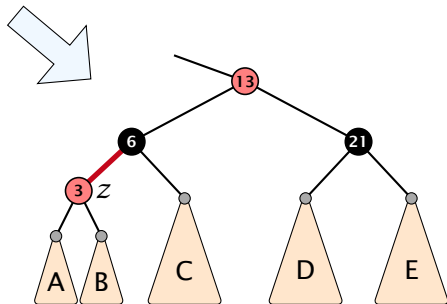
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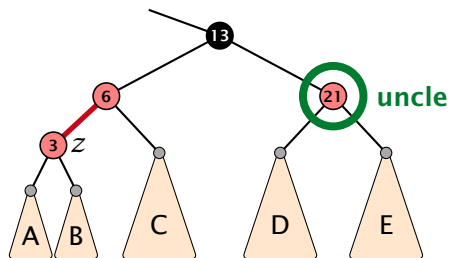
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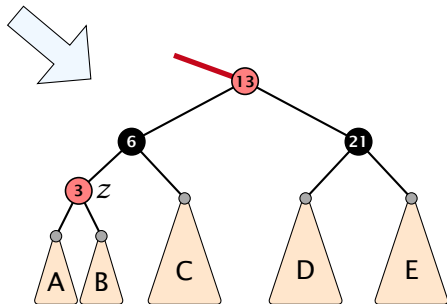
1. recolour



## Case 1: Red Uncle

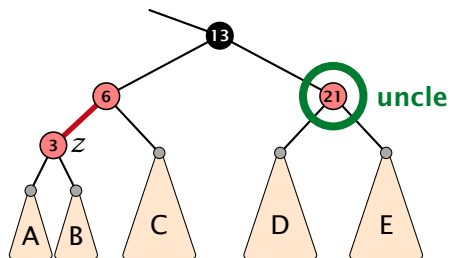


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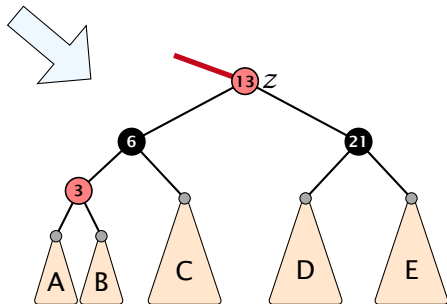




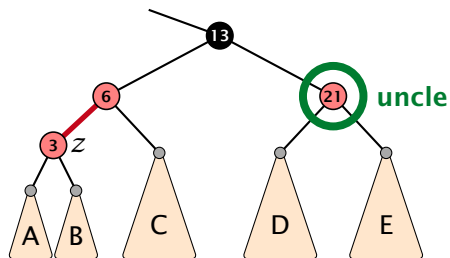
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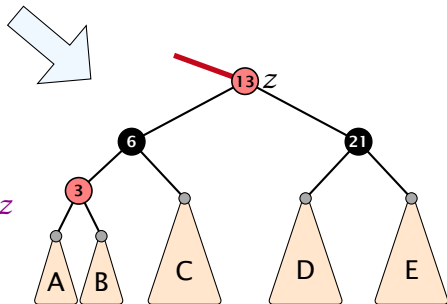
1. recolour
2. move  $z$  to grand-parent



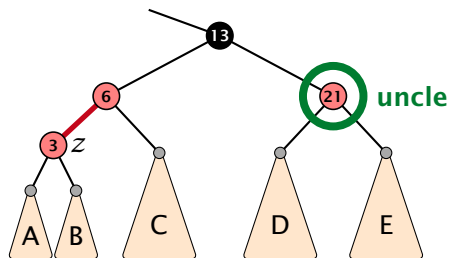
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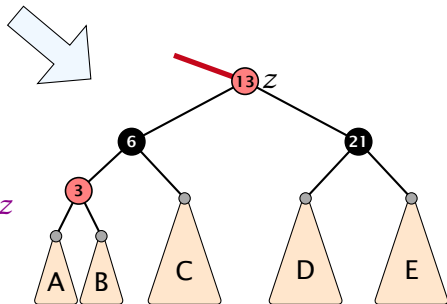
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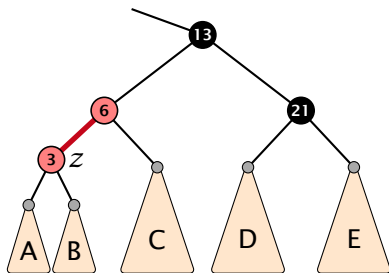
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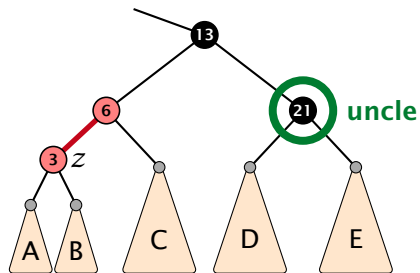
1. recolour
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$
4. you made progress



## Case 2b: Black uncle and z is left child

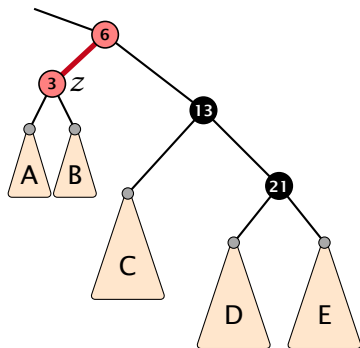
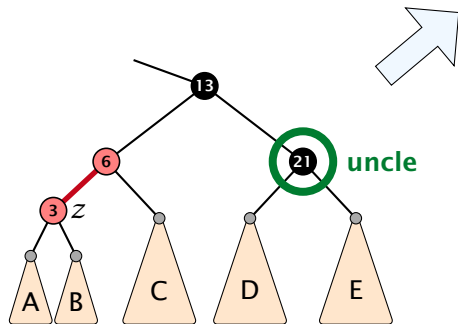


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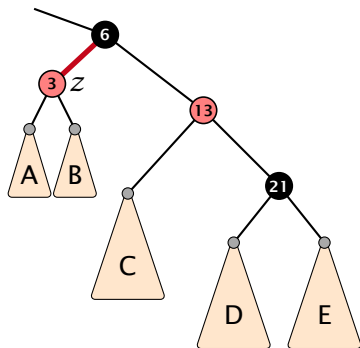
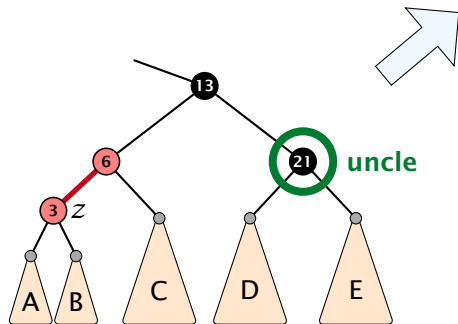
## Case 2b: Black uncle and z is left child

1. rotate around grandparent



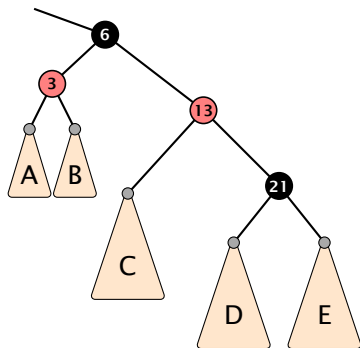
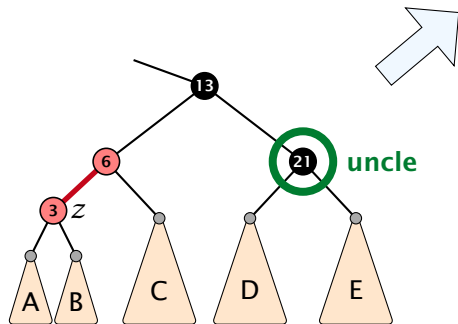
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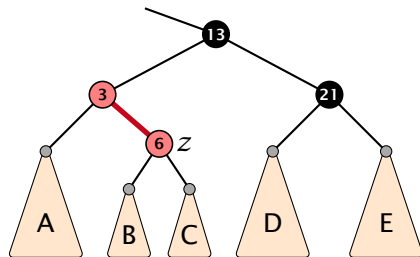
## Case 2b: Black uncle and z is left child

1. rotate around grandparent
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3. you have a red black tree

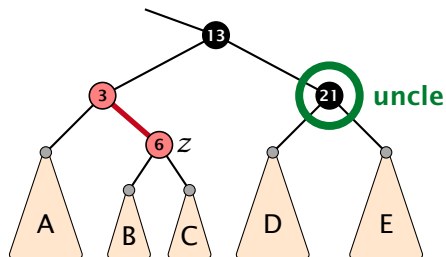




## Case 2a: Black uncle and z is right child

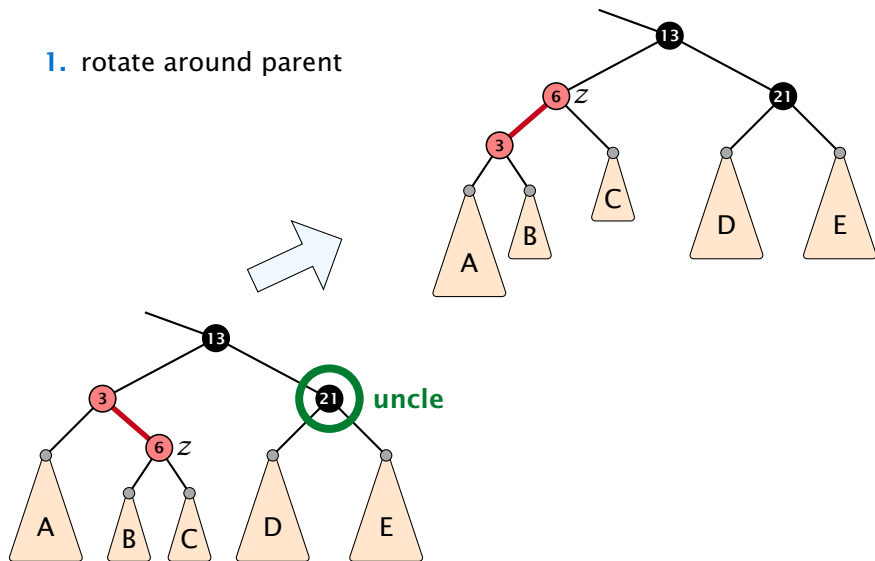


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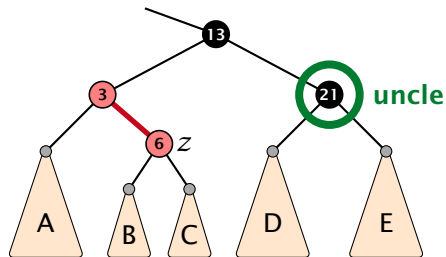
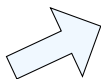
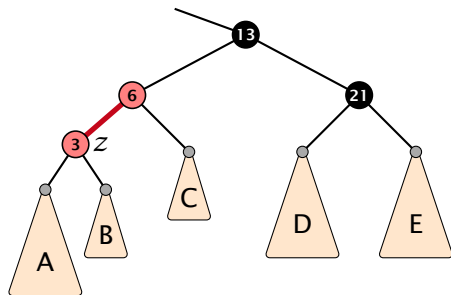
## Case 2a: Black uncle and z is right child

1. rotate around parent



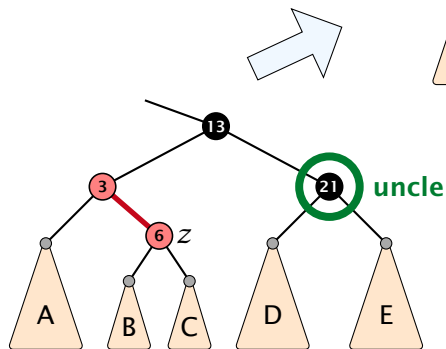
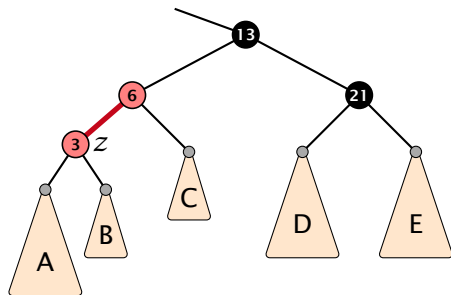
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## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.

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- ▶ Case 2b  $\rightarrow$  red-black tree

Performing Case 1 at most  $\mathcal{O}(\log n)$  times and every other case at most once, we get a red-black tree. Hence  $\mathcal{O}(\log n)$  re-colorings and at most 2 rotations.

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- ▶ If you delete the root, the root may now be red.

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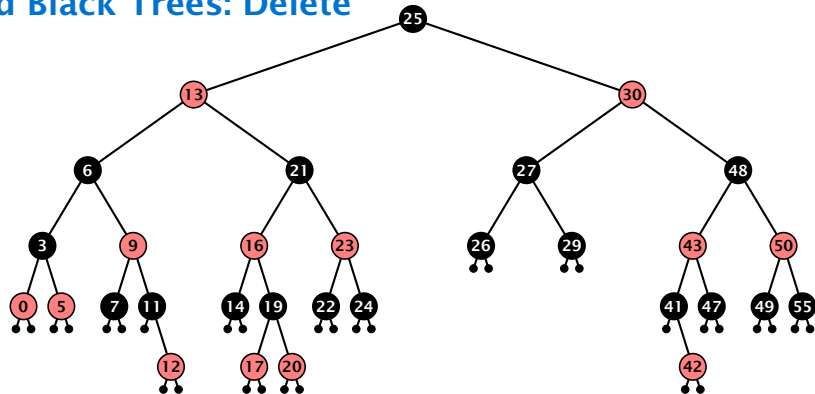
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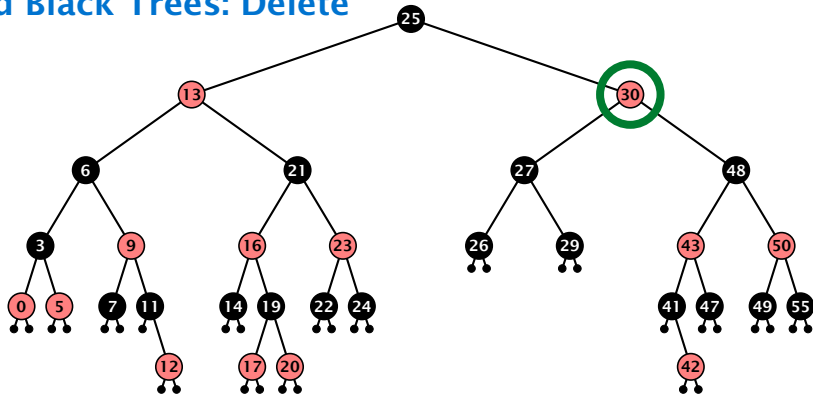
- ▶ Parent and child of  $x$  were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of  $x$  to a descendant leaf of  $x$  changes the number of black nodes. Black height property might be violated.



## Red Black Trees: Delete



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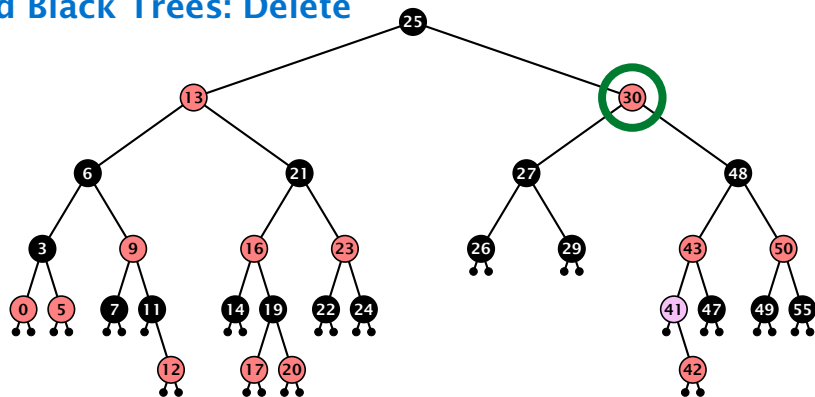


### Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

## Red Black Trees: Delete

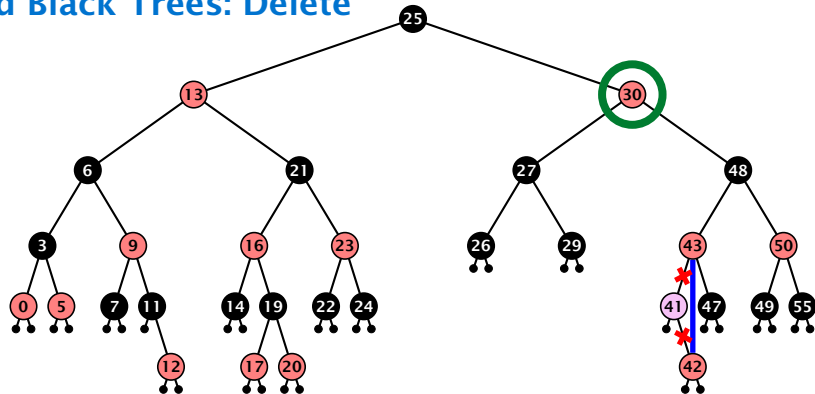


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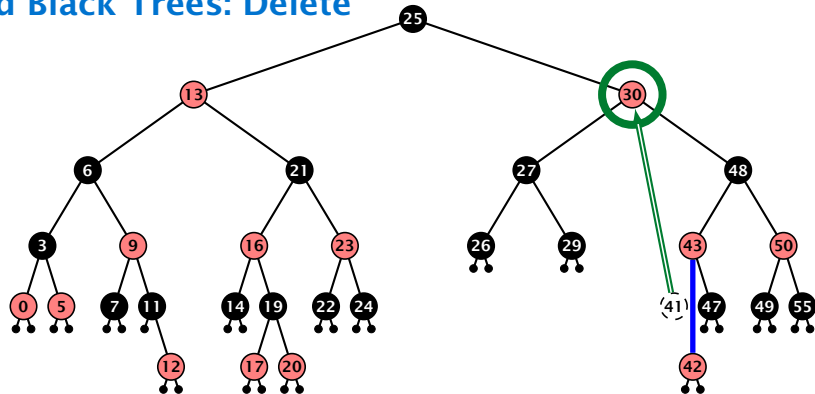


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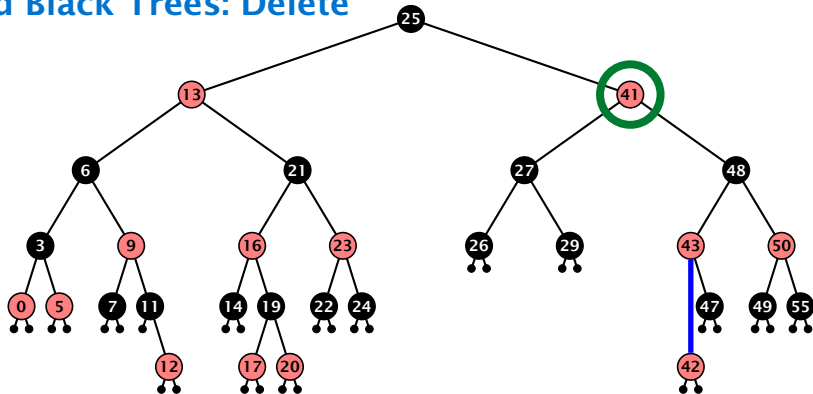


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## Red Black Trees: Delete

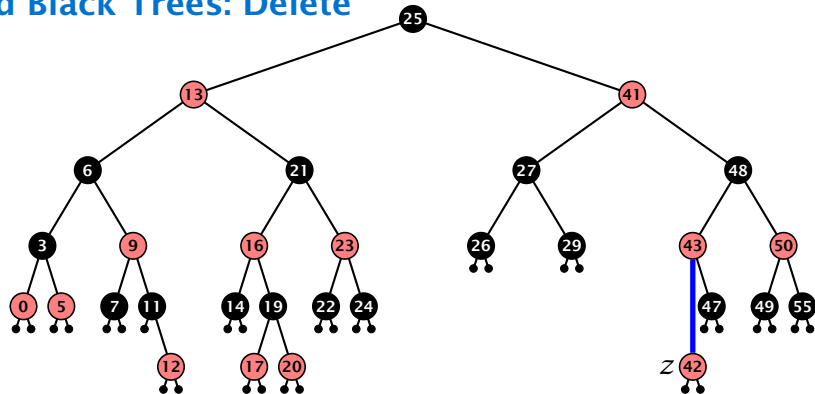


### Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

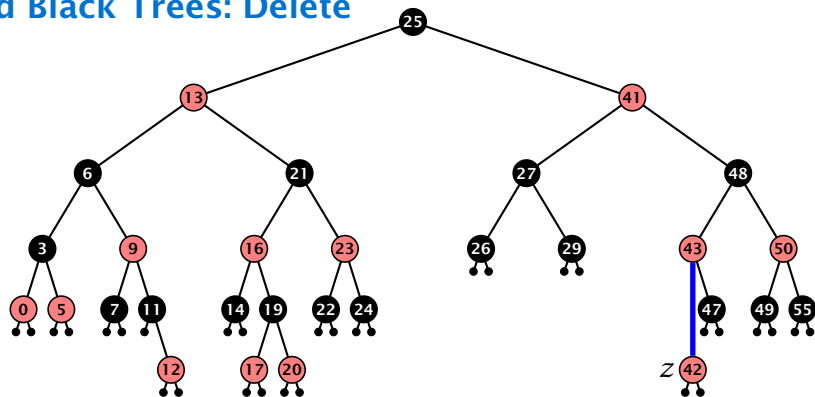
## Red Black Trees: Delete



Delete:

- ▶ deleting black node messes up black-height property

## Red Black Trees: Delete

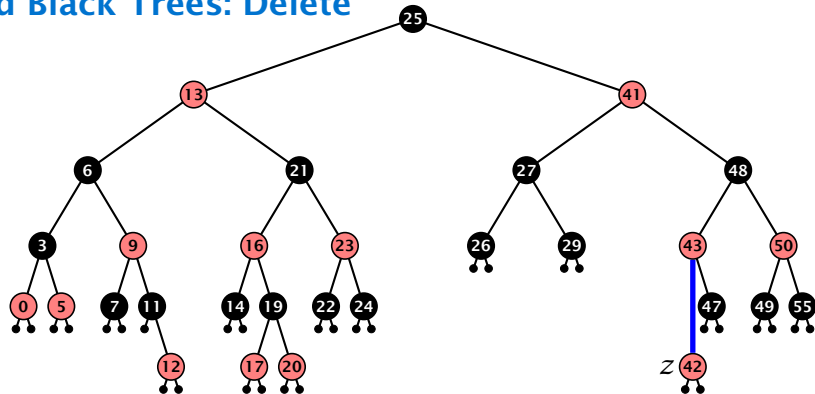


### Delete:

- ▶ deleting black node messes up black-height property
- ▶ if  $z$  is red, we can simply color it black and everything is fine



## Red Black Trees: Delete



### Delete:

- ▶ deleting black node messes up black-height property
- ▶ if  $z$  is red, we can simply color it black and everything is fine
- ▶ the problem is if  $z$  is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

# Red Black Trees: Delete

## Invariant of the fix-up algorithm

- ▶ the node  $z$  is black

# Red Black Trees: Delete

## Invariant of the fix-up algorithm

- ▶ the node  $z$  is black
- ▶ if we “assign” a fake black unit to the edge from  $z$  to its parent then the black-height property is fulfilled

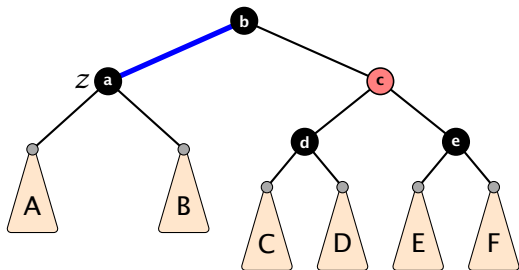
# Red Black Trees: Delete

## Invariant of the fix-up algorithm

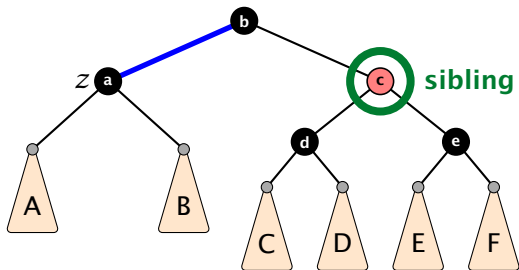
- ▶ the node  $z$  is black
- ▶ if we “assign” a fake black unit to the edge from  $z$  to its parent then the black-height property is fulfilled

**Goal:** make rotations in such a way that you at some point can remove the fake black unit from the edge.

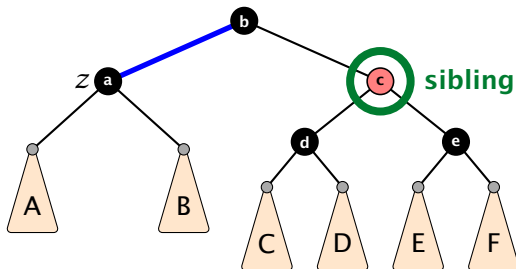
## Case 1: Sibling of $z$ is red



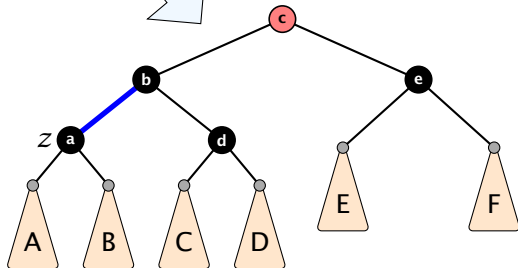
## Case 1: Sibling of z is red



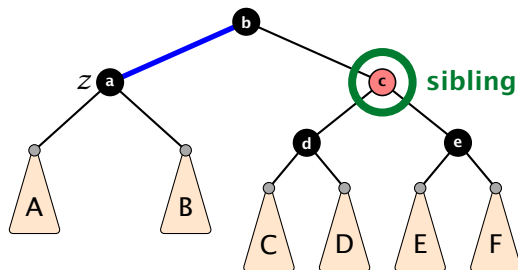
## Case 1: Sibling of $z$ is red



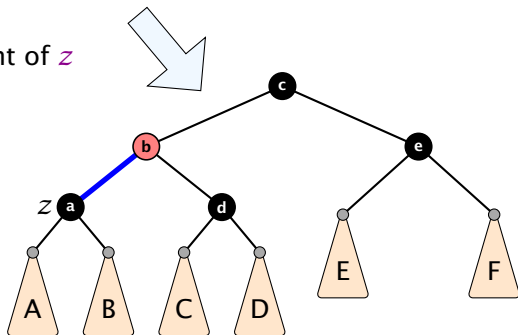
1. left-rotate around parent of  $z$



## Case 1: Sibling of $z$ is red

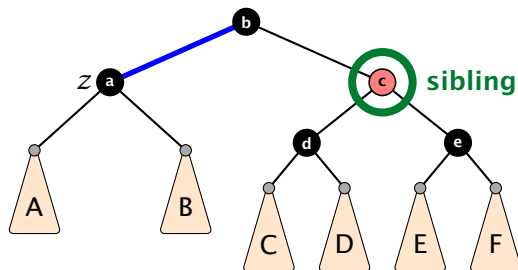


1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$

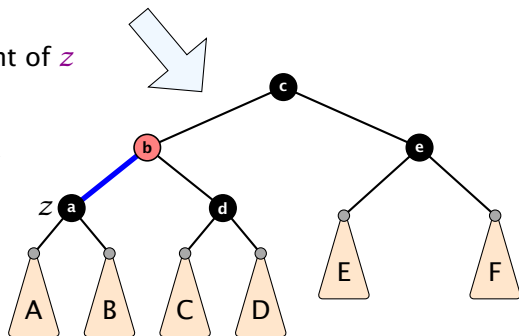




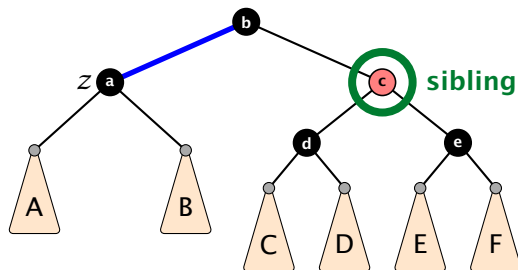
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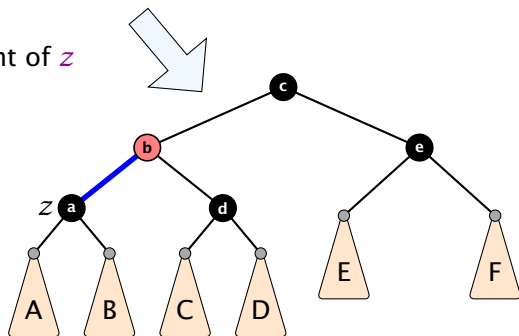
1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)



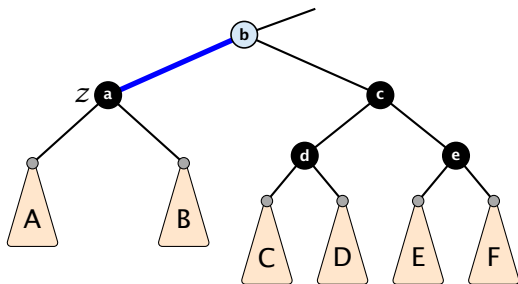
## Case 1: Sibling of $z$ is red



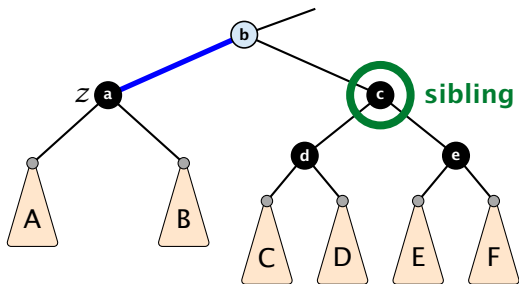
1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)
4. Case 2 (special), or Case 3, or Case 4



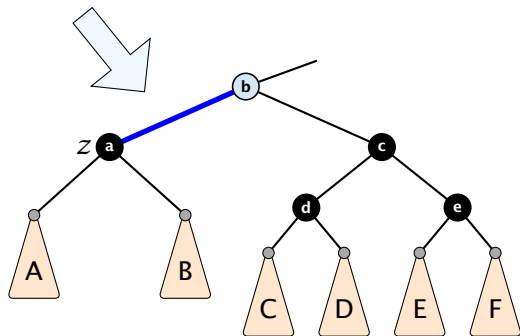
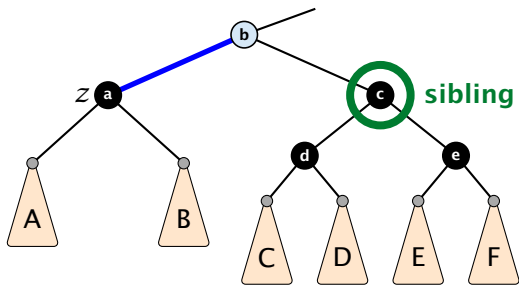
## Case 2: Sibling is black with two black children



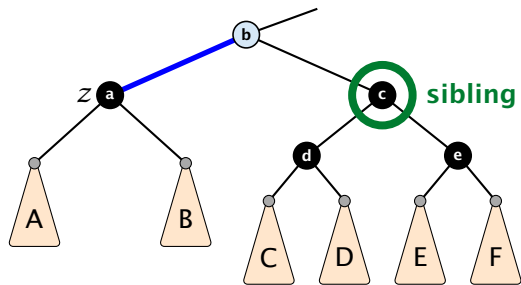
## Case 2: Sibling is black with two black children



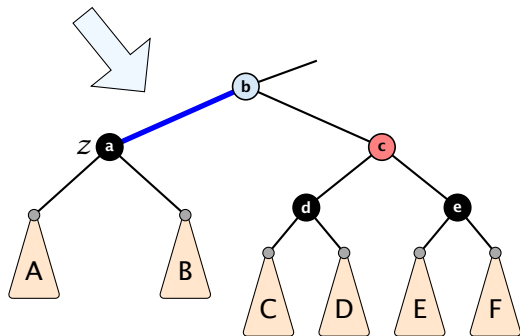
## Case 2: Sibling is black with two black children



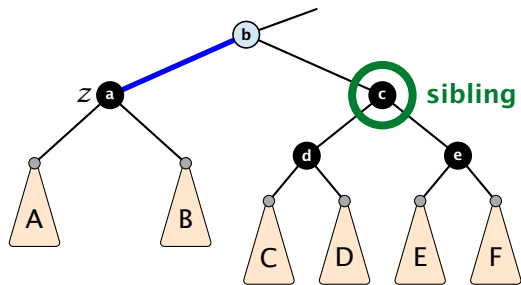
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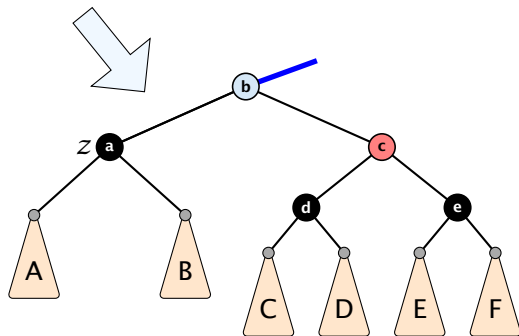
1. re-color node **c**



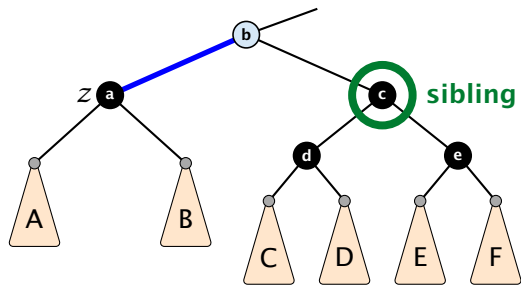
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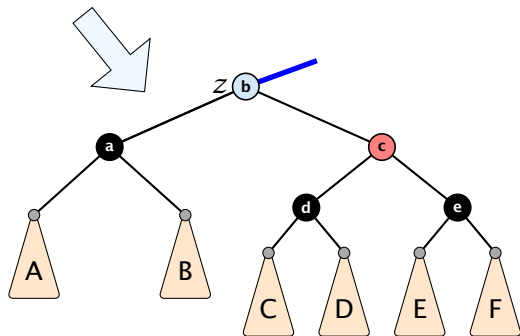
1. re-color node **c**
2. move fake black unit upwards



## Case 2: Sibling is black with two black children

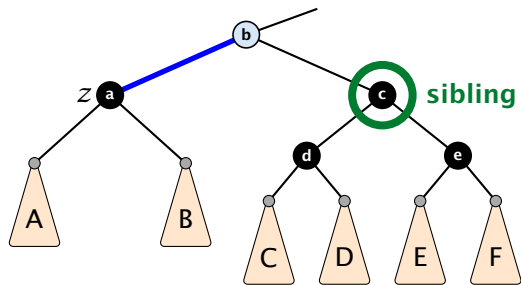


1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards

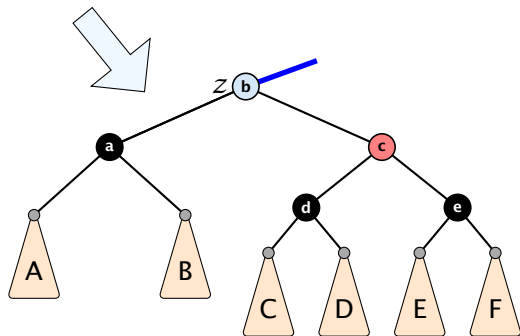




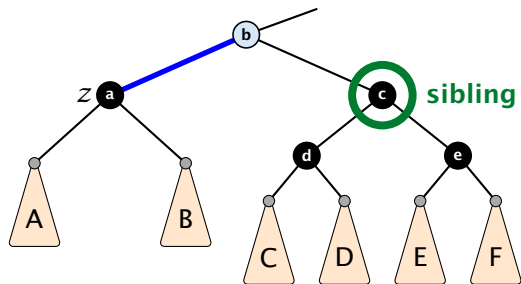
## Case 2: Sibling is black with two black children



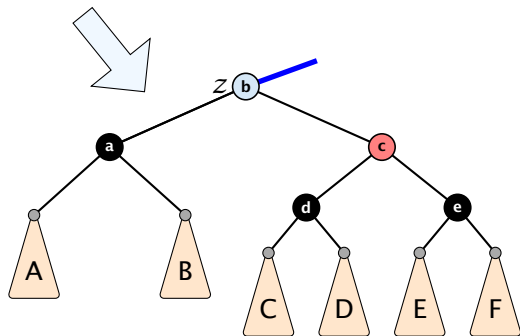
1. re-color node **c**
2. move fake black unit upwards
3. move **z** upwards
4. we made progress



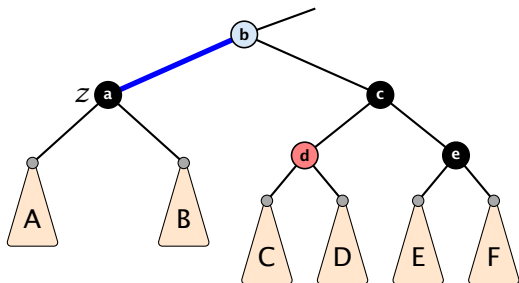
## Case 2: Sibling is black with two black children



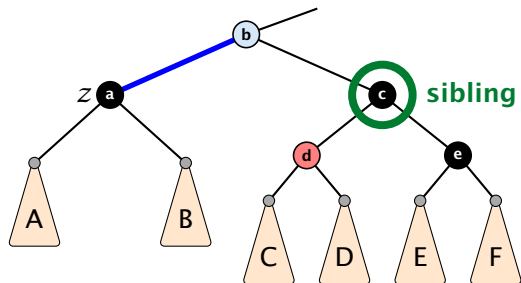
1. re-color node  $c$
2. move fake black unit upwards
3. move  $z$  upwards
4. we made progress
5. if  $b$  is red we color it black and are done



## Case 3: Sibling black with one black child to the right

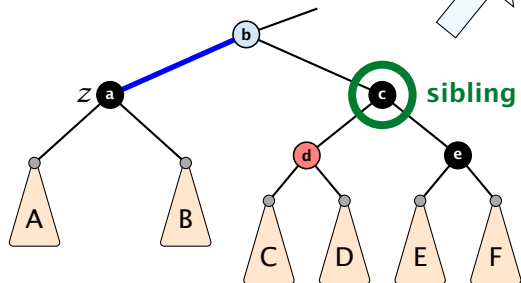
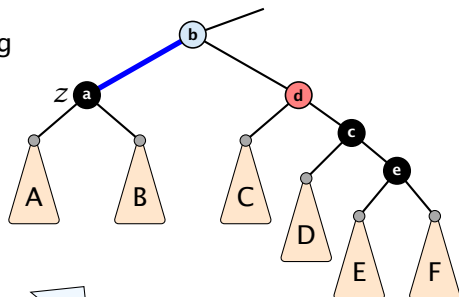


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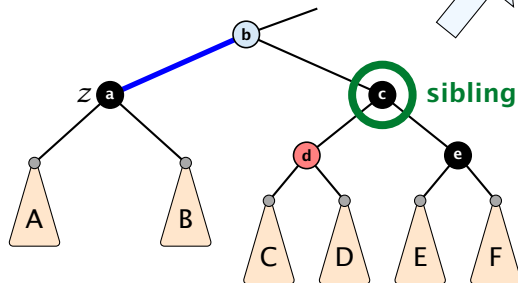
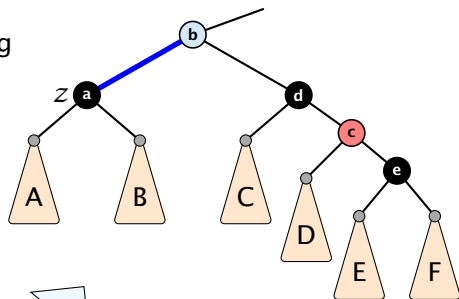
## Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling



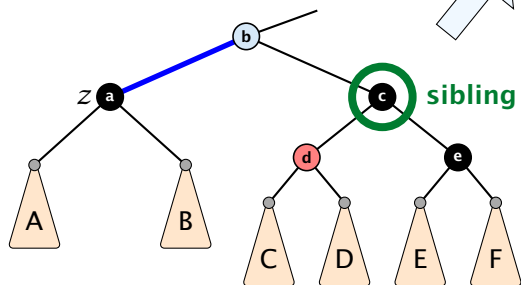
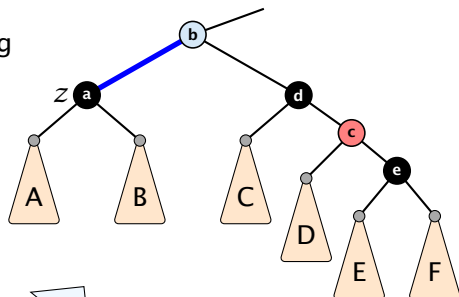
## Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor  $c$  and  $d$

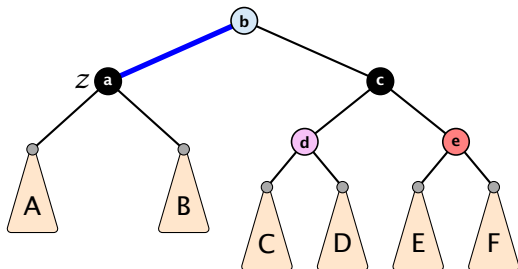


## Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor  $c$  and  $d$
3. new sibling is black with red right child (Case 4)

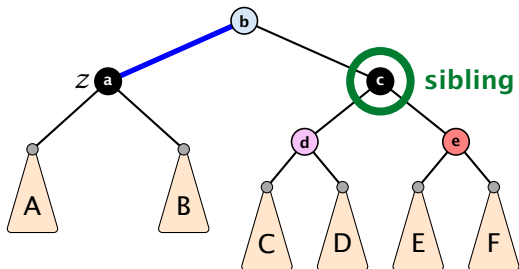


## Case 4: Sibling is black with red right child

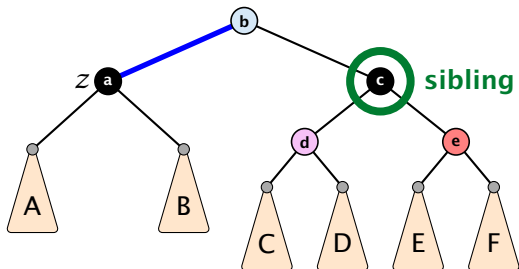




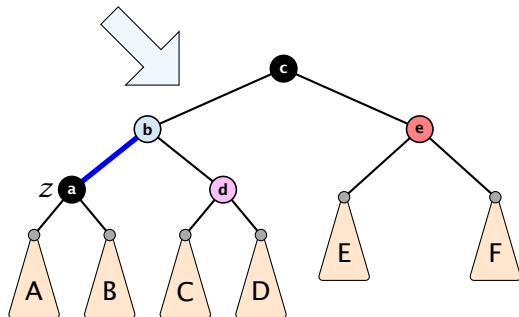
## Case 4: Sibling is black with red right child



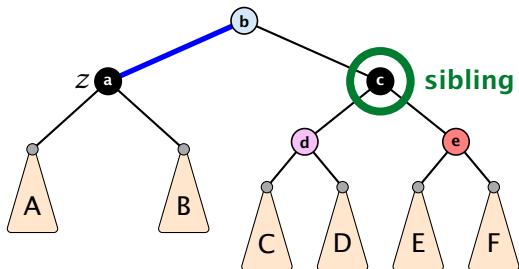
## Case 4: Sibling is black with red right child



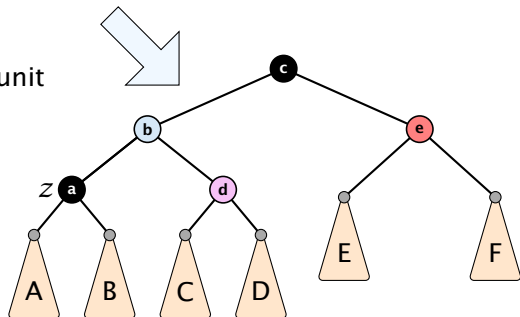
1. left-rotate around **b**



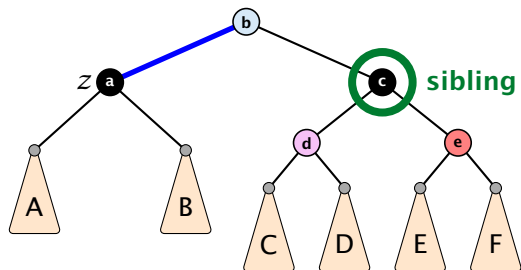
## Case 4: Sibling is black with red right child



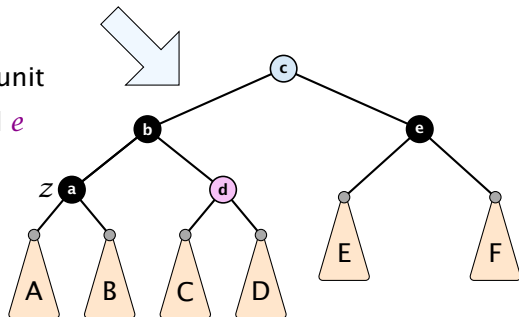
1. left-rotate around *b*
2. remove the fake black unit



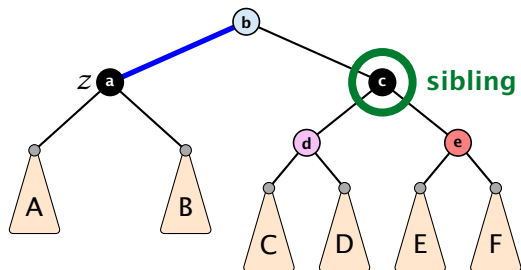
## Case 4: Sibling is black with red right child



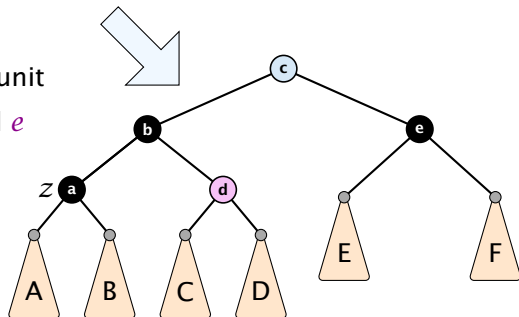
1. left-rotate around  $b$
2. remove the fake black unit
3. recolor nodes  $b$ ,  $c$ , and  $e$



## Case 4: Sibling is black with red right child



1. left-rotate around  $b$
2. remove the fake black unit
3. recolor nodes  $b$ ,  $c$ , and  $e$
4. you have a valid red black tree



## Running time:

- ▶ only Case 2 can repeat; but only  $h$  many steps, where  $h$  is the height of the tree

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Case 1 → Case 3 → Case 4 → red black tree  
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## Running time:

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Case 1 → Case 3 → Case 4 → red black tree  
Case 1 → Case 4 → red black tree
- ▶ Case 3 → Case 4 → red black tree
- ▶ Case 4 → red black tree

Performing Case 2 at most  $\mathcal{O}(\log n)$  times and every other step at most once, we get a red black tree. Hence,  $\mathcal{O}(\log n)$  re-colorings and at most 3 rotations.

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repeated accesses are faster



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- only amortized guarantee
- read-operations change the tree

# Splay Trees

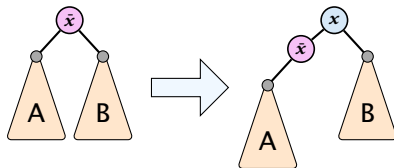
## **find( $x$ )**

- ▶ search for  $x$  according to a search tree
- ▶ let  $\tilde{x}$  be last element on search-path
- ▶  $\text{splay}(\tilde{x})$

# Splay Trees

## insert( $x$ )

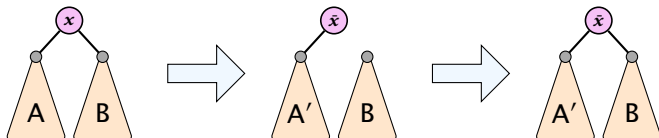
- ▶ search for  $x$ ;  $\bar{x}$  is last visited element during search (successor or predecessor of  $x$ )
- ▶ splay( $\bar{x}$ ) moves  $\bar{x}$  to the root
- ▶ insert  $x$  as new root



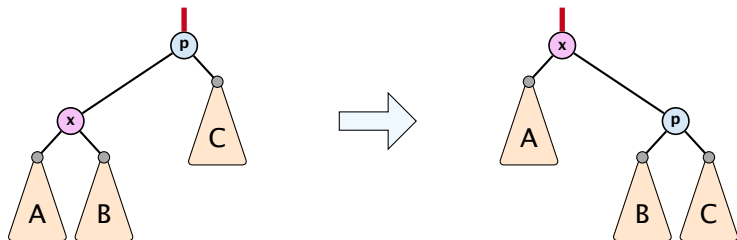
# Splay Trees

## delete( $x$ )

- ▶ search for  $x$ ; splay( $x$ ); remove  $x$
- ▶ search largest element  $\bar{x}$  in  $A$
- ▶ splay( $\bar{x}$ ) (on subtree  $A$ )
- ▶ connect root of  $B$  as right child of  $\bar{x}$



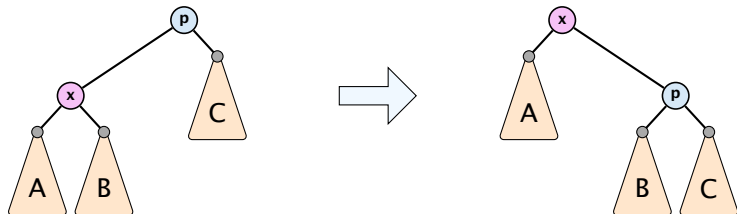
# Move to Root



## How to bring element to root?

- ▶ one (bad) option: `moveToRoot(x)`
- ▶ iteratively do rotation around parent of  $x$  until  $x$  is root
- ▶ if  $x$  is left child do right rotation otw. left rotation

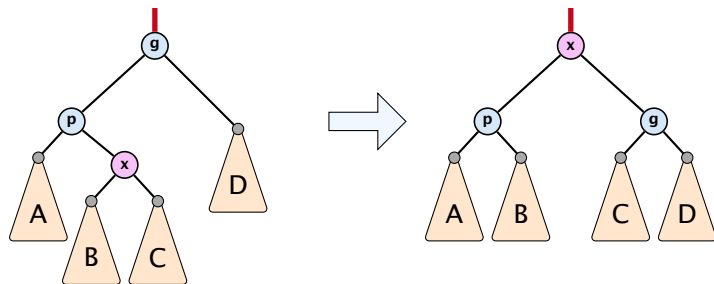
## Splay: Zig Case



**better option splay( $x$ ):**

- ▶ zig case: if  $x$  is child of root do left rotation or right rotation around parent

## Splay: Zigzag Case

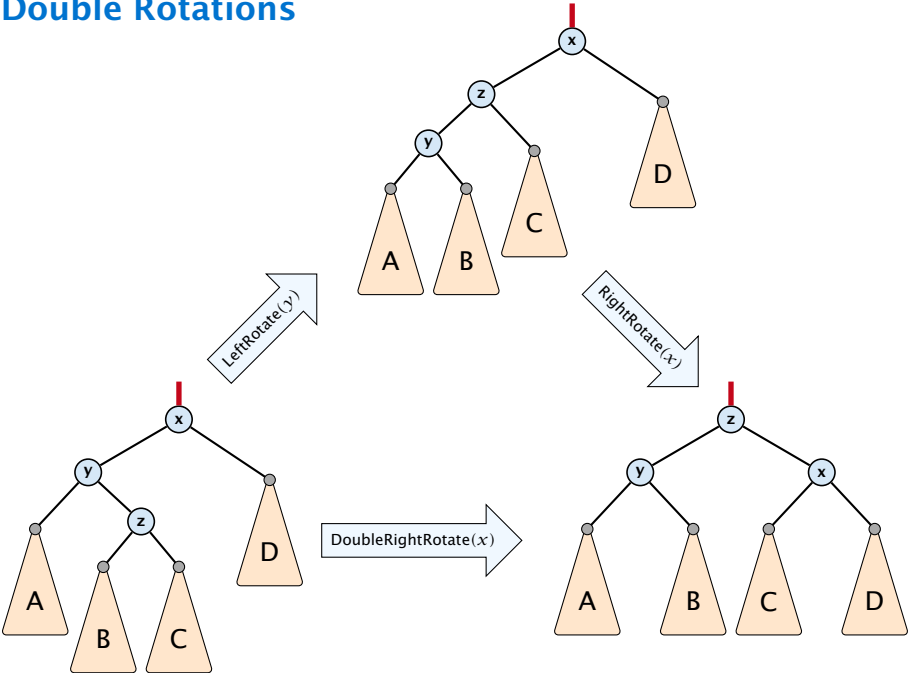


### better option $\text{splay}(x)$ :

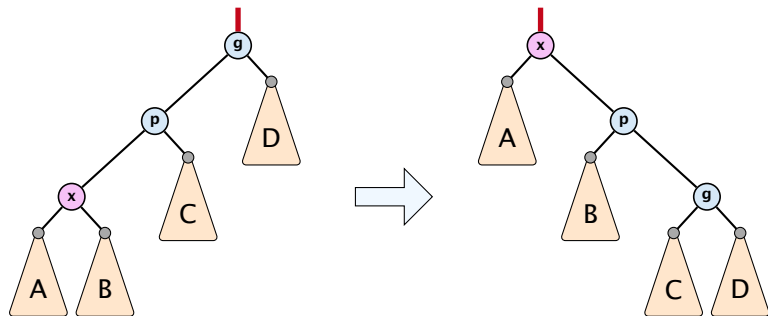
- ▶ zigzag case: if  $x$  is right child and parent of  $x$  is left child (or  $x$  left child parent of  $x$  right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)



# Double Rotations



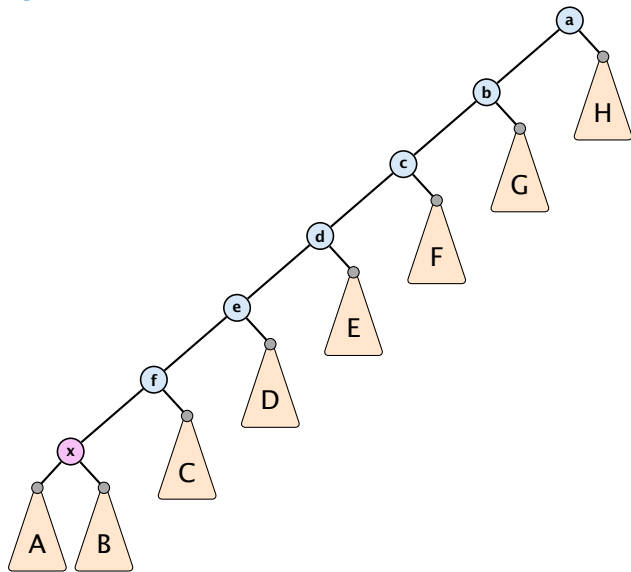
## Splay: Zigzig Case



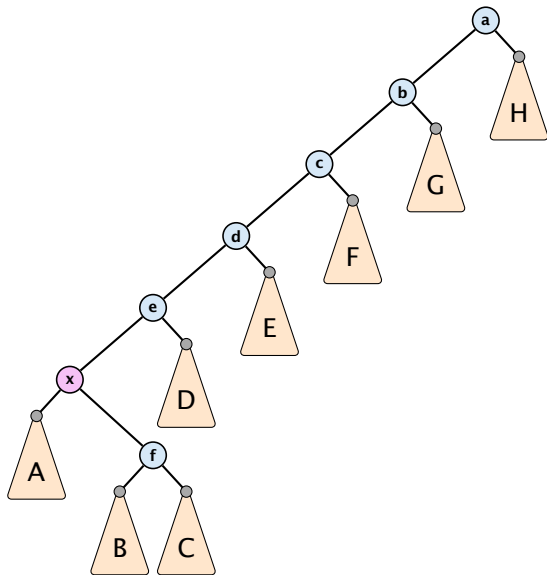
### better option $\text{splay}(x)$ :

- ▶ zigzig case: if  $x$  is left child and parent of  $x$  is left child (or  $x$  right child, parent of  $x$  right child)
- ▶ do right rotation around grand-parent followed by right rotation around parent (resp. left rotations)

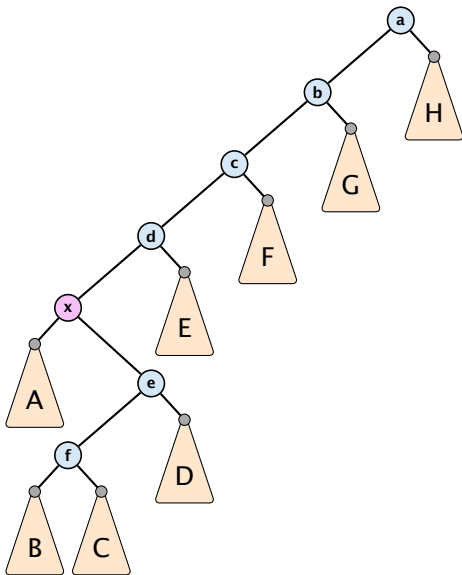
# Splay vs. Move to Root



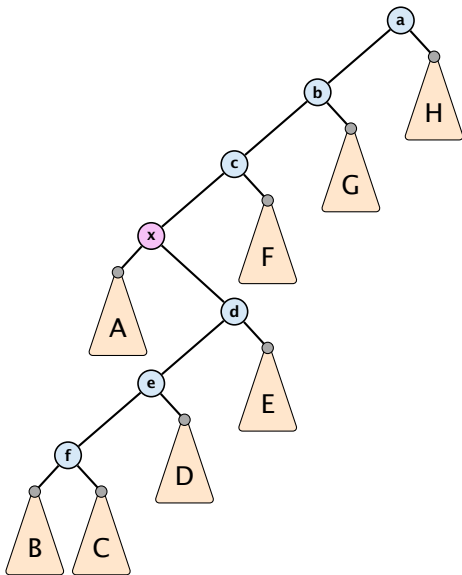
# Splay vs. Move to Root



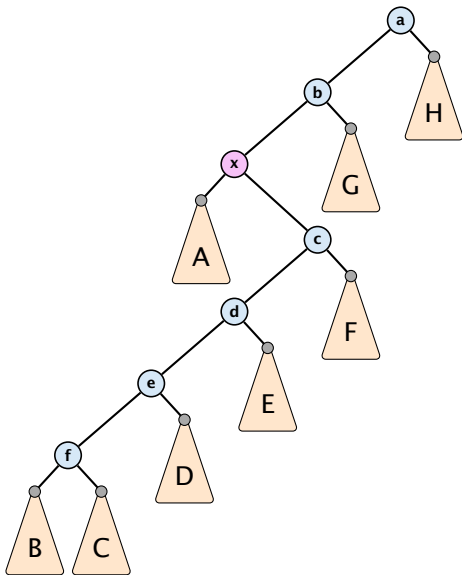
# Splay vs. Move to Root



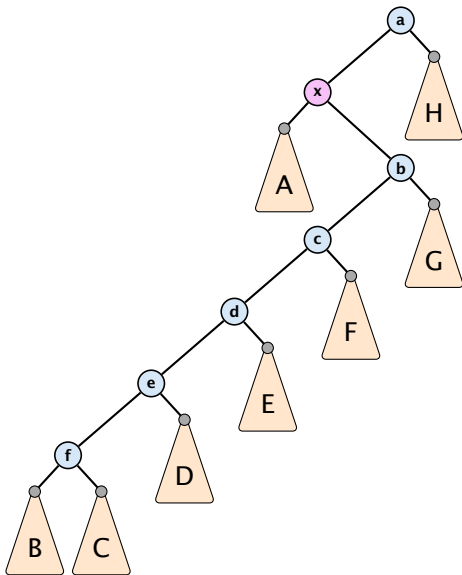
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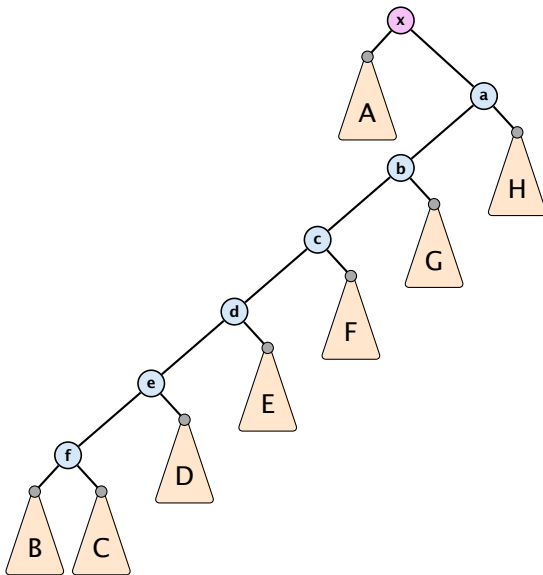


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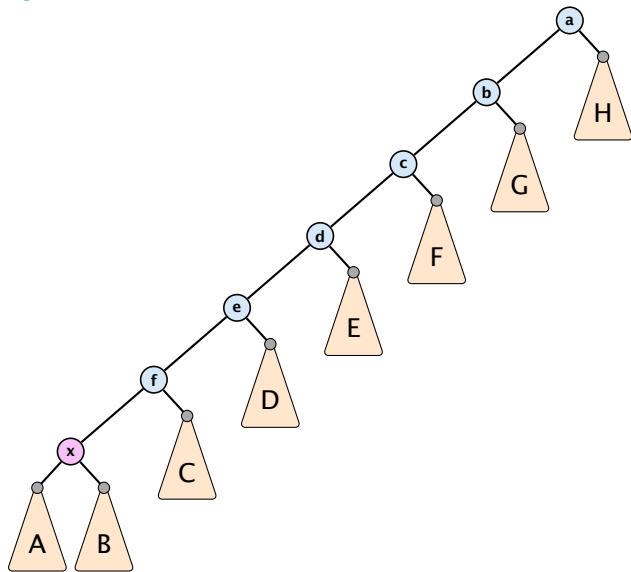




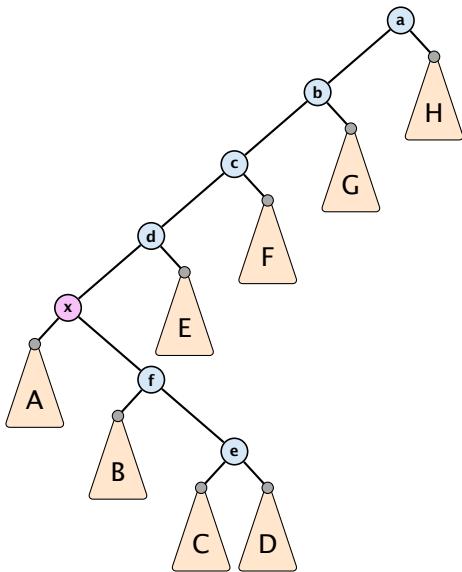
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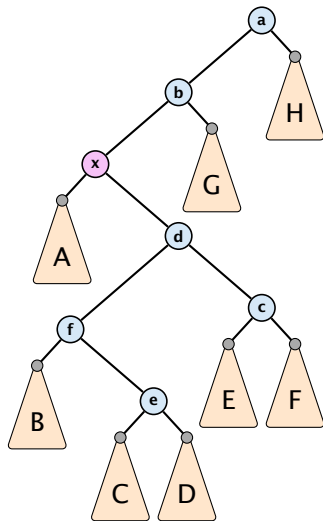
# Splay vs. Move to Root



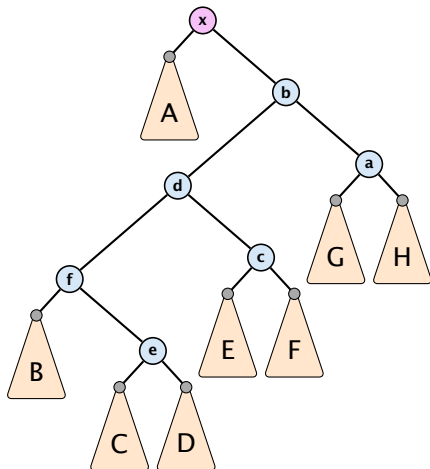
# Splay vs. Move to Root



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# Static Optimality

Suppose we have a sequence of  $m$  find-operations.  $\text{find}(x)$  appears  $h_x$  times in this sequence.

The cost of a **static** search tree  $T$  is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is  $\mathcal{O}(\text{cost}(T_{\min}))$ , where  $T_{\min}$  is an **optimal static search tree**.

# Dynamic Optimality

Let  $S$  be a sequence with  $m$  find-operations.

Let  $A$  be a data-structure based on a search tree:

- ▶ the cost for accessing element  $x$  is  $1 + \text{depth}(x)$ ;
- ▶ after accessing  $x$  the tree may be re-arranged through rotations;

## Conjecture:

A splay tree that only contains elements from  $S$  has cost  $\mathcal{O}(\text{cost}(A, S))$ , for processing  $S$ .

## Lemma 16

*Splay Trees have an **amortized** running time of  $\mathcal{O}(\log n)$  for all operations.*



# Amortized Analysis

## Definition 17

A data structure with operations  $\text{op}_1(), \dots, \text{op}_k()$  has amortized running times  $t_1, \dots, t_k$  for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most  $n$  elements, and let  $k_i$  denote the number of occurrences of  $\text{op}_i()$  within this sequence. Then the actual running time must be at most  $\sum_i k_i \cdot t_i(n)$ .

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Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.



# Example: Stack

## Stack

- ▶  $S.$  push()
- ▶  $S.$  pop()
- ▶  $S.$  multipop( $k$ ): removes  $k$  items from the stack. If the stack currently contains less than  $k$  items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

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- ▶ The user has to ensure that pop and multipop do not generate an underflow.

## Actual cost:

- ▶  $S.$  push(): cost 1.
- ▶  $S.$  pop(): cost 1.
- ▶  $S.$  multipop( $k$ ): cost  $\min\{\text{size}, k\} = k$ .

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- ▶  **$S$ . pop():** cost

$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \leq 0 .$$

- ▶  **$S$ . multipop( $k$ ):** cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 .$$

## Example: Binary Counter

### **Incrementing a binary counter:**

Consider a computational model where each bit-operation costs one time-unit.

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### Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ Increment: cost is  $k + 1$ , where  $k$  is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has  $k = 1$ ).

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Choose potential function  $\Phi(x) = k$ , where  $k$  denotes the number of ones in the binary representation of  $x$ .

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- ▶ Changing bit from 1 to 0:

$$\hat{C}_{1 \rightarrow 0} = C_{1 \rightarrow 0} + \Delta\Phi = 1 - 1 \leq 0 .$$

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- ▶ **Increment:** Let  $k$  denotes the number of consecutive ones in the least significant bit-positions. An increment involves  $k$   $(1 \rightarrow 0)$ -operations, and one  $(0 \rightarrow 1)$ -operation.

Hence, the amortized cost is  $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$ .

# Splay Trees

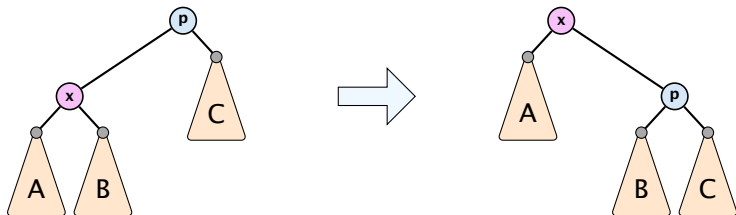
## potential function for splay trees:

- ▶ size  $s(x) = |T_x|$
- ▶ rank  $r(x) = \log_2(s(x))$
- ▶  $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

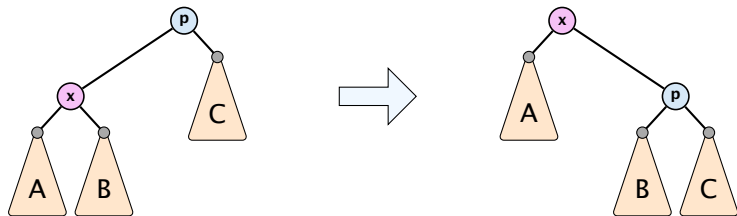
## Splay: Zig Case



$$\Delta\Phi =$$

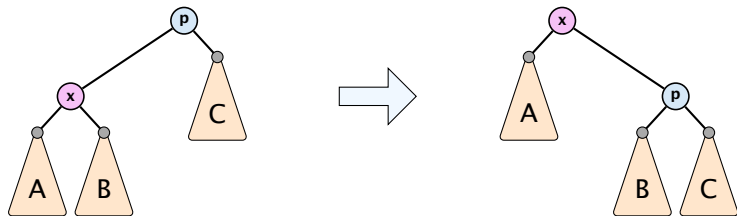


## Splay: Zig Case



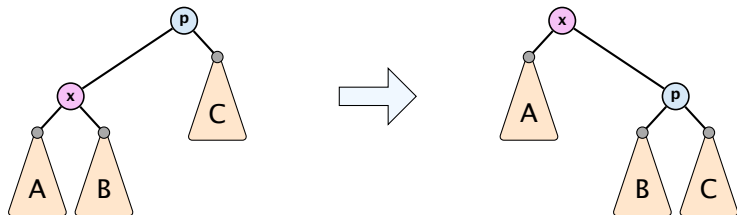
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

## Splay: Zig Case



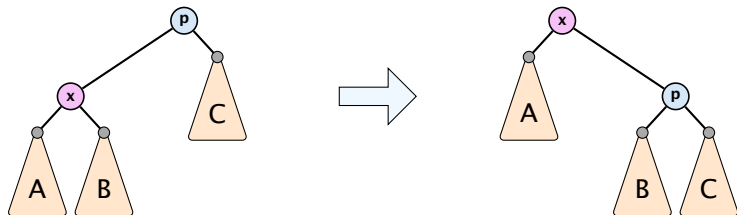
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x)\end{aligned}$$

## Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

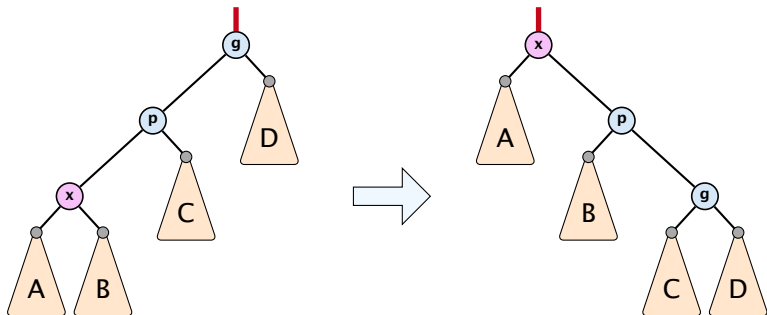
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$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

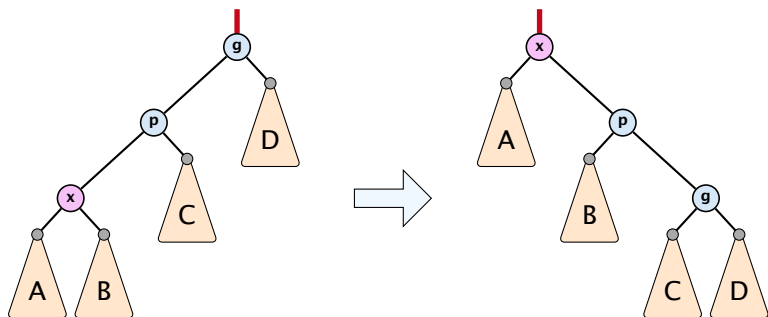
$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

## Splay: Zigzig Case



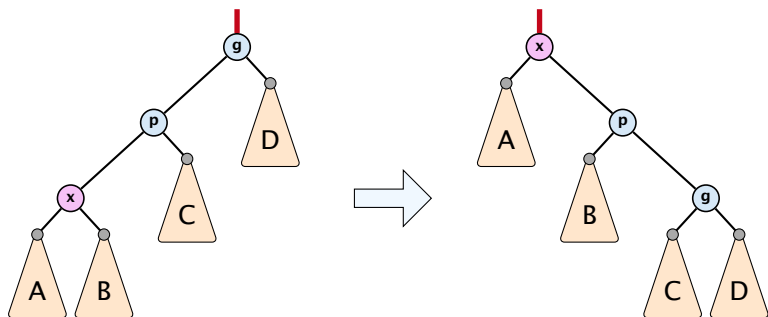
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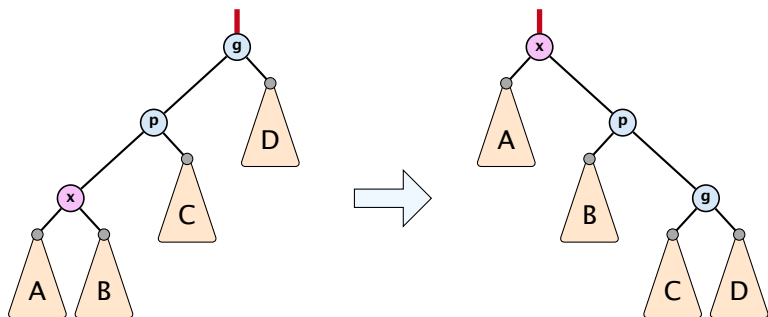
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

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$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p)\end{aligned}$$

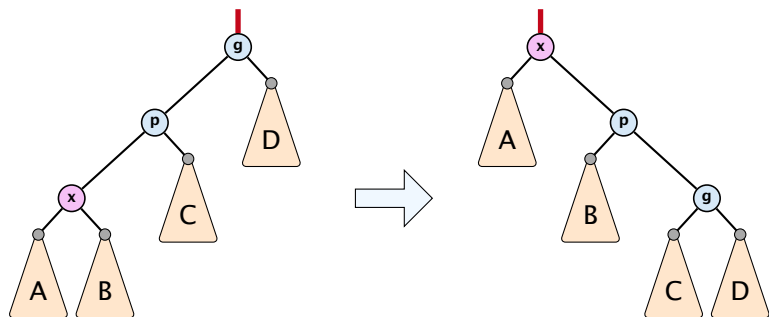
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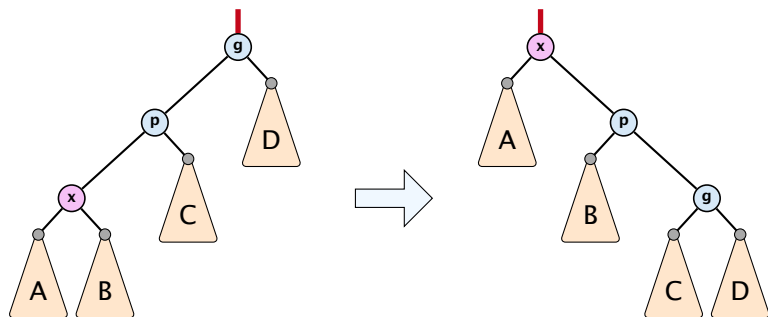


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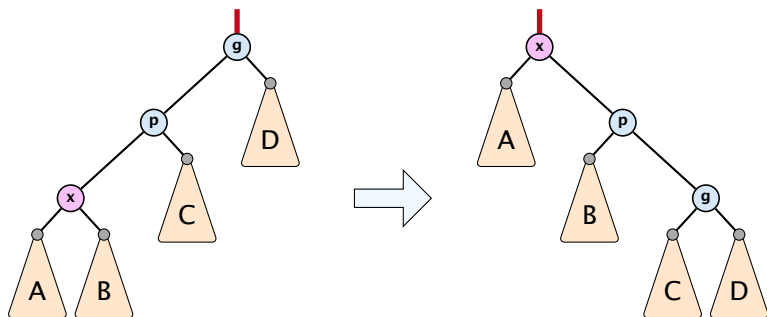
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x)\end{aligned}$$

## Splay: Zigzig Case



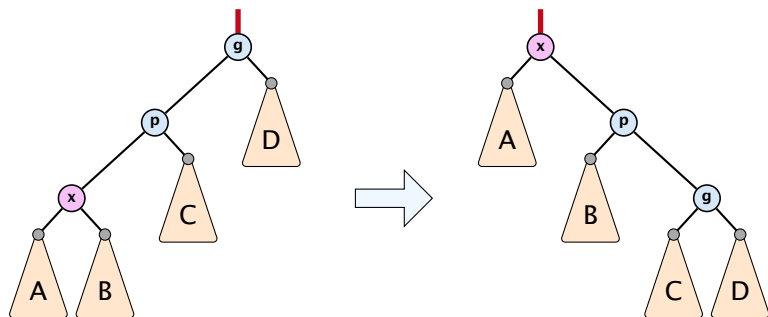
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzig Case



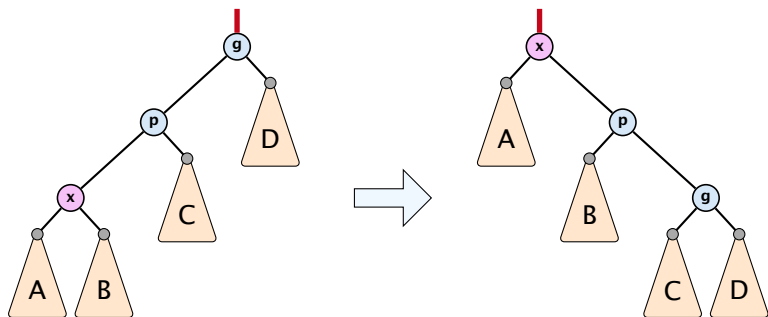
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\ &\leq -2 + 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzig Case



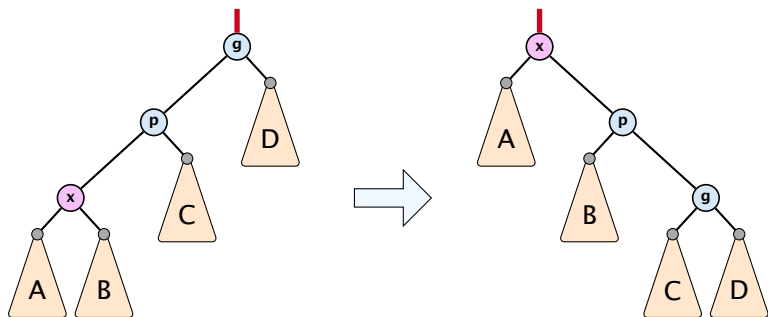
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\ &\leq -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{COST}_{\text{zigzig}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzig Case



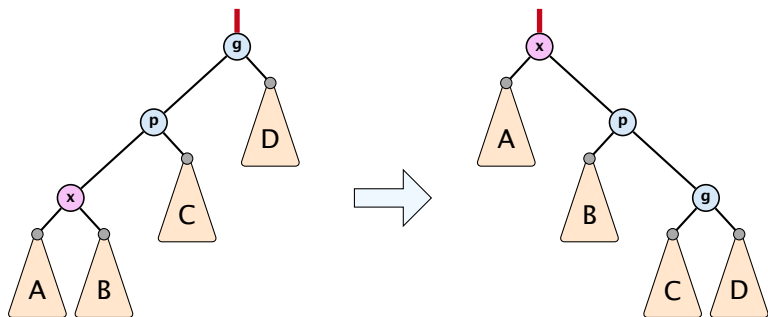
$$\frac{1}{2}(r(x) + r'(g) - 2r'(x))$$

## Splay: Zigzig Case



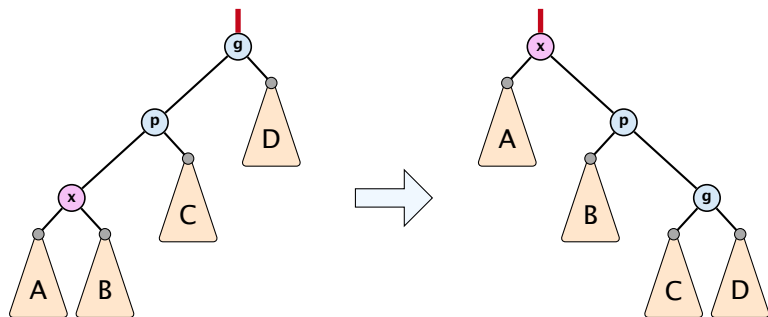
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \end{aligned}$$

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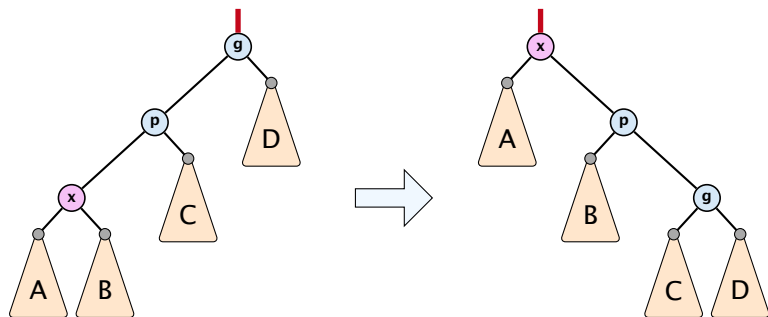
## Splay: Zigzig Case



$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \end{aligned}$$

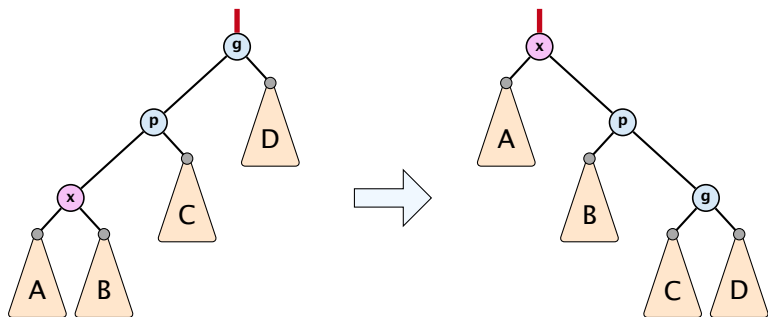


## Splay: Zigzig Case



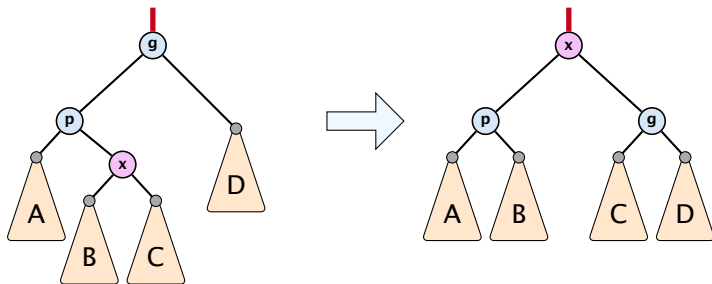
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) \end{aligned}$$

## Splay: Zigzig Case



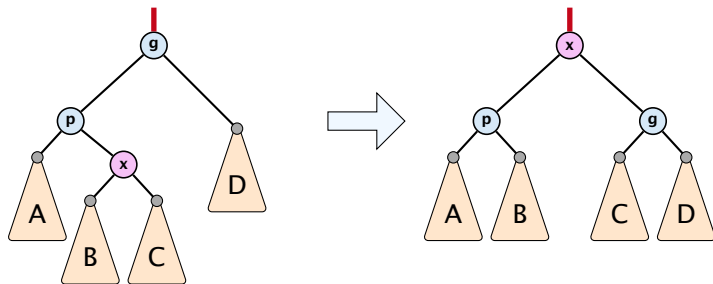
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \\ &\leq \log\left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

## Splay: Zigzag Case



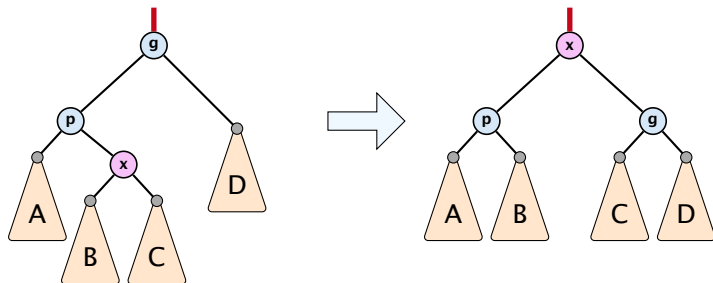
$\Delta\Phi =$

## Splay: Zigzag Case



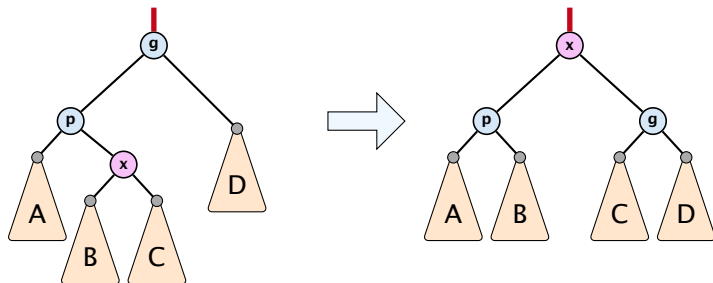
$$\Delta\Phi = r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g)$$

## Splay: Zigzag Case



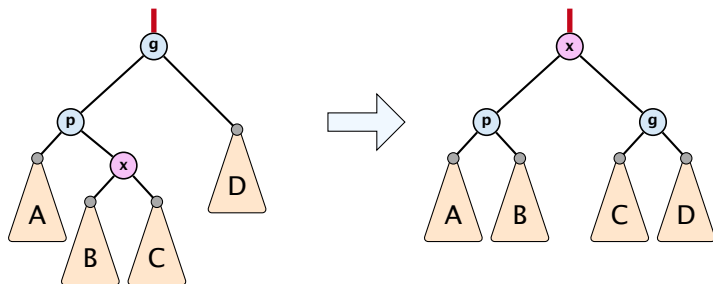
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p)\end{aligned}$$

## Splay: Zigzag Case



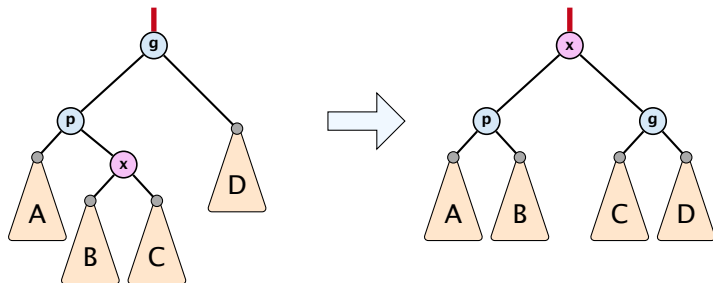
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x)\end{aligned}$$

## Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x)\end{aligned}$$

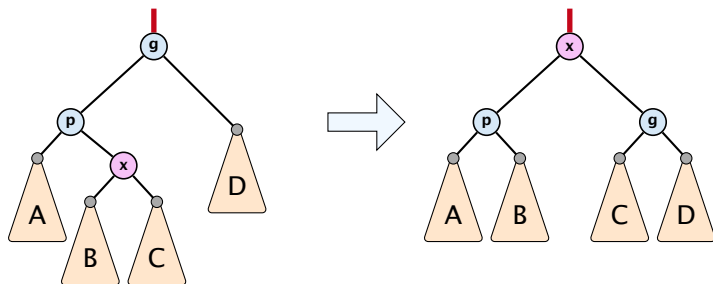
## Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x))\end{aligned}$$

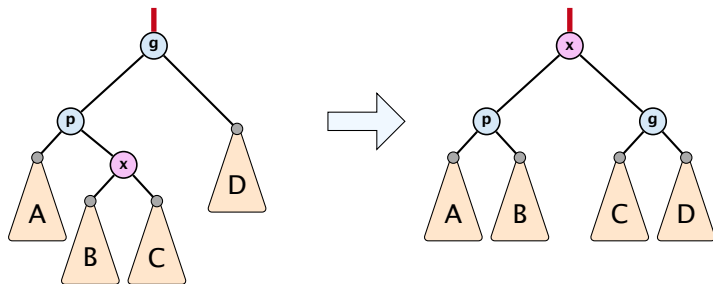


## Splay: Zigzag Case



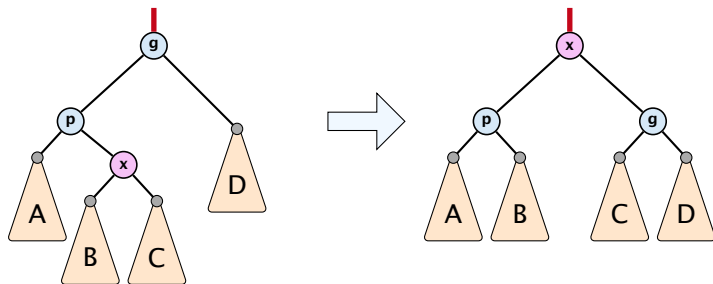
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{COST}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzag Case



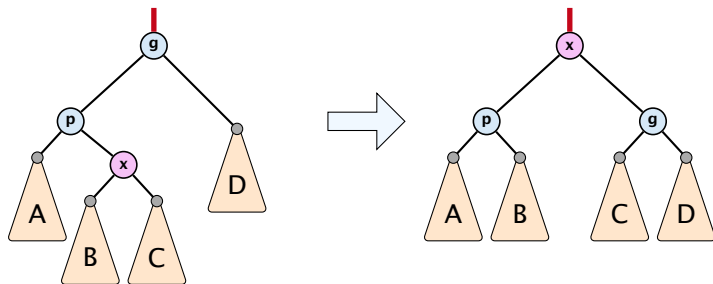
$$\frac{1}{2}(r'(p) + r'(g) - 2r'(x))$$

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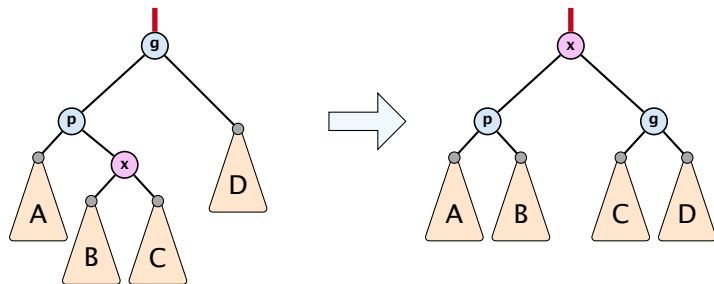
$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \end{aligned}$$

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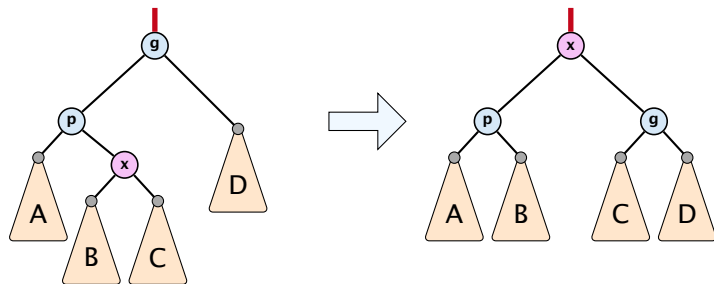
$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \\ &\leq \log\left(\frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \end{aligned}$$

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Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + 3(r(\text{root}) - r_0(x)) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

## 7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert( $x$ )**: insert element  $x$ .
- ▶ **Search( $k$ )**: search for element with key  $k$ .
- ▶ **Delete( $x$ )**: delete element referenced by pointer  $x$ .
- ▶ **find-by-rank( $\ell$ )**: return the  $\ell$ -th element; return “error” if the data-structure contains less than  $\ell$  elements.



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**Augment an existing data-structure instead of developing a new one.**

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## 7.4 Augmenting Data Structures

**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

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2. We store in each node  $v$  the size of the sub-tree rooted at  $v$ .
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...



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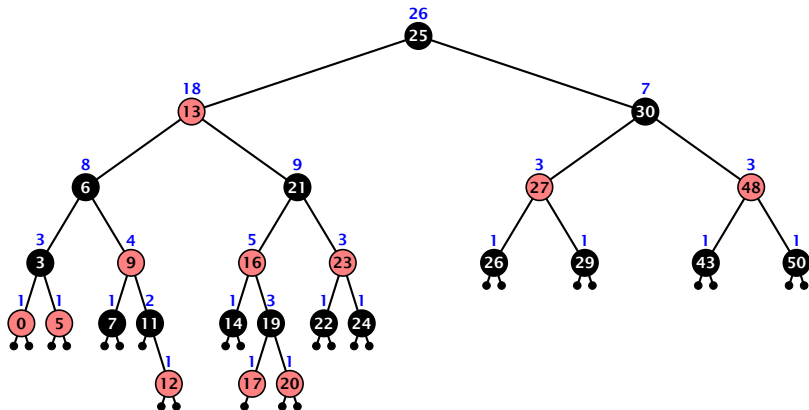
4. How does find-by-rank work?

Find-by-rank( $k$ ) := Select( $\text{root}, k$ ) with

**Algorithm 1** Select( $x, i$ )

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

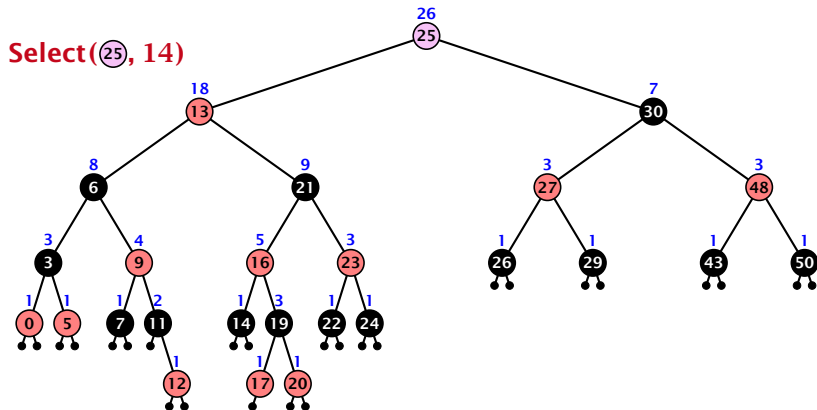
## Select( $x, i$ )



### Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
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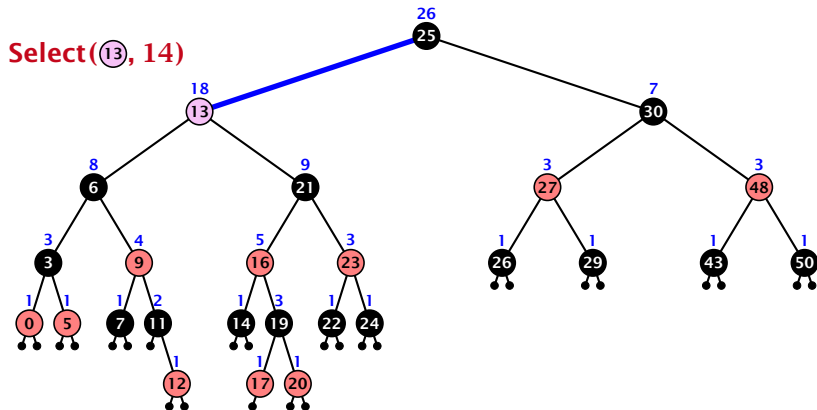
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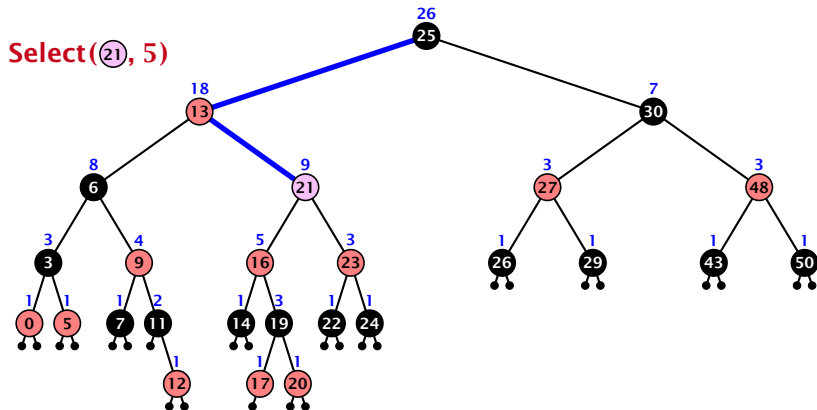
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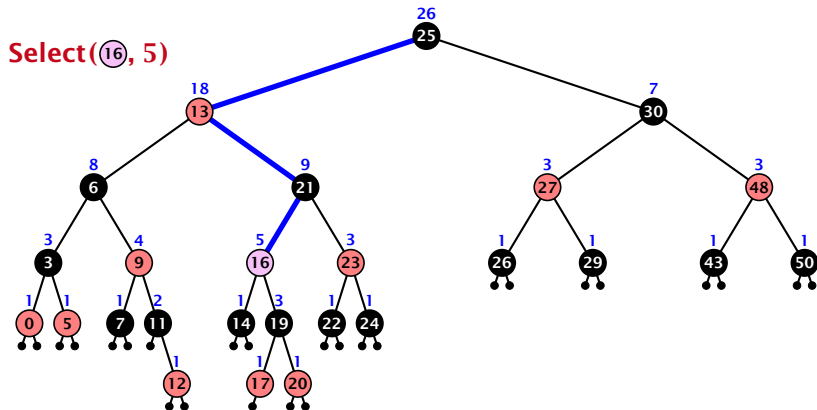
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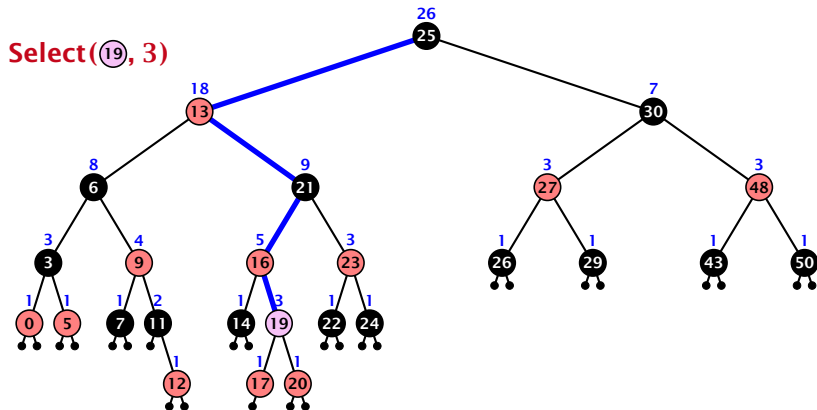
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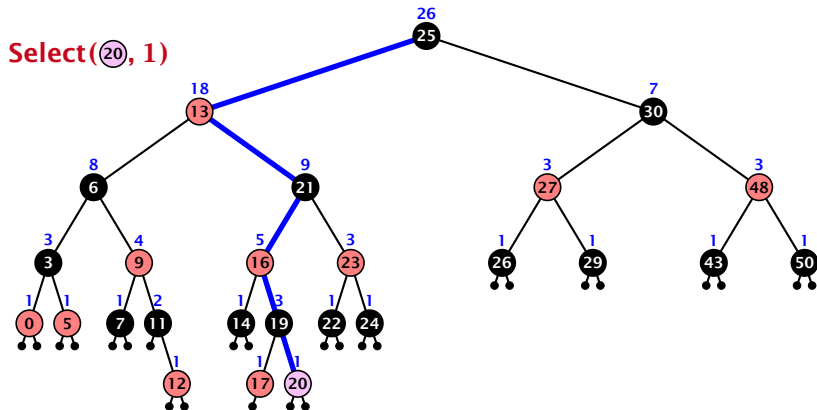
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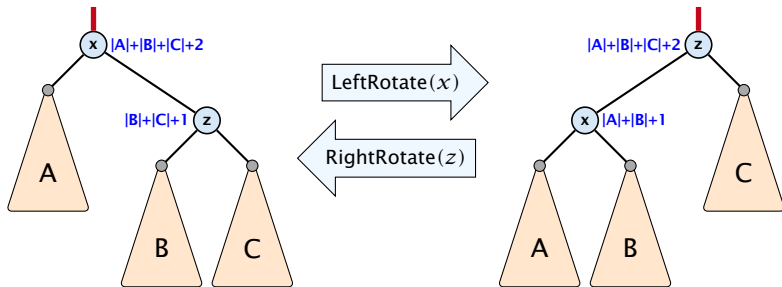
**Search( $k$ ):** Nothing to do.

**Insert( $x$ ):** When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

**Delete( $x$ ):** Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. **Maintain the size field during rotations.**

# Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes  $x$  and  $z$  are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

## 7.5 Skip Lists

**Why do we not use a list for implementing the ADT Dynamic Set?**

## 7.5 Skip Lists

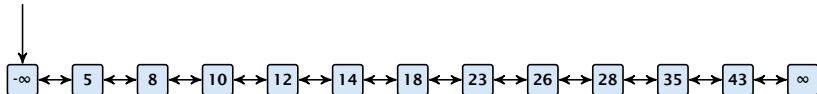
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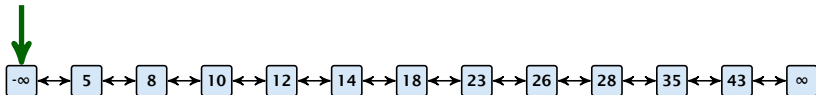




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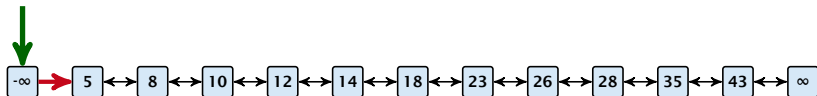
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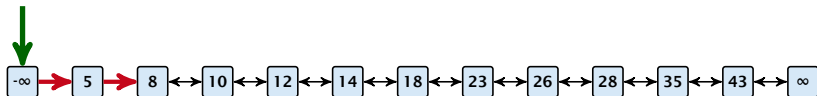
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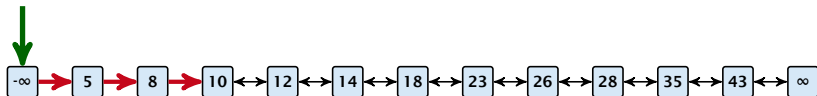
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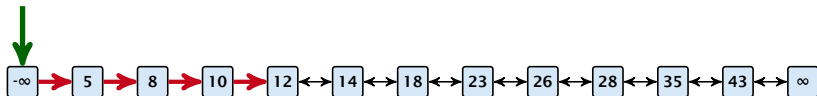
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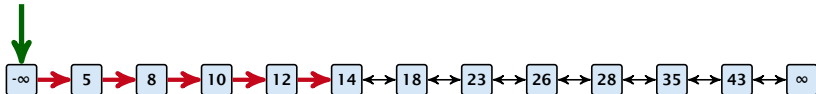
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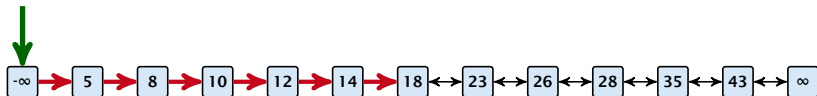
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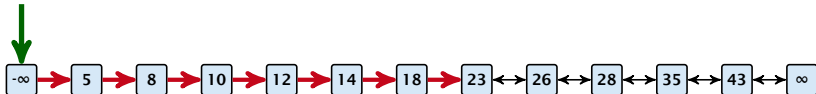
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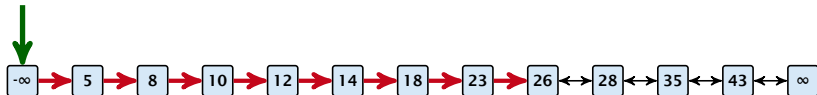




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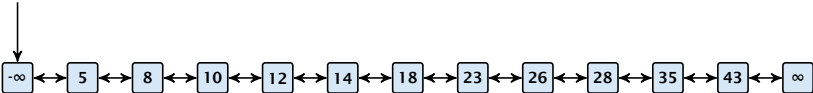
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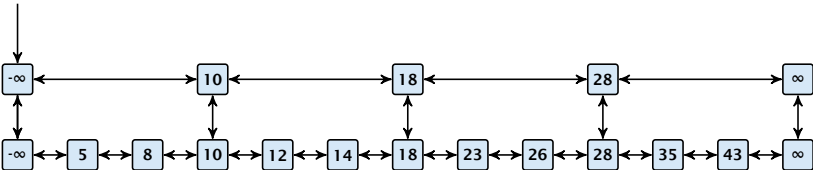
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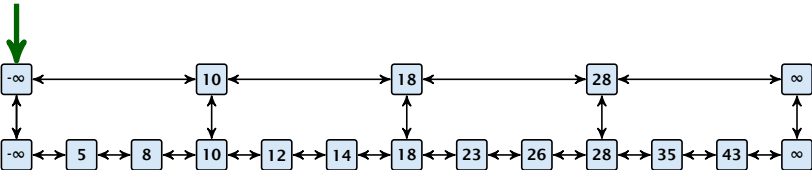
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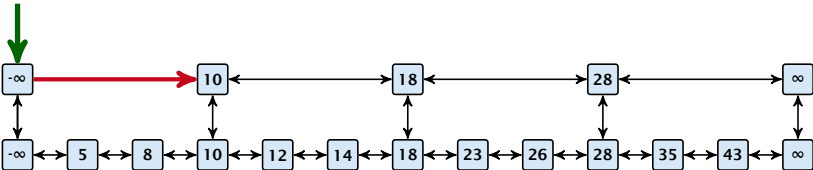
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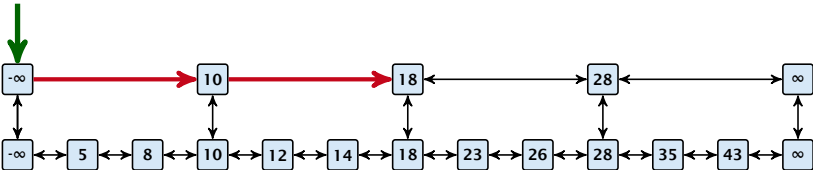
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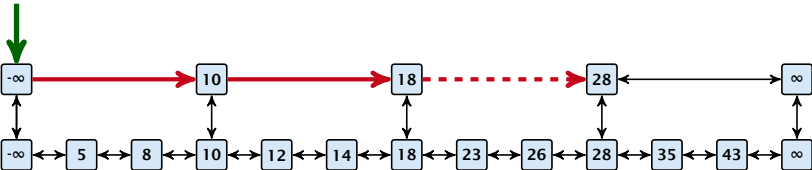




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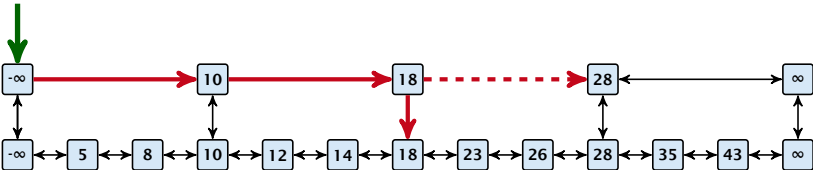
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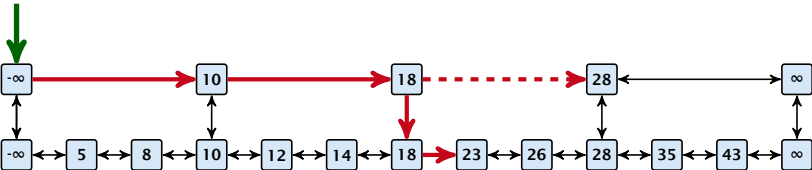
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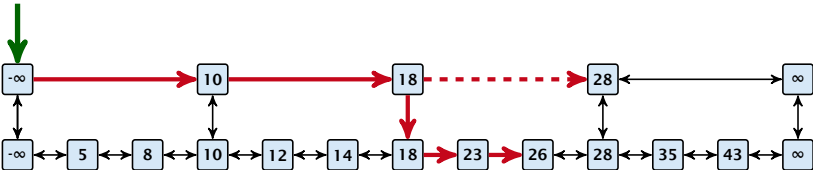
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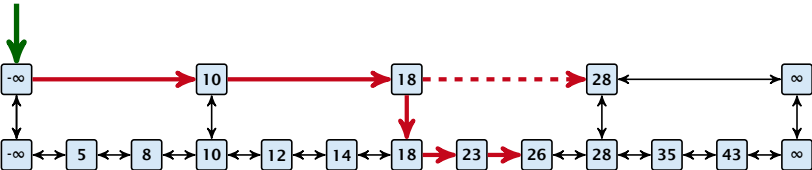
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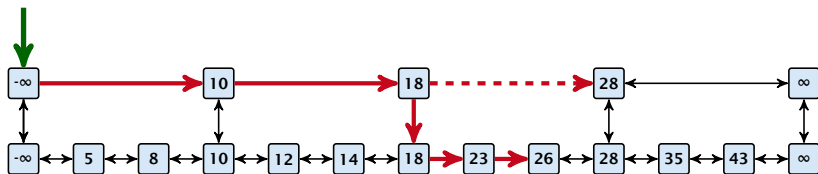


Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).

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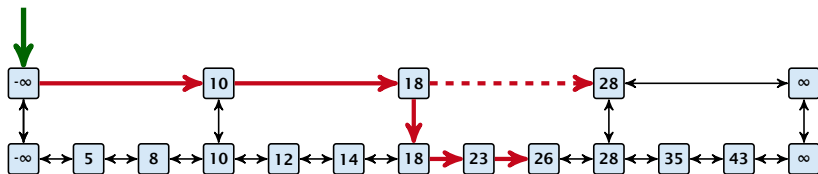
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Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

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## 7.5 Skip Lists

Add more express lanes. Lane  $L_i$  contains roughly every  $\frac{L_{i-1}}{L_i}$ -th item from list  $L_{i-1}$ .

**Search(x) ( $k + 1$  lists  $L_0, \dots, L_k$ )**

- ▶ Find the largest item in list  $L_k$  that is smaller than  $x$ . At most  $|L_k| + 2$  steps.
- ▶ Find the largest item in list  $L_{k-1}$  that is smaller than  $x$ . At most  $\lceil \frac{|L_{k-1}|}{|L_k|+1} \rceil + 2$  steps.
- ▶ Find the largest item in list  $L_{k-2}$  that is smaller than  $x$ . At most  $\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \rceil + 2$  steps.
- ▶ ...
- ▶ At most  $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$  steps.

## 7.5 Skip Lists

Choose ratios between list-lengths evenly, i.e.,  $\frac{|L_{i-1}|}{|L_i|} = r$ , and, hence,  $L_k \approx r^{-k}n$ .

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Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.

## 7.5 Skip Lists

**How to do insert and delete?**

## 7.5 Skip Lists

### How to do insert and delete?

- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

## 7.5 Skip Lists

### How to do insert and delete?

- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

**Use randomization instead!**



## 7.5 Skip Lists

**Insert:**

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- ▶ A search operation gives you the insert position for element  $x$  in every list.

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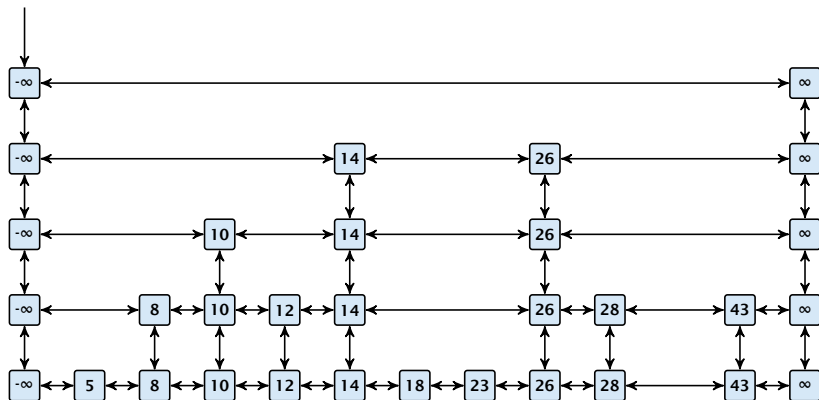
- ▶ You get all predecessors via backward pointers.
- ▶ Delete  $x$  in all lists it actually appears in.

**The time for both operations is dominated by the search time.**



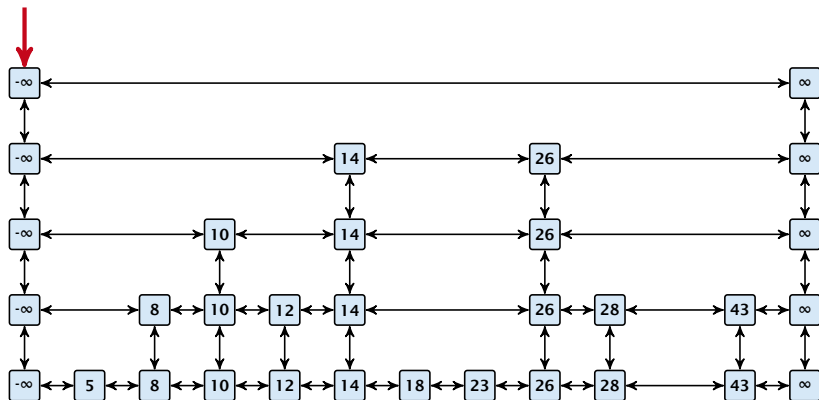
## 7.5 Skip Lists

Insert (35):



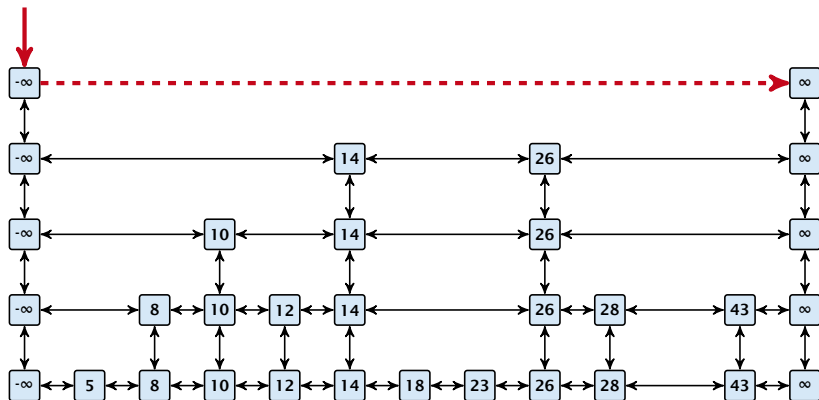
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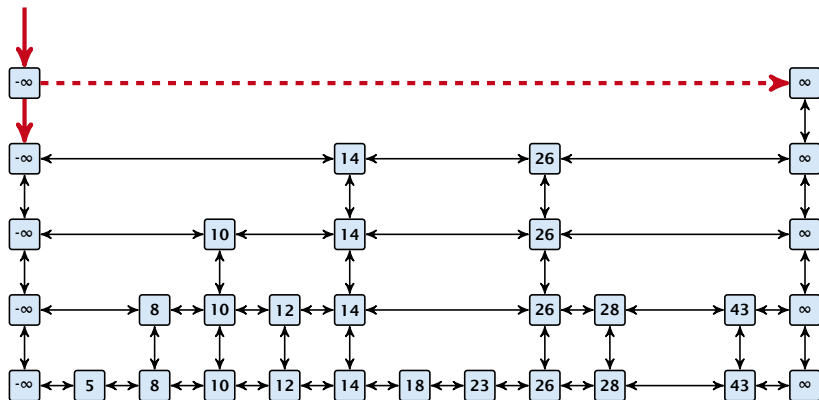
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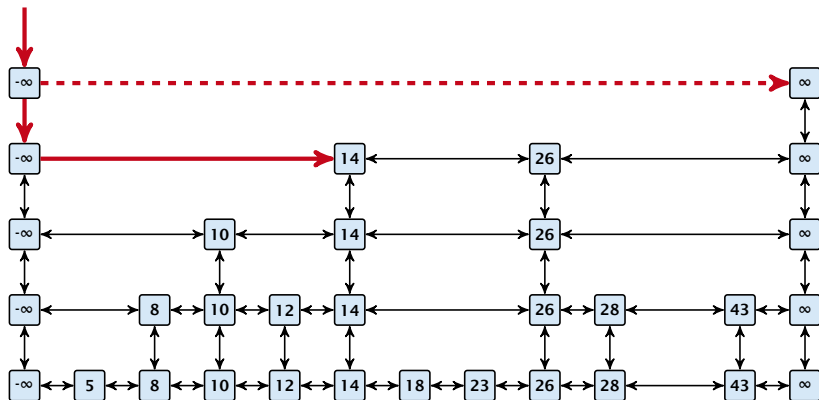
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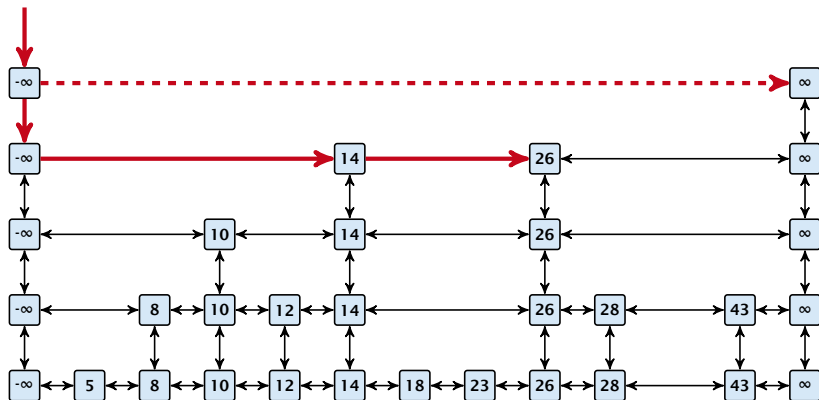
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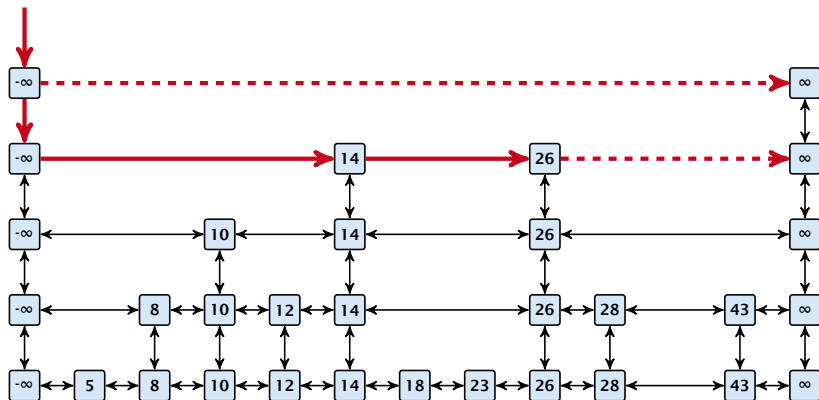
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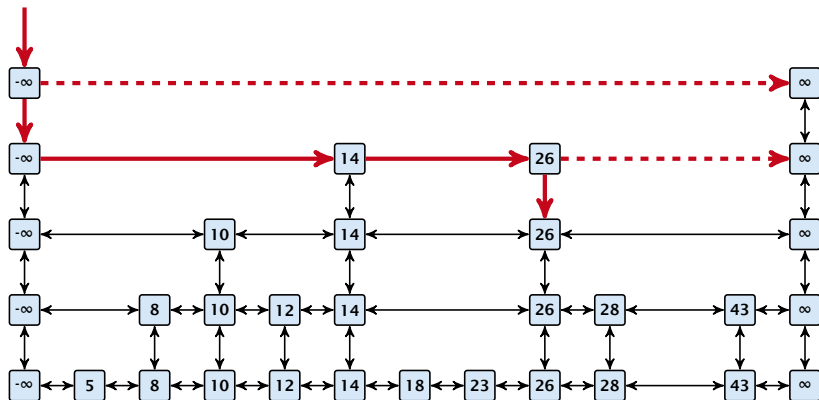
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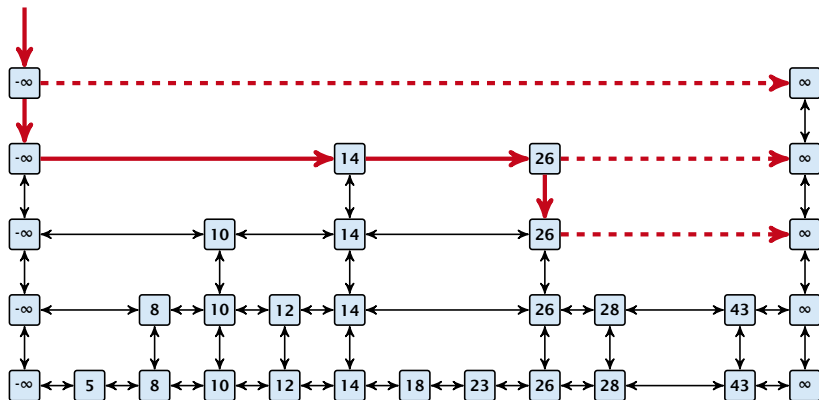
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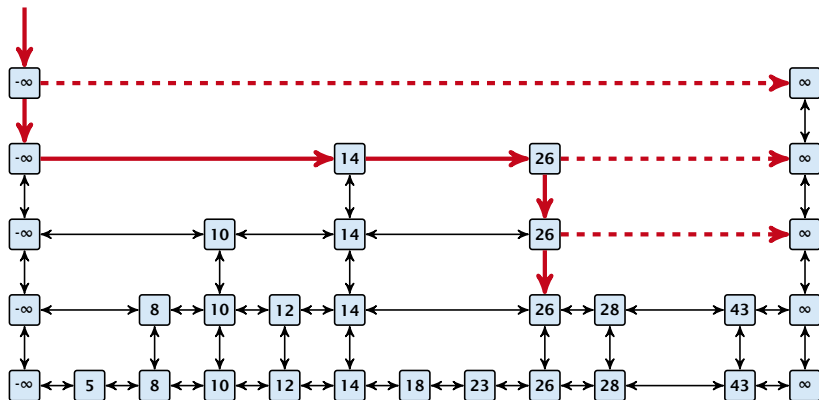
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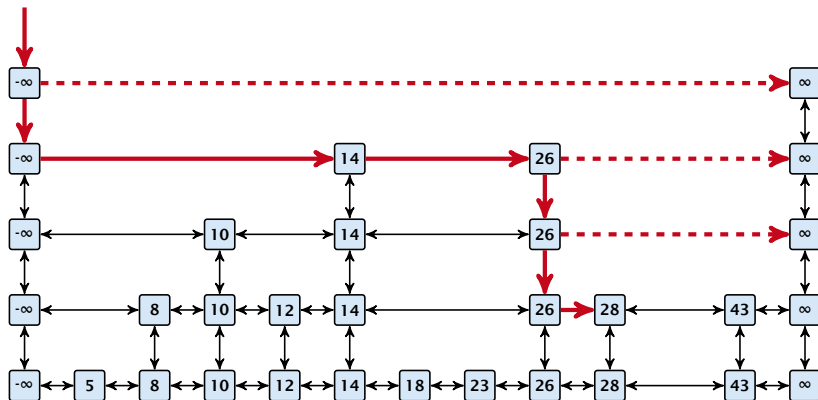
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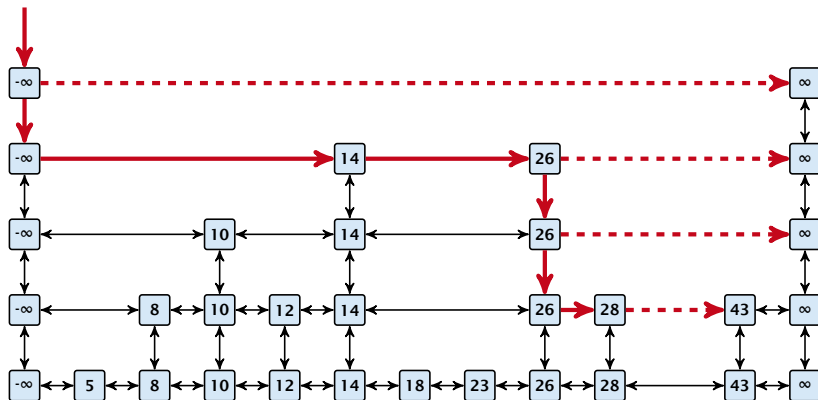
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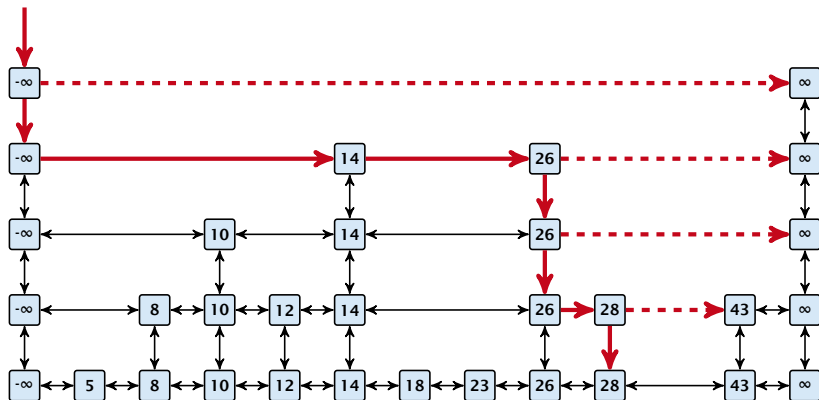
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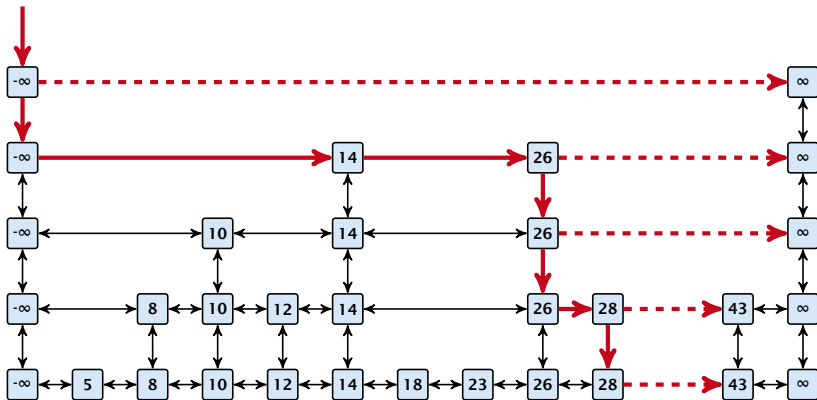
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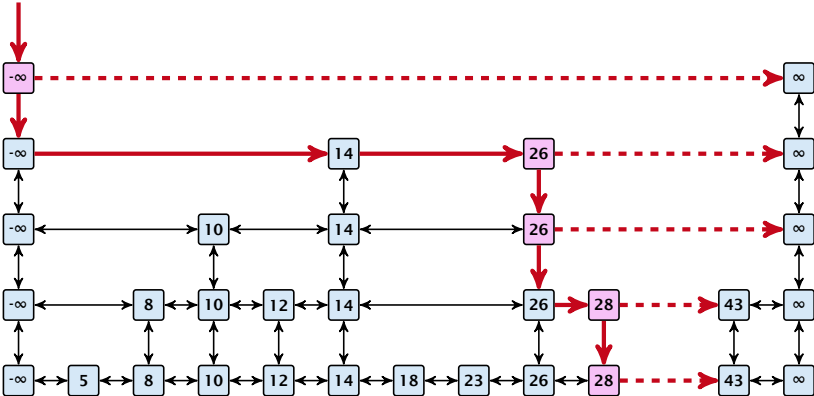
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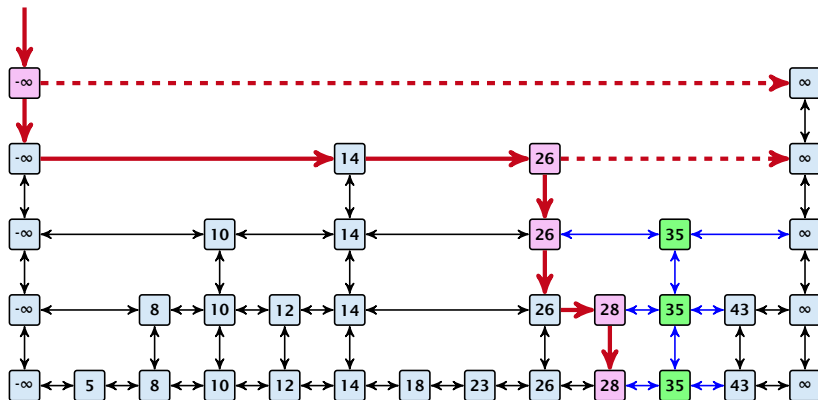
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# High Probability

## Definition 18 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

# High Probability

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Here the  $\mathcal{O}$ -notation hides a constant that may depend on  $\alpha$ .

# High Probability

Suppose there are **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

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This means  $\Pr[E_1 \wedge \dots \wedge E_\ell]$  holds with high probability.



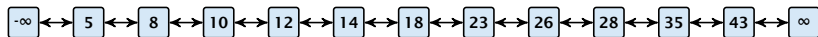
## 7.5 Skip Lists

### Lemma 19

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*

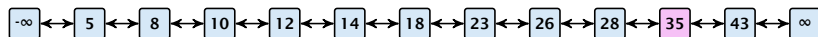
## 7.5 Skip Lists

Backward analysis:



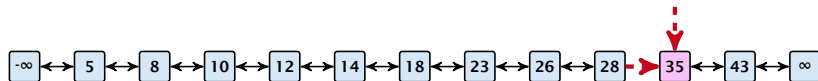
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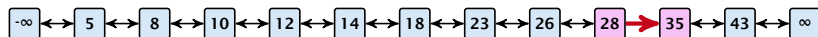
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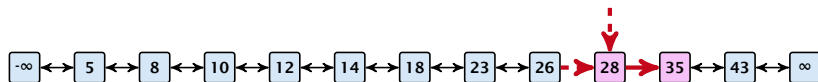
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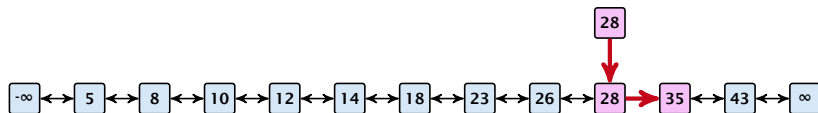
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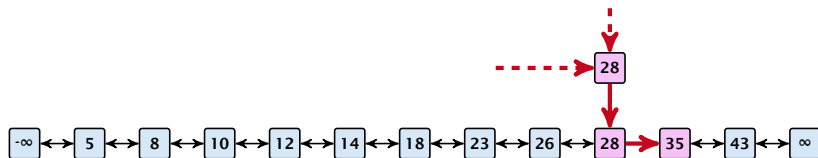
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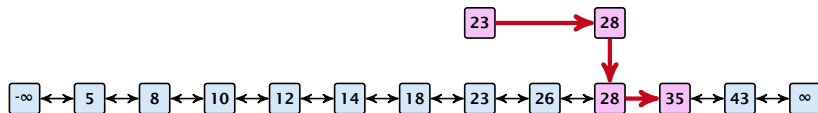
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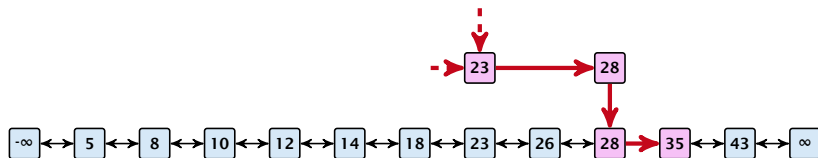
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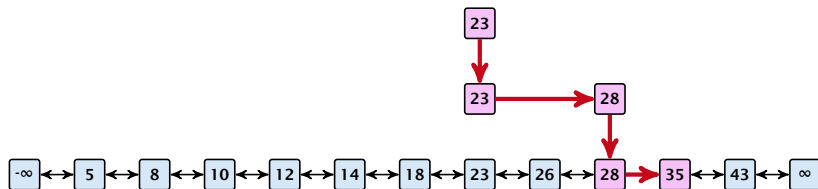
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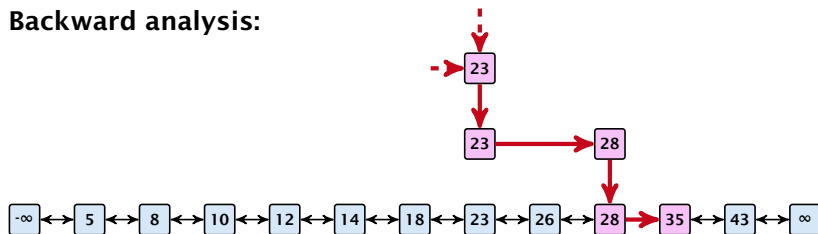
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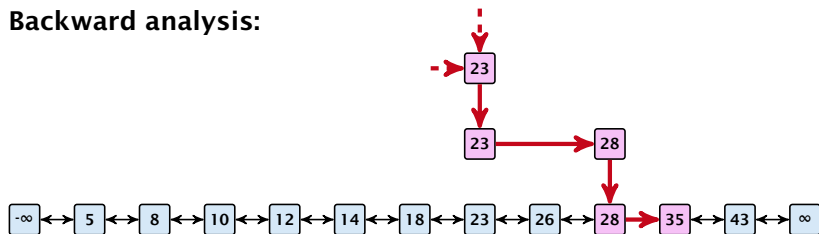
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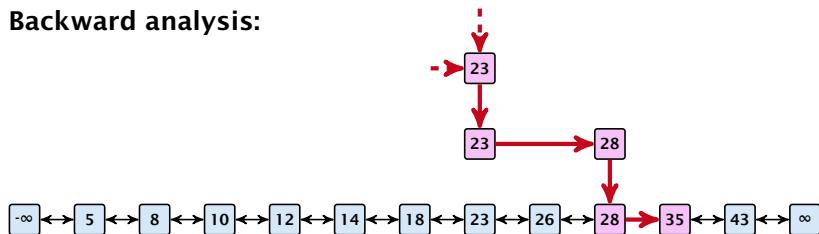
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At each point the path goes up with probability  $1/2$  and left with probability  $1/2$ .

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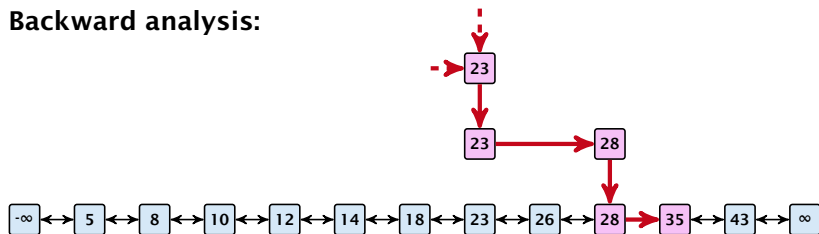
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We show that w.h.p:

- ▶ A “long” search path must also go very high.

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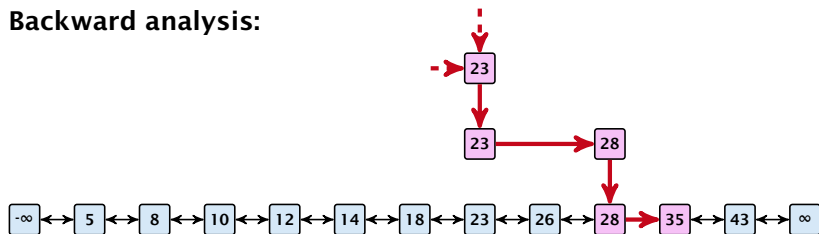
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We show that w.h.p:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.



## 7.5 Skip Lists

### Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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$$\begin{aligned}\binom{n}{k} &= \frac{n \cdot \dots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!} \\ &= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!}\end{aligned}$$

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$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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$$\binom{n}{k} = \frac{n \cdot \dots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}$$

$$= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{n}{k}\right)^k \cdot \sum_{i \geq 0} \frac{k^i}{i!}$$

## 7.5 Skip Lists

### Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \dots \cdot (n-k+1)}{k \cdot \dots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\begin{aligned} \binom{n}{k} &= \frac{n \cdot \dots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!} \\ &= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{n}{k}\right)^k \cdot \sum_{i \geq 0} \frac{k^i}{i!} = \left(\frac{en}{k}\right)^k \end{aligned}$$

## 7.5 Skip Lists

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Let  $E_{z,k}$  denote the event that a search path is of length  $z$  (number of edges) but does not visit a list above  $L_k$ .

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In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

## 7.5 Skip Lists

$$\Pr[E_{z,k}]$$



## 7.5 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

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for  $\alpha \geq 1$ .

## 7.5 Skip Lists

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Let  $A_{k+1}$  denote the event that the list  $L_{k+1}$  is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} .$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

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Hence,

$$\Pr[\text{search requires } z \text{ steps}]$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}]$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

This means, the search requires at most  $z$  steps, w. h. p.

## 7.6 van Emde Boas Trees

### Dynamic Set Data Structure $S$ :

- ▶  $S.insert(x)$
- ▶  $S.delete(x)$
- ▶  $S.search(x)$
- ▶  $S.min()$
- ▶  $S.max()$
- ▶  $S.succ(x)$
- ▶  $S.pred(x)$

## 7.6 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

- ▶  **$S$ . insert( $x$ ):** Inserts  $x$  into  $S$ .
- ▶  **$S$ . delete( $x$ ):** Deletes  $x$  from  $S$ . Usually assumes that  $x \in S$ .
- ▶  **$S$ . member( $x$ ):** Returns 1 if  $x \in S$  and 0 otherwise.
- ▶  **$S$ . min():** Returns the value of the minimum element in  $S$ .
- ▶  **$S$ . max():** Returns the value of the maximum element in  $S$ .
- ▶  **$S$ . succ( $x$ ):** Returns successor of  $x$  in  $S$ . Returns **null** if  $x$  is maximum or larger than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .
- ▶  **$S$ . pred( $x$ ):** Returns the predecessor of  $x$  in  $S$ . Returns **null** if  $x$  is minimum or smaller than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .

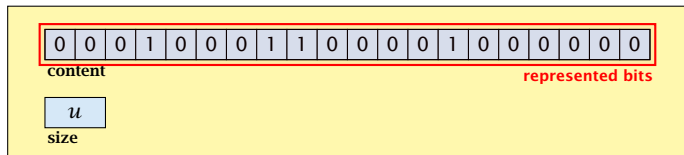


## 7.6 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from  $\{0, 1, \dots, u - 1\}$ , where  $u$  denotes the size of the universe.

# Implementation 1: Array



one array of  $u$  bits

Use an array that encodes the indicator function of the dynamic set.

# Implementation 1: Array

**Algorithm 1** `array.insert( $x$ )`

1: `content[ $x$ ] ← 1;`

**Algorithm 2** `array.delete( $x$ )`

1: `content[ $x$ ] ← 0;`

**Algorithm 3** `array.member( $x$ )`

1: **return** `content[ $x$ ];`

- ▶ Note that we assume that  $x$  is valid, i.e., it falls within the array boundaries.
- ▶ Obviously(?) the running time is constant.

## Implementation 1: Array

### Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

# Implementation 1: Array

## Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
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## Algorithm 5 `array.min()`

```
1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

# Implementation 1: Array

## Algorithm 4 `array.max()`

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1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
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## Algorithm 5 `array.min()`

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1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.

## Implementation 1: Array

### Algorithm 6 `array.succ(x)`

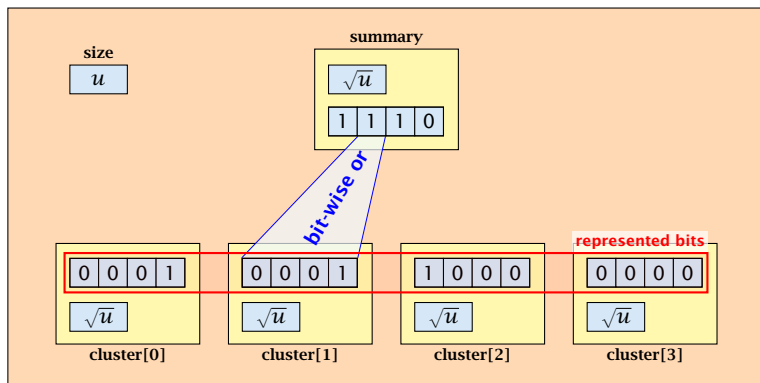
```
1: for ( $i = x + 1$ ;  $i < \text{size}$ ;  $i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

### Algorithm 7 `array.pred(x)`

```
1: for ( $i = x - 1$ ;  $i \geq 0$ ;  $i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.

## Implementation 2: Summary Array



- ▶  $\sqrt{u}$  cluster-arrays of  $\sqrt{u}$  bits.
- ▶ One summary-array of  $\sqrt{u}$  bits. The  $i$ -th bit in the summary array stores the bit-wise or of the bits in the  $i$ -th cluster.



# Implementation 2: Summary Array

## Implementation 2: Summary Array

The bit for a key  $x$  is contained in cluster number  $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$ .

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Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

## Implementation 2: Summary Array

The bit for a key  $x$  is contained in cluster number  $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$ .

Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

For simplicity we assume that  $u = 2^{2k}$  for some  $k \geq 1$ . Then we can compute the cluster-number for an entry  $x$  as  $\text{high}(x)$  (the upper half of the dual representation of  $x$ ) and the position of  $x$  within its cluster as  $\text{low}(x)$  (the lower half of the dual representation).

## Implementation 2: Summary Array

**Algorithm 8**  $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

### Algorithm 9 $\text{insert}(x)$

1:  $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x));$

2:  $\text{summary}.\text{insert}(\text{high}(x));$

## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

```
1: return cluster[high(x)].member(low(x));
```

### Algorithm 9 $\text{insert}(x)$

```
1: cluster[high(x)].insert(low(x));  
2: summary.insert(high(x));
```

- ▶ The running times are constant, because the corresponding array-functions have constant running times.

## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

- 1: cluster[high( $x$ )].delete(low( $x$ ));
- 2: **if** cluster[high( $x$ )].min() = null **then**
- 3:     summary.delete(high( $x$ ));



## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

- ▶ The running time is dominated by the cost of a minimum computation on an array of size  $\sqrt{u}$ . Hence,  $\mathcal{O}(\sqrt{u})$ .

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}()$   
4: return  $\text{maxcluster} \circ \text{offs}$ ;
```

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs};$ 
```

### Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs}$ ;
```

### Algorithm 12 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs}$ ;
```

The operator  $\circ$  stands for the concatenation of two bitstrings.

This means if  $x = 0111_2$  and  $y = 0001_2$  then  $x \circ y = 01110001_2$ .

- ▶ Running time is roughly  $2\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 2: Summary Array

### Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

## Implementation 2: Summary Array

### Algorithm 13 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 2: Summary Array

### Algorithm 14 $\text{pred}(x)$

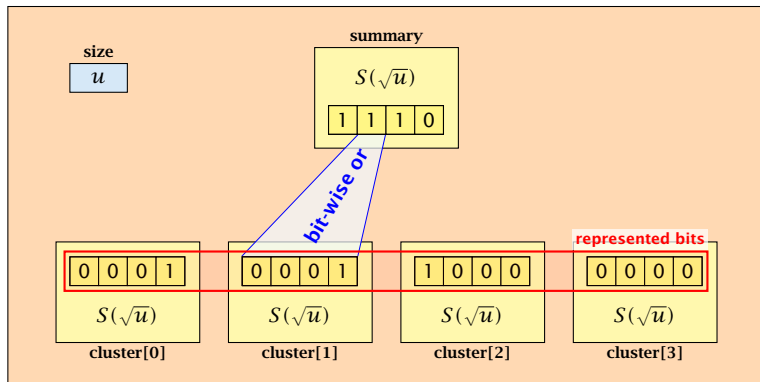
```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$ ;
4: if  $\text{predcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{predcluster}].\text{max}()$ ;
6:     return  $\text{predcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$  is a dynamic set data-structure representing  $u$  bits:





## Implementation 3: Recursion

We assume that  $u = 2^{2^k}$  for some  $k$ .

The data-structure  $S(2)$  is defined as an array of 2-bits (end of the recursion).

# Implementation 3: Recursion

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The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an  $S(4)$  will contain  $S(2)$ 's as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure  $S(4)$  is **not** a recursive call as it will call the function `array.min()`.

## Implementation 3: Recursion

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an  $S(4)$  will contain  $S(2)$ 's as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure  $S(4)$  is **not** a recursive call as it will call the function `array.min()`.

This means that the non-recursive case is been dealt with while initializing the data-structure.

## Implementation 3: Recursion

**Algorithm 15** `member(x)`

---

1: **return** `cluster[high(x)].member(low(x));`

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 16 `insert(x)`

```
1: cluster[high(x)].insert(low(x));  
2: summary.insert(high(x));
```

►  $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 17 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

►  $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$



## Implementation 3: Recursion

### Algorithm 18 $\text{min}()$

```
1: mincluster  $\leftarrow$  summary.min();  
2: if mincluster = null return null;  
3: offs  $\leftarrow$  cluster[mincluster].min();  
4: return mincluster  $\circ$  offs;
```

►  $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 19 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

►  $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ .

## Implementation 3: Recursion

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$$X(\ell)$$

## Implementation 3: Recursion

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## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + \mathbf{1}:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u)$$



## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

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## Implementation 3: Recursion

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

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## Implementation 3: Recursion

$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) &= T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X\left(\frac{\ell}{2}\right) + 1 . \end{aligned}$$

Using Master theorem gives  $X(\ell) = \mathcal{O}(\log \ell)$ , and hence  $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

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$$X(\ell)$$



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## Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{ins}}(2^\ell)$ . Then

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\ell)$ , and hence  $T_{\text{ins}}(u) = \mathcal{O}(\log u)$ .

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\ell)$ , and hence  $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$ .

The same holds for  $T_{\text{max}}(\mathbf{u})$  and  $T_{\text{min}}(\mathbf{u})$ .

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + \mathbf{c} \log(\mathbf{u}).$$



## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ .

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

$$X(\ell) = T_{\text{del}}(2^\ell) = T_{\text{del}}(u)$$

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

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## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell \end{aligned}$$

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

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Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

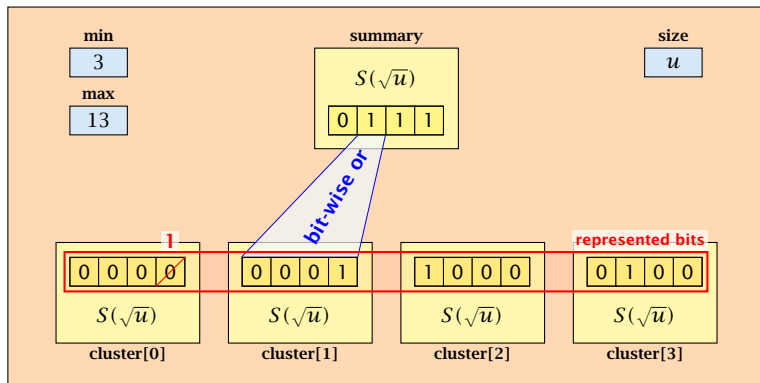
Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) &= T_{\text{del}}(2^\ell) = T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + c \log u \\ &= 2T_{\text{del}}(2^{\frac{\ell}{2}}) + c\ell = 2X\left(\frac{\ell}{2}\right) + c\ell . \end{aligned}$$

Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

The same holds for  $T_{\text{pred}}(u)$  and  $T_{\text{succ}}(u)$ .

# Implementation 4: van Emde Boas Trees



- ▶ The bit referenced by **min** is **not** set within sub-datastructures.
- ▶ The bit referenced by **max** is set within sub-datastructures (if  $\text{max} \neq \text{min}$ ).

# Implementation 4: van Emde Boas Trees

**Advantages of having max/min pointers:**

## Implementation 4: van Emde Boas Trees

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# Implementation 4: van Emde Boas Trees

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## Implementation 4: van Emde Boas Trees

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- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.

## Implementation 4: van Emde Boas Trees

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## Implementation 4: van Emde Boas Trees

### Advantages of having max/min pointers:

- ▶ Recursive calls for **min** and **max** are constant time.
- ▶ **min = null** means that the data-structure is empty.
- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting **min = max =  $x$** .
- ▶ We can delete from a data-structure that just contains one element in constant time by setting **min = max = null**.

## Implementation 4: van Emde Boas Trees

**Algorithm 20** max()

---

1: **return** max;

**Algorithm 21** min()

---

1: **return** min;

- ▶ Constant time.

## Implementation 4: van Emde Boas Trees

### Algorithm 22 `member(x)`

```
1: if  $x = \min$  then return 1; // TRUE  
2: return cluster[high(x)].member(low(x));
```

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Rightarrow T(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 23 $\text{succ}(x)$

```
1: if  $\text{min} \neq \text{null} \wedge x < \text{min}$  then return  $\text{min}$ ;  
2:  $\text{maxincluster} \leftarrow \text{cluster}[\text{high}(x)].\text{max}()$ ;  
3: if  $\text{maxincluster} \neq \text{null} \wedge \text{low}(x) < \text{maxincluster}$  then  
4:    $\text{offs} \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ ;  
5:   return  $\text{high}(x) \circ \text{offs}$ ;  
6: else  
7:    $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;  
8:   if  $\text{succcluster} = \text{null}$  then return  $\text{null}$ ;  
9:    $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;  
10:  return  $\text{succcluster} \circ \text{offs}$ ;
```

►  $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 35 insert( $x$ )

```
1: if min = null then
2:     min =  $x$ ; max =  $x$ ;
3: else
4:     if  $x < \text{min}$  then exchange  $x$  and min;
5:     if  $x > \text{max}$  then max =  $x$ ;
6:     if cluster[high( $x$ )].min = null; then
7:         summary.insert(high( $x$ ));
8:         cluster[high( $x$ )].insert(low( $x$ ));
9:     else
10:        cluster[high( $x$ )].insert(low( $x$ ));
```

►  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Rightarrow T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

Note that the recursive call in Line 7 takes constant time as the if-condition in Line 5 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 6 and in Line 9. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$ .

## Implementation 4: van Emde Boas Trees

- ▶ **Assumes that  $x$  is contained in the structure.**

**Algorithm 36** delete( $x$ )

```
1: if min = max then  
2:     min = max = null;  
3: else  
4:     if  $x$  = min then  
5:         firstcluster  $\leftarrow$  summary.min();  
6:         offs  $\leftarrow$  cluster[firstcluster].min();  
7:          $x \leftarrow$  firstcluster  $\circ$  offs;  
8:         min  $\leftarrow$   $x$ ;  
9:     cluster[high( $x$ )].delete(low( $x$ ));  
                                     continued...
```

## Implementation 4: van Emde Boas Trees

- Assumes that  $x$  is contained in the structure.

**Algorithm 36** delete( $x$ )

```
1: if min = max then
2:     min = max = null;
3: else
4:     if  $x = \text{min}$  then find new minimum
5:          $\text{firstcluster} \leftarrow \text{summary.min}()$ ;
6:          $\text{offs} \leftarrow \text{cluster}[\text{firstcluster}].\text{min}()$ ;
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
continued...
```



## Implementation 4: van Emde Boas Trees

- ▶ **Assumes that  $x$  is contained in the structure.**

**Algorithm 36** delete( $x$ )

```
1: if min = max then  
2:     min = max = null;  
3: else  
4:     if  $x$  = min then  
5:         firstcluster  $\leftarrow$  summary.min();  
6:         offs  $\leftarrow$  cluster[firstcluster].min();  
7:          $x \leftarrow$  firstcluster  $\circ$  offs;  
8:         min  $\leftarrow$   $x$ ;  
9:     cluster[high( $x$ )].delete(low( $x$ )); delete
```

continued...

## Implementation 4: van Emde Boas Trees

### Algorithm 36 delete( $x$ )

...continued

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:   if  $x$  = max then
13:       summax  $\leftarrow$  summary.max();
14:       if summax = null then max  $\leftarrow$  min;
15:       else
16:           offs  $\leftarrow$  cluster[summax].max();
17:           max  $\leftarrow$  summax  $\circ$  offs
18:   else
19:       if  $x$  = max then
20:           offs  $\leftarrow$  cluster[high( $x$ )].max();
21:           max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```

## Implementation 4: van Emde Boas Trees

### Algorithm 36 delete( $x$ )

...continued

fix maximum

```
10:   if cluster[high( $x$ )].min() = null then
11:       summary.delete(high( $x$ ));
12:       if  $x$  = max then
13:           summax  $\leftarrow$  summary.max();
14:           if summax = null then max  $\leftarrow$  min;
15:           else
16:               offs  $\leftarrow$  cluster[summax].max();
17:               max  $\leftarrow$  summax  $\circ$  offs
18:       else
19:           if  $x$  = max then
20:               offs  $\leftarrow$  cluster[high( $x$ )].max();
21:               max  $\leftarrow$  high( $x$ )  $\circ$  offs;
```

## Implementation 4: van Emde Boas Trees

Note that only one of the possible recursive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where  $x$  was deleted is now empty. But this means that the call in Line 9 deleted the last element in  $\text{cluster}[\text{high}(x)]$ . Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c .$$

This gives  $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$ .

## 7.6 van Emde Boas Trees

### Space requirements:

- ▶ The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) .$$

- ▶ Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- ▶ One can show by induction that the space requirement is  $S(u) = \mathcal{O}(u)$ . Exercise.

- ▶ Let the “real” recurrence relation be

$$S(k^2) = (k + 1)S(k) + c_1 \cdot k; S(4) = c_2$$

- ▶ Replacing  $S(k)$  by  $R(k) := S(k)/c_2$  gives the recurrence

$$R(k^2) = (k + 1)R(k) + ck; R(4) = 1$$

where  $c = c_1/c_2 < 1$ .

- ▶ Now, we show  $R(k^2) \leq k^2 - 2$  for  $k^2 \geq 4$ .
  - ▶ Obviously, this holds for  $k^2 = 4$ .
  - ▶ For  $k^2 > 4$  we have

$$\begin{aligned} R(k^2) &= (1 + k)R(k) + ck \\ &\leq (1 + k)(k - 2) + k \leq k^2 - 2 \end{aligned}$$

- ▶ This shows that  $R(k)$  and, hence,  $S(k)$  grows linearly.

## 7.7 Hashing

### Dictionary:

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Then the memory location of an object  $x$  with key  $k$  is determined by successively comparing  $k$  to split-elements.

**Hashing** tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

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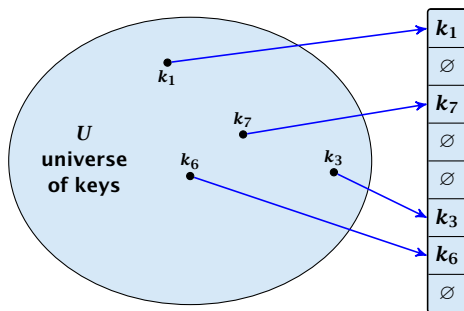
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- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

# Direct Addressing

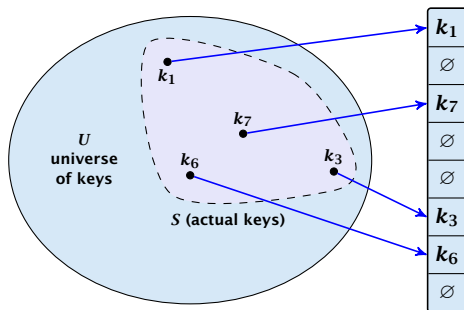
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

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If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

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Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

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Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .



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## Lemma 20

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

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## Uniform hashing:

Choose a hash function uniformly at random from all functions  $f : U \rightarrow [0, \dots, n-1]$ .

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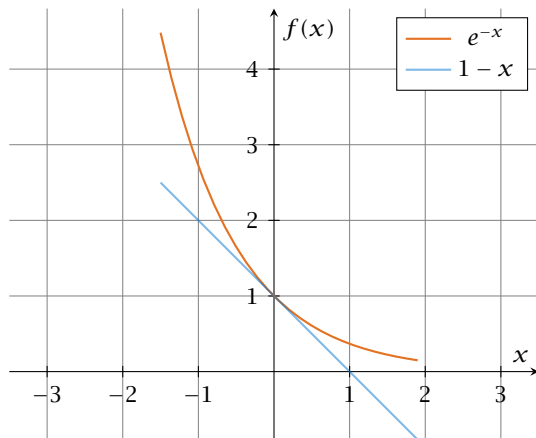
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Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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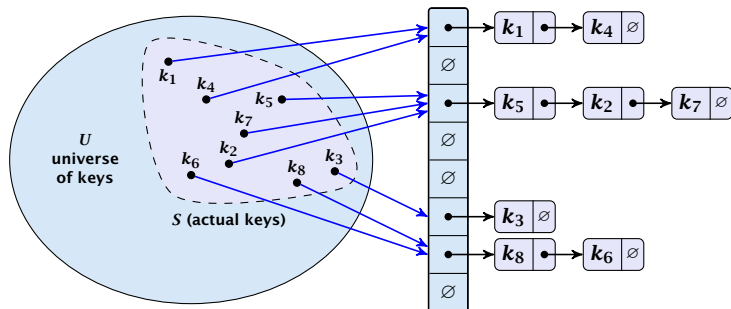
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- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.



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We assume **uniform hashing** for the following analysis.

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$$A^- = 1 + \alpha .$$

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Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

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**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

(sometimes:  $h(k, i) = h(k) + ci \bmod n$ ).

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes:  $h(k, i) = h(k) + ci \pmod n$ ).

- ▶ **Quadratic probing:**

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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (**teilerfremd**); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

# Linear Probing

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## Lemma 21

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

# Quadratic Probing

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## Lemma 22

Let  $Q$  be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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## Lemma 23

Let  $D$  be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right)$$

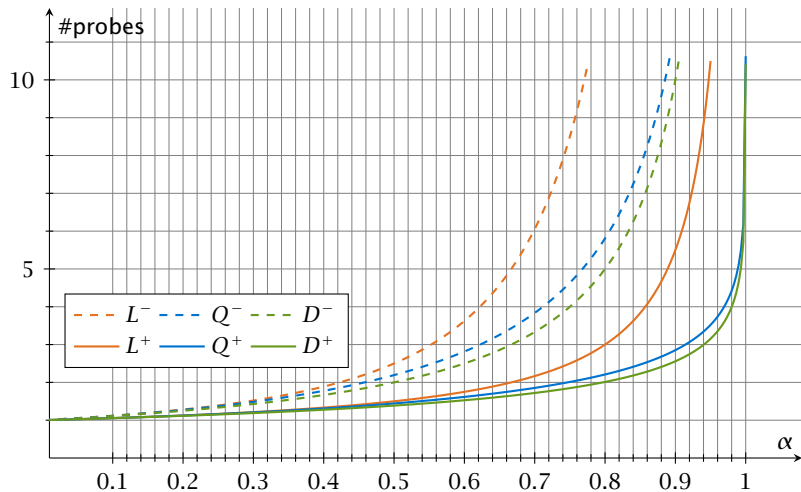
$$D^- \approx \frac{1}{1 - \alpha}$$

# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing





# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

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$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

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$$\begin{aligned} & \Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ & \quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}] \end{aligned}$$

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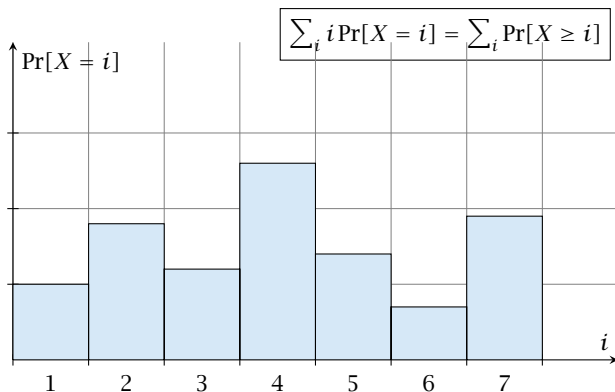
$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

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$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

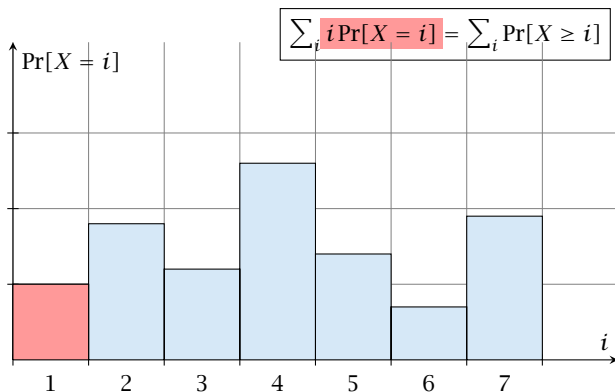
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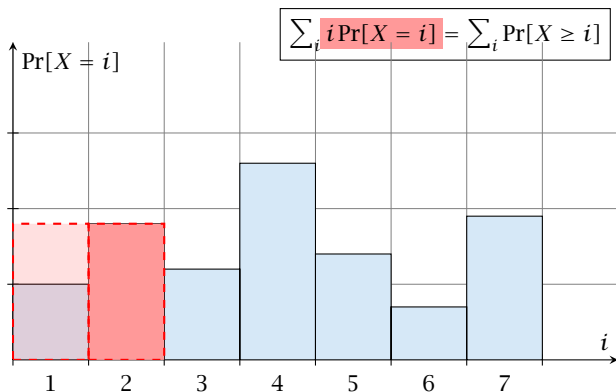
# Analysis of Idealized Open Address Hashing

$i = 1$



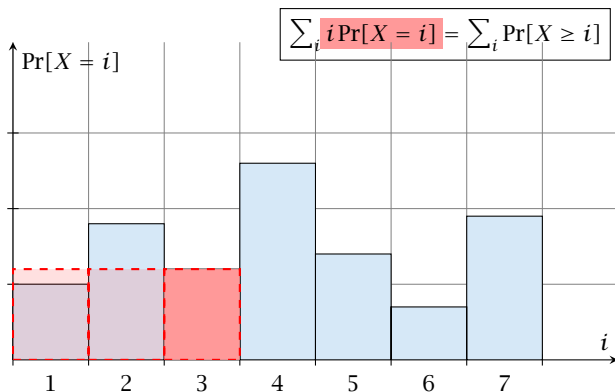
# Analysis of Idealized Open Address Hashing

$i = 2$



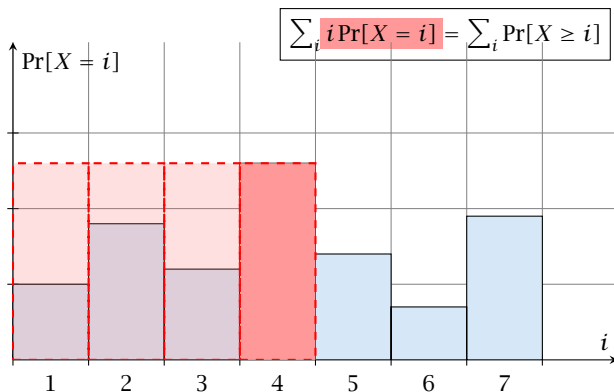
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$i = 3$



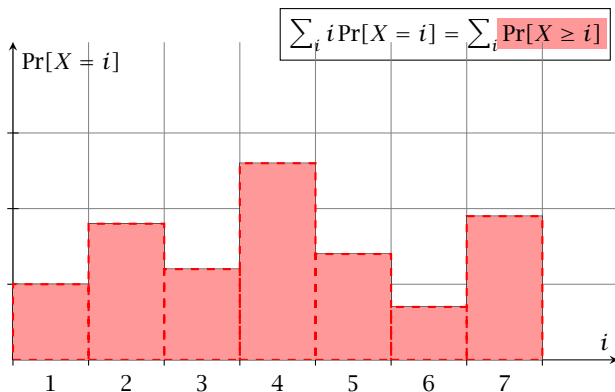
# Analysis of Idealized Open Address Hashing

$i = 4$



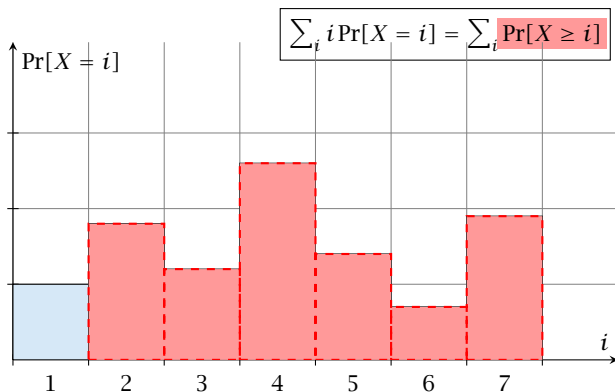
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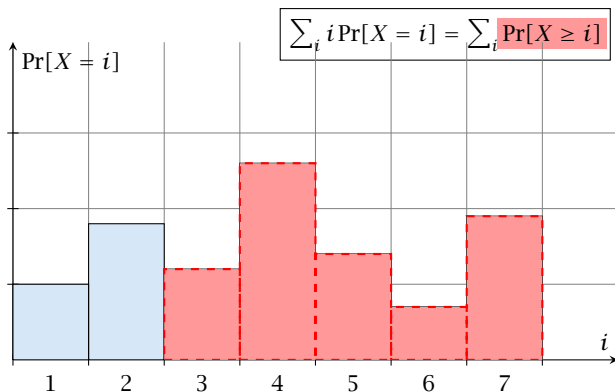
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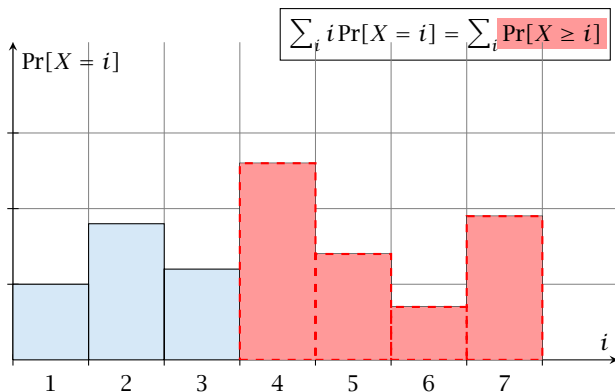
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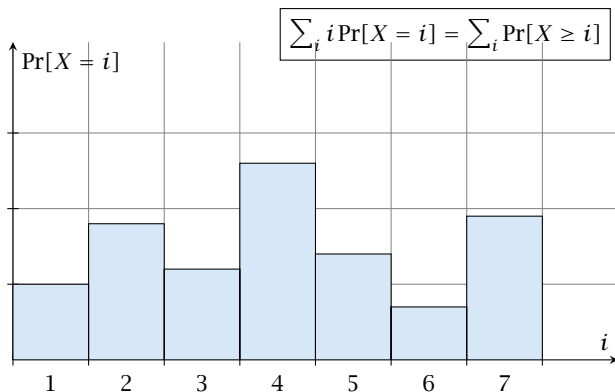
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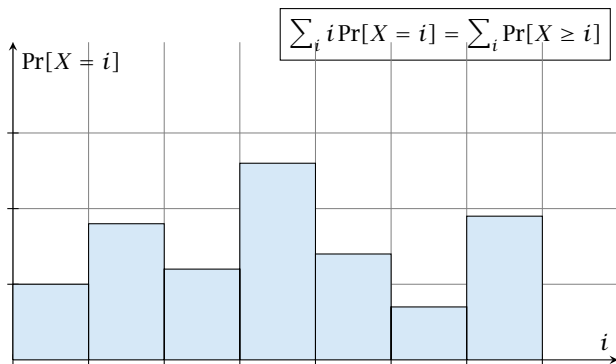




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The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

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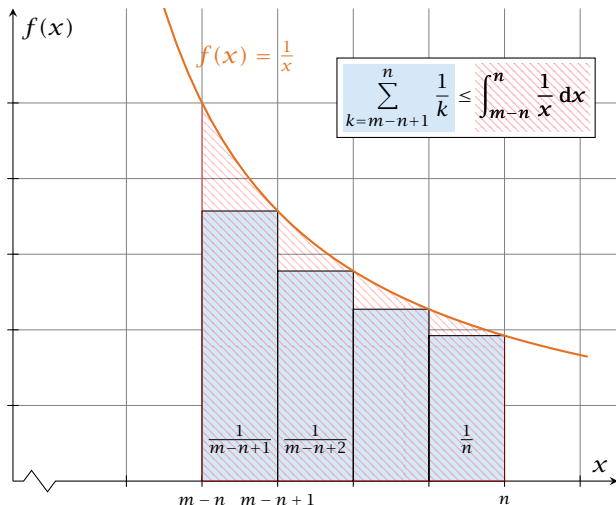
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# Analysis of Idealized Open Address Hashing



## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- ▶ For open addressing this is difficult.

# Deletions in Hashtables

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  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

## Deletions for Linear Probing

### Algorithm 37 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:      $y \leftarrow T[p]$ 
5:      $T[p] \leftarrow \text{null}$ 
6:      $p \leftarrow \text{succ}(p)$ 
7:     insert( $y$ )
```

$p$  is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.



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However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

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Universal hashing tries to define a set  $\mathcal{H}$  of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from  $\mathcal{H}$ .

# Universal Hashing

## Definition 24

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

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Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

# Universal Hashing

## Definition 25

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ ,  
i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$



# Universal Hashing

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A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ , i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

This requirement clearly implies a universal hash-function.

## Definition 26

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

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A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  $(\mu, k)$ -independent if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

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Let  $U := \{0, \dots, p - 1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p - 1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

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$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

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## Lemma 28

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

*is a universal class of hash-functions from  $U$  to  $\{0, \dots, n-1\}$ .*

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## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

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where we use that  $\mathbb{Z}_p$  is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

## Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

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$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

# Universal Hashing

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There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

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What happens when we do the  $\text{mod } n$  operation?

Fix a value  $t_x$ . There are  $p - 1$  possible values for choosing  $t_y$ .

From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.



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It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$



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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)}$$

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Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 29

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

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For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

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The polynomial is defined by  $d+1$  distinct points.

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Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.



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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

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Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.

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$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose  $\bar{a}$  such that  $h_{\bar{a}} \in A_\ell$ .

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

$$\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}}$$



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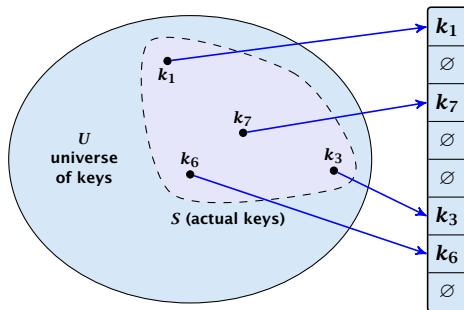
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This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



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Using a universal hash-function the expected number of collisions is

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The probability of having **1** or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

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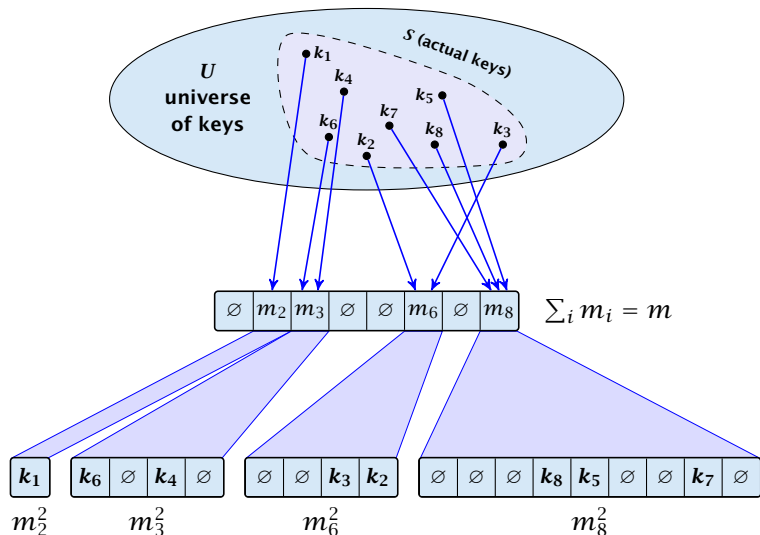
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However, a hash-table size of  $n = m^2$  is very very high.

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Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

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The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ .  
Note that  $m_j$  is a random variable.

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$$\mathbb{E} \left[ \sum_j m_j^2 \right] = \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right]$$

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$$\begin{aligned} \mathbb{E} \left[ \sum_j m_j^2 \right] &= \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[ \sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[ \sum_j m_j \right] \end{aligned}$$

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$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$



# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

# Cuckoo Hashing

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## Goal:

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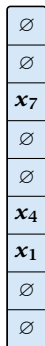
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- ▶ An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .
- ▶ A search clearly takes constant time if the above constraint is met.

# Cuckoo Hashing

Insert:



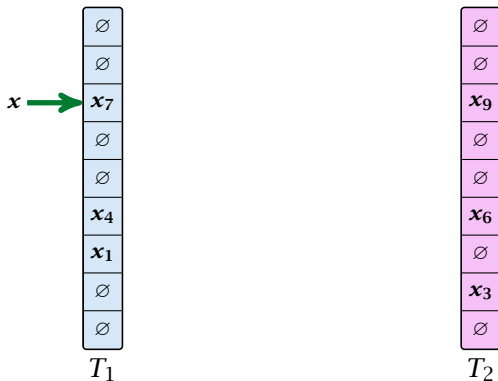
$T_1$



$T_2$

# Cuckoo Hashing

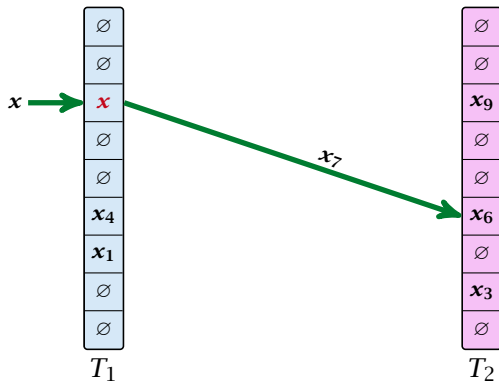
Insert:





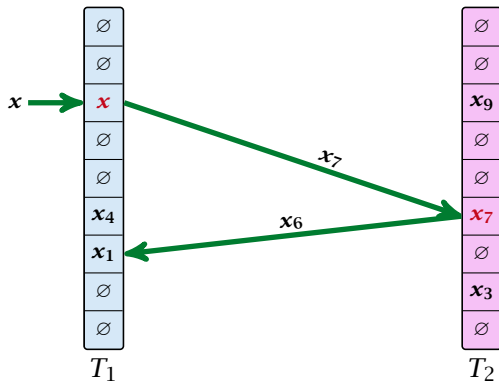
# Cuckoo Hashing

Insert:



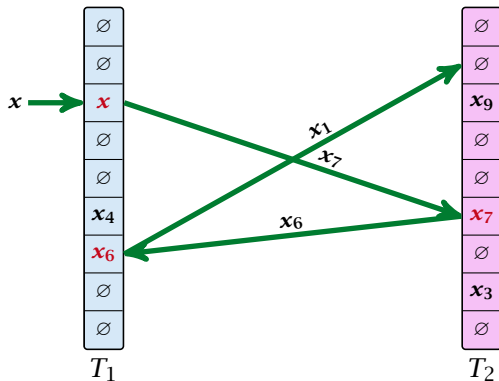
# Cuckoo Hashing

Insert:



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Insert:



## Algorithm 38 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.

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- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because  $x = \text{null}$ .

# Cuckoo Hashing



**What is the expected time for an insert-operation?**

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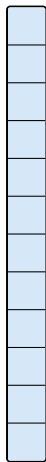
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## What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after  $\text{maxsteps}$  steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

# Cuckoo Hashing: Insert

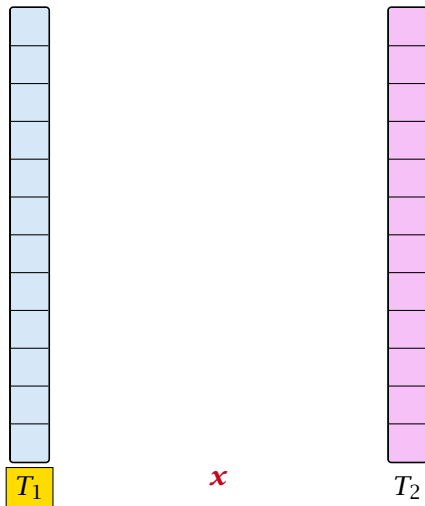


$T_1$

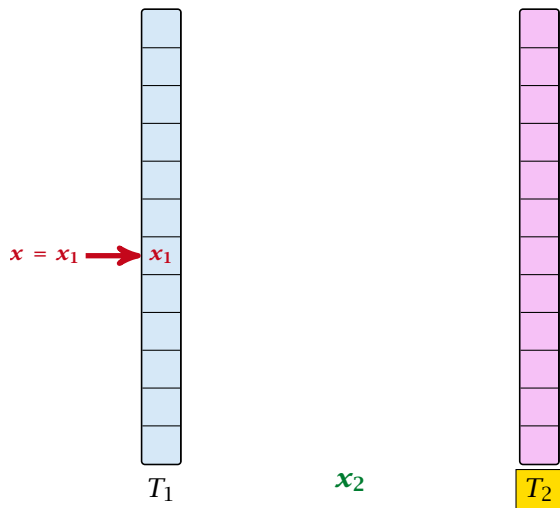


$T_2$

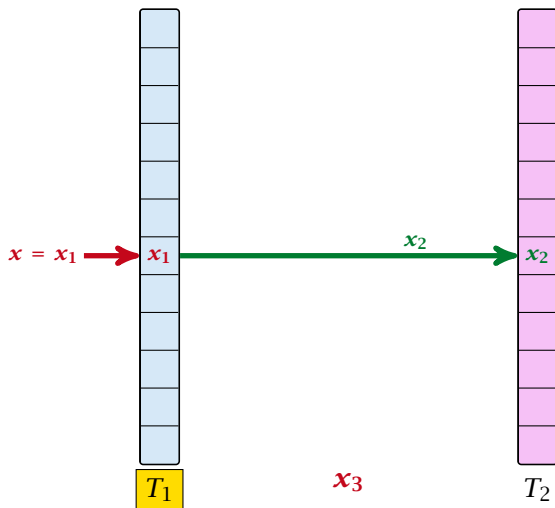
# Cuckoo Hashing: Insert



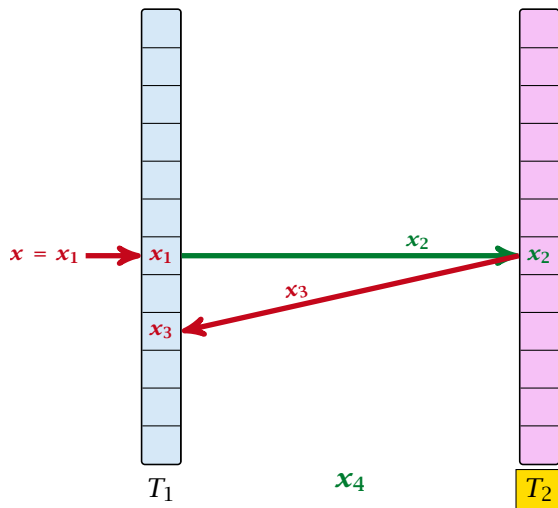
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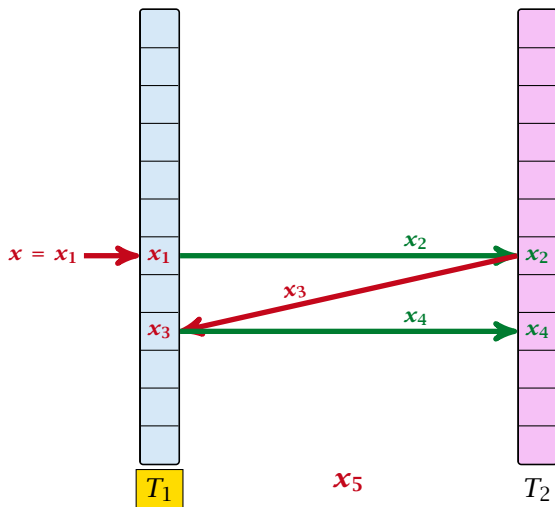


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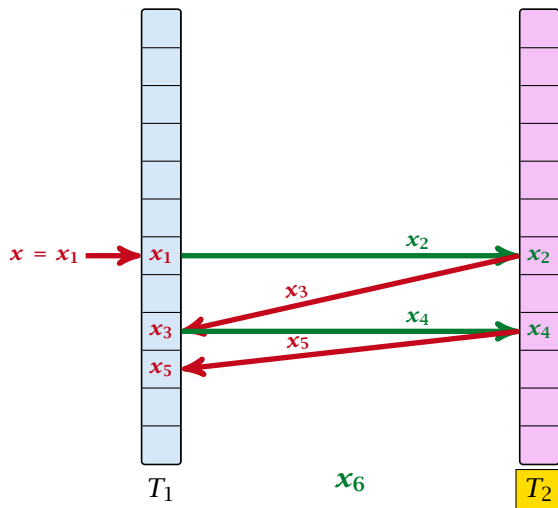




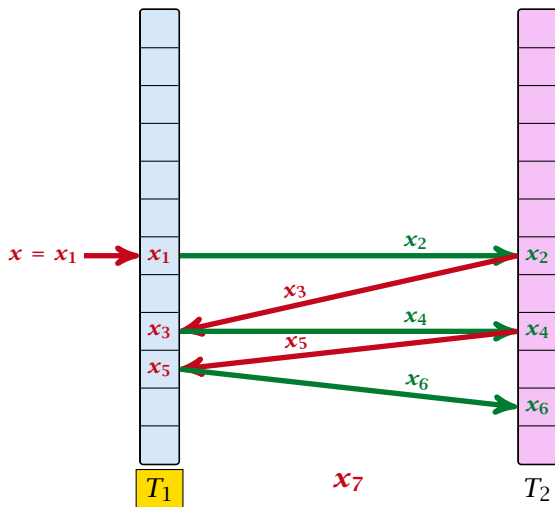
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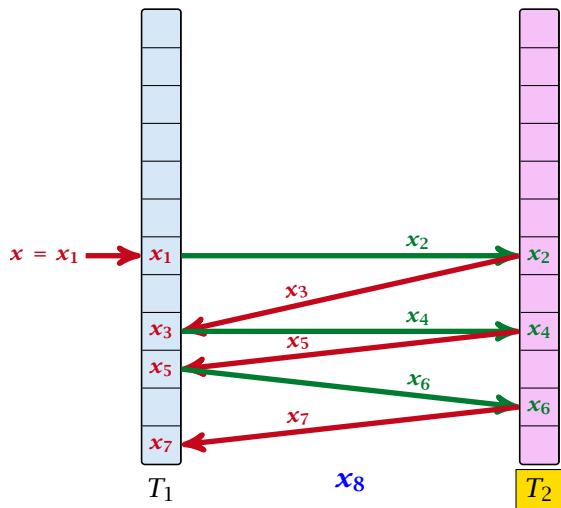
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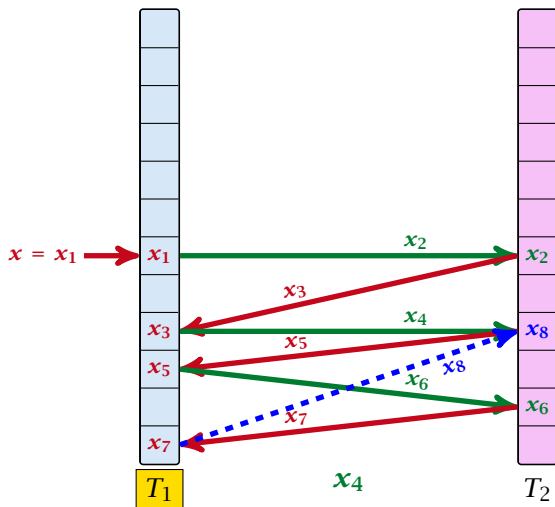
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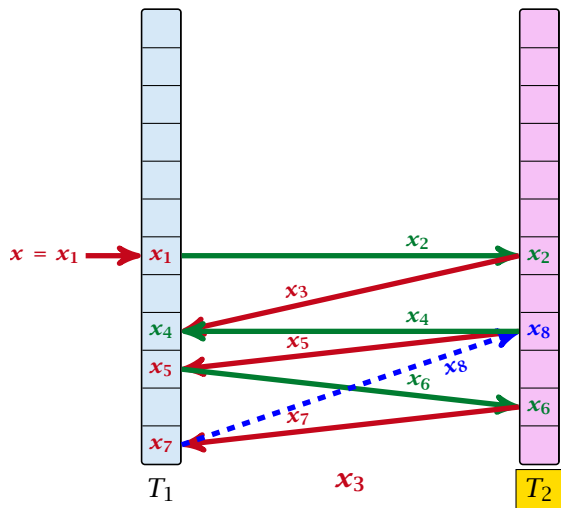
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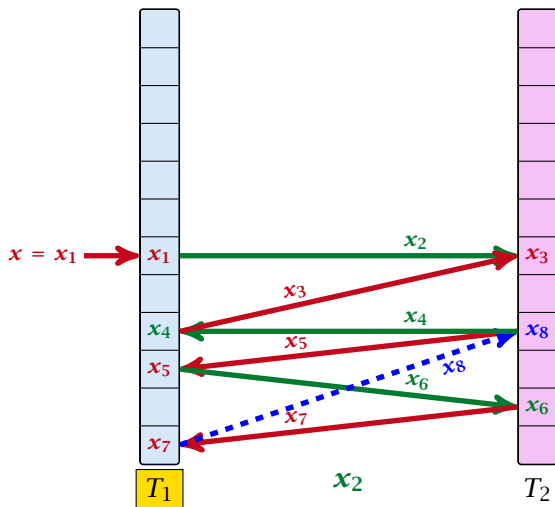
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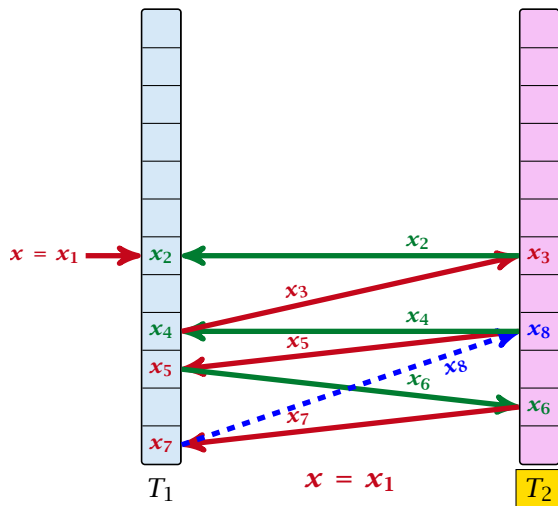
# Cuckoo Hashing: Insert



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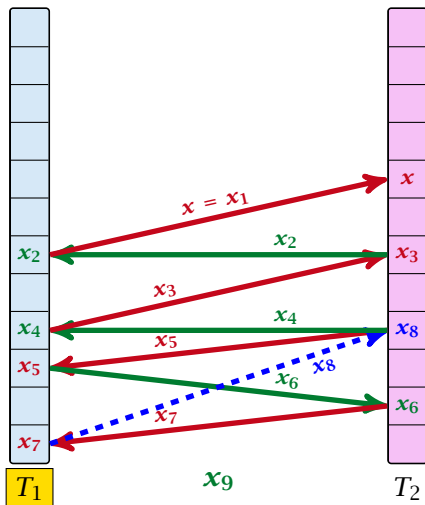


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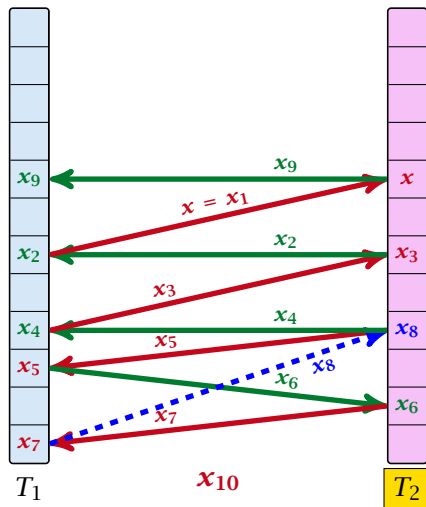




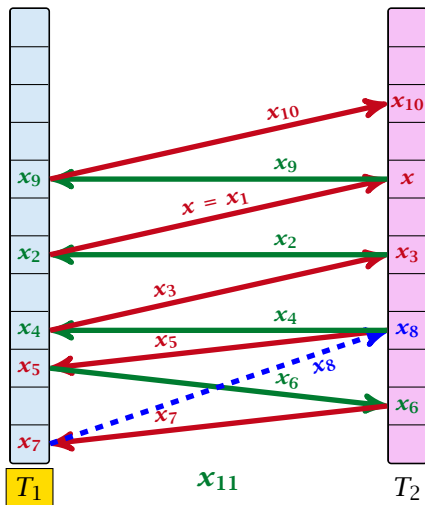
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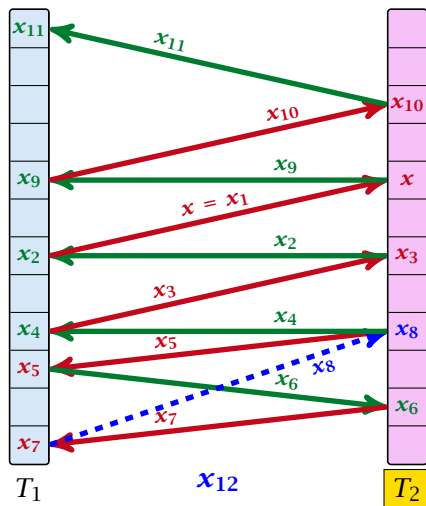
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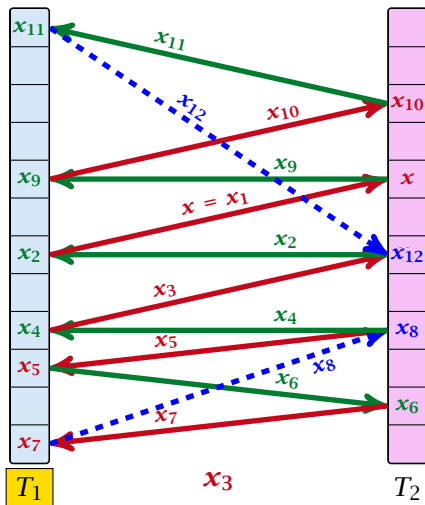
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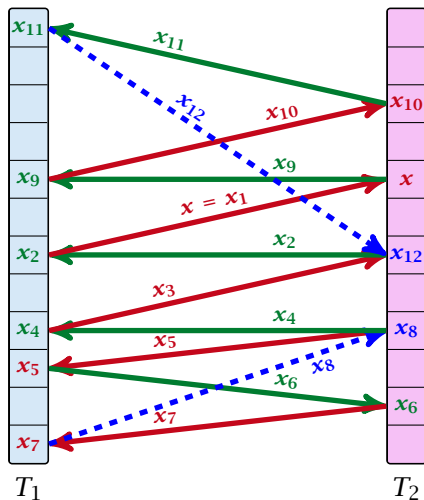
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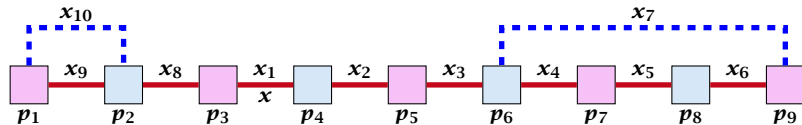
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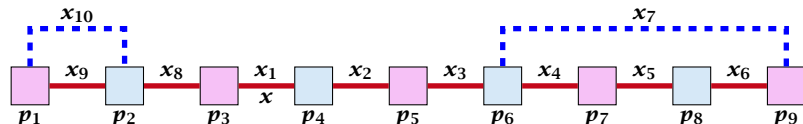


# Cuckoo Hashing



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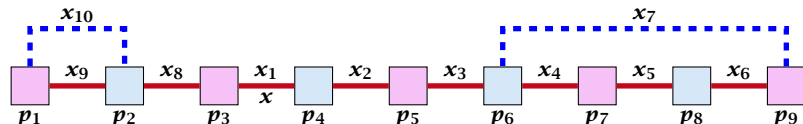


A cycle-structure of size  $s$  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).



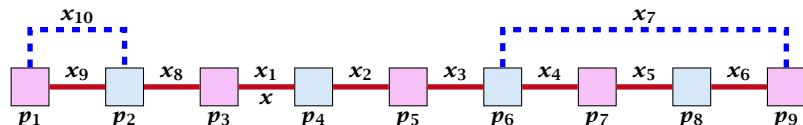
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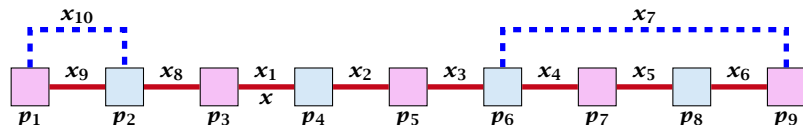
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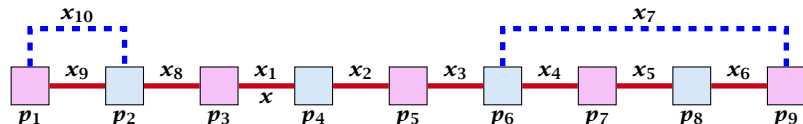
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- ▶ The leftmost cell is “linked forward” to some cell on the right.
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- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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## Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .

# Cuckoo Hashing

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This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.



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These events are independent.

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The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

What is the probability that **there exists** an active cycle structure of size  $s$ ?

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- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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Here we used the fact that  $(1 + \epsilon)m \leq n$ .

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Hence,

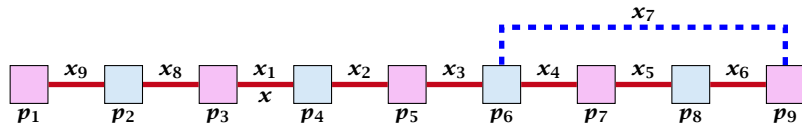
$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$



# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

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## Lemma 30

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

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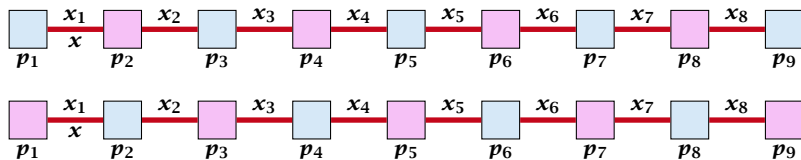
$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

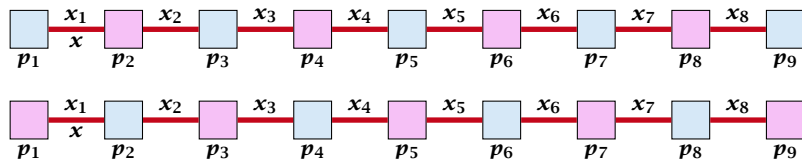
Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

# Cuckoo Hashing



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# Cuckoo Hashing

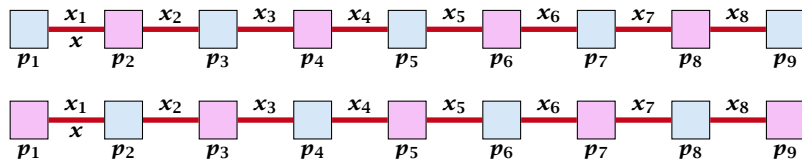


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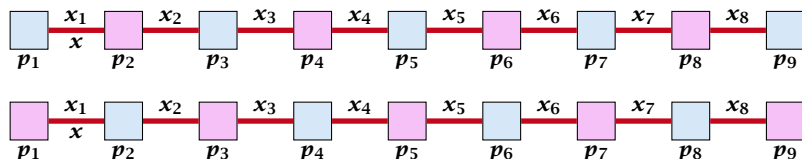
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A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

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This gives  $\text{maxsteps} = \Theta(\log m)$ .

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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for a suitable constant  $c > 0$ .

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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Therefore the expected cost for re-hashes is  $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$ .

## Formal Proof

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$$\begin{aligned} \mathbb{E}[X_i^s] &= \mathbb{E}[\text{steps} \mid \text{phase successful}] \cdot \Pr[\text{phase successful}] \\ &\quad + \text{maxsteps} \cdot \Pr[\text{not successful}] = \mathcal{O}(1) . \end{aligned}$$

The expected cost for all rehashes is

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Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

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How do we make sure that  $n \geq (1 + \epsilon)m$ ?

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- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.



# Cuckoo Hashing

## Lemma 31

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .

## 8 Priority Queues

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Sometimes we also have

- ▶  **$S$ . merge( $S'$ )**:  $S := S \cup S'$ ;  $S' := \emptyset$ .

## 8 Priority Queues

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- ▶ **handle  $S$ . insert( $x$ )**: Adds element  $x$  to the data-structure, and returns a **handle** to the object for future reference.
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## 8 Priority Queues

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- ▶  **$S$ . decrease-key( $h, k$ )**: Decreases the key of the element specified by handle  $h$  to  $k$ . Assumes that the key is at least  $k$  before the operation.

# Dijkstra's Shortest Path Algorithm

## Algorithm 39 Shortest-Path( $G = (V, E, d), s \in V$ )

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
6:      $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.is-empty() = false$  do  
9:      $v \leftarrow S.delete-min()$ ;  
10:    for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:        if  $x.key > v.key + d(v, x)$  then  
12:             $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:             $x.key \leftarrow v.key + d(v, x)$ ;
```

# Prim's Minimum Spanning Tree Algorithm

**Algorithm 40** Prim-MST( $G = (V, E, d), s \in V$ )

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:      $v.key \leftarrow \infty$ ;  
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13:             $x.key \leftarrow d(v, x)$ ;  
14:             $x.pred \leftarrow v$ ;
```

# Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶  $|V|$  insert() operations
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- ▶  $|E|$  decrease-key() operations



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**How good a running time can we obtain?**

## 8 Priority Queues

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	$n$	$n \log n$	$n \log n$	$n$
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	$n$	$n \log n$	$\log n$	1

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Note that most applications use **build()** only to create an empty heap which then costs time 1.

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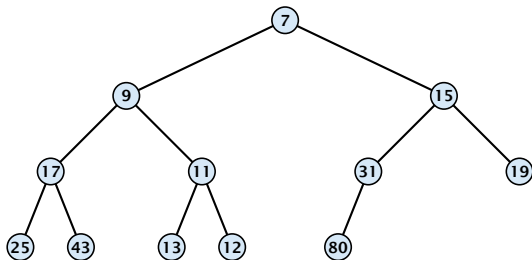
Fibonacci heaps only give an **amortized** guarantee.

## 8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time  $\mathcal{O}((|V| + |E|) \log |V|)$ .

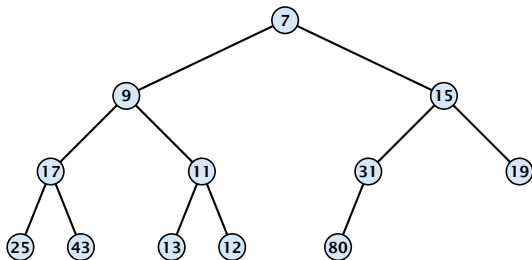
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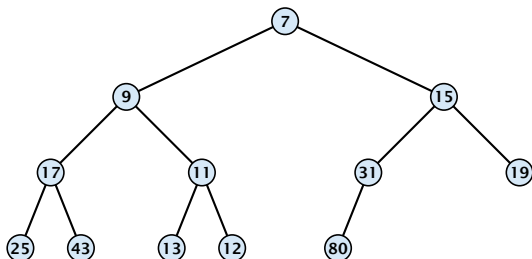
- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.





## 8.1 Binary Heaps

- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- ▶ **Heap property:** A node's key is not larger than the key of one of its children.



**Operations:**

# Binary Heaps

## Operations:

- ▶ **minimum()**: return the root-element. Time  $\mathcal{O}(1)$ .

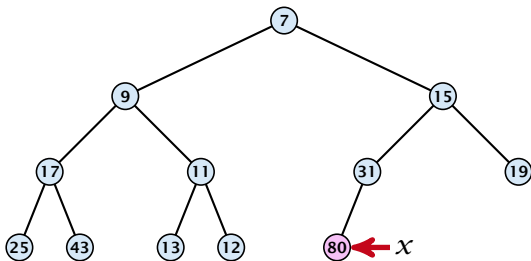
# Binary Heaps

## Operations:

- ▶ **minimum()**: return the root-element. Time  $\mathcal{O}(1)$ .
- ▶ **is-empty()**: check whether root-pointer is **null**. Time  $\mathcal{O}(1)$ .

## 8.1 Binary Heaps

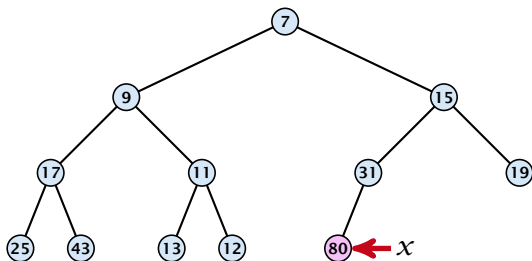
Maintain a pointer to the **last element**  $x$ .



## 8.1 Binary Heaps

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- ▶ We can compute the predecessor of  $x$  (last element when  $x$  is deleted) in time  $\mathcal{O}(\log n)$ .



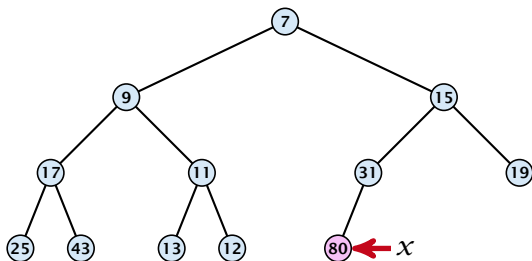
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go left; go right until you reach a leaf



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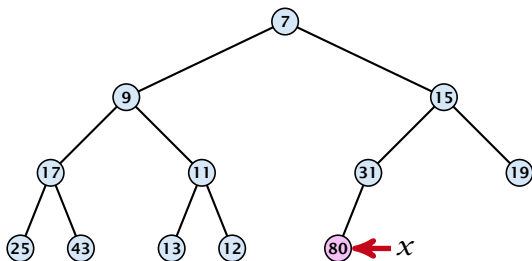
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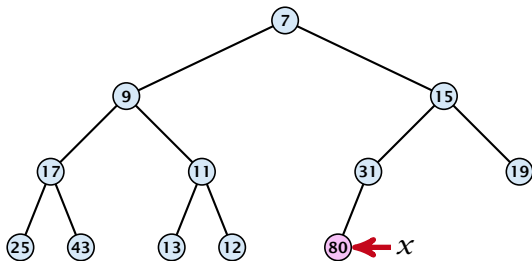
if you hit the root on the way up, go to the rightmost element





## 8.1 Binary Heaps

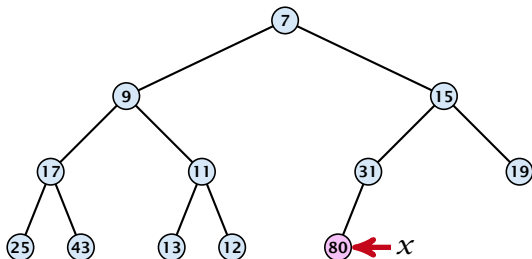
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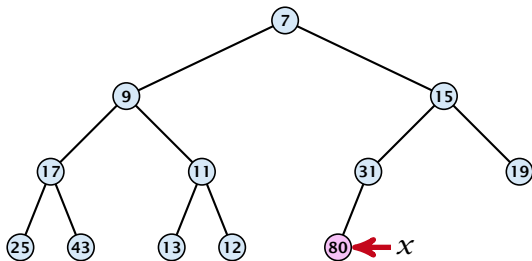
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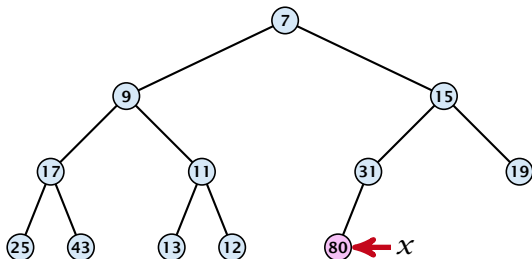
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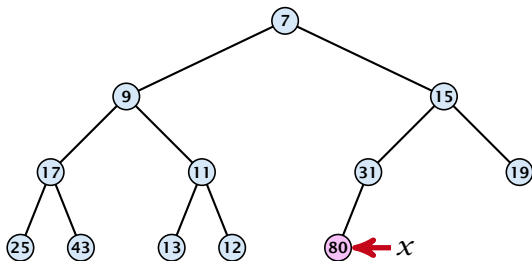
if you hit the root on the way up, go to the leftmost element;

insert a new element as a left child;



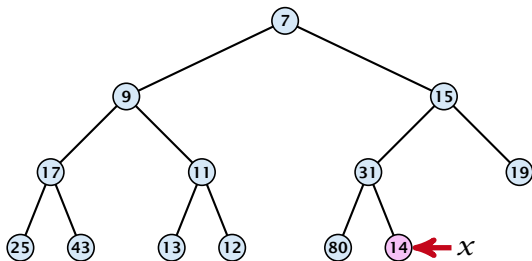
# Insert

1. Insert element at successor of  $x$ .



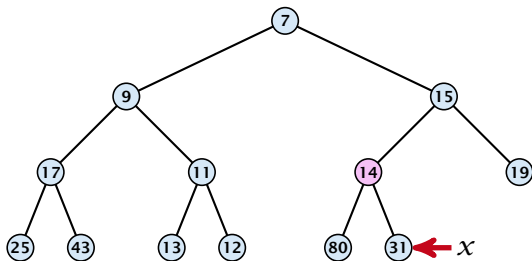
# Insert

1. Insert element at successor of  $x$ .
2. Exchange with parent until heap property is fulfilled.



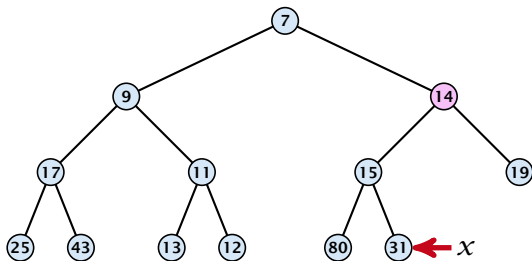
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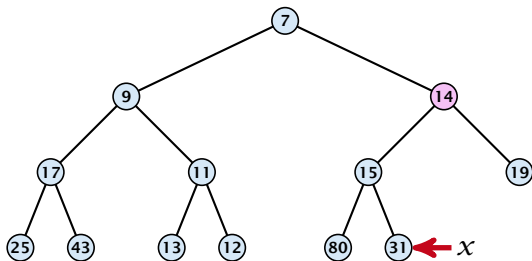
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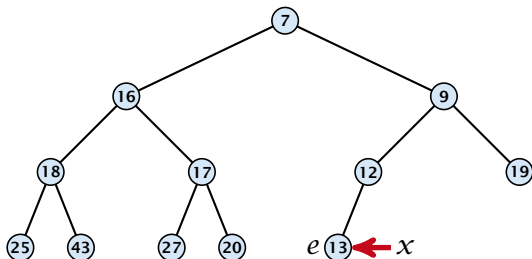
1. Insert element at successor of  $x$ .
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Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

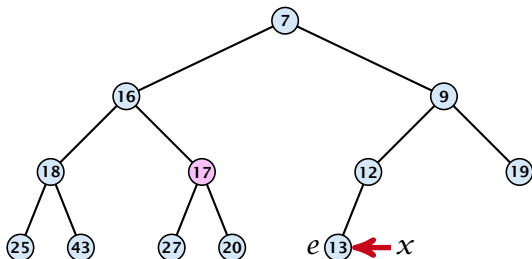
# Delete

1. Exchange the element to be deleted with the element  $e$  pointed to by  $x$ .



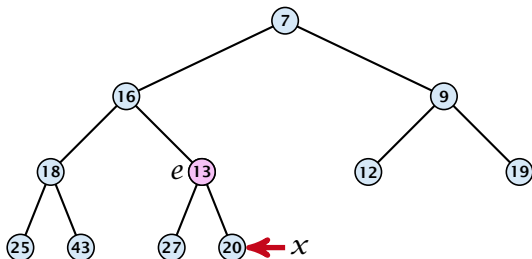
# Delete

1. Exchange the element to be deleted with the element  $e$  pointed to by  $x$ .
2. Restore the heap-property for the element  $e$ .



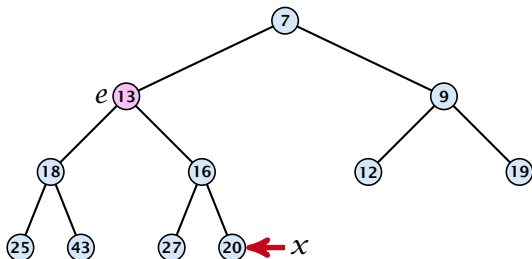
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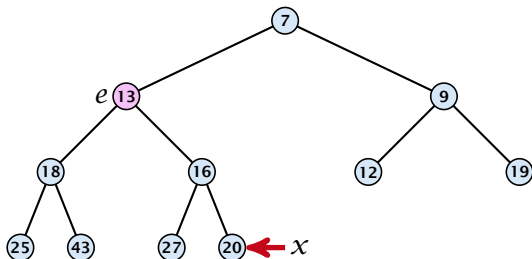
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At its new position  $e$  may either travel up or down in the tree (but not both directions).

# Binary Heaps

## Operations:

- ▶ **minimum()**: return the root-element. Time  $\mathcal{O}(1)$ .
- ▶ **is-empty()**: check whether root-pointer is **null**. Time  $\mathcal{O}(1)$ .
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- ▶ **build( $x_1, \dots, x_n$ )**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time  $\mathcal{O}(n)$ .



# Binary Heaps

# Binary Heaps

The standard implementation of binary heaps is via arrays. Let  $A[0, \dots, n - 1]$  be an array

- ▶ The parent of  $i$ -th element is at position  $\lfloor \frac{i-1}{2} \rfloor$ .
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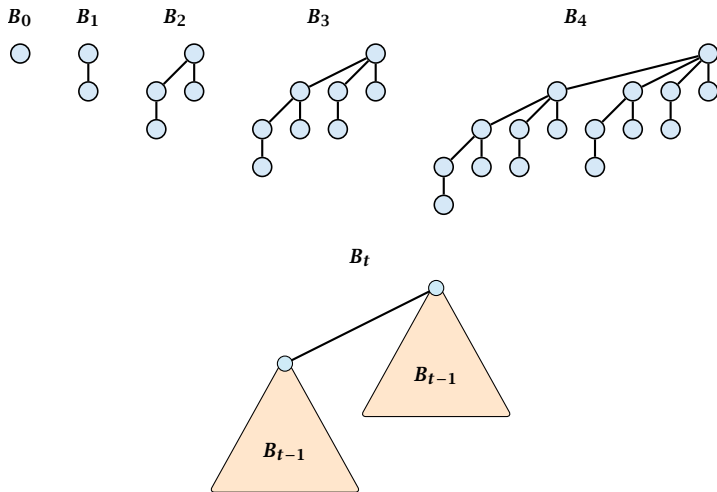
Finding the successor of  $x$  is much easier than in the description on the previous slide. Simply increase or decrease  $x$ .

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

## 8.2 Binomial Heaps

<i>Operation</i>	<i>Binary Heap</i>	<i>BST</i>	<i>Binomial Heap</i>	<i>Fibonacci Heap*</i>
build	$n$	$n \log n$	$n \log n$	$n$
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	$n$	$n \log n$	<b><math>\log n</math></b>	1

# Binomial Trees



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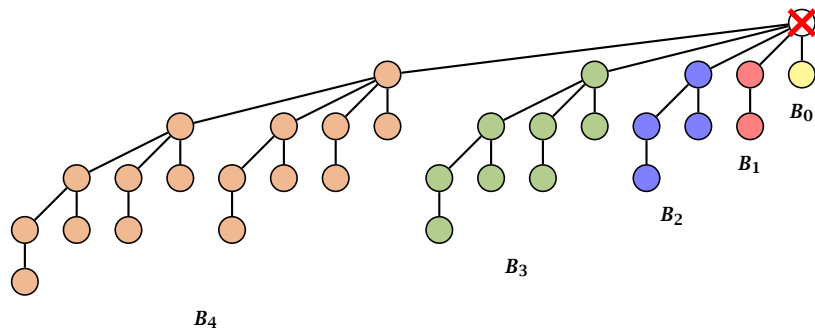
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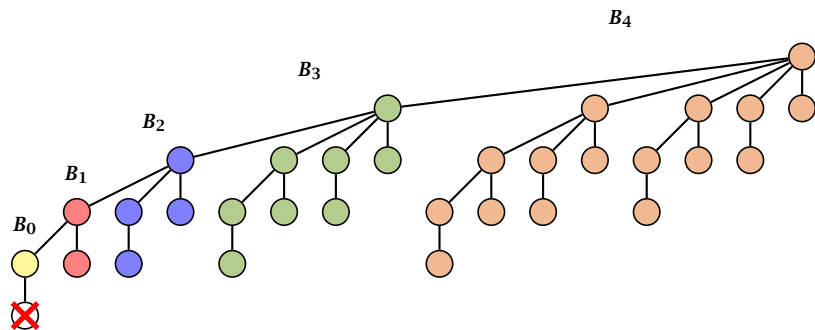
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- ▶  $B_k$  has  $\binom{k}{\ell}$  nodes on level  $\ell$ .
- ▶ Deleting the root of  $B_k$  gives trees  $B_0, B_1, \dots, B_{k-1}$ .

# Binomial Trees



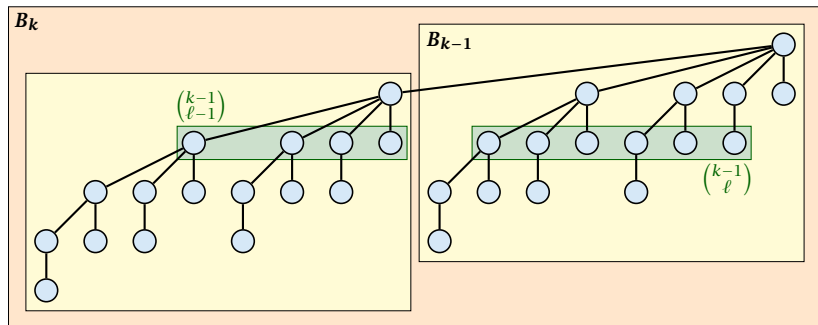
Deleting the root of  $B_5$  leaves sub-trees  $B_4$ ,  $B_3$ ,  $B_2$ ,  $B_1$ , and  $B_0$ .

# Binomial Trees



Deleting the leaf furthest from the root (in  $B_5$ ) leaves a path that connects the roots of sub-trees  $B_4$ ,  $B_3$ ,  $B_2$ ,  $B_1$ , and  $B_0$ .

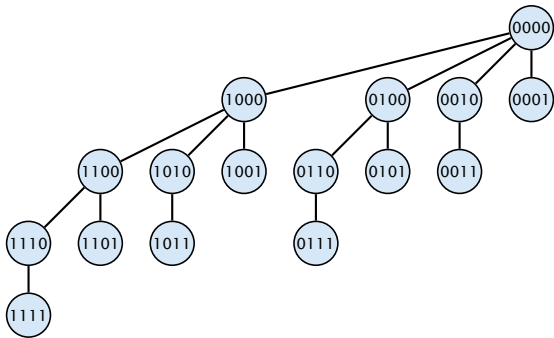
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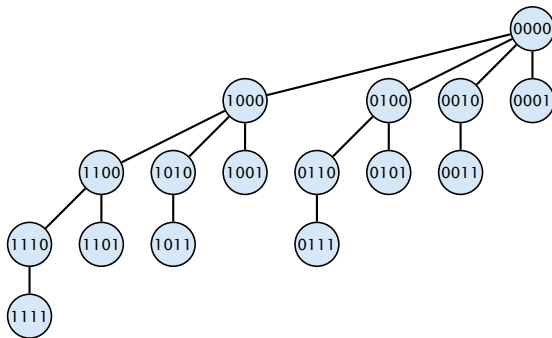
The number of nodes on level  $\ell$  in tree  $B_k$  is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

# Binomial Trees



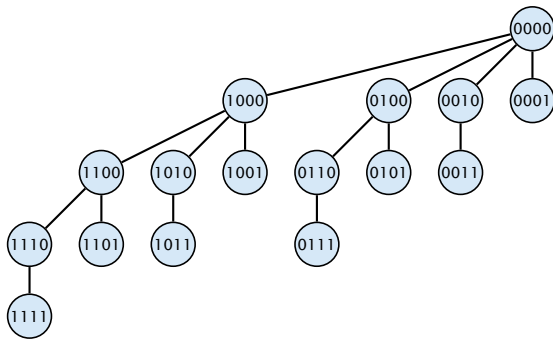
# Binomial Trees



The binomial tree  $B_k$  is a sub-graph of the hypercube  $H_k$ .



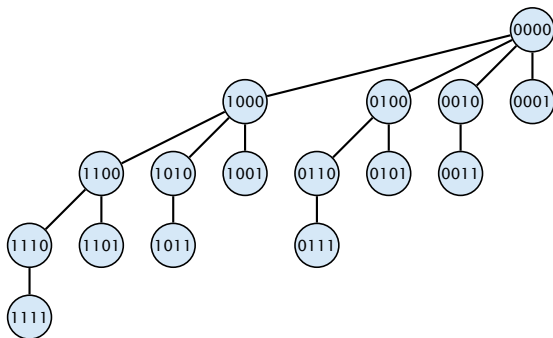
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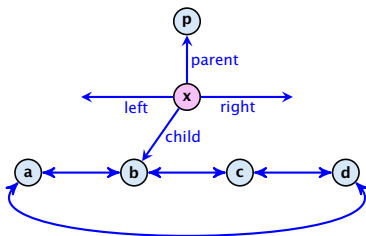
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The  $\ell$ -th level contains nodes that have  $\ell$  1's in their label.

## 8.2 Binomial Heaps

How do we implement trees with non-constant degree?

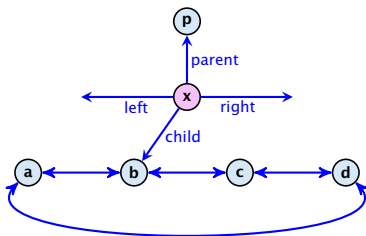
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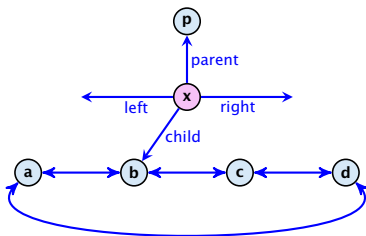
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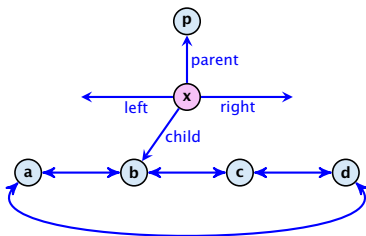
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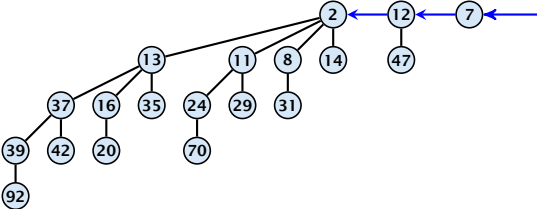
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers  $x.\text{left}$  and  $x.\text{right}$  point to the left and right sibling of  $x$  (if  $x$  does not have siblings then  $x.\text{left} = x.\text{right} = x$ ).



## 8.2 Binomial Heaps

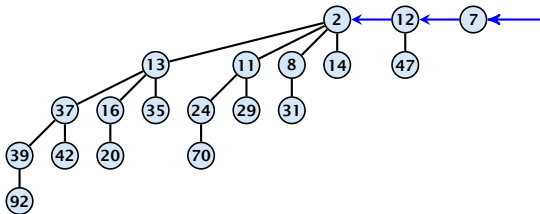
- ▶ Given a pointer to a node  $x$  we can splice out the sub-tree rooted at  $x$  in constant time.
- ▶ We can add a child-tree  $T$  to a node  $x$  in constant time if we are given a pointer to  $x$  and a pointer to the root of  $T$ .

# Binomial Heap



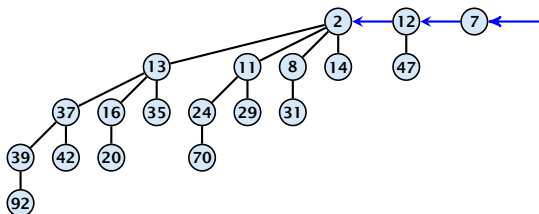


# Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

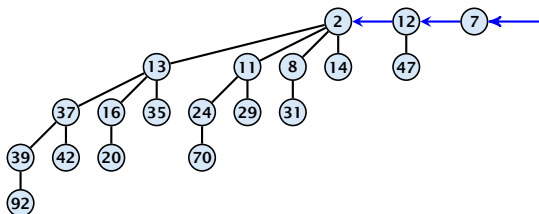
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There is at most one tree for every dimension/order. For example the above heap contains trees  $B_0$ ,  $B_1$ , and  $B_4$ .

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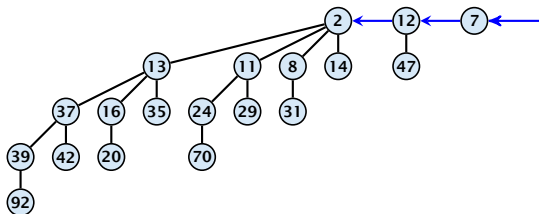
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Then  $n = \sum_i 2^{k_i}$  must hold. But since the  $k_i$  are all distinct this means that the  $k_i$  define the non-zero bit-positions in the binary representation of  $n$ .

# Binomial Heap

Properties of a heap with  $n$  keys:

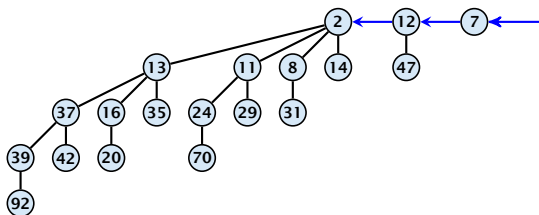




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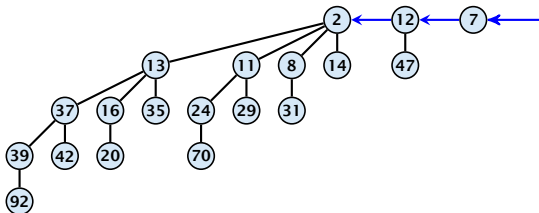
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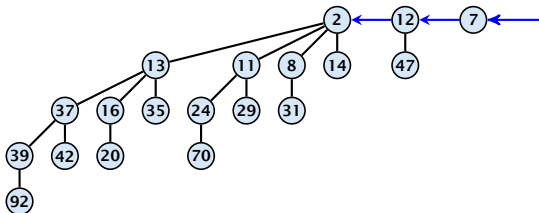
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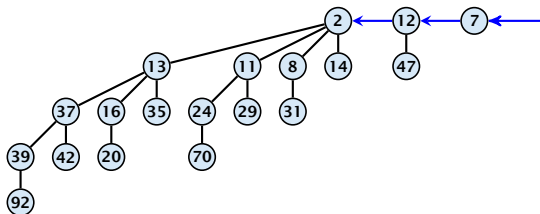
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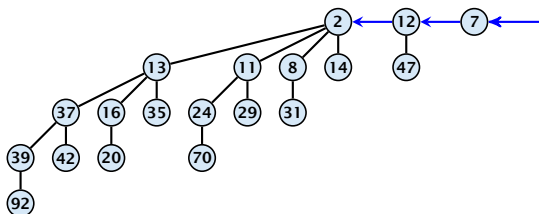
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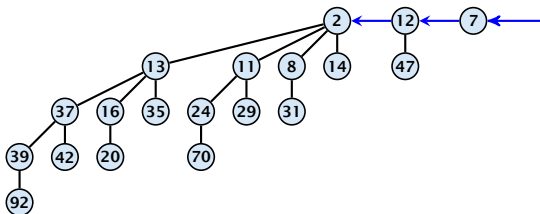
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- ▶ The minimum must be contained in one of the roots.
- ▶ The height of the largest tree is at most  $\lfloor \log n \rfloor$ .
- ▶ The trees are stored in a single-linked list; ordered by dimension/size.



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Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

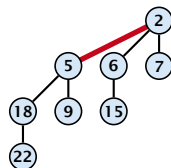
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Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.



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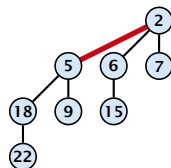
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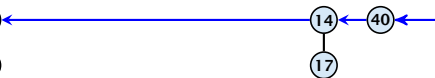
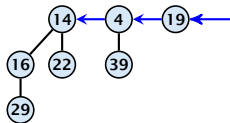
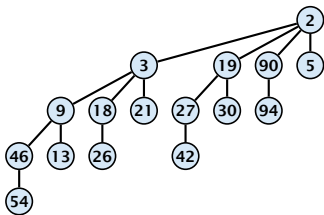
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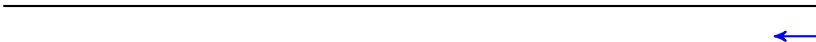
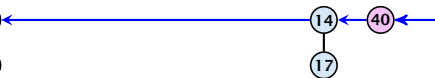
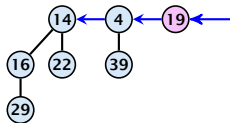
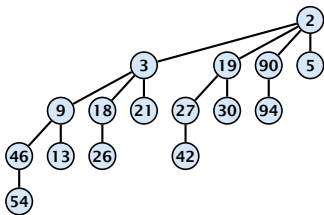
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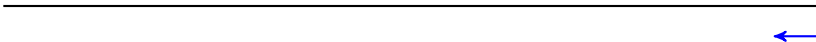
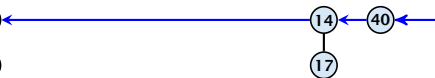
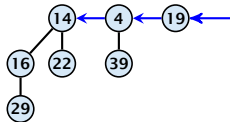
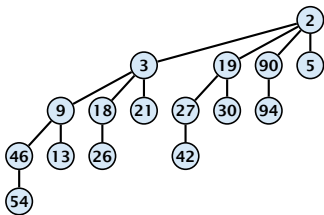
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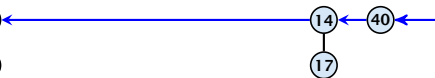
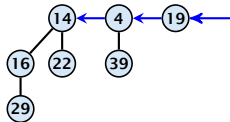
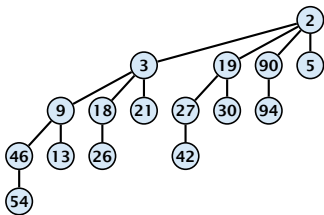
For more trees the technique is analogous to binary addition.

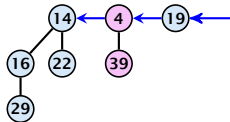
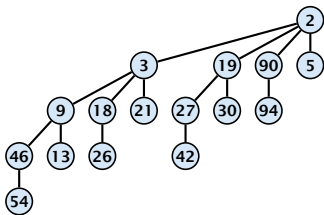




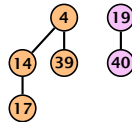
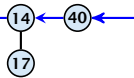
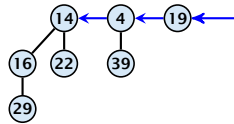
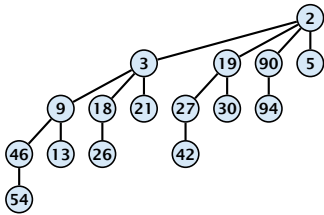


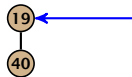
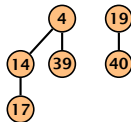
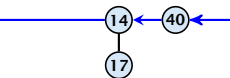
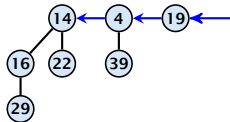
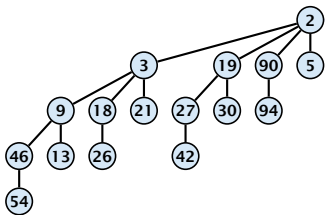


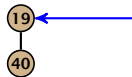
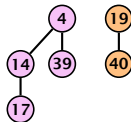
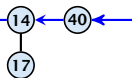
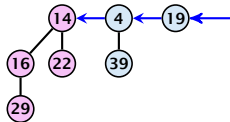
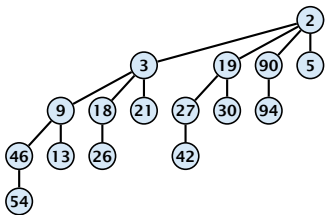


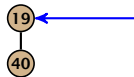
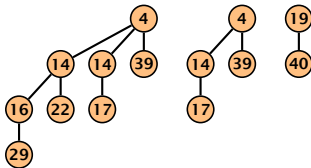
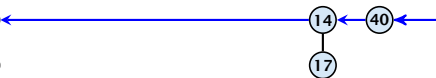
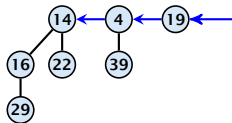
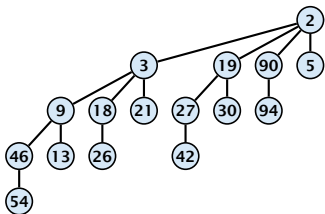


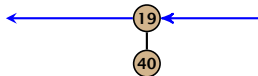
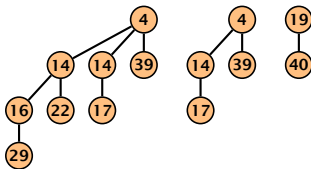
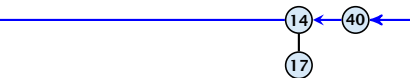
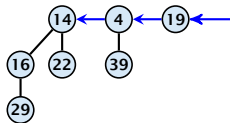
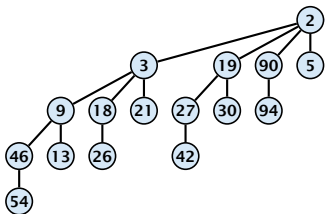


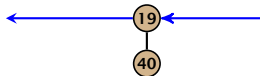
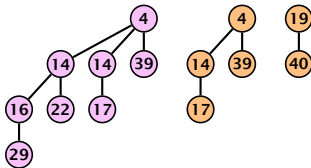
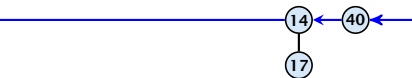
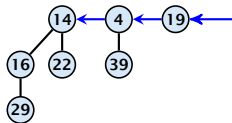
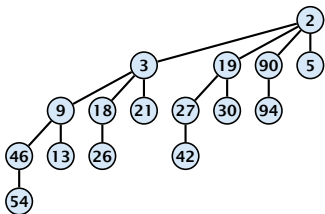




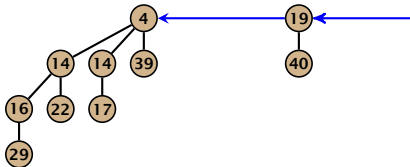
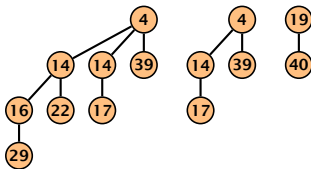
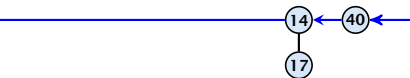
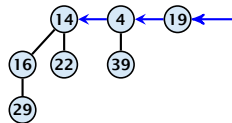
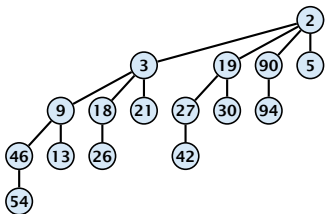




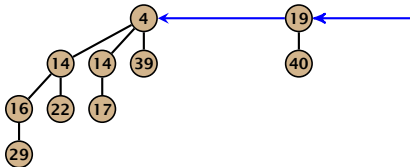
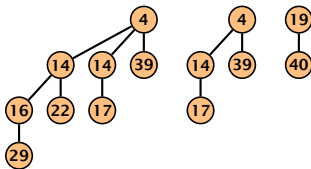
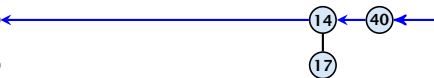
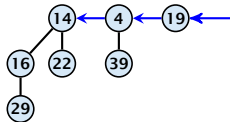
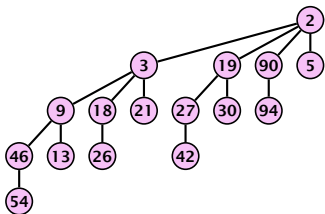




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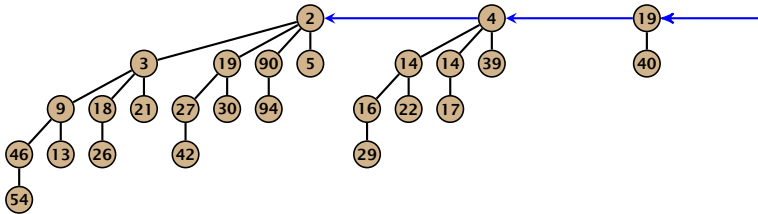
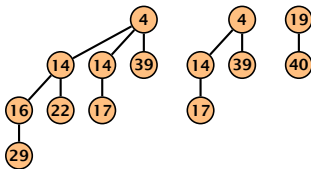
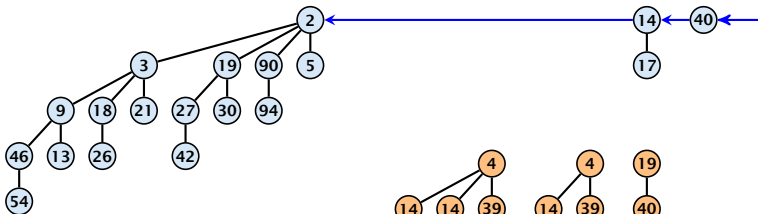


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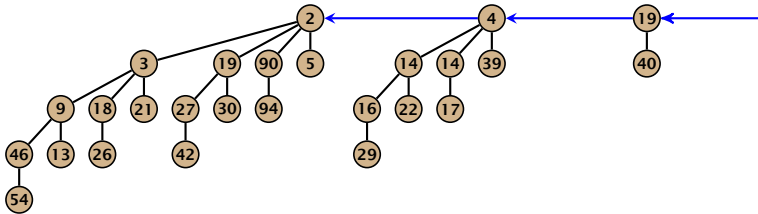
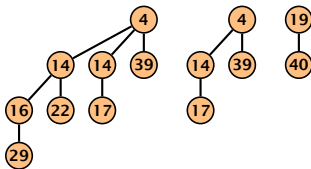
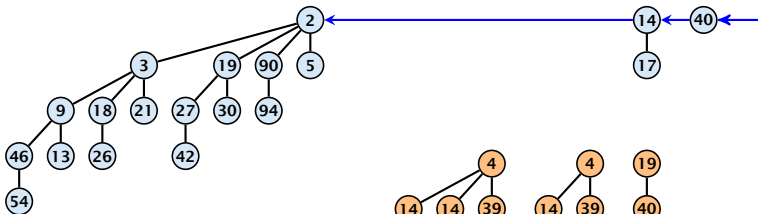




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## 8.2 Binomial Heaps

$S_1$ . merge( $S_2$ ):

- ▶ Analogous to binary addition.

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- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

All other operations can be reduced to `merge()`.

**S.insert( $x$ ):**

- ▶ Create a new heap  $S'$  that contains just the element  $x$ .

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## 8.2 Binomial Heaps

### **S. minimum():**

- ▶ Find the minimum key-value among all roots.
- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

**S. delete-min():**

## 8.2 Binomial Heaps

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## 8.2 Binomial Heaps

**S. decrease-key(handle  $h$ ):**



## 8.2 Binomial Heaps

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## 8.2 Binomial Heaps

### S. decrease-key(handle $h$ ):

- ▶ Decrease the key of the element pointed to by  $h$ .
- ▶ Bubble the element up in the tree until the heap property is fulfilled.
- ▶ Time:  $\mathcal{O}(\log n)$  since the trees have height  $\mathcal{O}(\log n)$ .

## 8.2 Binomial Heaps

***S.* delete(handle *h*):**

## 8.2 Binomial Heaps

**$S$ . delete(handle  $h$ ):**

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## 8.2 Binomial Heaps

***S*. delete(handle *h*):**

- ▶ Execute *S*. decrease-key(*h*,  $-\infty$ ).
- ▶ Execute *S*. delete-min().

## 8.2 Binomial Heaps

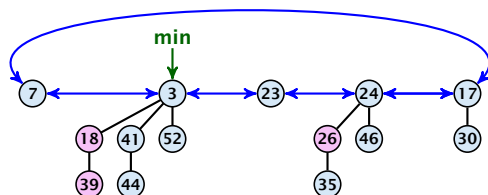
**$S$ . delete(handle  $h$ ):**

- ▶ Execute  $S$ . decrease-key( $h, -\infty$ ).
- ▶ Execute  $S$ . delete-min().
- ▶ Time:  $\mathcal{O}(\log n)$ .

## 8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.





## 8.3 Fibonacci Heaps

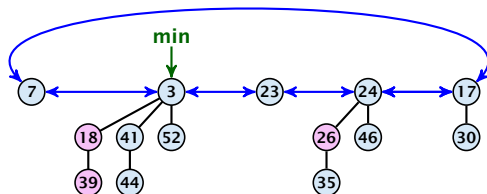
### Additional implementation details:

- ▶ Every node  $x$  stores its degree in a field  $x.degree$ . Note that this can be updated in constant time when adding a child to  $x$ .
- ▶ Every node stores a boolean value  $x.marked$  that specifies whether  $x$  is **marked** or not.

## 8.3 Fibonacci Heaps

### The potential function:

- ▶  $t(S)$  denotes the number of trees in the heap.
- ▶  $m(S)$  denotes the number of marked nodes.
- ▶ We use the potential function  $\Phi(S) = t(S) + 2m(S)$ .



The potential is  $\Phi(S) = 5 + 2 \cdot 3 = 11$ .

## 8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use  $c$  to denote the amount of work that a unit of potential can pay for.

## 8.3 Fibonacci Heaps

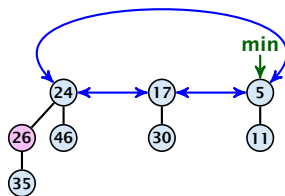
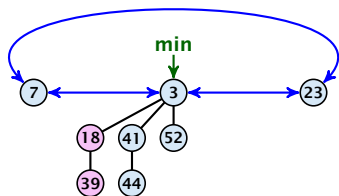
### S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Amortized cost  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

### $S$ . merge( $S'$ )

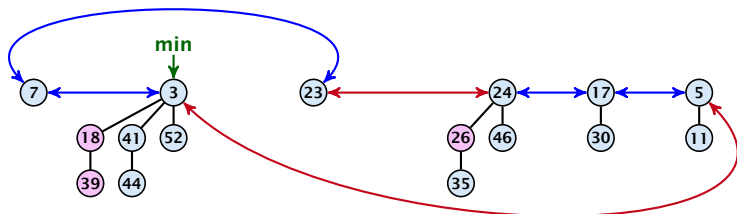
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



## 8.3 Fibonacci Heaps

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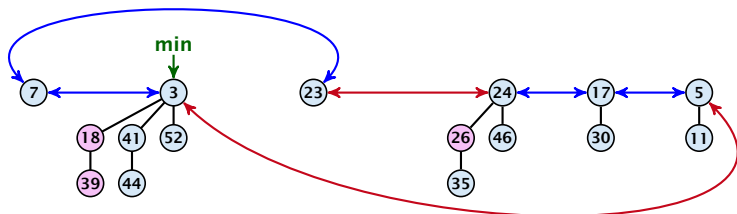
### Running time:

- ▶ Actual cost  $\mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

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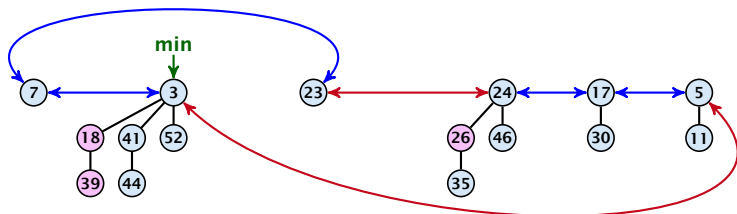
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### Running time:

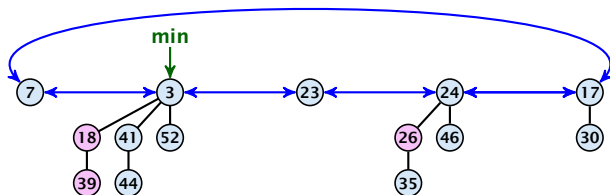
- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ No change in potential.
- ▶ Hence, amortized cost is  $\mathcal{O}(1)$ .



## 8.3 Fibonacci Heaps

### S. insert( $x$ )

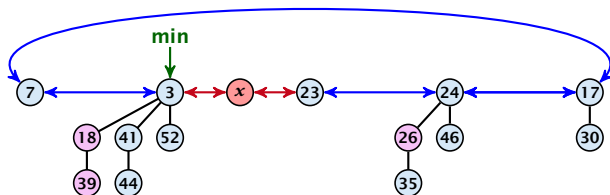
- ▶ Create a new tree containing  $x$ .
- ▶ Insert  $x$  into the root-list.
- ▶ Update min-pointer, if necessary.



## 8.3 Fibonacci Heaps

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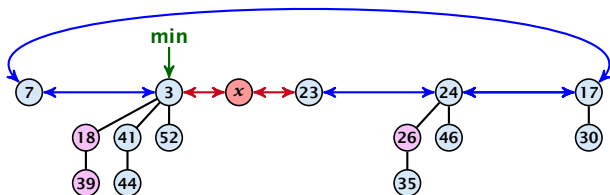
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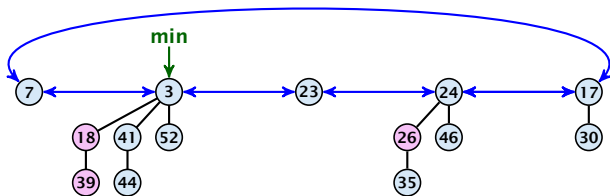


### Running time:

- ▶ Actual cost  $\mathcal{O}(1)$ .
- ▶ Change in potential is  $+1$ .
- ▶ Amortized cost is  $c + \mathcal{O}(1) = \mathcal{O}(1)$ .

## 8.3 Fibonacci Heaps

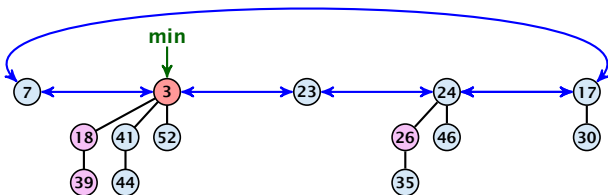
S. delete-min( $x$ )



## 8.3 Fibonacci Heaps

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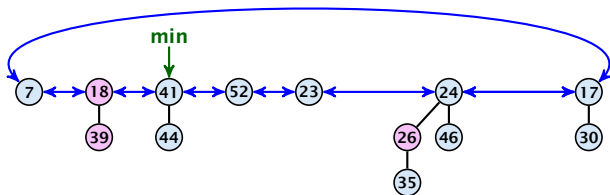
- ▶ Delete minimum; add child-trees to heap;  
time:  $D(\min) \cdot \mathcal{O}(1)$ .



## 8.3 Fibonacci Heaps

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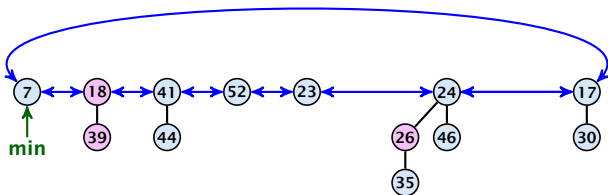
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## 8.3 Fibonacci Heaps

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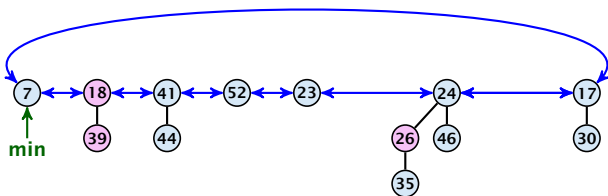
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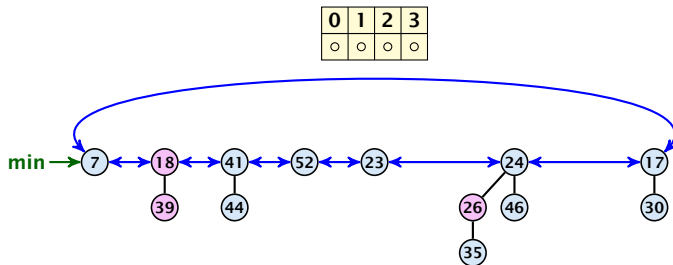


- ▶ Consolidate root-list so that no roots have the same degree. Time  $t \cdot \mathcal{O}(1)$  (see next slide).



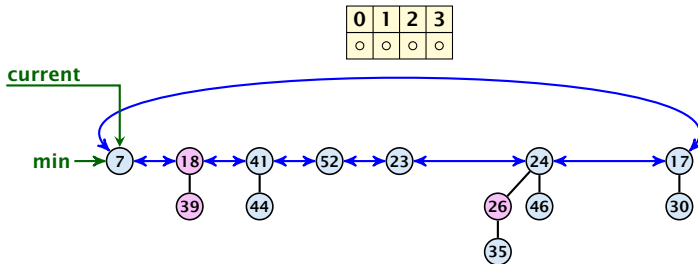
## 8.3 Fibonacci Heaps

Consolidate:



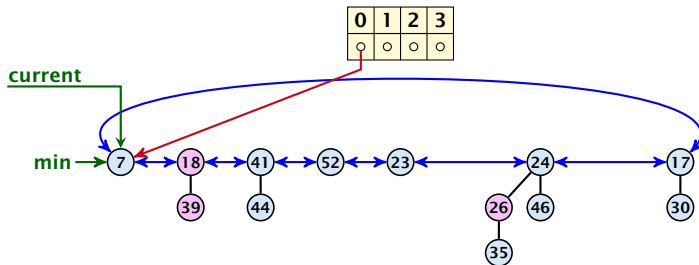
## 8.3 Fibonacci Heaps

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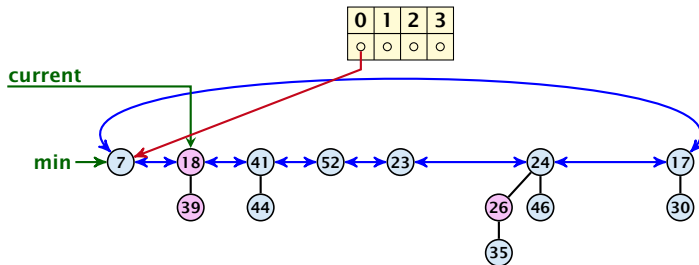
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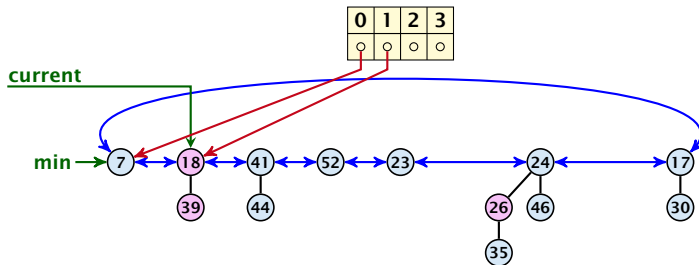
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Consolidate:



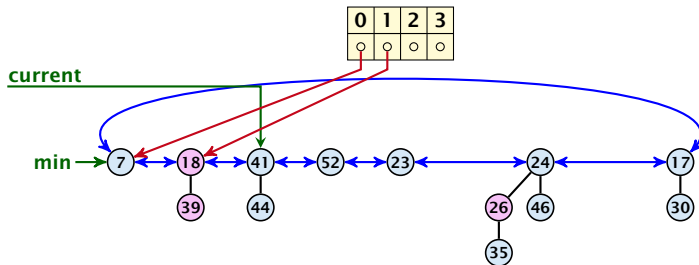
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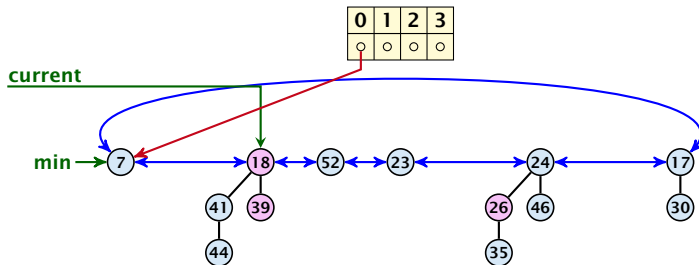
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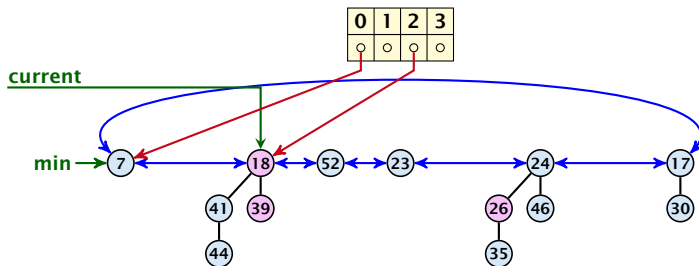
## 8.3 Fibonacci Heaps

Consolidate:



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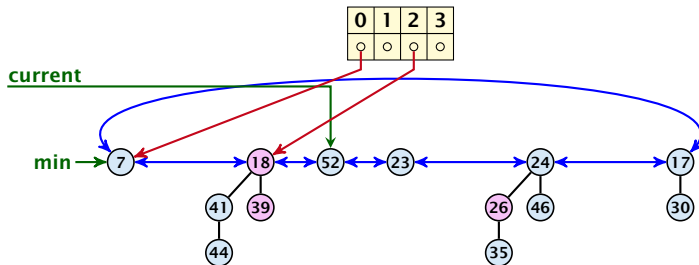
Consolidate:





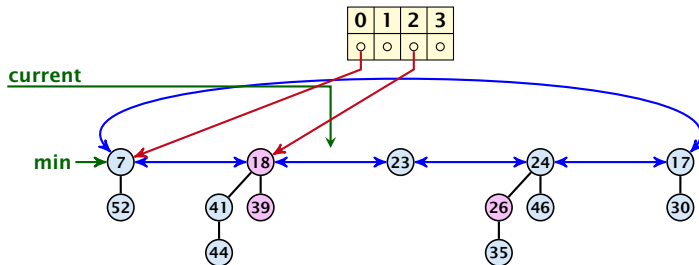
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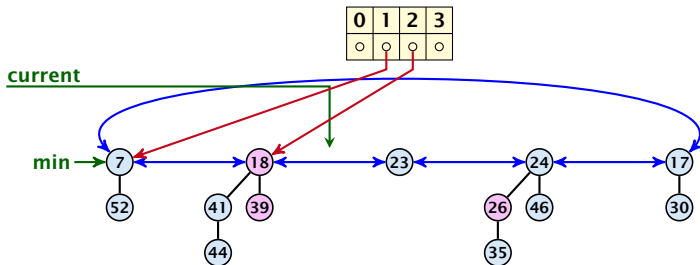
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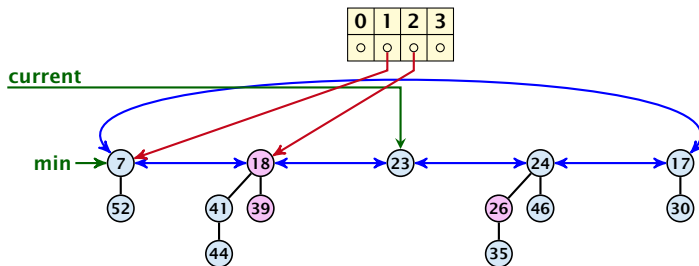
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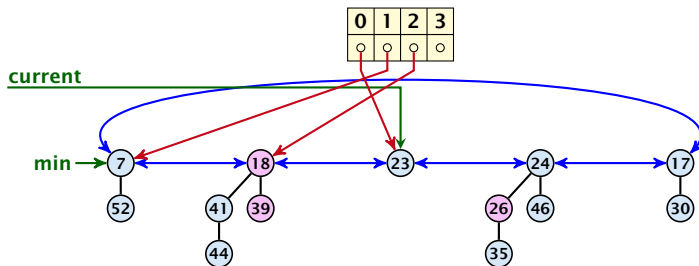
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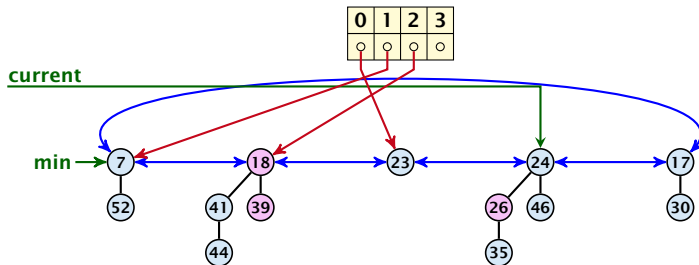
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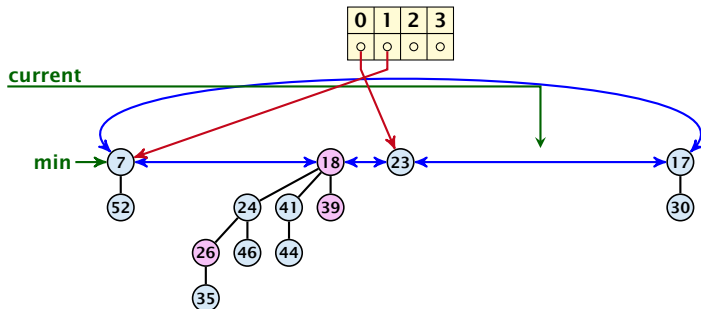
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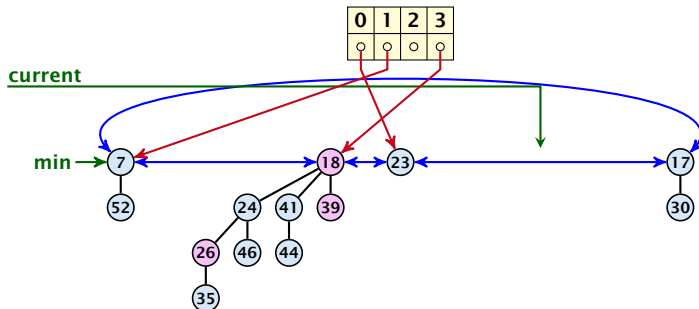
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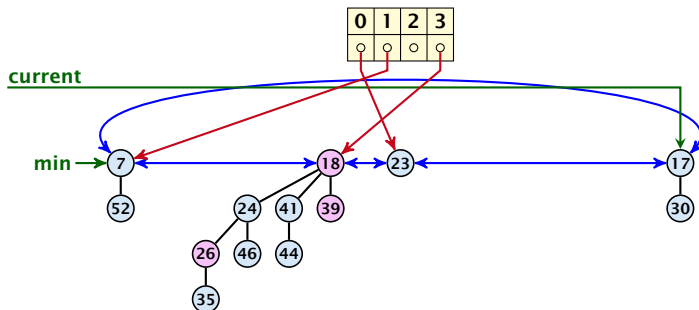
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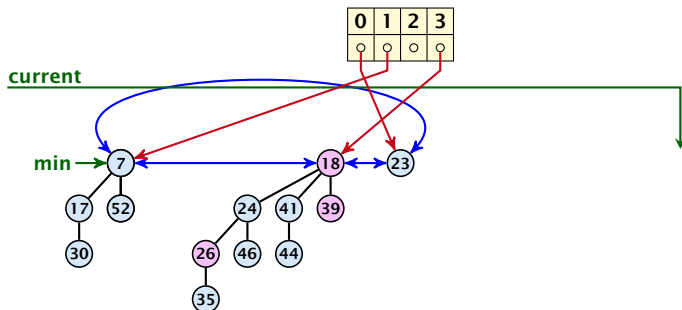
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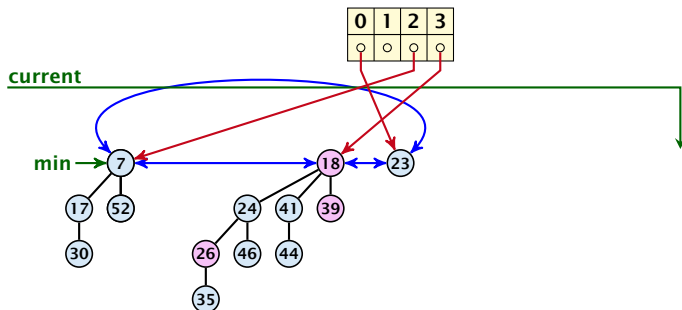
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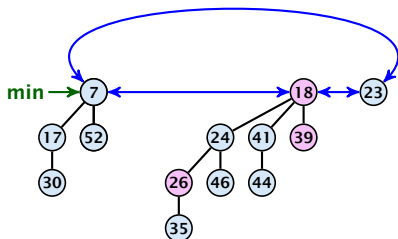
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for  $c \geq c_1$  .

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If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

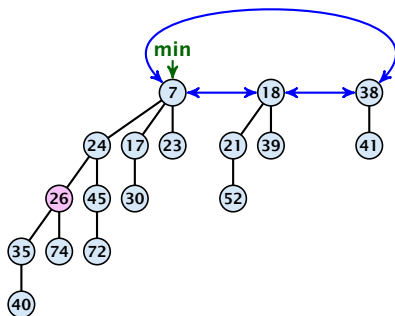


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If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then  
 $D_n \leq \log n$ .

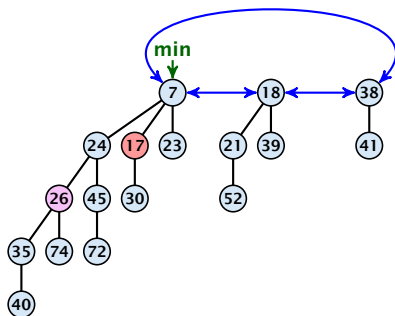
## Fibonacci Heaps: decrease-key(handle $h, v$ )



### Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by  $h$ . Nothing else to do.

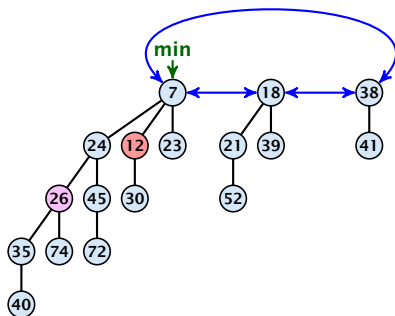
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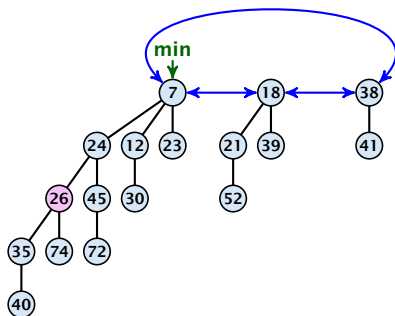
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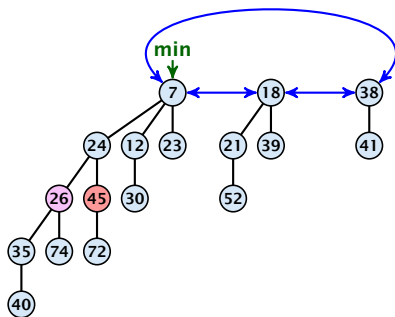
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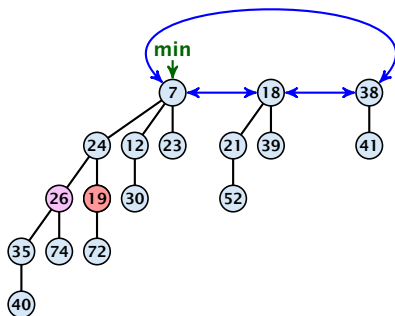
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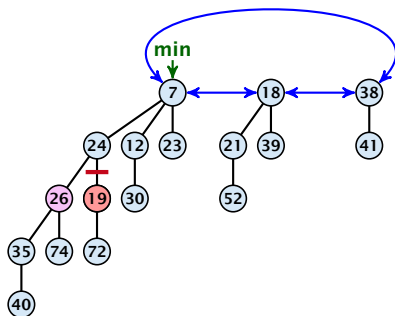
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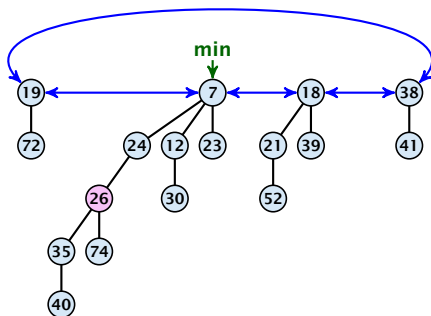


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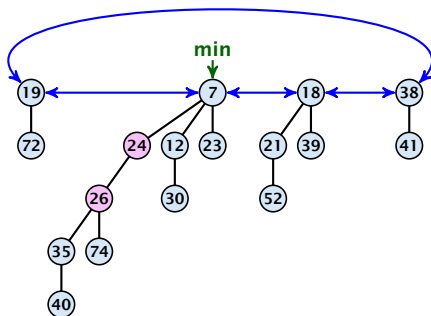
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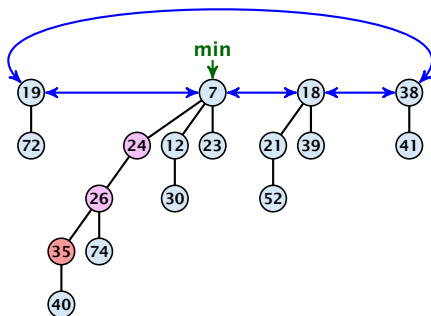
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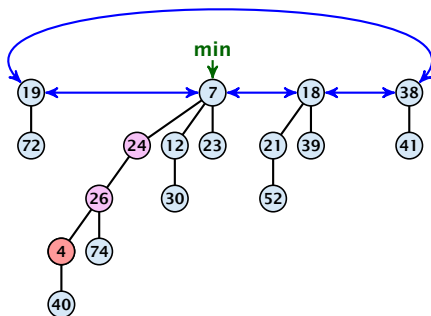
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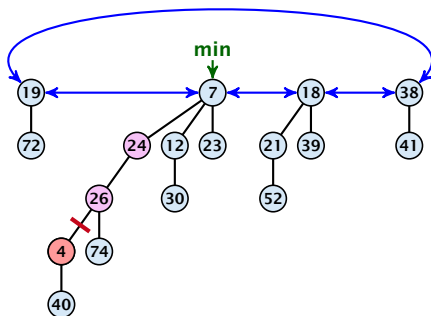
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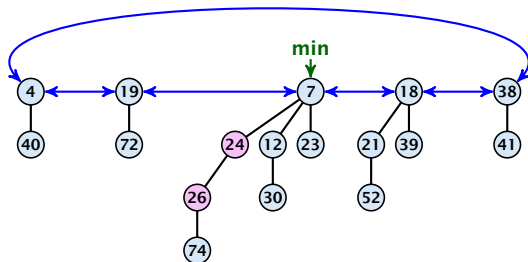
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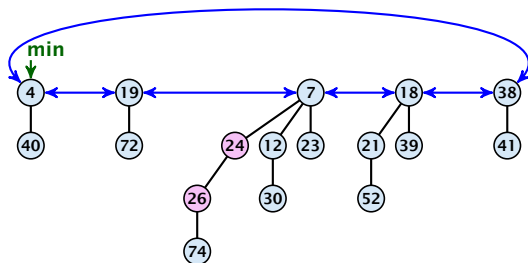
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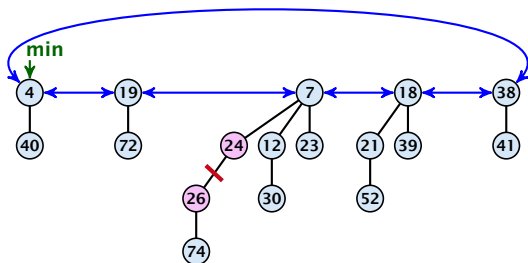
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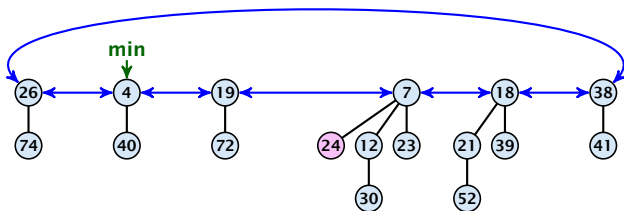


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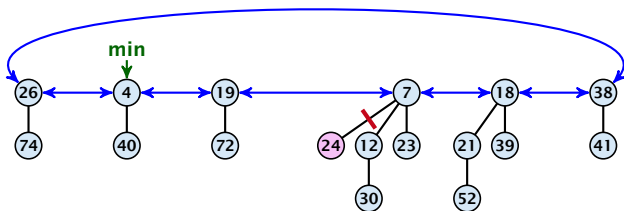
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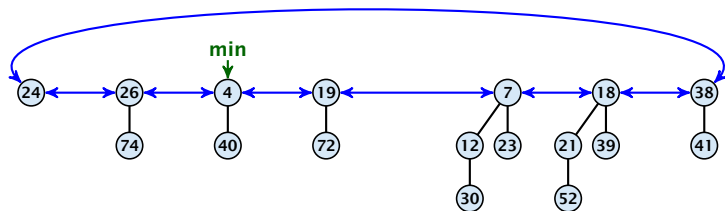
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- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

# Fibonacci Heaps: decrease-key(handle $h, v$ )

**Actual cost:**

## Fibonacci Heaps: decrease-key(handle $h, v$ )

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- ▶ Constant cost for decreasing the value.

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- ▶  $t' = t + \ell$ , as every cut creates one new root.
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$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c + c_2 = \mathcal{O}(1),$$
if  $c \geq c_2$ .

# Delete node

***H. delete( $x$ ):***

- ▶ decrease value of  $x$  to  $-\infty$ .
- ▶ delete-min.

**Amortized cost:  $\mathcal{O}(D_n)$**

- ▶  $\mathcal{O}(1)$  for decrease-key.
- ▶  $\mathcal{O}(D_n)$  for delete-min.



## 8.3 Fibonacci Heaps

### Lemma 32

Let  $x$  be a node with degree  $k$  and let  $y_1, \dots, y_k$  denote the children of  $x$  in the order that they were linked to  $x$ . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

## 8.3 Fibonacci Heaps

### Proof

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Let  $x$  be a degree  $k$  node of size  $s_k$  and let  $y_1, \dots, y_k$  be its children.

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$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \\ &= 2 + \sum_{i=0}^{k-2} s_i \end{aligned}$$

## 8.3 Fibonacci Heaps

### Definition 33

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

#### Facts:

1.  $F_k \geq \phi^k$ .
2. For  $k \geq 2$ :  $F_k = 2 + \sum_{i=0}^{k-2} F_i$ .

The above facts can be easily proved by induction. From this it follows that  $s_k \geq F_k \geq \phi^k$ , which gives that the maximum degree in a Fibonacci heap is logarithmic.

$$k=0: \quad 1 = F_0 \geq \Phi^0 = 1$$

$$k=1: \quad 2 = F_1 \geq \Phi^1 \approx 1.61$$

$$k-2, k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2} \underbrace{(\Phi + 1)}_{\Phi^2} = \Phi^k$$

$$k=2: \quad 3 = F_2 = 2 + 1 = 2 + F_0$$

$$k-1 \rightarrow k: \quad F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$$

## 9 Union Find

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- ▶  **$\mathcal{P}$ . find( $x$ ):** Given a handle for an element  $x$ ; find the set that contains  $x$ . Returns a representative/identifier for this set.
- ▶  **$\mathcal{P}$ . union( $x, y$ ):** Given two elements  $x$ , and  $y$  that are currently in sets  $S_x$  and  $S_y$ , respectively, the function replaces  $S_x$  and  $S_y$  by  $S_x \cup S_y$  and returns an identifier for the new set.

# 9 Union Find

## Applications:

- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.

# 9 Union Find

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- ▶ Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- ▶ Kruskals Minimum Spanning Tree Algorithm

## 9 Union Find

### Algorithm 1 Kruskal-MST( $G = (V, E), w$ )

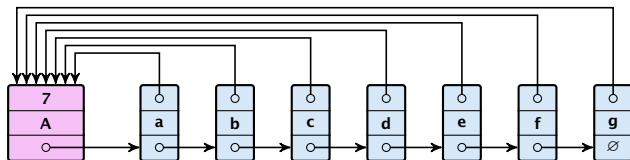
```
1:  $A \leftarrow \emptyset$ ;  
2: for all  $v \in V$  do  
3:    $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$   
4: sort edges in non-decreasing order of weight  $w$   
5: for all  $(u, v) \in E$  in non-decreasing order do  
6:   if  $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$  then  
7:      $A \leftarrow A \cup \{(u, v)\}$   
8:      $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$ 
```

# List Implementation

- ▶ The elements of a set are stored in a list; each node has a backward pointer to the head.

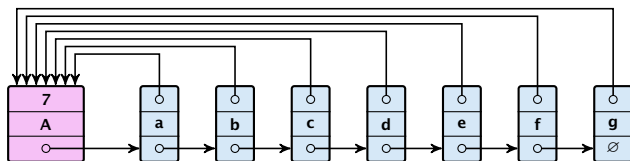
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- ▶ The elements of a set are stored in a list; each node has a backward pointer to the head.
- ▶ The head of the list contains the identifier for the set and a field that stores the **size** of the set.



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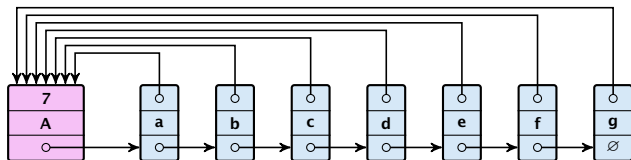
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- ▶ **makeset( $x$ )** can be performed in constant time.

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- ▶ **makeset**( $x$ ) can be performed in constant time.
- ▶ **find**( $x$ ) can be performed in constant time.



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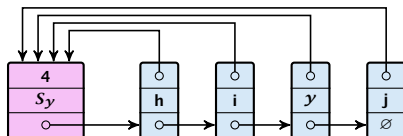
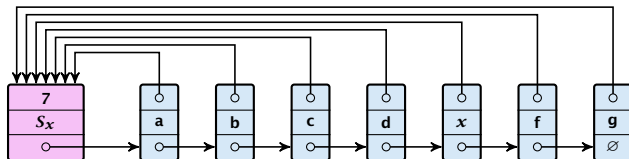
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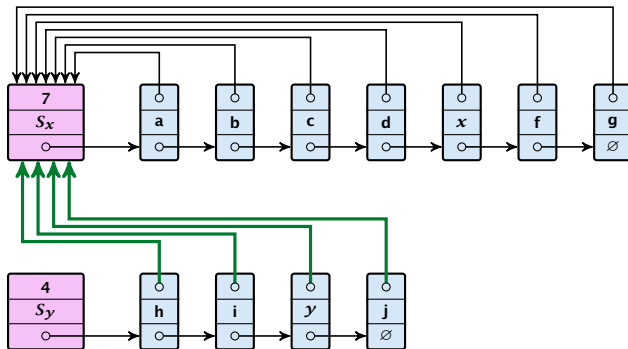
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- ▶ Time:  $\min\{|S_x|, |S_y|\}$ .

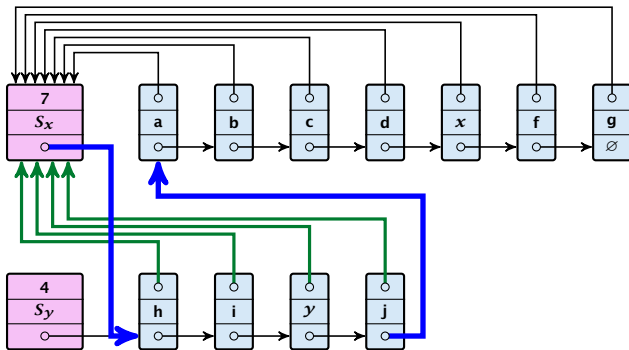
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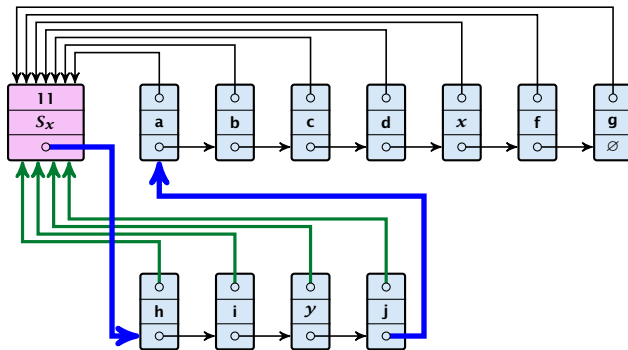


# List Implementation





# List Implementation



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## Running times:

- ▶  $\text{find}(x)$ : constant
- ▶  $\text{makeset}(x)$ : constant
- ▶  $\text{union}(x, y)$ :  $\mathcal{O}(n)$ , where  $n$  denotes the number of elements contained in the set system.

# List Implementation

## Lemma 34

*The list implementation for the ADT union find fulfills the following amortized time bounds:*

- ▶  $\text{find}(x): \mathcal{O}(1)$ .
- ▶  $\text{makeset}(x): \mathcal{O}(\log n)$ .
- ▶  $\text{union}(x, y): \mathcal{O}(1)$ .

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- ▶ If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.



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- ▶ Later operations charge the account but the balance never drops below zero.

# List Implementation

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- ▶ Assume wlog. that  $S_x$  is the smaller set; let  $c$  denote the hidden constant, i.e., the actual cost is at most  $c \cdot |S_x|$ .
- ▶ Charge  $c$  to every element in set  $S_x$ .

# List Implementation

## Lemma 35

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*An element is charged at most  $\lceil \log_2 n \rceil$  times, where  $n$  is the total number of elements in the set system.*

## Proof.

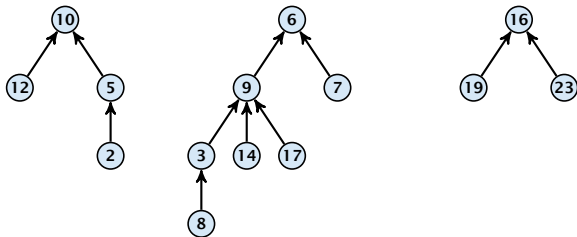
Whenever an element  $x$  is charged the number of elements in  $x$ 's set doubles. This can happen at most  $\lceil \log n \rceil$  times.  $\square$

# Implementation via Trees

- ▶ Maintain nodes of a set in a tree.
- ▶ The root of the tree is the label of the set.
- ▶ Only pointer to parent exists; we cannot list all elements of a given set.

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- ▶ The root of the tree is the label of the set.
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- ▶ Example:



Set system  $\{2, 5, 10, 12\}$ ,  $\{3, 6, 7, 8, 9, 14, 17\}$ ,  $\{16, 19, 23\}$ .

# Implementation via Trees

**makeset( $x$ )**

- ▶ Create a singleton tree. Return pointer to the root.



# Implementation via Trees

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- ▶  $\text{link}(a, b)$  attaches the **smaller** tree as the child of the larger.

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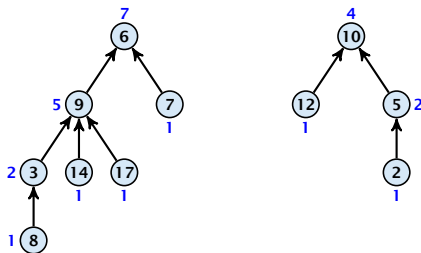
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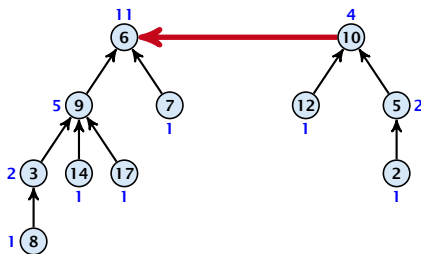


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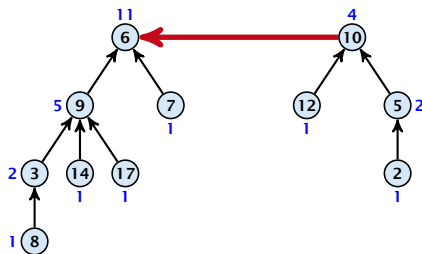


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- ▶ Time: constant for  $\text{link}(a, b)$  plus two find-operations.

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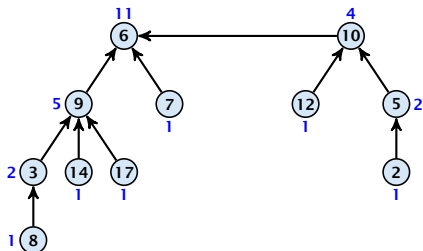
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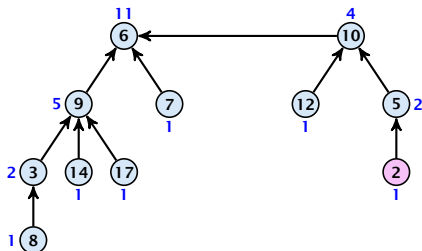
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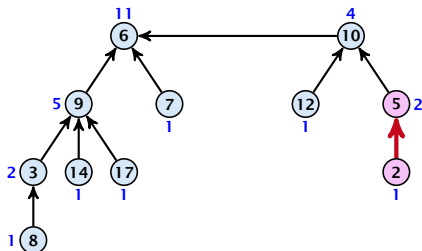
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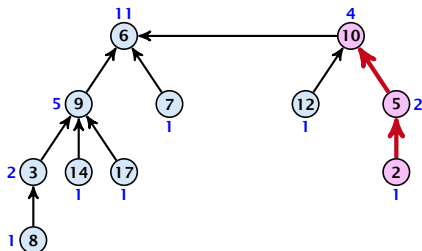
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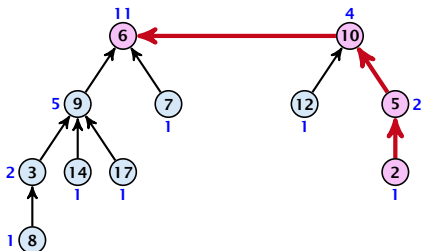
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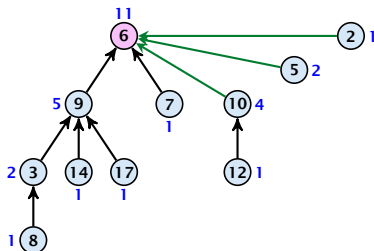
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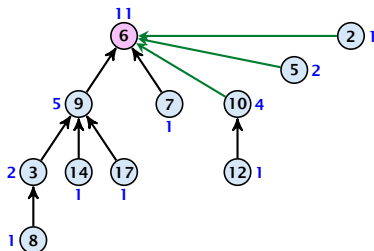




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- ▶ Note that the size-fields now only give an upper bound on the size of a sub-tree.

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However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time  $\mathcal{O}(\log n)$ .

# Amortized Analysis

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- ▶  $\text{size}(v)$  := the number of nodes that were in the sub-tree rooted at  $v$  when  $v$  became the child of another node (or the number of nodes if  $v$  is the root).

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## Lemma 37

*The rank of a parent must be strictly larger than the rank of a child.*



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## Lemma 38

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- ▶ A node  $v$  sees at most one node of rank  $s$  during the running time of the algorithm.
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- ▶ This holds because the rank-sequence of the roots of the different trees that contain  $v$  during the running time of the algorithm is a strictly increasing sequence.
- ▶ Hence, every node sees at most one rank  $s$  node, but every rank  $s$  node is seen by at least  $2^s$  different nodes. □

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## Theorem 39

*Union find with path compression fulfills the following amortized running times:*

- ▶  $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- ▶  $\text{find}(x) : \mathcal{O}(\log^*(n))$
- ▶  $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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- ▶ Hence, the total number of rank-groups is at most  $\log^* n$ .

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- ▶ Otherwise we charge the cost to the find-account.

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- ▶ The total charge made to a node in rank-group  $g$  is at most  $\text{tow}(g) - \text{tow}(g - 1) - 1 \leq \text{tow}(g)$ .

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- ▶ The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) ,$$

where  $n(g)$  is the number of nodes in group  $g$ .

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This means if we inflate the cost of **makeset** to  $\log^* n$  and add this to the node account of  $v$  then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

# Amortized Analysis

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is  $\mathcal{O}(\alpha(m, n))$ , where  $\alpha(m, n)$  is the inverse Ackermann function which grows a lot lot slower than  $\log^* n$ . (Here, we consider the average running time of  $m$  operations on at most  $n$  elements).

# Amortized Analysis

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There is also a lower bound of  $\Omega(\alpha(m, n))$ .

# Amortized Analysis

$$A(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otw.} \end{cases}$$

$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

# Amortized Analysis

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$$\alpha(m, n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- ▶  $A(0, y) = y + 1$
- ▶  $A(1, y) = y + 2$
- ▶  $A(2, y) = 2y + 3$
- ▶  $A(3, y) = 2^{y+3} - 3$
- ▶  $A(4, y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$

# 10 van Emde Boas Trees

## Dynamic Set Data Structure $S$ :

- ▶  $S.insert(x)$
- ▶  $S.delete(x)$
- ▶  $S.search(x)$
- ▶  $S.min()$
- ▶  $S.max()$
- ▶  $S.succ(x)$
- ▶  $S.pred(x)$

# 10 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

- ▶  **$S$ . insert( $x$ ):** Inserts  $x$  into  $S$ .
- ▶  **$S$ . delete( $x$ ):** Deletes  $x$  from  $S$ . Usually assumes that  $x \in S$ .
- ▶  **$S$ . member( $x$ ):** Returns 1 if  $x \in S$  and 0 otherwise.
- ▶  **$S$ . min():** Returns the value of the minimum element in  $S$ .
- ▶  **$S$ . max():** Returns the value of the maximum element in  $S$ .
- ▶  **$S$ . succ( $x$ ):** Returns successor of  $x$  in  $S$ . Returns **null** if  $x$  is maximum or larger than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .
- ▶  **$S$ . pred( $x$ ):** Returns the predecessor of  $x$  in  $S$ . Returns **null** if  $x$  is minimum or smaller than any element in  $S$ . Note that  $x$  needs not to be in  $S$ .

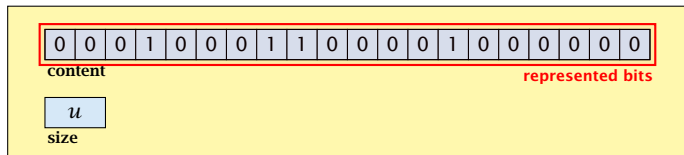


# 10 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from  $\{0, 1, \dots, u - 1\}$ , where  $u$  denotes the size of the universe.

# Implementation 1: Array



one array of  $u$  bits

Use an array that encodes the indicator function of the dynamic set.

# Implementation 1: Array

**Algorithm 1** `array.insert( $x$ )`

1: `content[ $x$ ] ← 1;`

**Algorithm 2** `array.delete( $x$ )`

1: `content[ $x$ ] ← 0;`

**Algorithm 3** `array.member( $x$ )`

1: **return** `content[ $x$ ];`

- ▶ Note that we assume that  $x$  is valid, i.e., it falls within the array boundaries.
- ▶ Obviously(?) the running time is constant.

## Implementation 1: Array

### Algorithm 4 `array.max()`

```
1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

# Implementation 1: Array

## Algorithm 4 `array.max()`

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1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

## Algorithm 5 `array.min()`

```
1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

## Implementation 1: Array

### Algorithm 4 `array.max()`

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1: for ( $i = \text{size} - 1; i \geq 0; i--$ ) do  
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### Algorithm 5 `array.min()`

```
1: for ( $i = 0; i < \text{size}; i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.

## Implementation 1: Array

### Algorithm 6 `array.succ(x)`

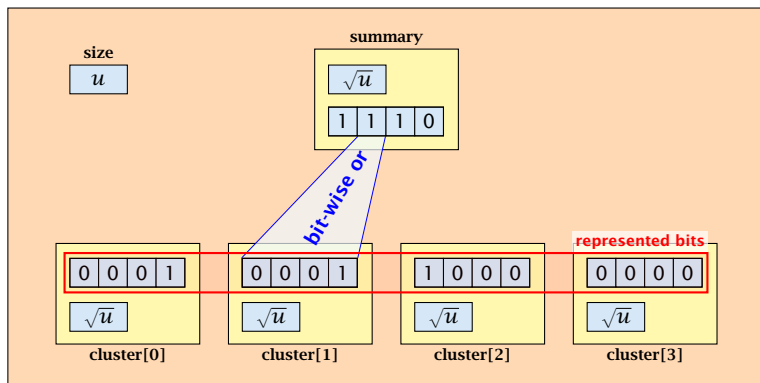
```
1: for ( $i = x + 1$ ;  $i < \text{size}$ ;  $i++$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

### Algorithm 7 `array.pred(x)`

```
1: for ( $i = x - 1$ ;  $i \geq 0$ ;  $i--$ ) do  
2:     if content[i] = 1 then return  $i$ ;  
3: return null;
```

- ▶ Running time is  $\mathcal{O}(u)$  in the worst case.

## Implementation 2: Summary Array



- ▶  $\sqrt{u}$  cluster-arrays of  $\sqrt{u}$  bits.
- ▶ One summary-array of  $\sqrt{u}$  bits. The  $i$ -th bit in the summary array stores the bit-wise or of the bits in the  $i$ -th cluster.



# Implementation 2: Summary Array

## Implementation 2: Summary Array

The bit for a key  $x$  is contained in cluster number  $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$ .

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Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

## Implementation 2: Summary Array

The bit for a key  $x$  is contained in cluster number  $\lfloor \frac{x}{\sqrt{u}} \rfloor$ .

Within the cluster-array the bit is at position  $x \bmod \sqrt{u}$ .

For simplicity we assume that  $u = 2^{2k}$  for some  $k \geq 1$ . Then we can compute the cluster-number for an entry  $x$  as  $\text{high}(x)$  (the upper half of the dual representation of  $x$ ) and the position of  $x$  within its cluster as  $\text{low}(x)$  (the lower half of the dual representation).

## Implementation 2: Summary Array

**Algorithm 8**  $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

1: **return**  $\text{cluster}[\text{high}(x)].\text{member}(\text{low}(x));$

### Algorithm 9 $\text{insert}(x)$

1:  $\text{cluster}[\text{high}(x)].\text{insert}(\text{low}(x));$

2:  $\text{summary}.\text{insert}(\text{high}(x));$

## Implementation 2: Summary Array

### Algorithm 8 $\text{member}(x)$

```
1: return cluster[high(x)].member(low(x));
```

### Algorithm 9 $\text{insert}(x)$

```
1: cluster[high(x)].insert(low(x));  
2: summary.insert(high(x));
```

- ▶ The running times are constant, because the corresponding array-functions have constant running times.

## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

- 1: cluster[high( $x$ )].delete(low( $x$ ));
- 2: **if** cluster[high( $x$ )].min() = null **then**
- 3:     summary.delete(high( $x$ ));



## Implementation 2: Summary Array

### Algorithm 10 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

- ▶ The running time is dominated by the cost of a minimum computation on an array of size  $\sqrt{u}$ . Hence,  $\mathcal{O}(\sqrt{u})$ .

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

- 1:  $\text{maxcluster} \leftarrow \text{summary.max}();$
- 2: **if**  $\text{maxcluster} = \text{null}$  **return**  $\text{null};$
- 3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$
- 4: **return**  $\text{maxcluster} \circ \text{offs};$

## Implementation 2: Summary Array

### Algorithm 11 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs};$ 
```

### Algorithm 23 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs};$ 
```

## Implementation 2: Summary Array

### Algorithm 22 $\text{max}()$

```
1:  $\text{maxcluster} \leftarrow \text{summary.max}();$   
2: if  $\text{maxcluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{maxcluster}].\text{max}();$   
4: return  $\text{maxcluster} \circ \text{offs}$ ;
```

### Algorithm 23 $\text{min}()$

```
1:  $\text{mincluster} \leftarrow \text{summary.min}();$   
2: if  $\text{mincluster} = \text{null}$  return  $\text{null}$ ;  
3:  $\text{offs} \leftarrow \text{cluster}[\text{mincluster}].\text{min}();$   
4: return  $\text{mincluster} \circ \text{offs}$ ;
```

The operator  $\circ$  stands for the concatenation of two bitstrings.

This means if  $x = 0111_2$  and  $y = 0001_2$  then  $x \circ y = 01110001_2$ .

- ▶ Running time is roughly  $2\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 2: Summary Array

### Algorithm 24 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

## Implementation 2: Summary Array

### Algorithm 24 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
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6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 2: Summary Array

### Algorithm 25 $\text{pred}(x)$

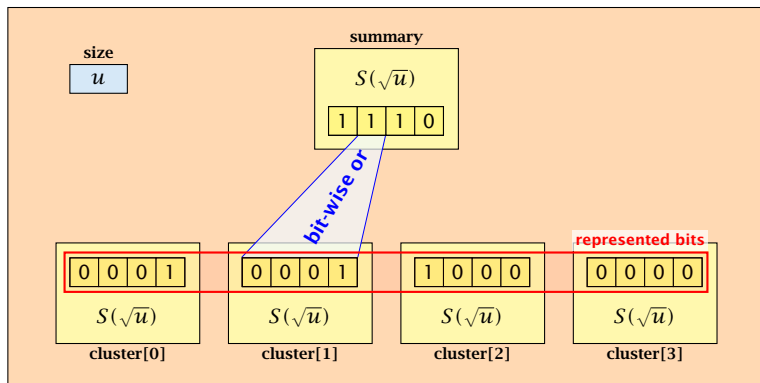
```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{pred}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{predcluster} \leftarrow \text{summary}.\text{pred}(\text{high}(x))$ ;
4: if  $\text{predcluster} \neq \text{null}$  then
5:    $\text{offs} \leftarrow \text{cluster}[\text{predcluster}].\text{max}()$ ;
6:   return  $\text{predcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

- ▶ Running time is roughly  $3\sqrt{u} = \mathcal{O}(\sqrt{u})$  in the worst case.

## Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$  is a dynamic set data-structure representing  $u$  bits:





## Implementation 3: Recursion

We assume that  $u = 2^{2^k}$  for some  $k$ .

The data-structure  $S(2)$  is defined as an array of 2-bits (end of the recursion).

# Implementation 3: Recursion

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The code from Implementation 2 can be used **unchanged**. We only need to redo the analysis of the running time.

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Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an  $S(4)$  will contain  $S(2)$ 's as sub-datastructures, which are **arrays**. Hence, a call like `cluster[1].min()` from within the data-structure  $S(4)$  is **not** a recursive call as it will call the function `array.min()`.

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This means that the non-recursive case is been dealt with while initializing the data-structure.

## Implementation 3: Recursion

**Algorithm 26** `member(x)`

1: **return** `cluster[high(x)].member(low(x));`

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 27 insert( $x$ )

```
1: cluster[high( $x$ )].insert(low( $x$ ));  
2: summary.insert(high( $x$ ));
```

►  $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

## Implementation 3: Recursion

### Algorithm 28 delete( $x$ )

```
1: cluster[high( $x$ )].delete(low( $x$ ));  
2: if cluster[high( $x$ )].min() = null then  
3:     summary.delete(high( $x$ ));
```

►  $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$



## Implementation 3: Recursion

### Algorithm 29 $\text{min}()$

```
1: mincluster  $\leftarrow$  summary.min();  
2: if mincluster = null return null;  
3: offs  $\leftarrow$  cluster[mincluster].min();  
4: return mincluster  $\circ$  offs;
```

- ▶  $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$ .

## Implementation 3: Recursion

### Algorithm 30 $\text{succ}(x)$

```
1:  $m \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ 
2: if  $m \neq \text{null}$  then return  $\text{high}(x) \circ m$ ;
3:  $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;
4: if  $\text{succcluster} \neq \text{null}$  then
5:      $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;
6:     return  $\text{succcluster} \circ \text{offs}$ ;
7: return  $\text{null}$ ;
```

►  $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

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$$T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1:$$

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ .

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$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u)$$



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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) = T_{\text{mem}}(2^\ell) &= T_{\text{mem}}(\mathbf{u}) = T_{\text{mem}}(\sqrt{\mathbf{u}}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 \end{aligned}$$

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{mem}}(2^\ell)$ . Then

$$\begin{aligned} X(\ell) &= T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \\ &= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X\left(\frac{\ell}{2}\right) + 1 . \end{aligned}$$

Using Master theorem gives  $X(\ell) = \mathcal{O}(\log \ell)$ , and hence  $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 3: Recursion

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Set  $\ell := \log u$  and  $X(\ell) := T_{\text{ins}}(2^\ell)$ . Then

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\ell)$ , and hence  $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$ .

## Implementation 3: Recursion

$$T_{\text{ins}}(\mathbf{u}) = 2T_{\text{ins}}(\sqrt{\mathbf{u}}) + 1.$$

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Using Master theorem gives  $X(\ell) = \mathcal{O}(\ell)$ , and hence  $T_{\text{ins}}(\mathbf{u}) = \mathcal{O}(\log u)$ .

The same holds for  $T_{\text{max}}(\mathbf{u})$  and  $T_{\text{min}}(\mathbf{u})$ .

## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$



## Implementation 3: Recursion

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## Implementation 3: Recursion

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## Implementation 3: Recursion

$$T_{\text{del}}(\mathbf{u}) = 2T_{\text{del}}(\sqrt{\mathbf{u}}) + T_{\text{min}}(\sqrt{\mathbf{u}}) + 1 \leq 2T_{\text{del}}(\sqrt{\mathbf{u}}) + c \log(\mathbf{u}).$$

Set  $\ell := \log u$  and  $X(\ell) := T_{\text{del}}(2^\ell)$ . Then

$$X(\ell)$$

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## Implementation 3: Recursion

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## Implementation 3: Recursion

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Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

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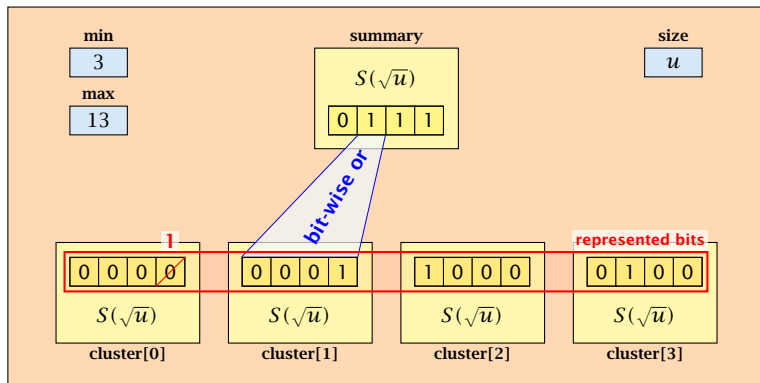
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Using Master theorem gives  $X(\ell) = \Theta(\ell \log \ell)$ , and hence  $T_{\text{del}}(u) = \mathcal{O}(\log u \log \log u)$ .

The same holds for  $T_{\text{pred}}(u)$  and  $T_{\text{succ}}(u)$ .

# Implementation 4: van Emde Boas Trees



- ▶ The bit referenced by **min** is **not** set within sub-datastructures.
- ▶ The bit referenced by **max** is set within sub-datastructures (if  $\text{max} \neq \text{min}$ ).

# Implementation 4: van Emde Boas Trees

**Advantages of having max/min pointers:**

## Implementation 4: van Emde Boas Trees

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- ▶ Recursive calls for **min** and **max** are constant time.

# Implementation 4: van Emde Boas Trees

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## Implementation 4: van Emde Boas Trees

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- ▶ Recursive calls for **min** and **max** are constant time.
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- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.

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- ▶ We can insert into an empty datastructure in constant time by only setting **min = max =  $x$** .



## Implementation 4: van Emde Boas Trees

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- ▶ **min = null** means that the data-structure is empty.
- ▶ **min = max  $\neq$  null** means that the data-structure contains exactly one element.
- ▶ We can insert into an empty datastructure in constant time by only setting **min = max =  $x$** .
- ▶ We can delete from a data-structure that just contains one element in constant time by setting **min = max = null**.

## Implementation 4: van Emde Boas Trees

**Algorithm 31** `max()`

1: **return** `max`;

**Algorithm 32** `min()`

1: **return** `min`;

- ▶ Constant time.

## Implementation 4: van Emde Boas Trees

### Algorithm 33 `member(x)`

```
1: if  $x = \min$  then return 1; // TRUE  
2: return cluster[high(x)].member(low(x));
```

- ▶  $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Rightarrow T(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 34 $\text{succ}(x)$

```
1: if  $\text{min} \neq \text{null} \wedge x < \text{min}$  then return  $\text{min}$ ;  
2:  $\text{maxincluster} \leftarrow \text{cluster}[\text{high}(x)].\text{max}()$ ;  
3: if  $\text{maxincluster} \neq \text{null} \wedge \text{low}(x) < \text{maxincluster}$  then  
4:    $\text{offs} \leftarrow \text{cluster}[\text{high}(x)].\text{succ}(\text{low}(x))$ ;  
5:   return  $\text{high}(x) \circ \text{offs}$ ;  
6: else  
7:    $\text{succcluster} \leftarrow \text{summary}.\text{succ}(\text{high}(x))$ ;  
8:   if  $\text{succcluster} = \text{null}$  then return  $\text{null}$ ;  
9:    $\text{offs} \leftarrow \text{cluster}[\text{succcluster}].\text{min}()$ ;  
10:  return  $\text{succcluster} \circ \text{offs}$ ;
```

►  $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \implies T_{\text{succ}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

### Algorithm 65 insert( $x$ )

```
1: if min = null then
2:     min =  $x$ ; max =  $x$ ;
3: else
4:     if  $x < \text{min}$  then exchange  $x$  and min;
5:     if cluster[high( $x$ )].min = null; then
6:         summary.insert(high( $x$ ));
7:         cluster[high( $x$ )].insert(low( $x$ ));
8:     else
9:         cluster[high( $x$ )].insert(low( $x$ ));
10:    if  $x > \text{max}$  then max =  $x$ ;
```

►  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Rightarrow T_{\text{ins}}(u) = \mathcal{O}(\log \log u)$ .

## Implementation 4: van Emde Boas Trees

Note that the recursive call in Line 7 takes constant time as the if-condition in Line 5 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 6 and in Line 9. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that  $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$ .

## Implementation 4: van Emde Boas Trees

- ▶ Assumes that  $x$  is contained in the structure.

### Algorithm 65 delete( $x$ )

```
1: if min = max then
2:     min = null; max = null;
3: else
4:     if  $x$  = min then
5:         firstcluster  $\leftarrow$  summary.min();
6:         offs  $\leftarrow$  cluster[firstcluster].min();
7:          $x \leftarrow$  firstcluster  $\circ$  offs;
8:         min  $\leftarrow$   $x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
10:    if cluster[high( $x$ )].min() = null then
11:        summary.delete(high( $x$ ));
```

continued...

## Implementation 4: van Emde Boas Trees

- ▶ Assumes that  $x$  is contained in the structure.

### Algorithm 65 delete( $x$ )

```
1: if min = max then
2:     min = null; max = null;
3: else
4:     if  $x = \text{min}$  then find new minimum
5:         firstcluster  $\leftarrow$  summary.min();
6:         offs  $\leftarrow$  cluster[firstcluster].min();
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:         cluster[high( $x$ )].delete(low( $x$ ));
10:        if cluster[high( $x$ )].min() = null then
11:            summary.delete(high( $x$ ));
continued...
```



## Implementation 4: van Emde Boas Trees

- ▶ Assumes that  $x$  is contained in the structure.

### Algorithm 65 delete( $x$ )

```
1: if min = max then
2:     min = null; max = null;
3: else
4:     if  $x = \text{min}$  then
5:          $\text{firstcluster} \leftarrow \text{summary.min}()$ ;
6:          $\text{offs} \leftarrow \text{cluster}[\text{firstcluster}].\text{min}()$ ;
7:          $x \leftarrow \text{firstcluster} \circ \text{offs}$ ;
8:         min  $\leftarrow x$ ;
9:     cluster[high( $x$ )].delete(low( $x$ )); delete
10:    if cluster[high( $x$ )].min() = null then
11:        summary.delete(high( $x$ ));
continued...
```

## Implementation 4: van Emde Boas Trees

**Algorithm 65** delete( $x$ )

...continued

```
10:   if  $x = \text{max}$  then
11:        $\text{summax} \leftarrow \text{summary.max}()$ ;
12:       if  $\text{summax} = \text{null}$  then  $\text{max} \leftarrow \text{min}$ ;
13:       else
14:            $\text{offs} \leftarrow \text{cluster}[\text{summax}].\text{max}()$ ;
15:            $\text{max} \leftarrow \text{summax} \circ \text{offs}$ 
```

## Implementation 4: van Emde Boas Trees

**Algorithm 65** delete( $x$ )

...continued

fix maximum

```
10:   if  $x = \text{max}$  then
11:        $\text{summax} \leftarrow \text{summary.max}()$ ;
12:       if  $\text{summax} = \text{null}$  then  $\text{max} \leftarrow \text{min}$ ;
13:       else
14:            $\text{offs} \leftarrow \text{cluster}[\text{summax}].\text{max}()$ ;
15:            $\text{max} \leftarrow \text{summax} \circ \text{offs}$ 
```

## Implementation 4: van Emde Boas Trees

Note that only one of the possible recursive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where  $x$  was deleted is now empty. But this means that the call in Line 9 deleted the last element in  $\text{cluster}[\text{high}(x)]$ . Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c .$$

This gives  $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$ .

# 10 van Emde Boas Trees

## Space requirements:

- ▶ The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u} + 1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) .$$

- ▶ Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- ▶ One can show by induction that the space requirement is  $S(u) = \mathcal{O}(u)$ . Exercise.

- ▶ Let the “real” recurrence relation be

$$S(k^2) = (k + 1)S(k) + c_1 \cdot k; S(4) = c_2$$

- ▶ Replacing  $S(k)$  by  $R(k) := S(k)/c_2$  gives the recurrence

$$R(k^2) = (k + 1)R(k) + ck; R(4) = 1$$

where  $c = c_1/c_2 < 1$ .

- ▶ Now, we show  $R(k) \leq k - 2$  for squares  $k \geq 4$ .
  - ▶ Obviously, this holds for  $k = 4$ .
  - ▶ For  $k = \ell^2 > 4$  with  $\ell$  integral we have

$$\begin{aligned} R(k) &= (1 + \ell)R(\ell) + c\ell \\ &\leq (1 + \ell)(\ell - 2) + \ell \leq k - 2 \end{aligned}$$

- ▶ This shows that  $R(k)$  and, hence,  $S(k)$  grows linearly.

# Part IV

## Flows and Cuts

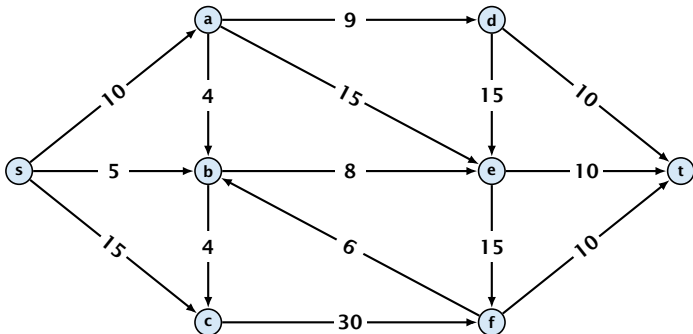
The following slides are partially based on slides by Kevin Wayne.



# 11 Introduction

## Flow Network

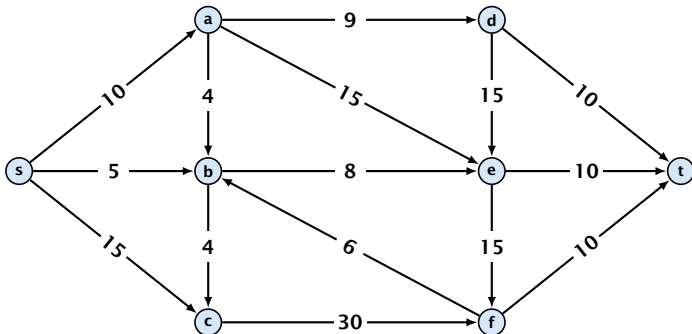
- ▶ directed graph  $G = (V, E)$ ; edge capacities  $c(e)$



# 11 Introduction

## Flow Network

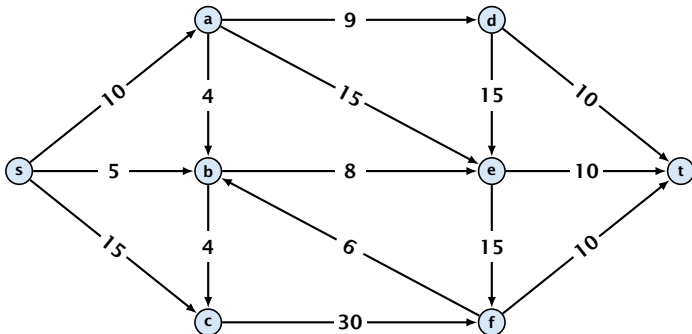
- ▶ directed graph  $G = (V, E)$ ; edge capacities  $c(e)$
- ▶ two special nodes: source  $s$ ; target  $t$ ;



# 11 Introduction

## Flow Network

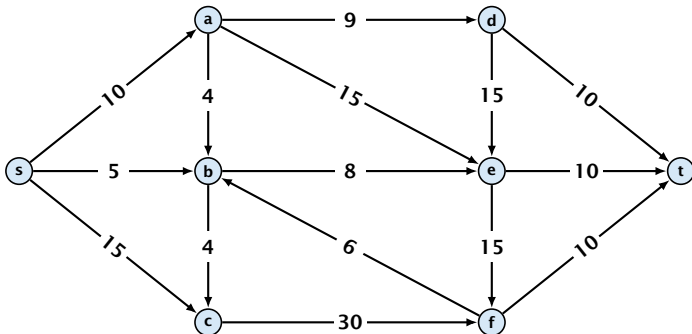
- ▶ directed graph  $G = (V, E)$ ; edge capacities  $c(e)$
- ▶ two special nodes: source  $s$ ; target  $t$ ;
- ▶ no edges entering  $s$  or leaving  $t$ ;



# 11 Introduction

## Flow Network

- ▶ directed graph  $G = (V, E)$ ; edge capacities  $c(e)$
- ▶ two special nodes: source  $s$ ; target  $t$ ;
- ▶ no edges entering  $s$  or leaving  $t$ ;
- ▶ at least for now: no parallel edges;



# Cuts

## Definition 40

An  $(s, t)$ -cut in the graph  $G$  is given by a set  $A \subset V$  with  $s \in A$  and  $t \in V \setminus A$ .

# Cuts

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## Definition 41

The **capacity** of a cut  $A$  is defined as

$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e) ,$$

where  $\text{out}(A)$  denotes the set of edges of the form  $A \times V \setminus A$  (i.e. edges leaving  $A$ ).

# Cuts

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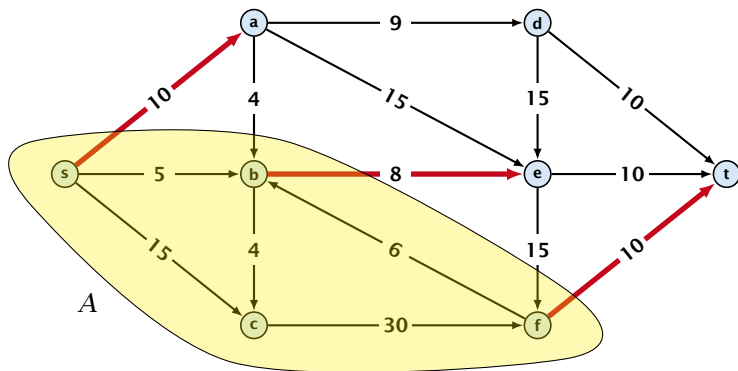
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**Minimum Cut Problem:** Find an  $(s, t)$ -cut with minimum capacity.

# Cuts

## Example 42



The capacity of the cut is  $\text{cap}(A, V \setminus A) = 28$ .



## Definition 43

An  $(s, t)$ -flow is a function  $f : E \mapsto \mathbb{R}^+$  that satisfies

1. For each edge  $e$

$$0 \leq f(e) \leq c(e) .$$

(capacity constraints)

## Definition 43

An  $(s, t)$ -flow is a function  $f : E \mapsto \mathbb{R}^+$  that satisfies

1. For each edge  $e$

$$0 \leq f(e) \leq c(e) .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) .$$

(flow conservation constraints)

## Definition 44

The **value of an  $(s, t)$ -flow  $f$**  is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

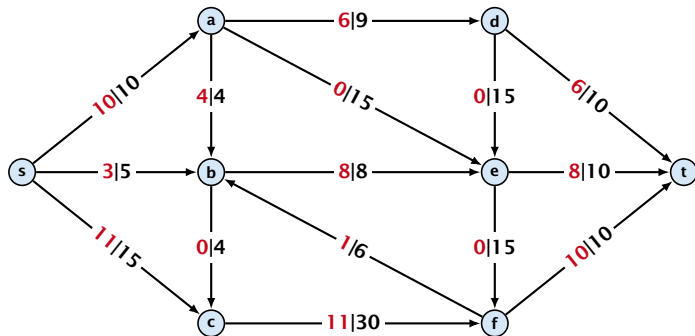
## Definition 44

The **value of an  $(s, t)$ -flow  $f$**  is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

**Maximum Flow Problem:** Find an  $(s, t)$ -flow with maximum value.

## Example 45



The value of the flow is  $\text{val}(f) = 24$ .

## Lemma 46 (Flow value lemma)

Let  $f$  be a flow, and let  $A \subseteq V$  be an  $(s, t)$ -cut. Then the *net-flow* across the cut is equal to the amount of flow leaving  $s$ , i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) .$$

**Proof.**

$\text{val}(f)$

**Proof.**

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e)$$



## Proof.

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)\end{aligned}$$

## Proof.

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) && = 0 \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)\end{aligned}$$

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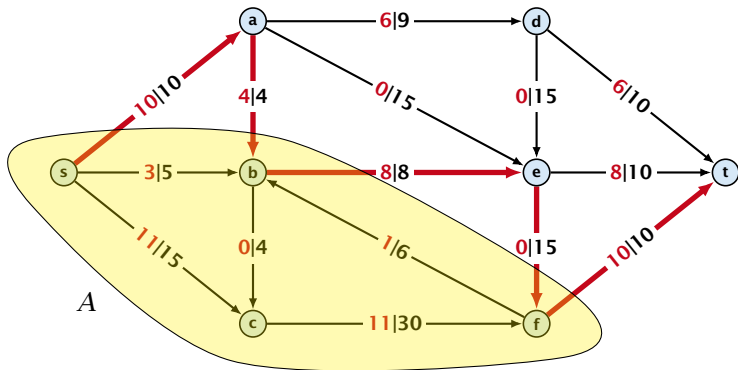
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## Proof.

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right) \\ &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)\end{aligned}$$

The last equality holds since every edge with both end-points in  $A$  contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering  $A$ .  $\square$

## Example 47



The net-flow across the cut is  $\text{val}(f) = 24$ .

## Corollary 48

Let  $f$  be an  $(s, t)$ -flow and let  $A$  be an  $(s, t)$ -cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then  $f$  is a maximum flow.

## Corollary 48

Let  $f$  be an  $(s, t)$ -flow and let  $A$  be an  $(s, t)$ -cut, such that

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### Proof.

Suppose that there is a flow  $f'$  with larger value. Then





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Suppose that there is a flow  $f'$  with larger value. Then

$$\text{cap}(A, V \setminus A) < \text{val}(f')$$



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$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then  $f$  is a maximum flow.

### Proof.

Suppose that there is a flow  $f'$  with larger value. Then

$$\begin{aligned} \text{cap}(A, V \setminus A) &< \text{val}(f') \\ &= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \end{aligned}$$



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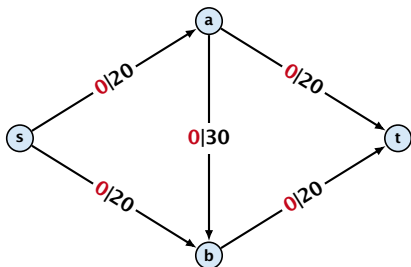
$$\begin{aligned} \text{cap}(A, V \setminus A) &< \text{val}(f') \\ &= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \\ &\leq \sum_{e \in \text{out}(A)} f'(e) \\ &\leq \text{cap}(A, V \setminus A) \end{aligned}$$

□

# 12 Augmenting Path Algorithms

## Greedy-algorithm:

- ▶ start with  $f(e) = 0$  everywhere
- ▶ find an  $s$ - $t$  path with  $f(e) < c(e)$  on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible

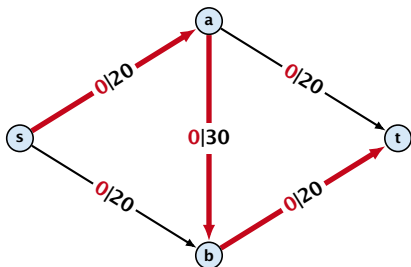


flow value: 0

# 12 Augmenting Path Algorithms

## Greedy-algorithm:

- ▶ start with  $f(e) = 0$  everywhere
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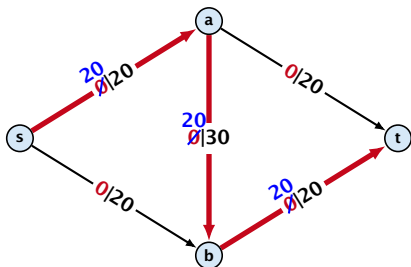


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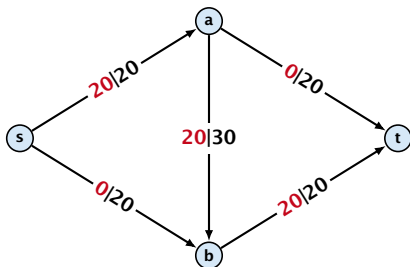


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## 12 Augmenting Path Algorithms

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flow value: 20



# The Residual Graph

From the graph  $G = (V, E, c)$  and the current flow  $f$  we construct an auxiliary graph  $G_f = (V, E_f, c_f)$  (the residual graph):

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# The Residual Graph

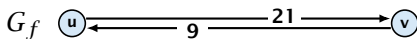
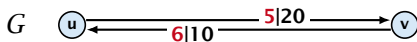
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- ▶ Suppose the original graph has edges  $e_1 = (u, v)$ , and  $e_2 = (v, u)$  between  $u$  and  $v$ .
- ▶  $G_f$  has edge  $e'_1$  with capacity  $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$  and  $e'_2$  with with capacity  $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$ .

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# Augmenting Path Algorithm

## Definition 49

An **augmenting path** with respect to flow  $f$ , is a path from  $s$  to  $t$  in the auxiliary graph  $G_f$  that contains only edges with non-zero capacity.

# Augmenting Path Algorithm

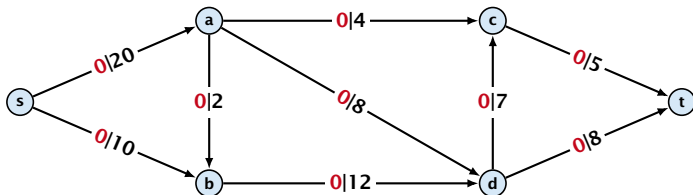
## Definition 49

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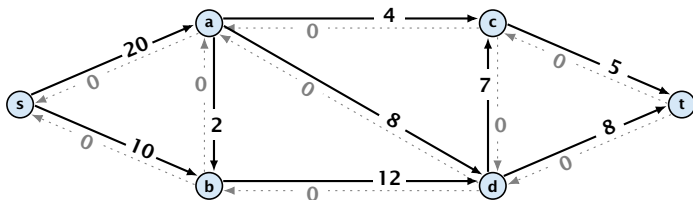
### Algorithm 1 FordFulkerson( $G = (V, E, c)$ )

- 1: Initialize  $f(e) \leftarrow 0$  for all edges.
- 2: **while**  $\exists$  augmenting path  $p$  in  $G_f$  **do**
- 3:     augment as much flow along  $p$  as possible.

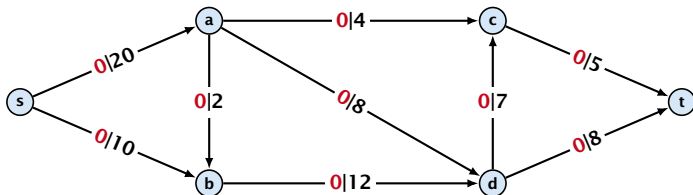
# Augmenting Paths



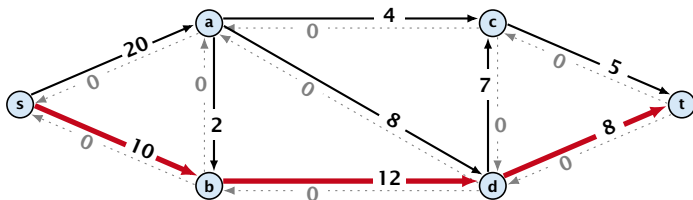
flow value: 0



# Augmenting Paths

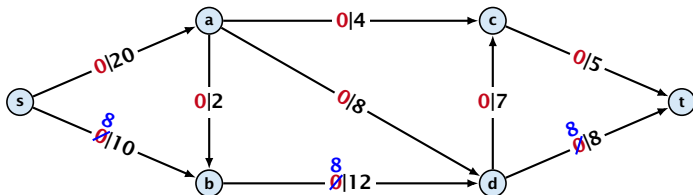


flow value: 0

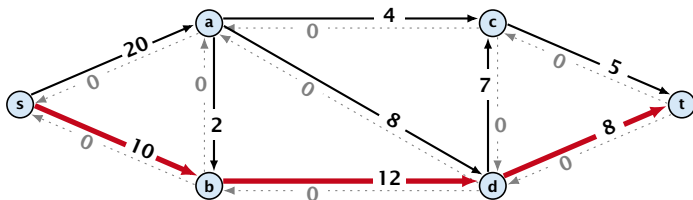




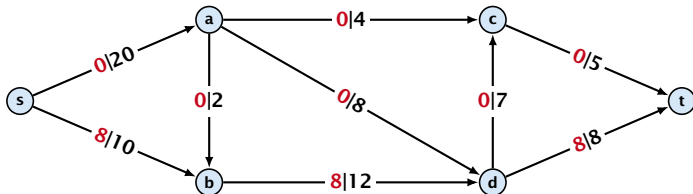
# Augmenting Paths



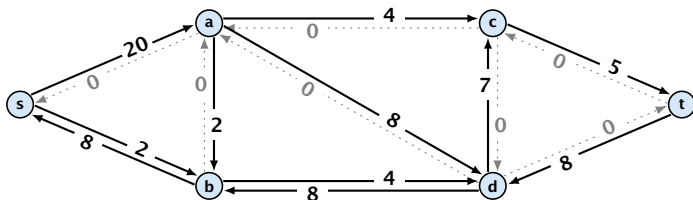
flow value: 0



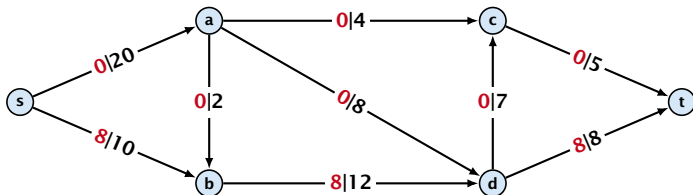
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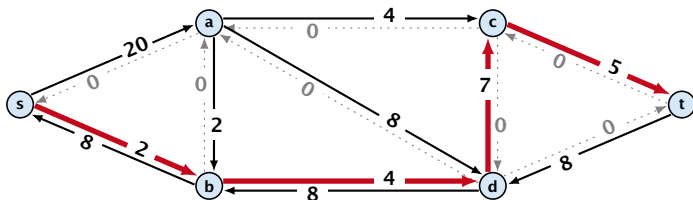
flow value: 8



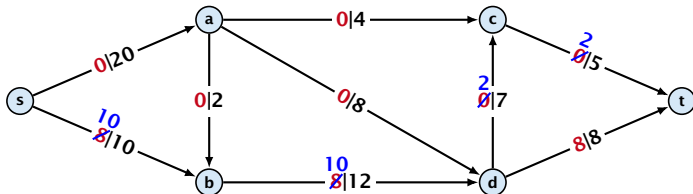
# Augmenting Paths



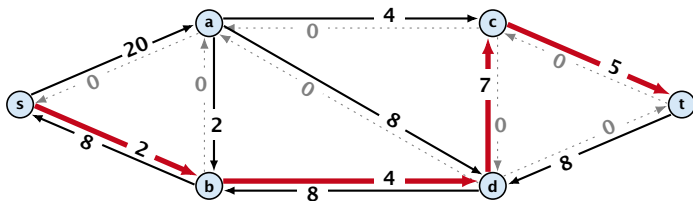
flow value: 8



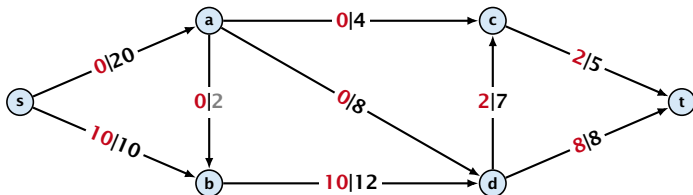
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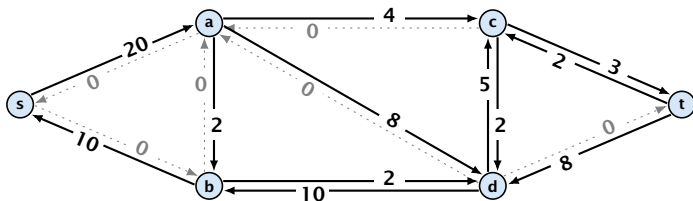
flow value: 8



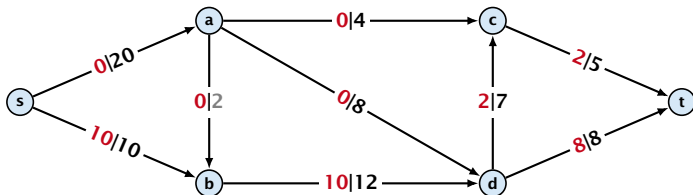
# Augmenting Paths



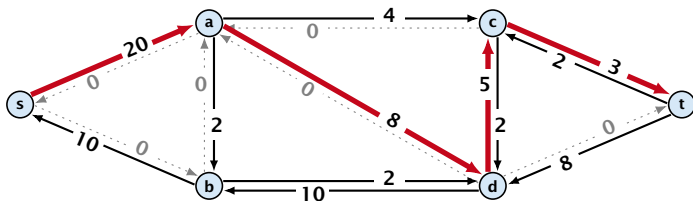
flow value: 10



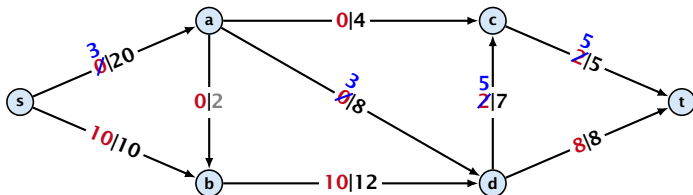
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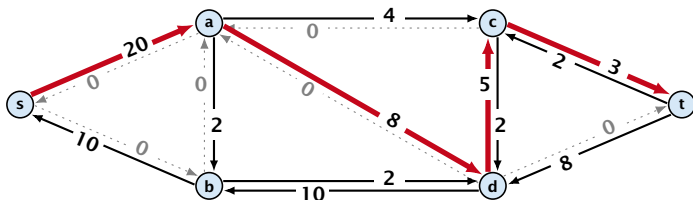
flow value: 10



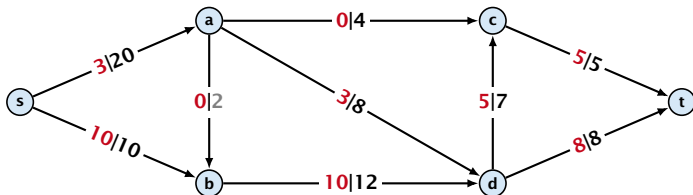
# Augmenting Paths



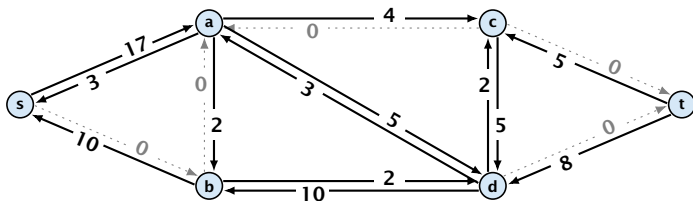
flow value: 10



# Augmenting Paths



flow value: 13





# Augmenting Path Algorithm

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## Theorem 50

A flow  $f$  is a maximum flow **iff** there are no augmenting paths.

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## Proof.

Let  $f$  be a flow. The following are equivalent:

1. There exists a cut  $A$  such that  $\text{val}(f) = \text{cap}(A, V \setminus A)$ .



# Augmenting Path Algorithm

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# Augmenting Path Algorithm

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The value of a maximum flow is equal to the value of a minimum cut.

## Proof.

Let  $f$  be a flow. The following are equivalent:

1. There exists a cut  $A$  such that  $\text{val}(f) = \text{cap}(A, V \setminus A)$ .
2. Flow  $f$  is a maximum flow.
3. There is no augmenting path w.r.t.  $f$ .



# Augmenting Path Algorithm

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1.  $\Rightarrow$  2.

This we already showed.



# Augmenting Path Algorithm

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If there were an augmenting path, we could improve the flow.

Contradiction.

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- ▶ Let  $f$  be a flow with no augmenting paths.
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3.  $\Rightarrow$  1.

- ▶ Let  $f$  be a flow with no augmenting paths.
- ▶ Let  $A$  be the set of vertices reachable from  $s$  in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have  $s \in A$  and  $t \notin A$ .

# Augmenting Path Algorithm

$\text{val}(f)$

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$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

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$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e)\end{aligned}$$

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving  $A$ .

**Assumption:**

All capacities are integers between 1 and  $C$ .

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**Invariant:**

Every flow value  $f(e)$  and every residual capacity  $c_f(e)$  remains integral throughout the algorithm.

## Lemma 52

The algorithm terminates in at most  $\text{val}(f^*) \leq nC$  iterations, where  $f^*$  denotes the maximum flow. Each iteration can be implemented in time  $\mathcal{O}(m)$ . This gives a total running time of  $\mathcal{O}(nmC)$ .

## Lemma 52

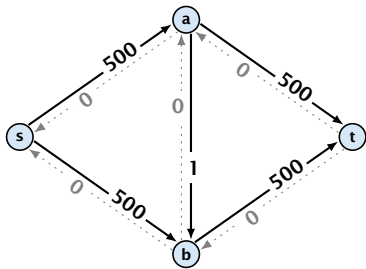
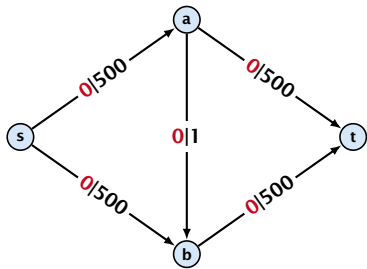
The algorithm terminates in at most  $\text{val}(f^*) \leq nC$  iterations, where  $f^*$  denotes the maximum flow. Each iteration can be implemented in time  $\mathcal{O}(m)$ . This gives a total running time of  $\mathcal{O}(nmC)$ .

## Theorem 53

If all capacities are integers, then there exists a maximum flow for which every flow value  $f(e)$  is integral.

# A Bad Input

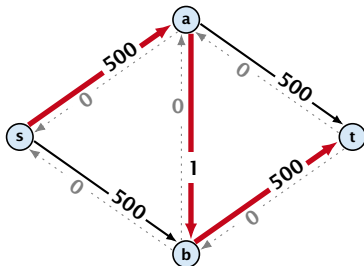
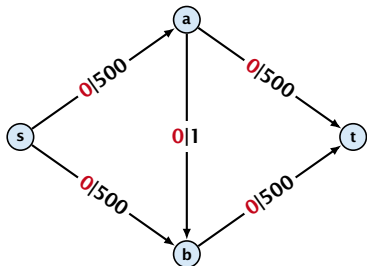
**Problem:** The running time may not be polynomial



flow value: 0

# A Bad Input

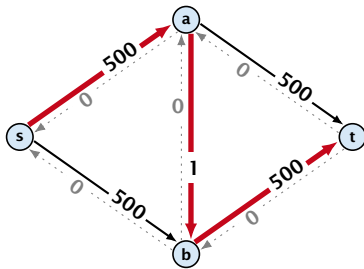
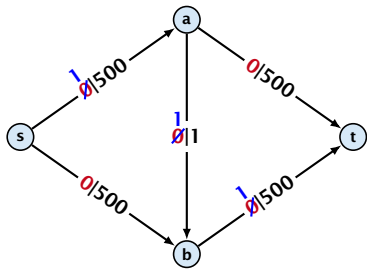
**Problem:** The running time may not be polynomial



flow value: 0

# A Bad Input

**Problem:** The running time may not be polynomial

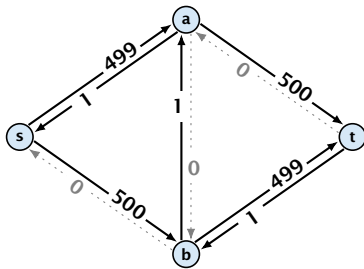
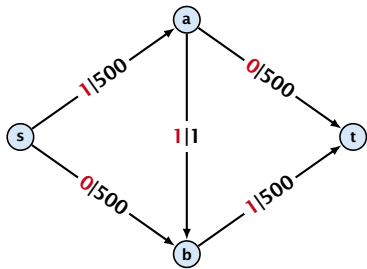


flow value: 0



# A Bad Input

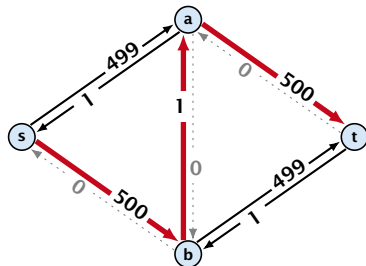
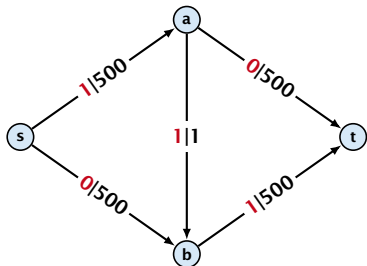
**Problem:** The running time may not be polynomial



flow value: 1

# A Bad Input

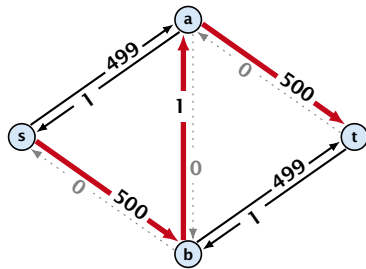
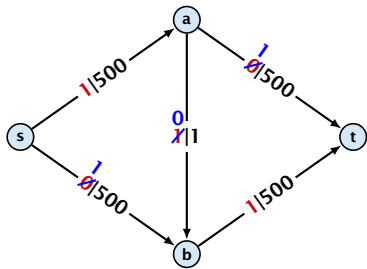
**Problem:** The running time may not be polynomial



flow value: 1

# A Bad Input

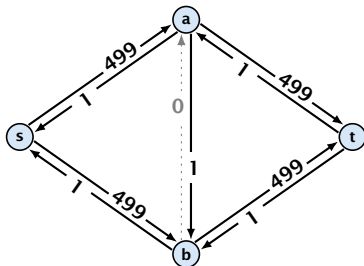
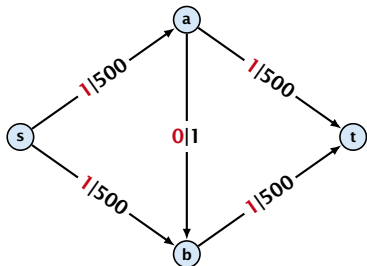
**Problem:** The running time may not be polynomial



flow value: 1

# A Bad Input

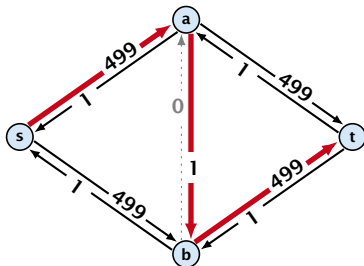
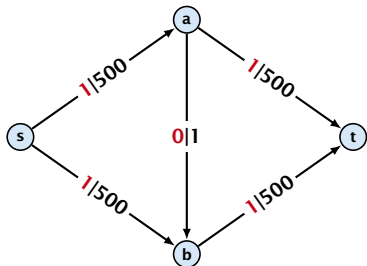
**Problem:** The running time may not be polynomial



flow value: 2

# A Bad Input

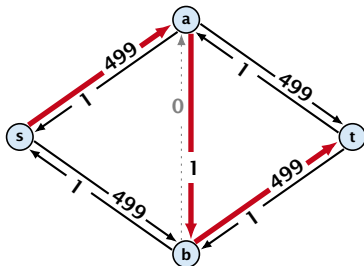
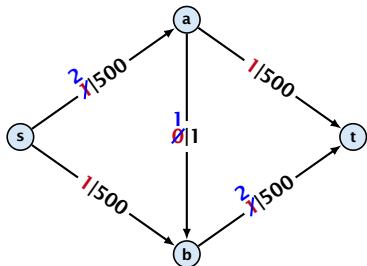
**Problem:** The running time may not be polynomial



flow value: 2

# A Bad Input

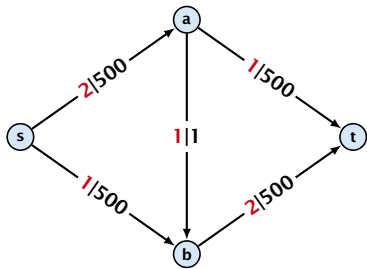
**Problem:** The running time may not be polynomial



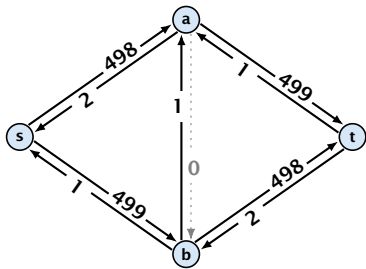
flow value: 2

# A Bad Input

**Problem:** The running time may not be polynomial

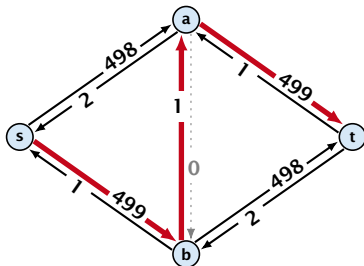
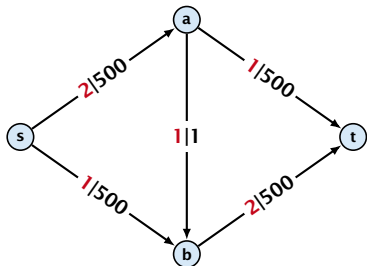


flow value: 3



# A Bad Input

**Problem:** The running time may not be polynomial

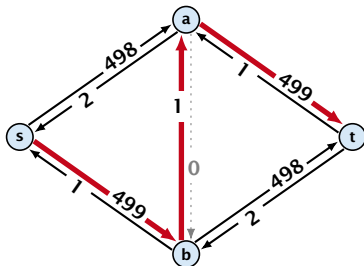
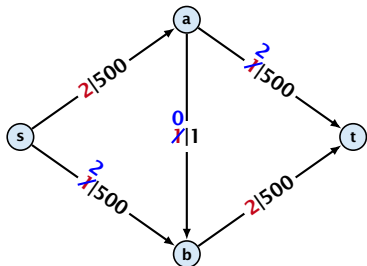


flow value: 3



# A Bad Input

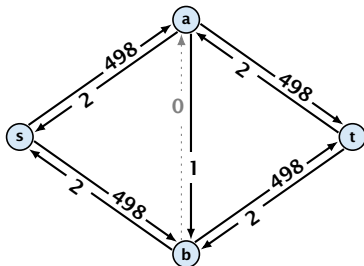
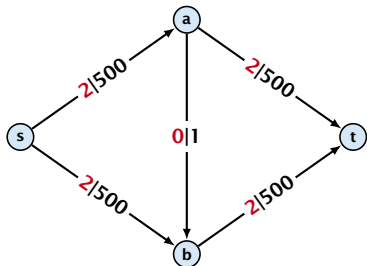
**Problem:** The running time may not be polynomial



flow value: 3

# A Bad Input

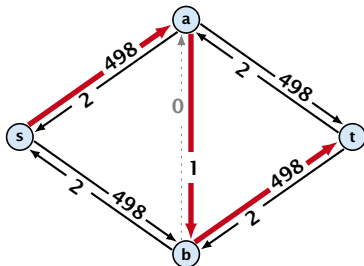
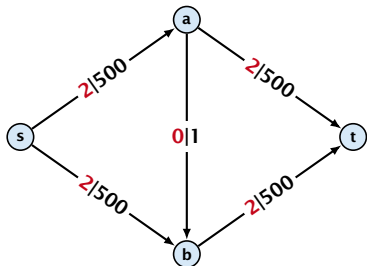
**Problem:** The running time may not be polynomial



flow value: 4

# A Bad Input

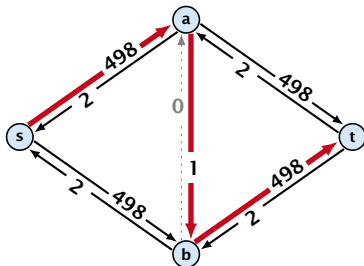
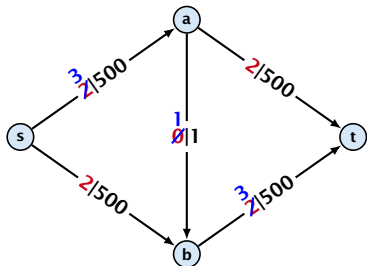
**Problem:** The running time may not be polynomial



flow value: 4

# A Bad Input

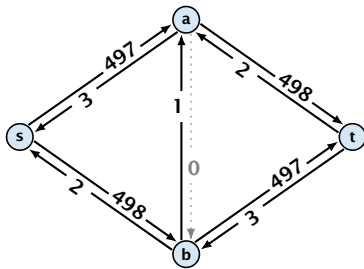
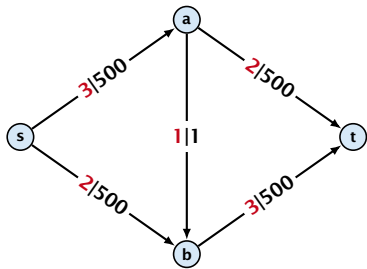
**Problem:** The running time may not be polynomial



flow value: 4

# A Bad Input

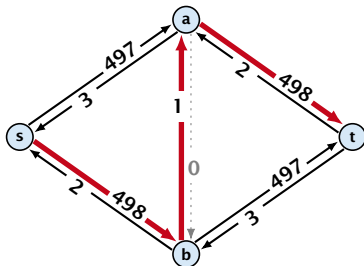
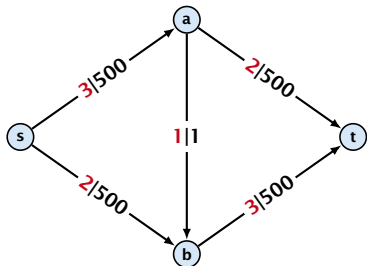
**Problem:** The running time may not be polynomial



flow value: 5

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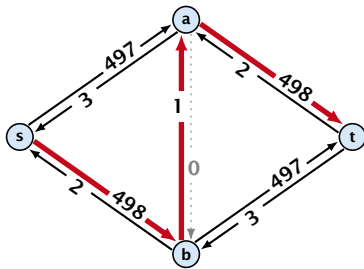
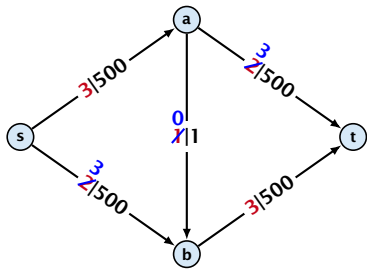
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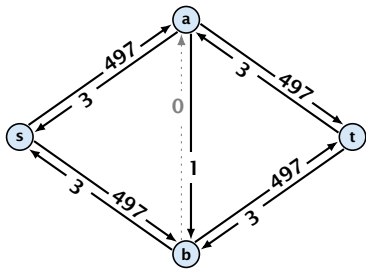
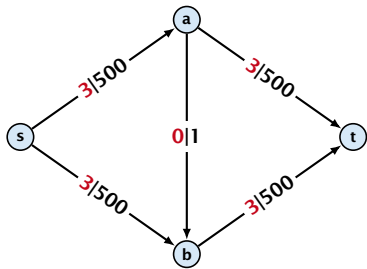
**Problem:** The running time may not be polynomial



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**Problem:** The running time may not be polynomial

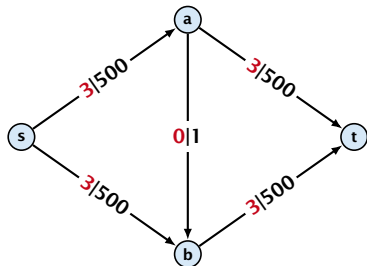


flow value: 6



# A Bad Input

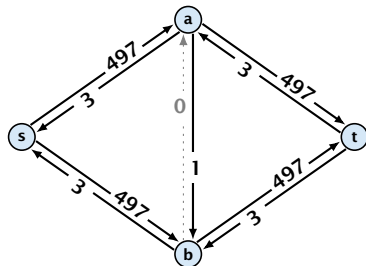
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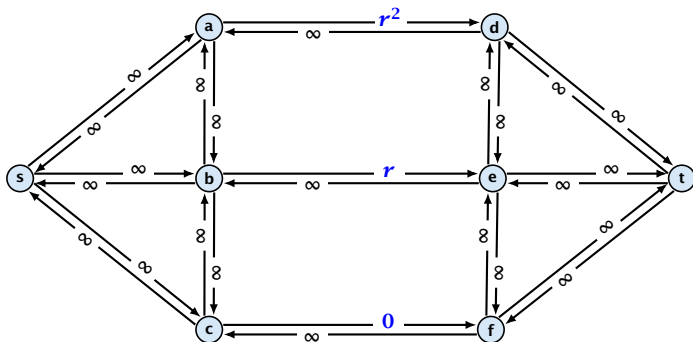
**Question:**

Can we tweak the algorithm so that the running time is polynomial in the input length?



# A Pathological Input

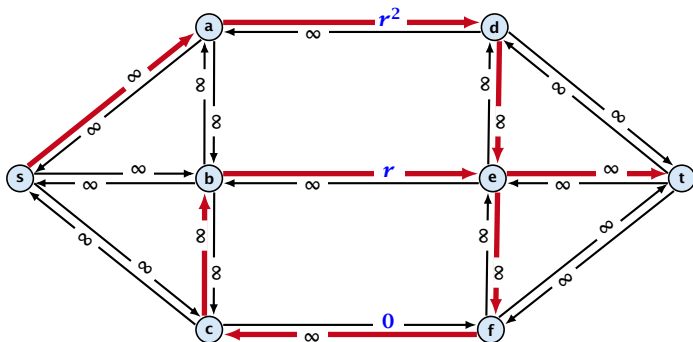
Let  $r = \frac{1}{2}(\sqrt{5} - 1)$ . Then  $r^{n+2} = r^n - r^{n+1}$ .



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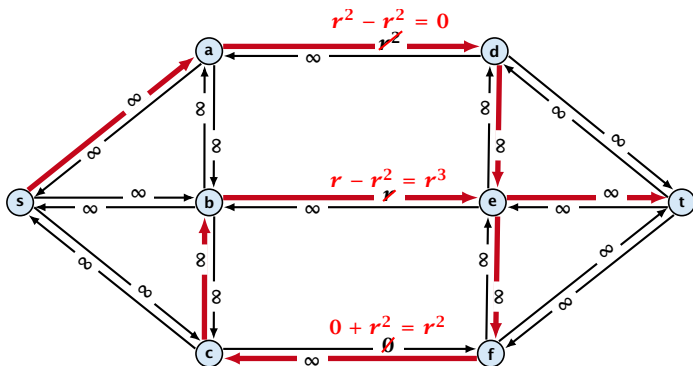
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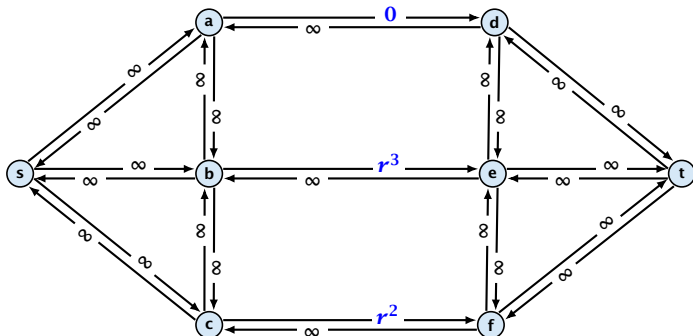
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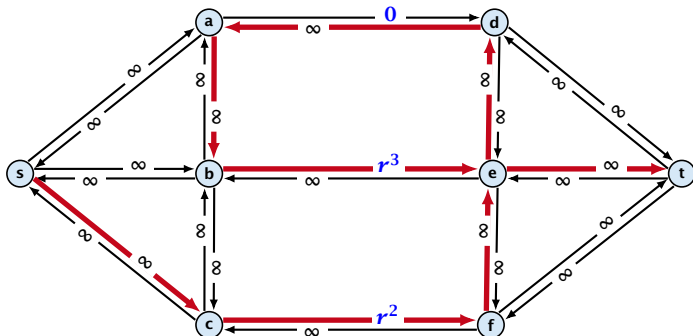
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flow value:  $r^2$

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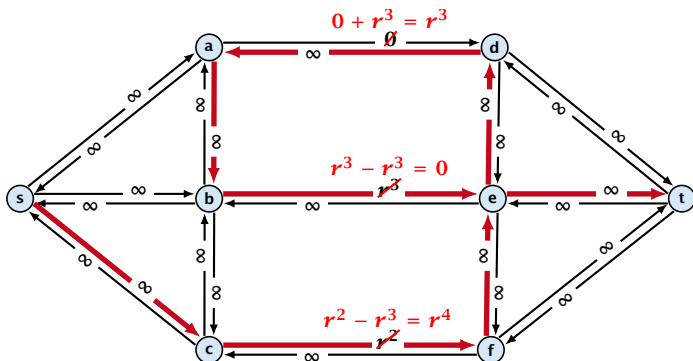
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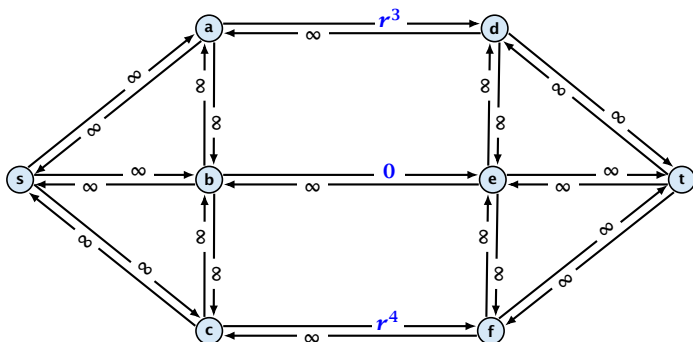
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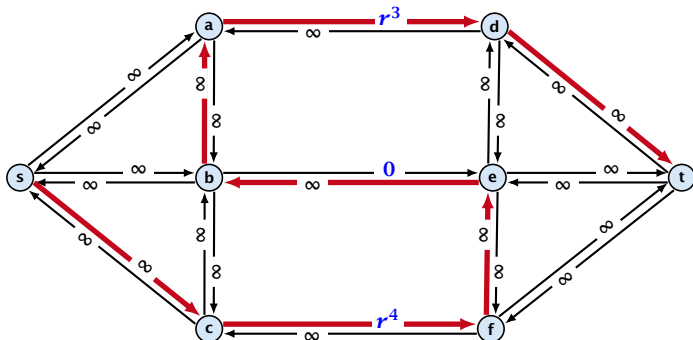


flow value:  $r^2 + r^3$



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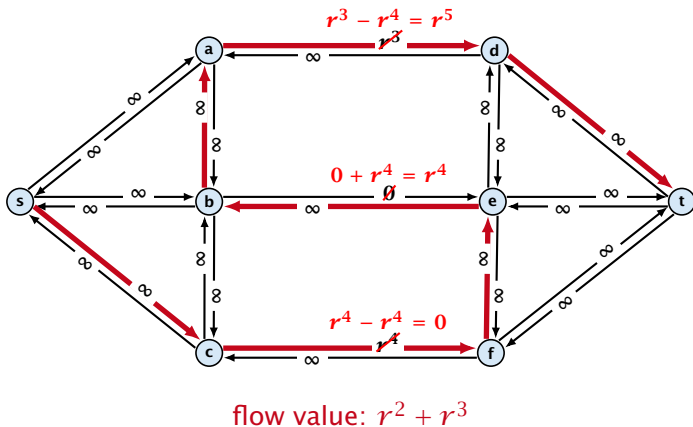
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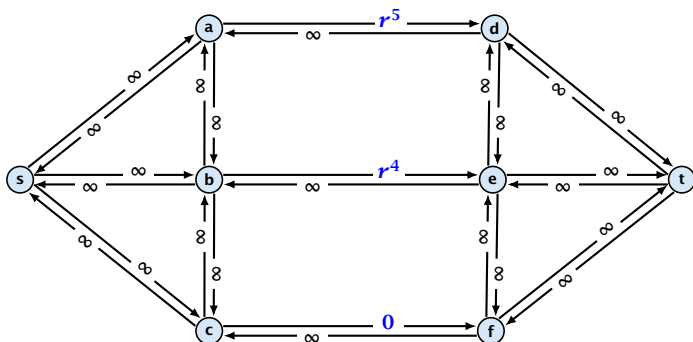
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flow value:  $r^2 + r^3 + r^4$

Running time may be infinite!!!



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- ▶ Choose the shortest augmenting path.

# Overview: Shortest Augmenting Paths

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## Lemma 54

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- ▶ We can find the shortest augmenting paths in time  $\mathcal{O}(m)$  via BFS.
- ▶  $\mathcal{O}(m)$  augmentations for paths of exactly  $k < n$  edges.



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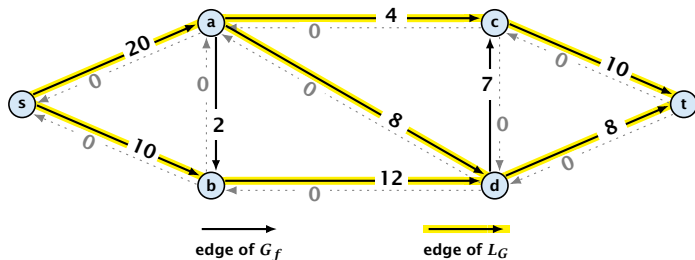
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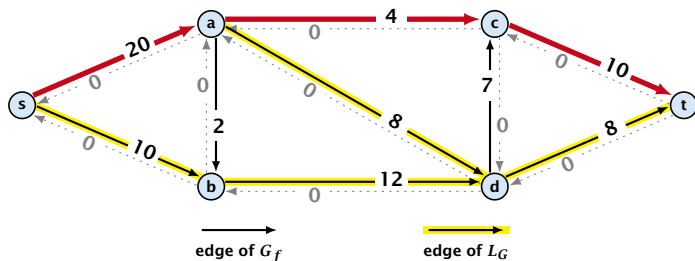


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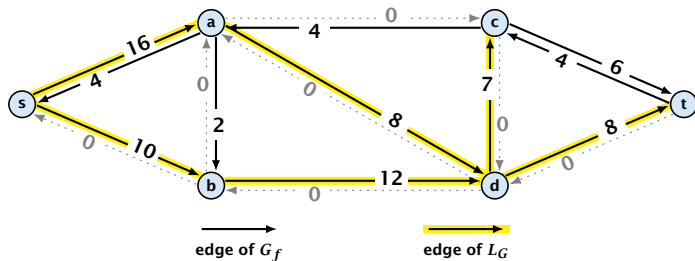


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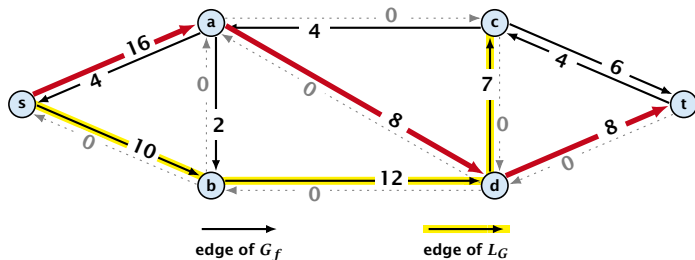


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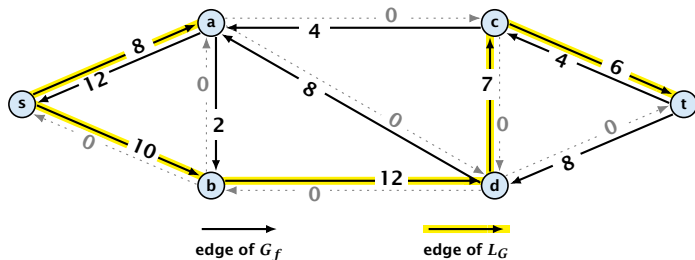


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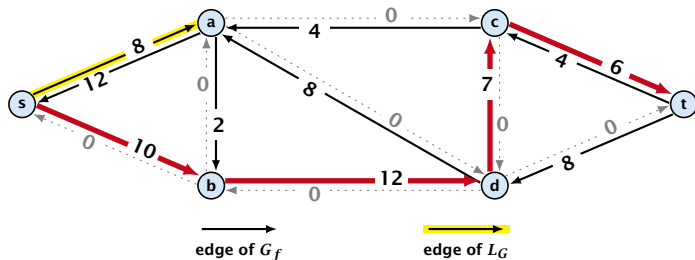


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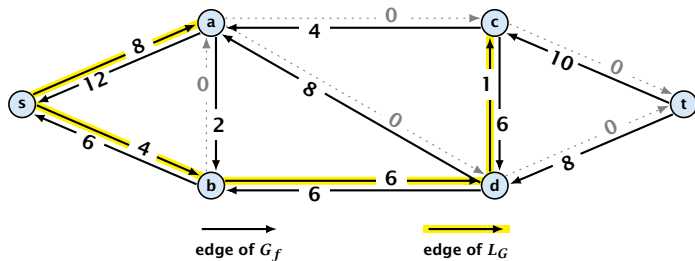


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In the following we assume that the residual graph  $G_f$  does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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The length of the shortest augmenting path never decreases.

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These changes cannot decrease the distance between  $s$  and  $t$ .



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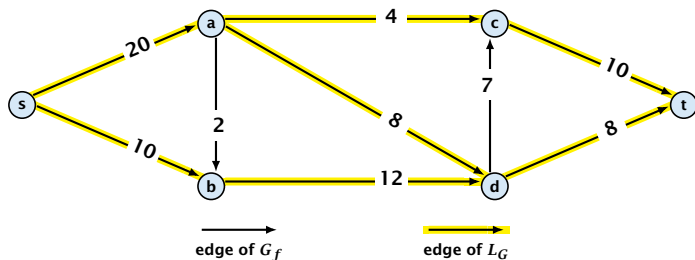
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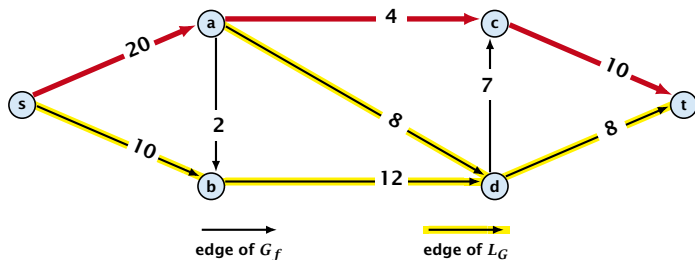
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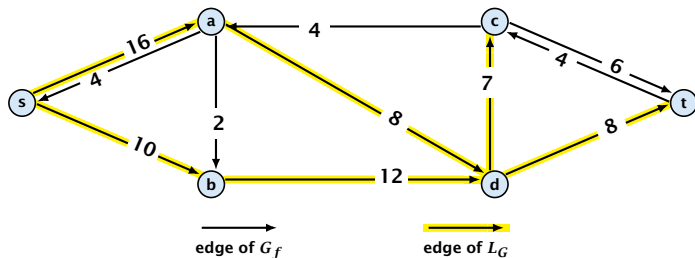
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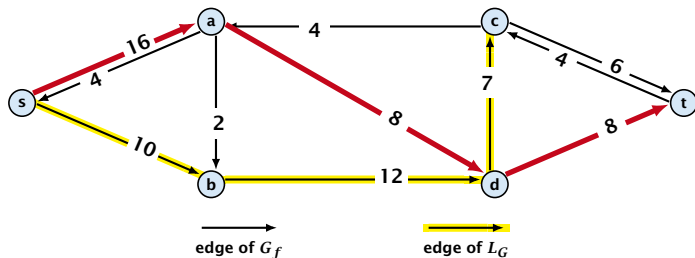
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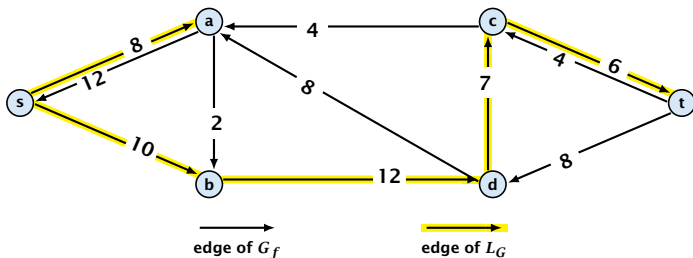
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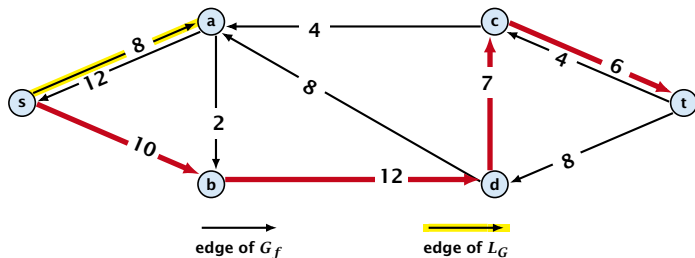
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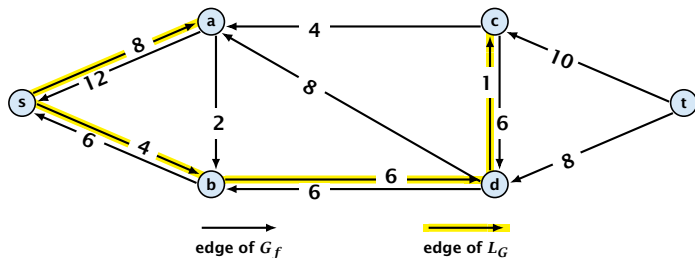
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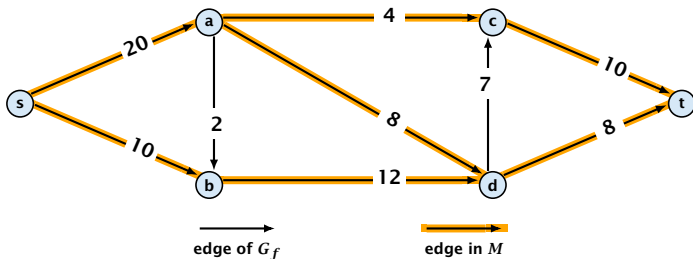
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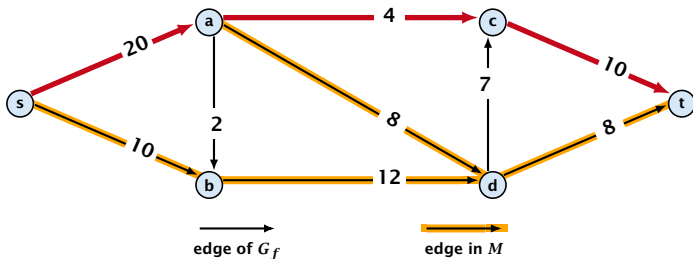
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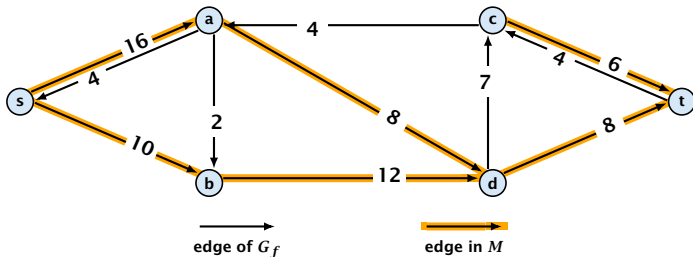
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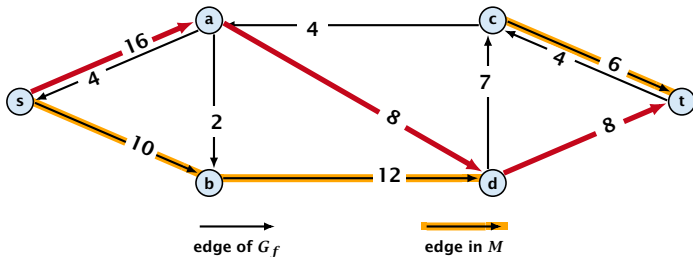
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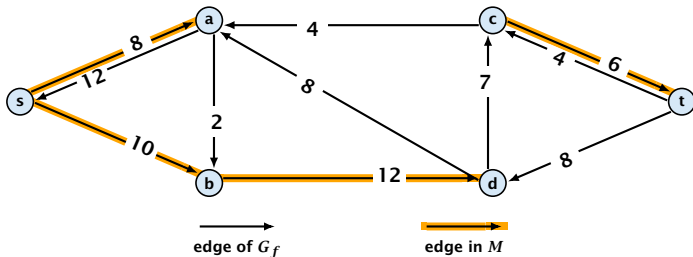
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## Theorem 58 (without proof)

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### Note:

There always exists a set of  $m$  augmentations that gives a maximum flow (why?).

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However, we can improve the running time to  $\mathcal{O}(mn^2)$  by improving the running time for finding an augmenting path (currently we assume  $\mathcal{O}(m)$  per augmentation for this).

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Note that  $M$  is not the set of edges of the level graph but a subset of level-graph edges.

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The total cost for searching for augmenting paths during a phase is at most  $\mathcal{O}(mn)$ , since every search (successful (i.e., reaching  $t$ ) or unsuccessful) decreases the number of edges in  $M$  and takes time  $\mathcal{O}(n)$ .

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There are at most  $n$  phases. Hence, total cost is  $\mathcal{O}(mn^2)$ .

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- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
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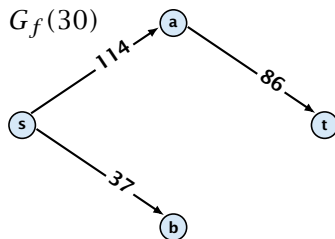
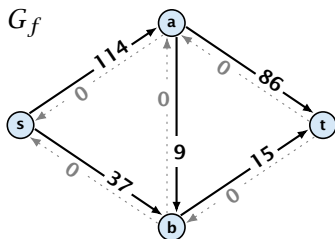
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# Capacity Scaling

## Algorithm 1 $\text{maxflow}(G, s, t, c)$

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
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- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

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- ▶ In  $G_f$  this cut can have capacity at most  $m\Delta$ .
- ▶ This gives me an upper bound on the flow that I can still add.

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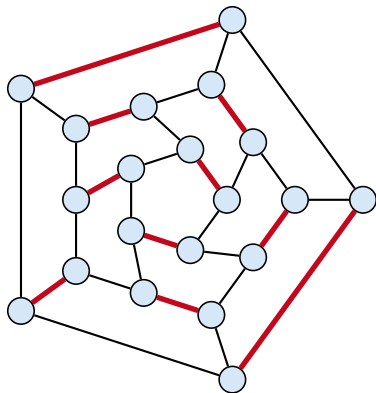
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## Theorem 62

*We need  $\mathcal{O}(m \log C)$  augmentations. The algorithm can be implemented in time  $\mathcal{O}(m^2 \log C)$ .*

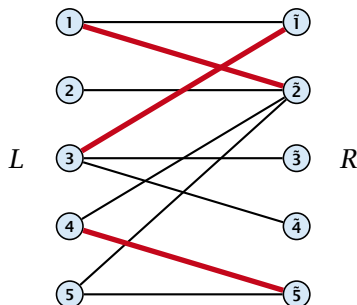
# Matching

- ▶ Input: undirected graph  $G = (V, E)$ .
- ▶  $M \subseteq E$  is a **matching** if each node appears in at most one edge in  $M$ .
- ▶ Maximum Matching: find a matching of maximum cardinality



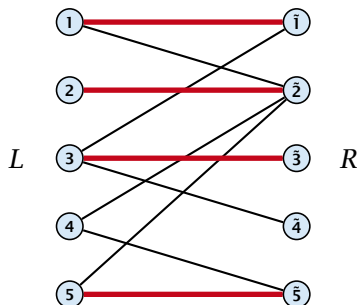
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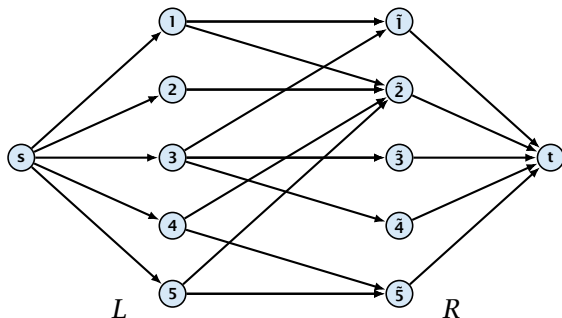
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# Maxflow Formulation

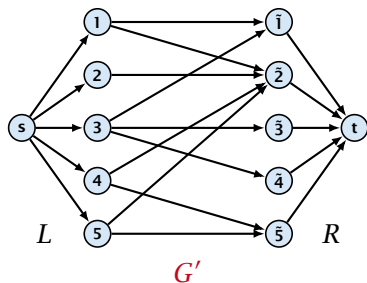
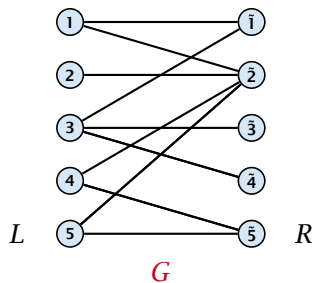
- ▶ Input: undirected, bipartite graph  $G = (L \uplus R \uplus \{s, t\}, E')$ .
- ▶ Direct all edges from  $L$  to  $R$ .
- ▶ Add source  $s$  and connect it to all nodes on the left.
- ▶ Add  $t$  and connect all nodes on the right to  $t$ .
- ▶ All edges have unit capacity.



# Proof

## Max cardinality matching in $G \leq$ value of maxflow in $G'$

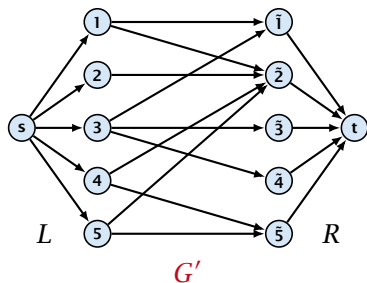
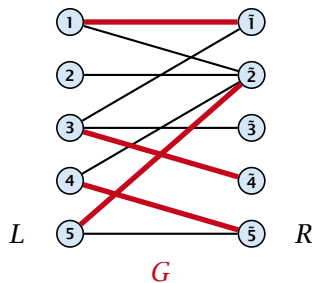
- ▶ Given a maximum matching  $M$  of cardinality  $k$ .
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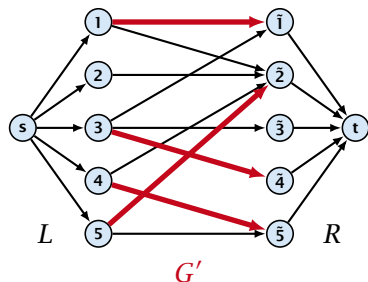
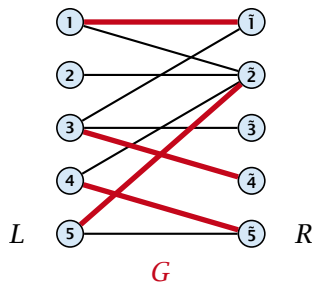
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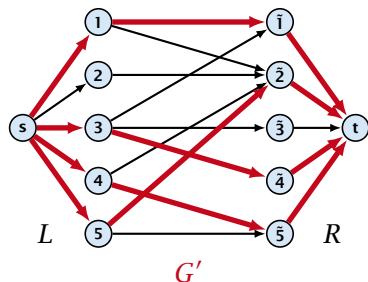
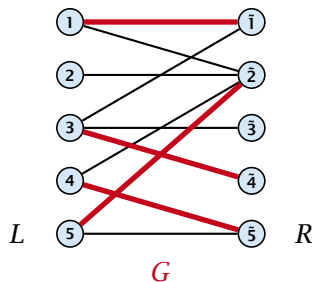




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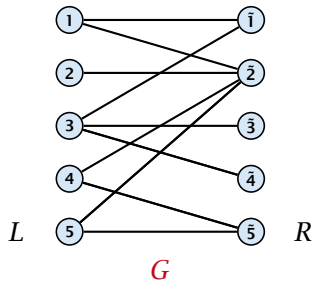
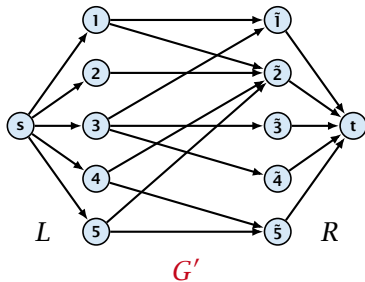
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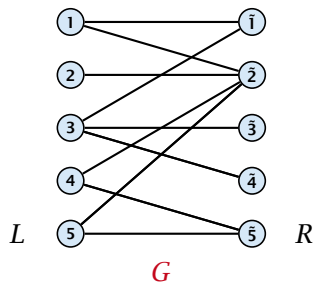
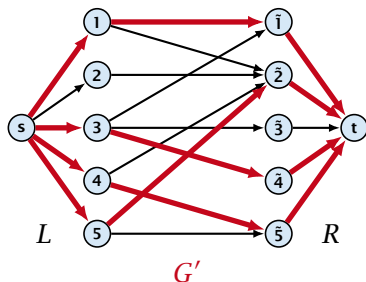
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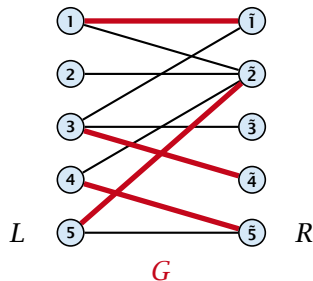
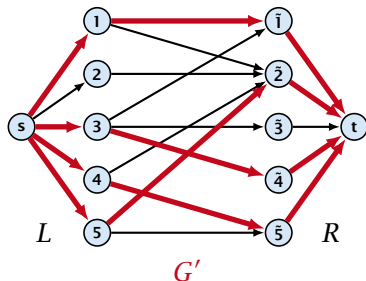
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# 13.1 Matching

## Which flow algorithm to use?

- ▶ Generic augmenting path:  $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$ .
- ▶ Capacity scaling:  $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$ .
- ▶ Shortest augmenting path:  $\mathcal{O}(mn^2)$ .

For **unit capacity simple graphs** shortest augmenting path can be implemented in time  $\mathcal{O}(m\sqrt{n})$ .

# Baseball Elimination

<i>team</i> <i>i</i>	<i>wins</i> $w_i$	<i>losses</i> $\ell_i$	<i>remaining games</i>			
			<i>Atl</i>	<i>Phi</i>	<i>NY</i>	<i>Mon</i>
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

**Which team can end the season with most wins?**

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

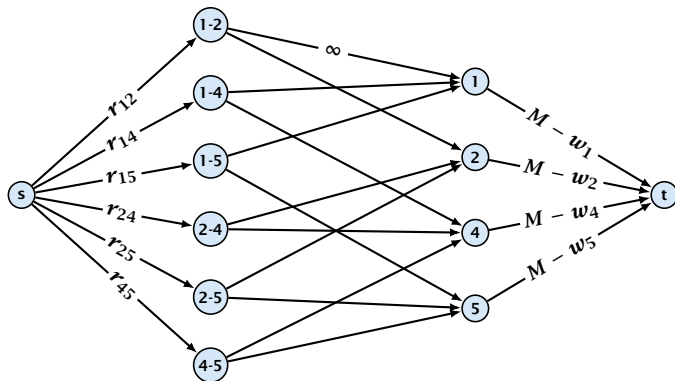
# Baseball Elimination

## Formal definition of the problem:

- ▶ Given a set  $S$  of teams, and one specific team  $z \in S$ .
- ▶ Team  $x$  has already won  $w_x$  games.
- ▶ Team  $x$  still has to play team  $y$ ,  $r_{xy}$  times.
- ▶ Does team  $z$  still have a chance to finish with the most number of wins.

# Baseball Elimination

Flow network for  $z = 3$ .  $M$  is number of wins Team 3 can still obtain.



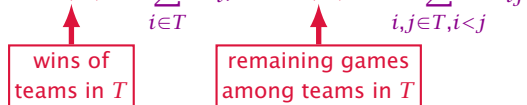
**Idea.** Distribute the results of remaining games in such a way that no team gets too many wins.



# Certificate of Elimination

Let  $T \subseteq S$  be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If  $\frac{w(T)+r(T)}{|T|} > M$  then one of the teams in  $T$  will have more than  $M$  wins in the end. A team that can win at most  $M$  games is therefore eliminated.

### Theorem 63

A team  $z$  is eliminated if and only if the flow network for  $z$  does not allow a flow of value  $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$ .

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### Proof ( $\Leftarrow$ )

- ▶ Consider the mincut  $A$  in the flow network. Let  $T$  be the set of team-nodes in  $A$ .

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- ▶ This gives  $M < (w(T) + r(T))/|T|$ , i.e.,  $z$  is eliminated.

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- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than  $M$  wins in total.
- ▶ Hence, team  $z$  is not eliminated.



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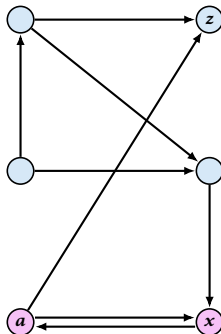
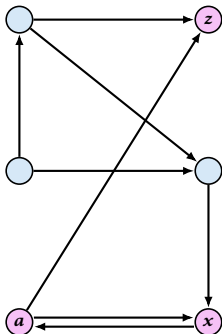
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**Goal:** Find a feasible set of projects that maximizes the profit.

# Project Selection

## The prerequisite graph:

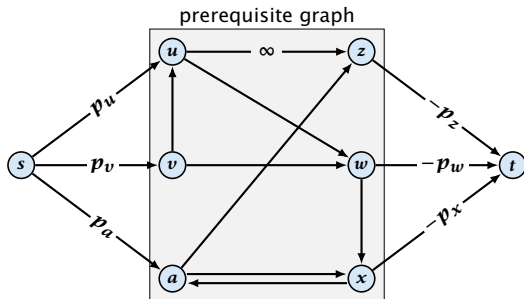
- ▶  $\{x, a, z\}$  is a feasible subset.
- ▶  $\{x, a\}$  is infeasible.



# Project Selection

## Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge  $(s, v)$  with capacity  $p_v$  for nodes  $v$  with positive profit.
- ▶ Create edge  $(v, t)$  with capacity  $-p_v$  for nodes  $v$  with negative profit.



## Theorem 64

$A$  is a mincut if  $A \setminus \{s\}$  is the optimal set of projects.

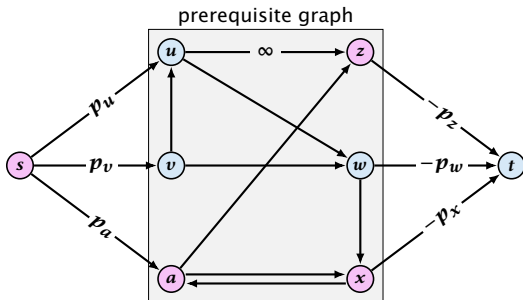


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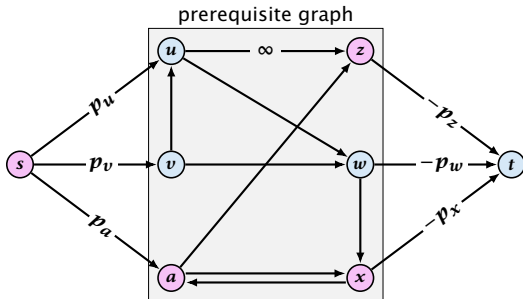


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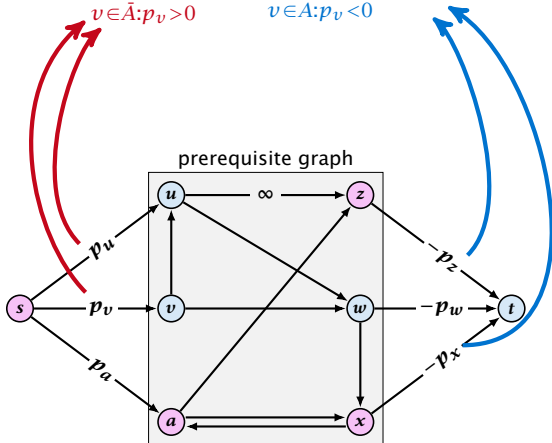
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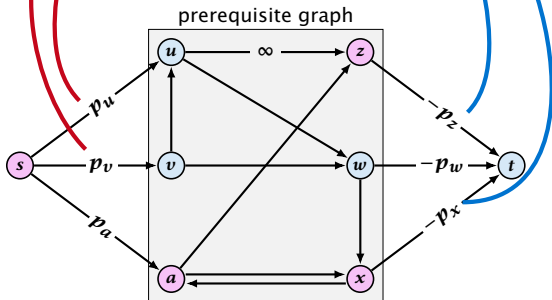
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$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$





# Preflows

## Definition 65

An  $(s, t)$ -preflow is a function  $f : E \mapsto \mathbb{R}^+$  that satisfies

1. For each edge  $e$

$$0 \leq f(e) \leq c(e) .$$

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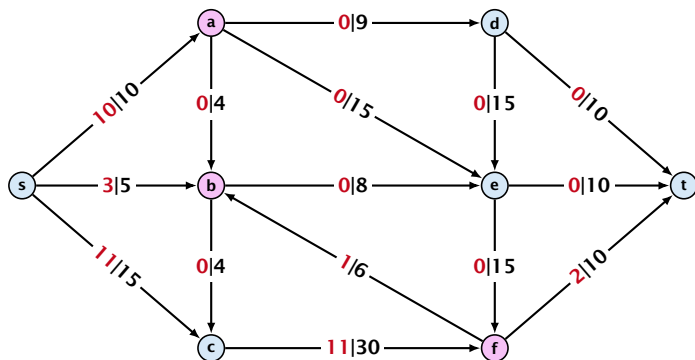
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2. For each  $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) \leq \sum_{e \in \text{into}(v)} f(e) .$$

# Preflows

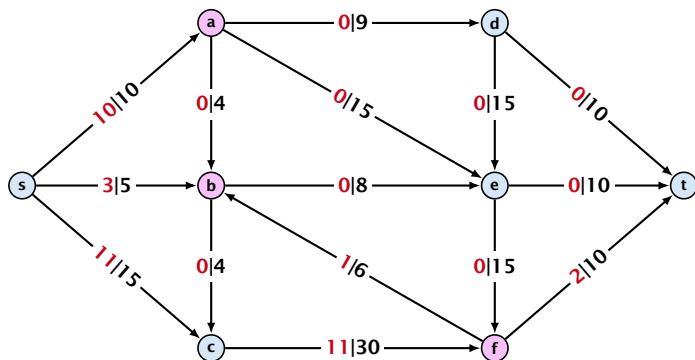
## Example 66





# Preflows

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A node that has  $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$  is called an **active node**.



# Preflows

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A **labelling** is a function  $\ell : V \rightarrow \mathbb{N}$ . It is **valid** for preflow  $f$  if

- ▶  $\ell(u) \leq \ell(v) + 1$  for all edges  $(u, v)$  in the residual graph  $G_f$   
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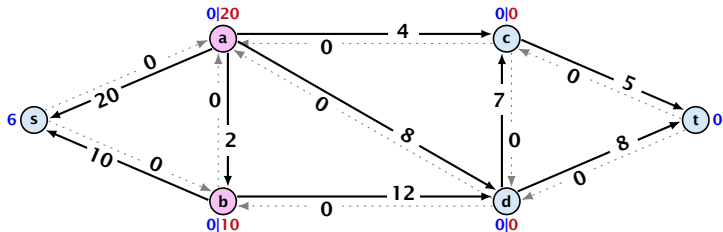
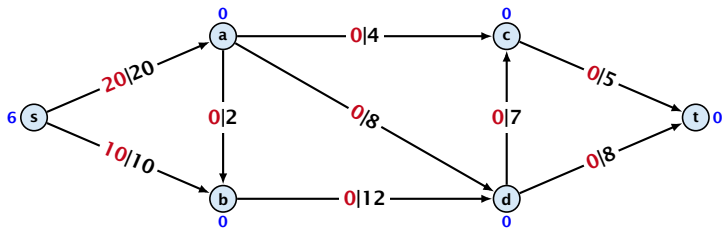
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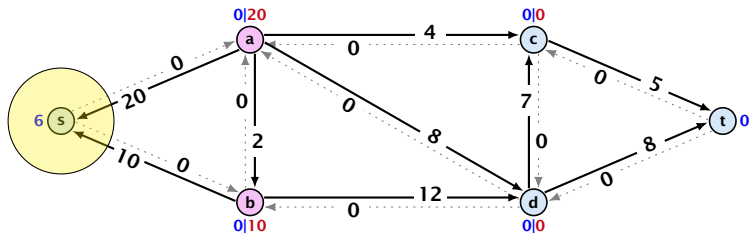
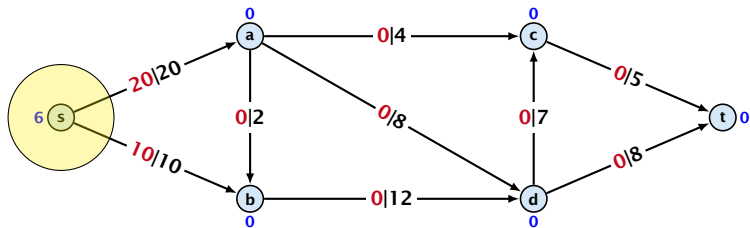
## Intuition:

The labelling can be viewed as a height function. Whenever the height from node  $u$  to node  $v$  decreases by more than 1 (i.e., it goes very steep downhill from  $u$  to  $v$ ), the corresponding edge must be saturated.

# Preflows



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A *flow* that has a valid labelling is a maximum flow.

# Push Relabel Algorithms



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- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

## Changing a Preflow

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An arc  $(u, v)$  with  $c_f(u, v) > 0$  in the residual graph is **admissible** if  $\ell(u) = \ell(v) + 1$  (i.e., it goes downwards w.r.t. labelling  $\ell$ ).

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### The push operation

Consider an active node  $u$  with **excess flow**

$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$  and suppose  $e = (u, v)$  is an admissible arc with residual capacity  $c_f(e)$ .

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- ▶ **saturating push**:  $\min\{f(u), c_f(e)\} = c_f(e)$   
the arc  $e$  is deleted from the residual graph
- ▶ **deactivating push**:  $\min\{f(u), c_f(e)\} = f(u)$   
the node  $u$  becomes inactive

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- ▶ Edges  $(w, u)$  incoming to  $u$  still fulfill their constraint  $\ell(w) \leq \ell(u) + 1$ .
- ▶ An outgoing edge  $(u, w)$  had  $\ell(u) < \ell(w) + 1$  before since it was not admissible. Now:  $\ell(u) \leq \ell(w) + 1$ .

# Push Relabel Algorithms

## Intuition:

We want to send flow downwards, since the source has a height/label of  $n$  and the target a height/label of  $0$ . If we see an active node  $u$  with an admissible arc we push the flow at  $u$  towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into  $u$  it should roughly mean that the level/height/label of  $u$  should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

# Reminder

- ▶ In a **preflow** nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- ▶ Such a node is called **active**.
- ▶ A labelling is **valid** if for every edge  $(u, v)$  in the residual graph  $\ell(u) \leq \ell(v) + 1$ .
- ▶ An arc  $(u, v)$  in residual graph is **admissible** if  $\ell(u) = \ell(v) + 1$ .
- ▶ A **saturating push** along  $e$  pushes an amount of  $c(e)$  flow along the edge, thereby saturating the edge (and making it disappear from the residual graph).
- ▶ A **deactivating push** along  $e = (u, v)$  pushes a flow of  $f(u)$ , where  $f(u)$  is the **excess flow** of  $u$ . This makes  $u$  inactive.



# Push Relabel Algorithms

## Algorithm 1 $\text{maxflow}(G, s, t, c)$

```
1: find initial preflow  $f$ 
2: while there is active node  $u$  do
3:     if there is admiss. arc  $e$  out of  $u$  then
4:          $\text{push}(G, e, f, c)$ 
5:     else
6:          $\text{relabel}(u)$ 
7: return  $f$ 
```

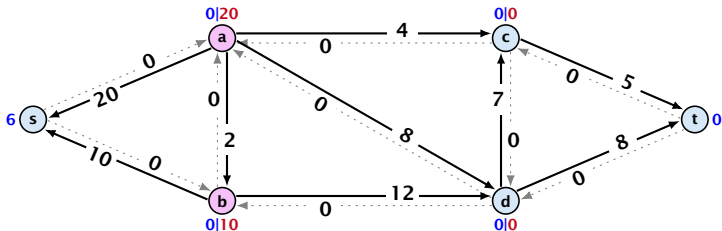
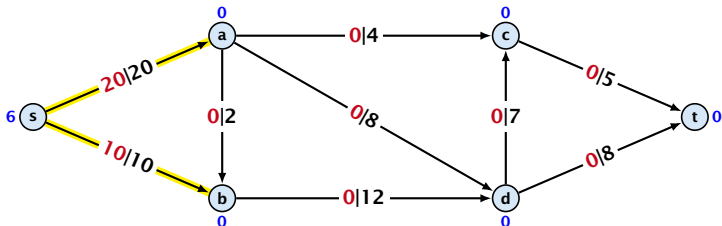
# Push Relabel Algorithms

## Algorithm 1 $\text{maxflow}(G, s, t, c)$

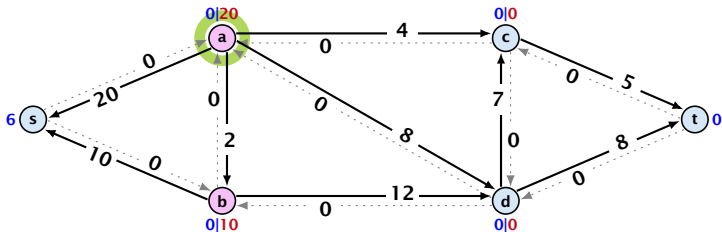
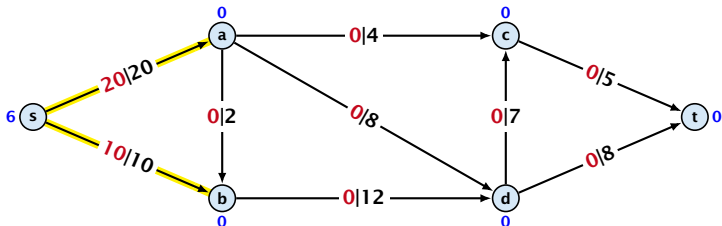
```
1: find initial preflow  $f$ 
2: while there is active node  $u$  do
3:     if there is admiss. arc  $e$  out of  $u$  then
4:          $\text{push}(G, e, f, c)$ 
5:     else
6:          $\text{relabel}(u)$ 
7: return  $f$ 
```

In the following example we always stick to the same active node  $u$  until it becomes inactive but this is not required.

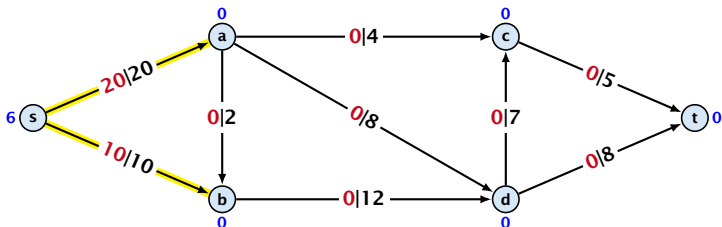
# Preflow Push



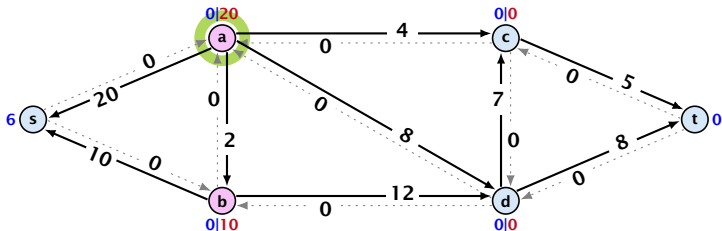
# Preflow Push



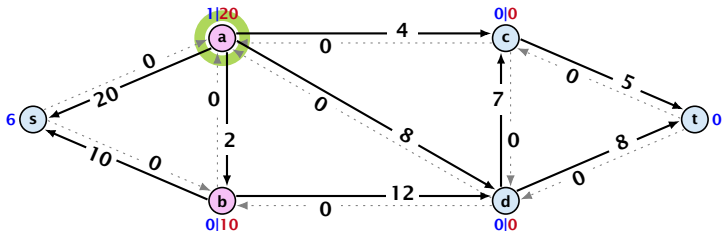
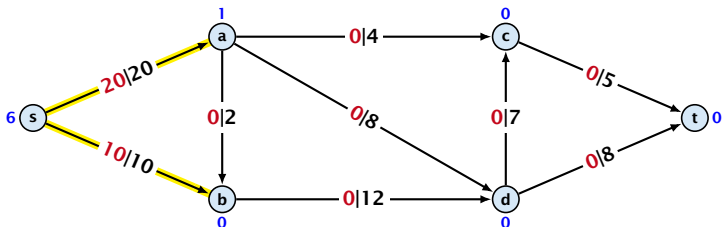
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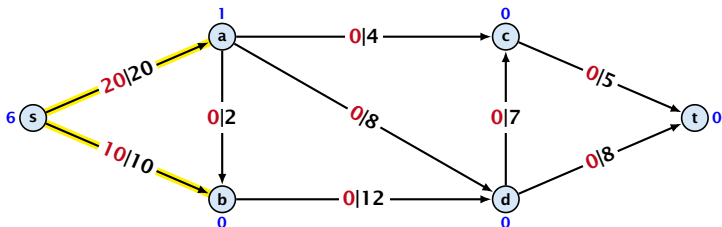
relabel to 1



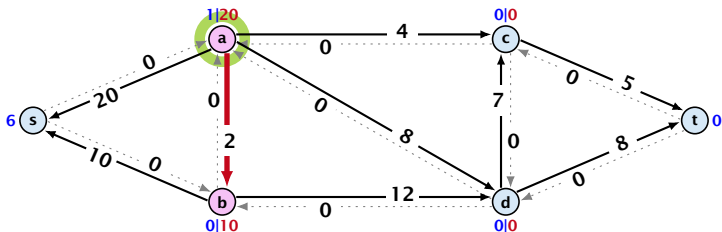
# Preflow Push



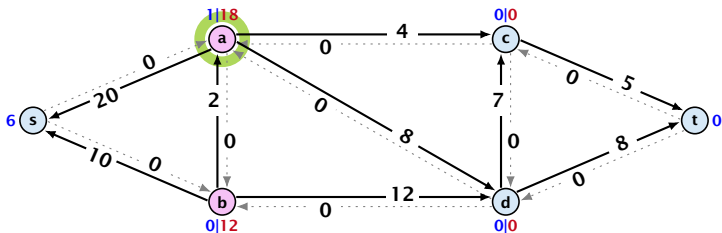
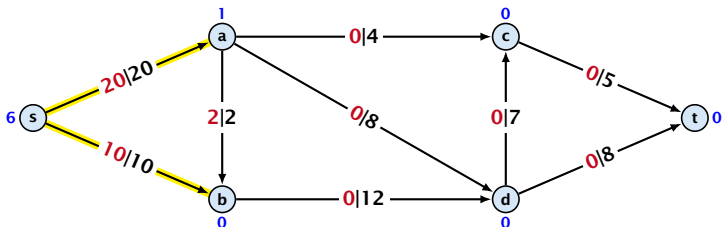
# Preflow Push



saturation push

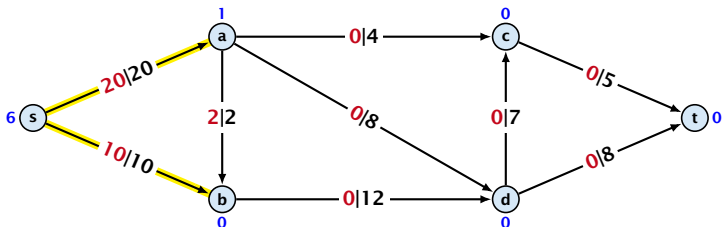


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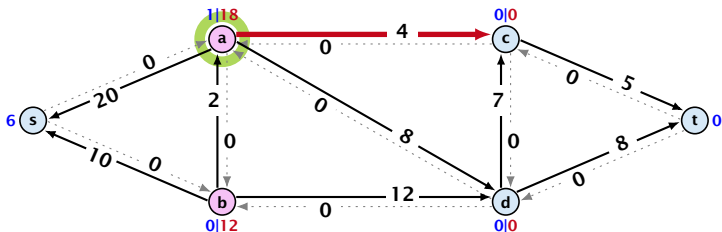




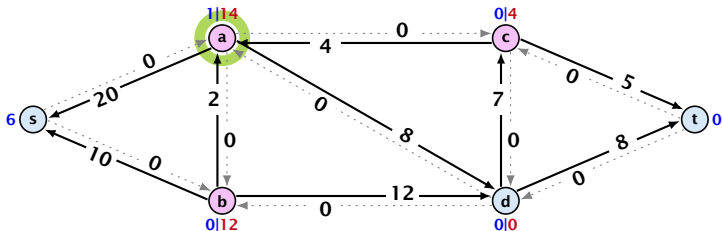
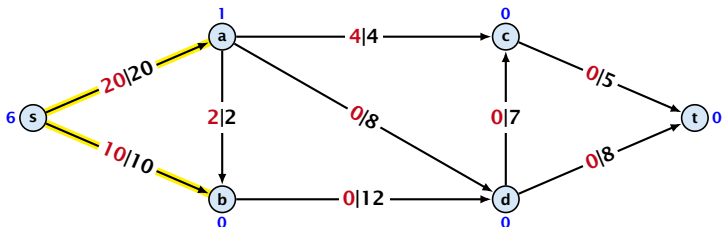
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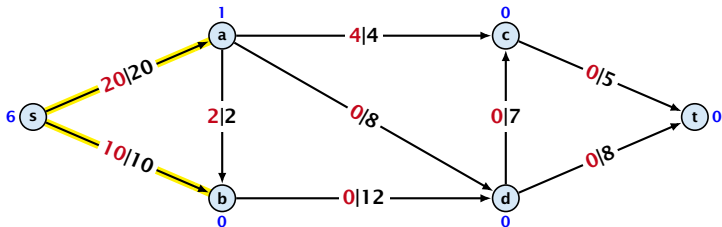
saturation push



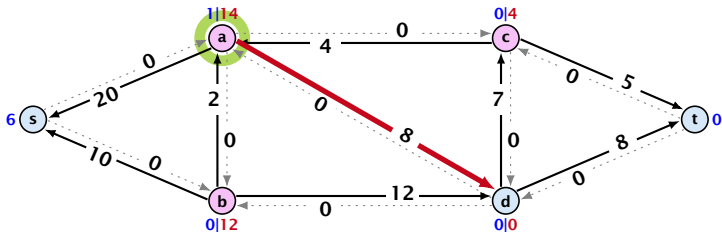
# Preflow Push



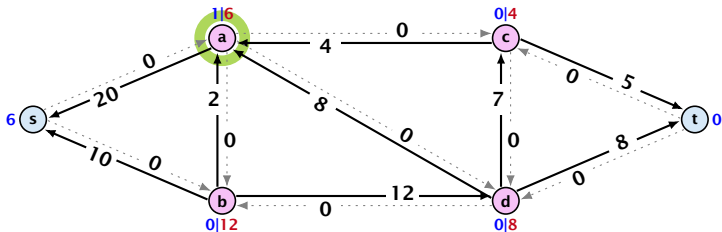
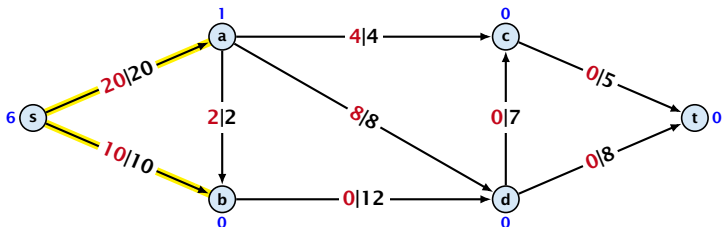
# Preflow Push



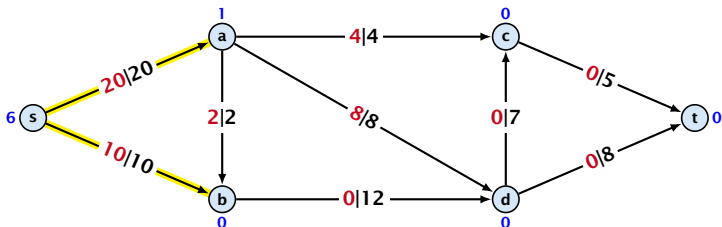
saturation push



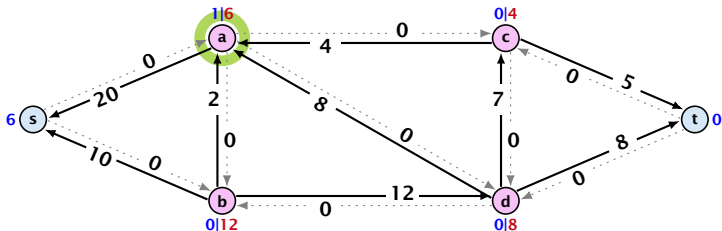
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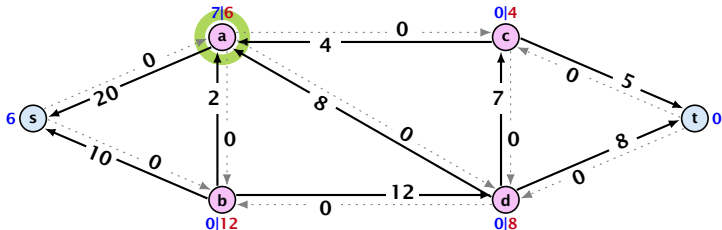
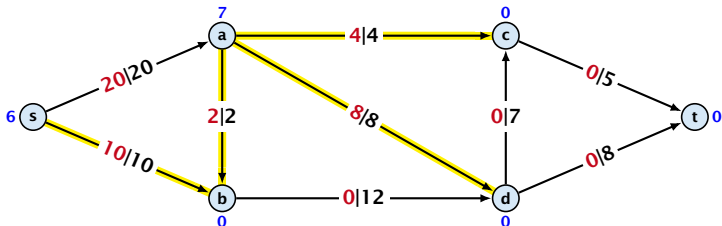
# Preflow Push



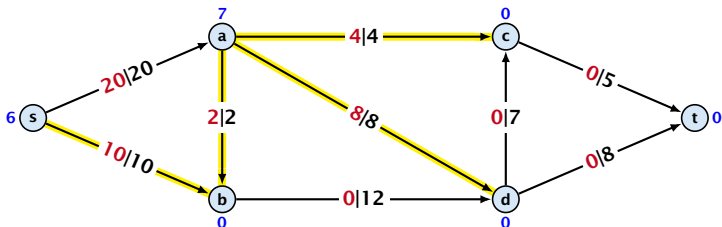
relabel to 7



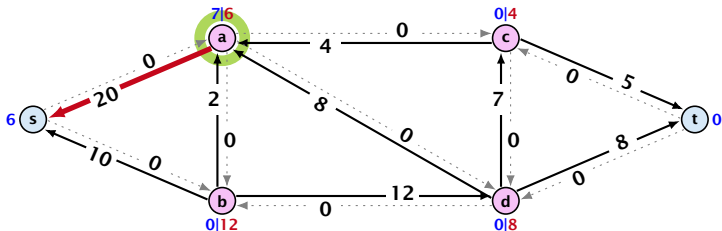
# Preflow Push



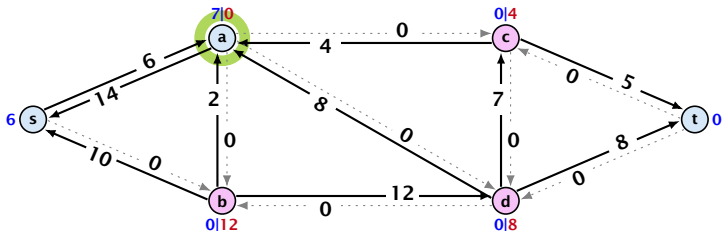
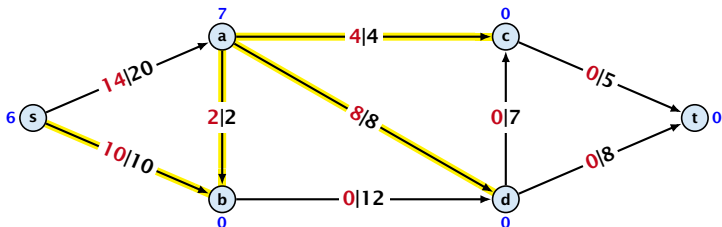
# Preflow Push



deactivating push

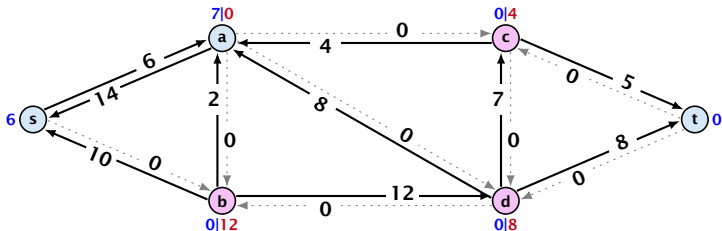
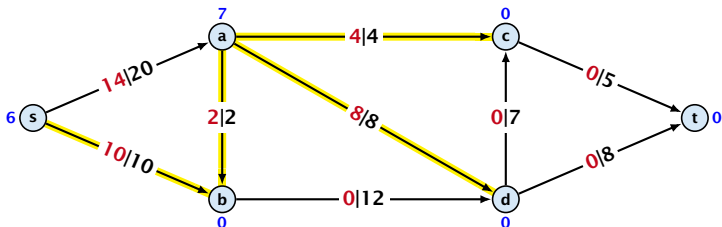


# Preflow Push

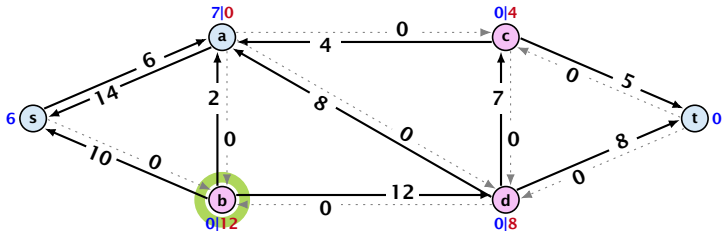
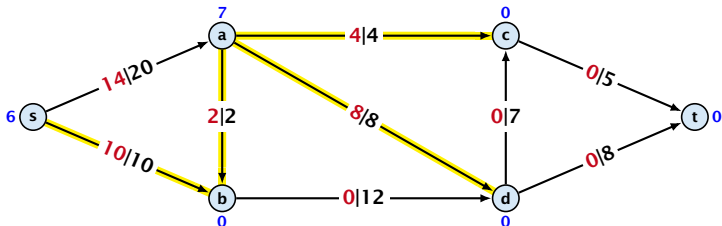




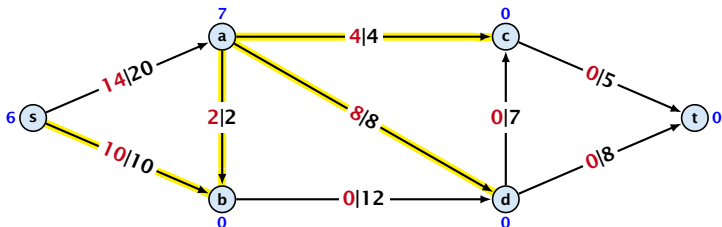
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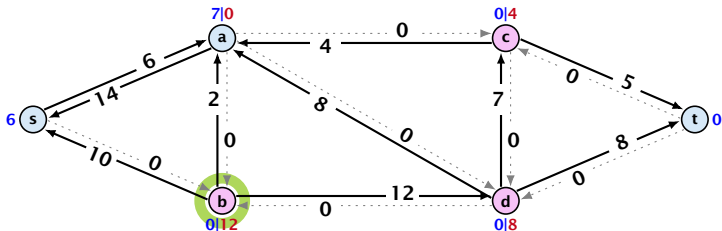
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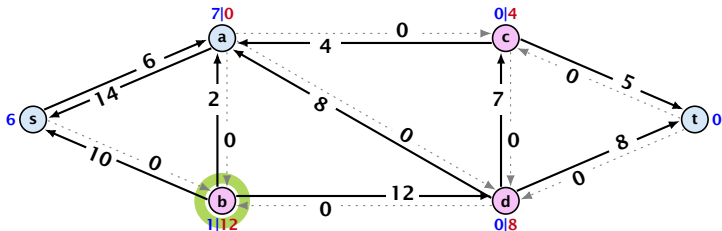
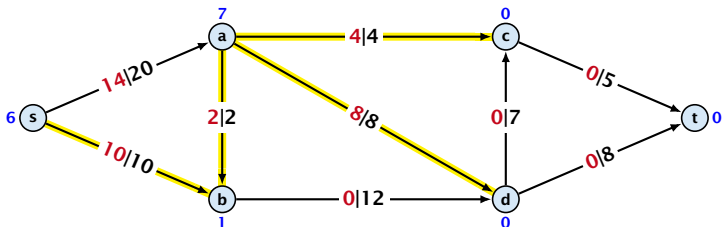
# Preflow Push



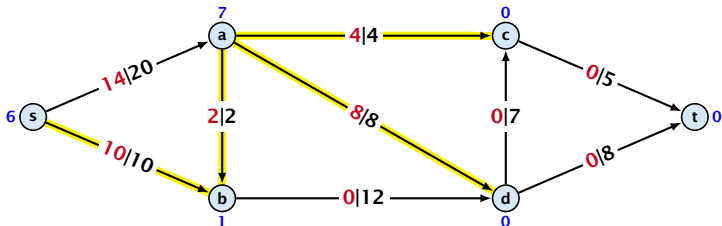
relabel to 1



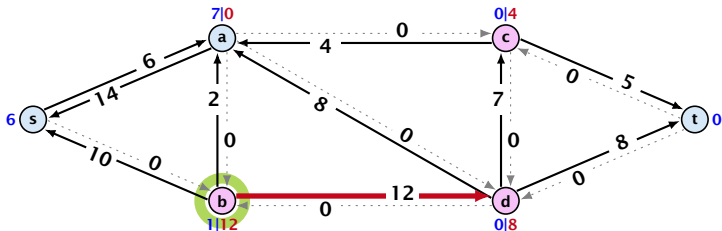
# Preflow Push



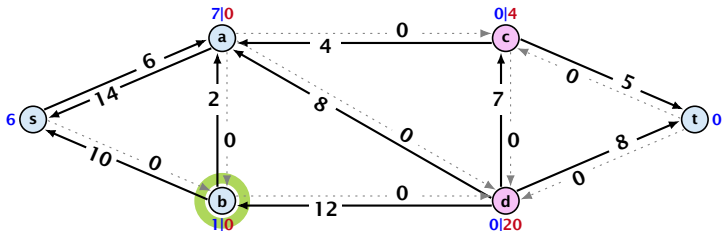
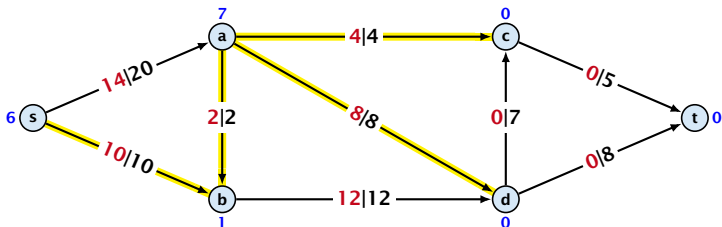
# Preflow Push



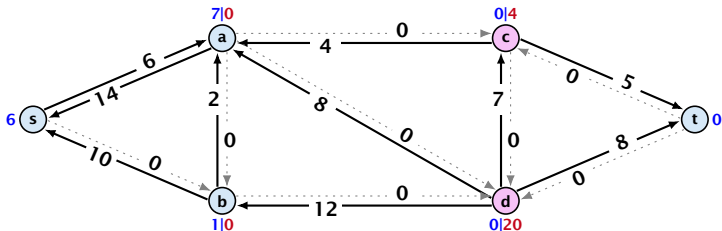
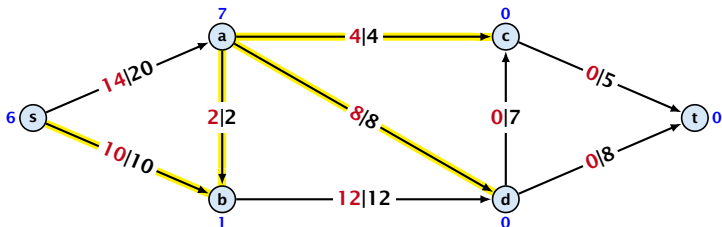
satürating and deactivating push



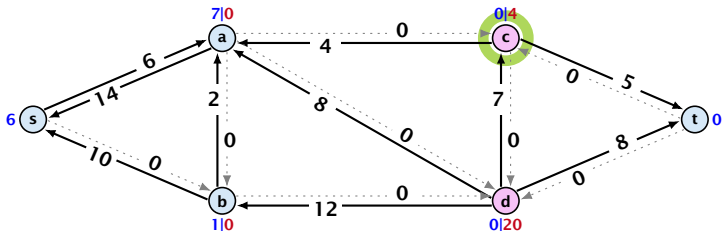
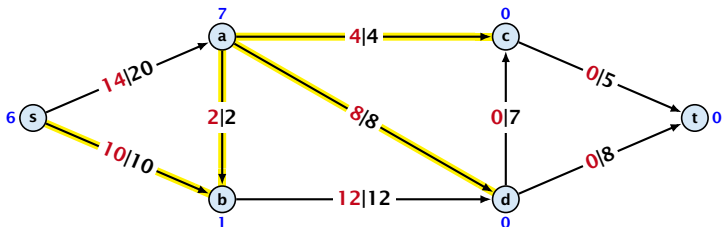
# Preflow Push



# Preflow Push

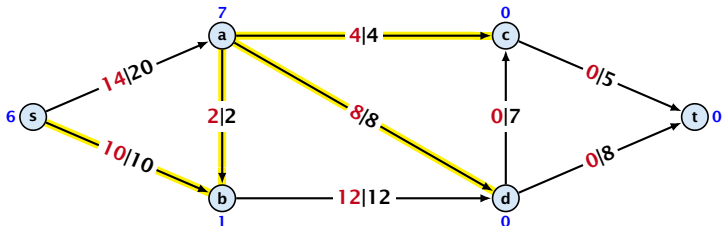


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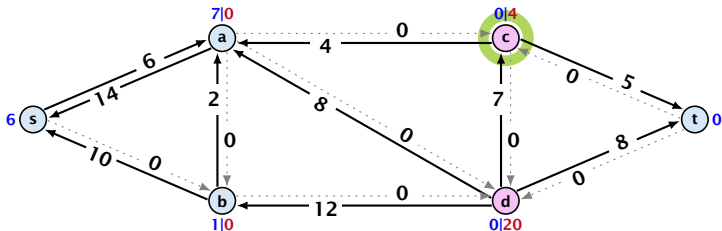




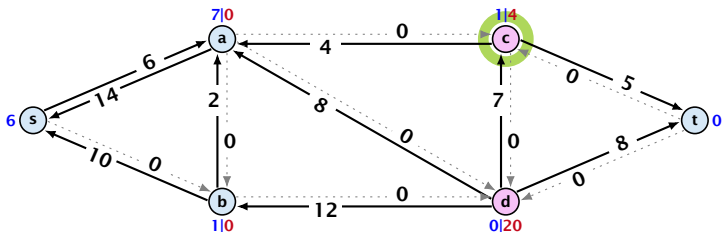
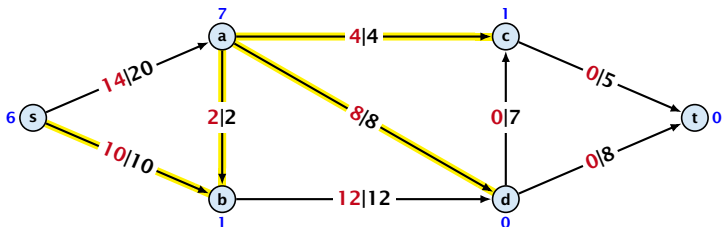
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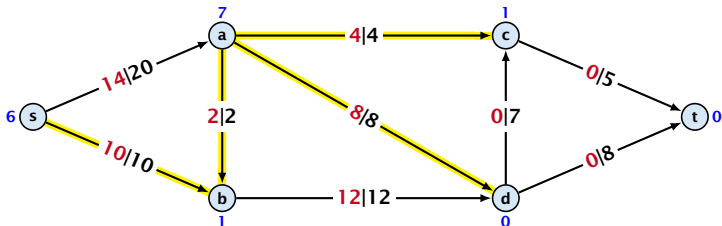
relabel to 1



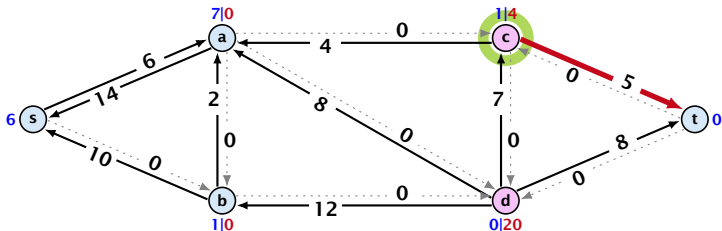
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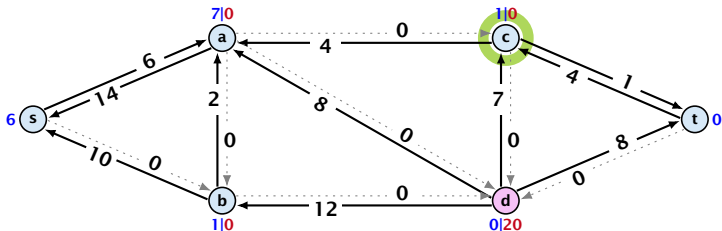
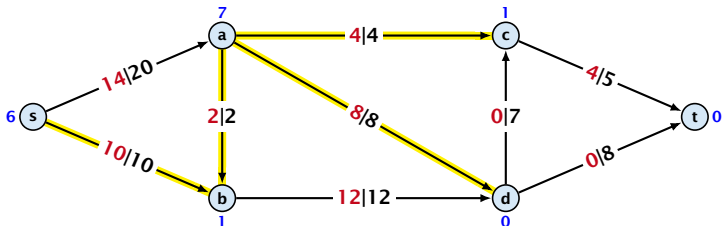
# Preflow Push



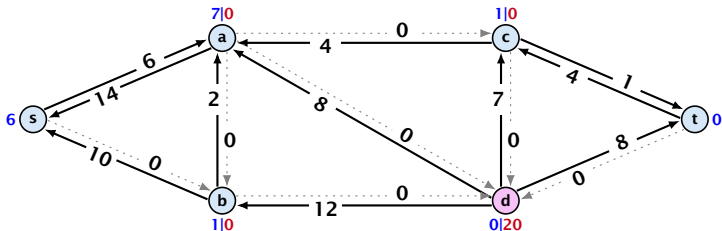
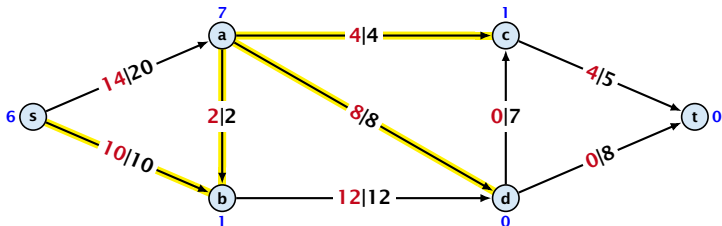
deactivating push



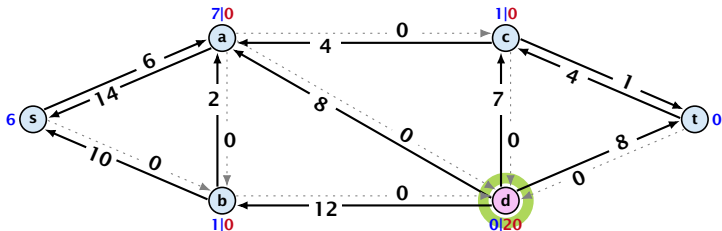
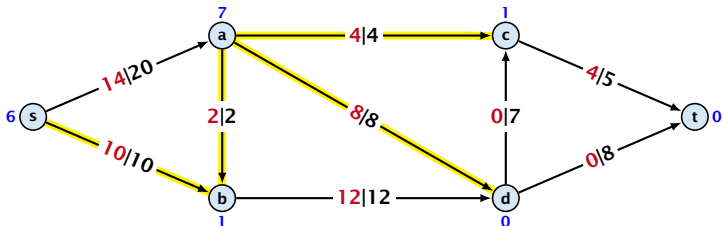
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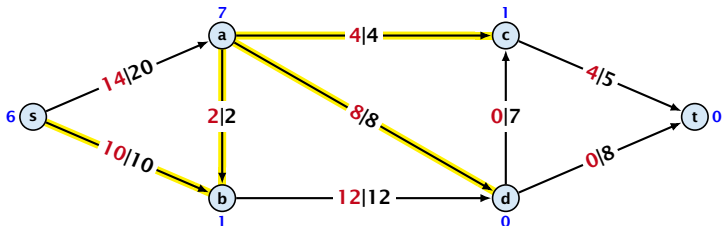
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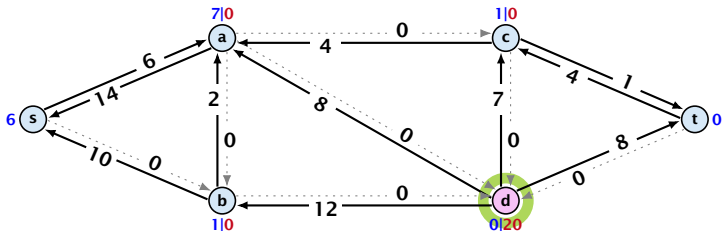
# Preflow Push



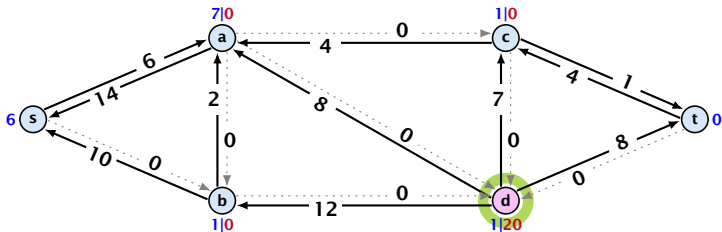
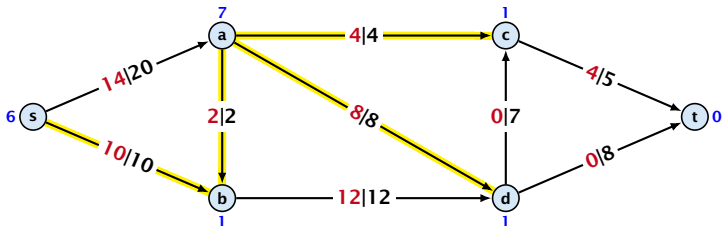
# Preflow Push



relabel to 1

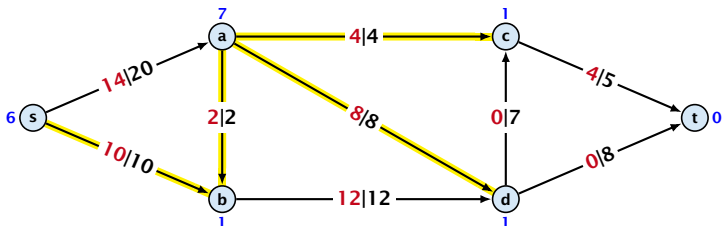


# Preflow Push

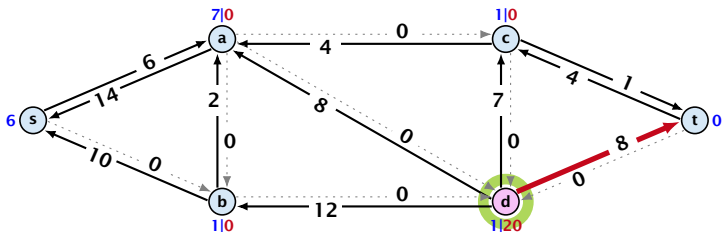




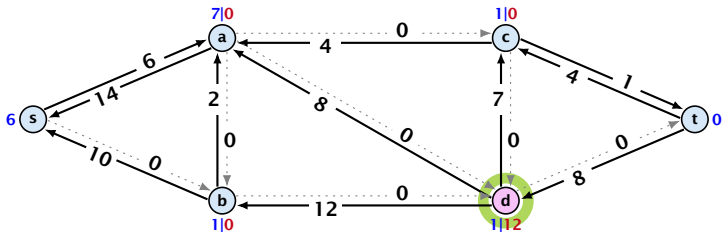
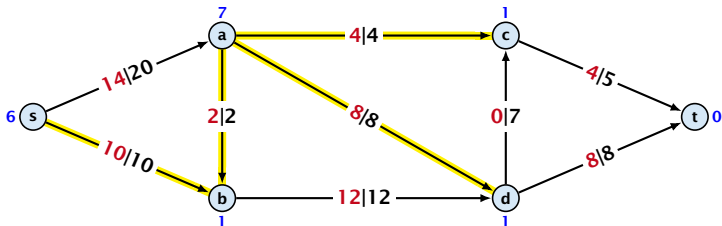
# Preflow Push



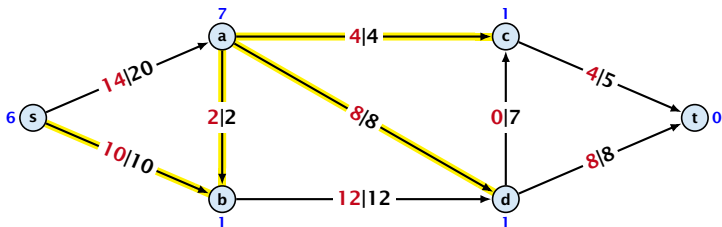
saturating push



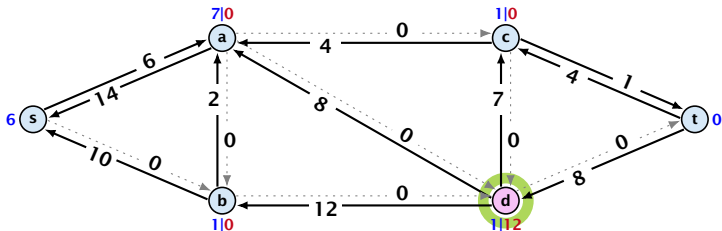
# Preflow Push



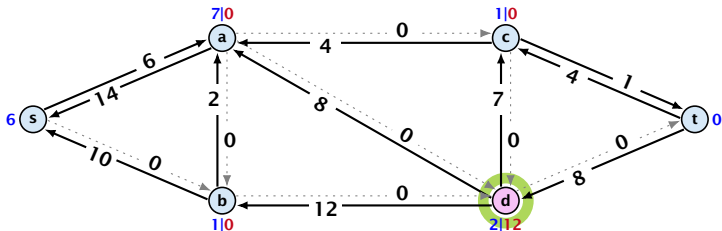
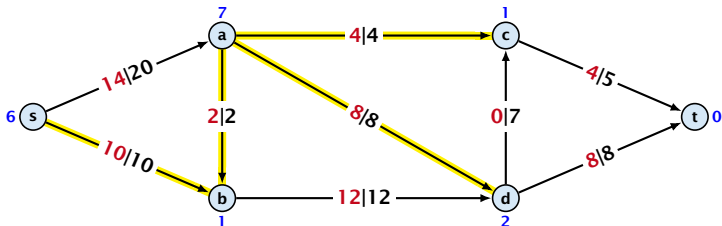
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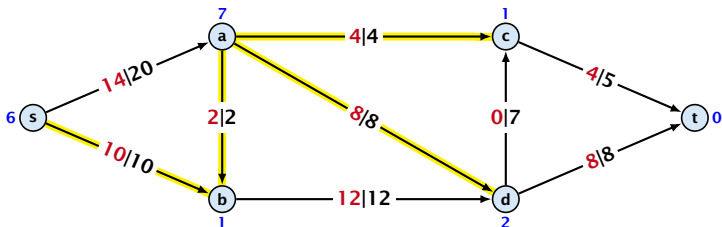
relabel to 2



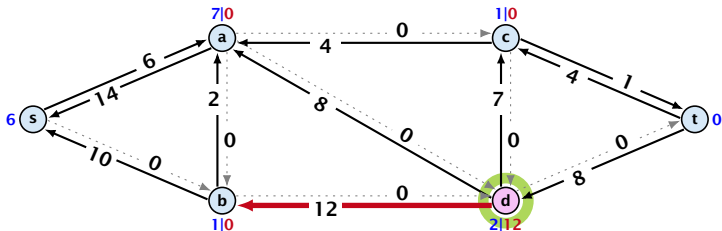
# Preflow Push



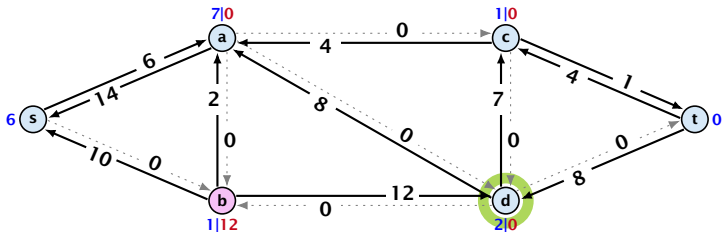
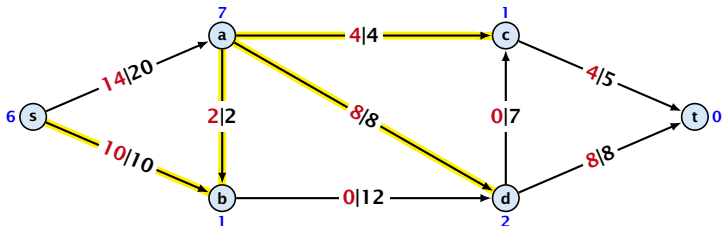
# Preflow Push



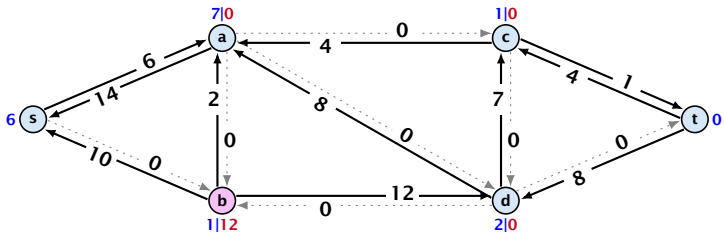
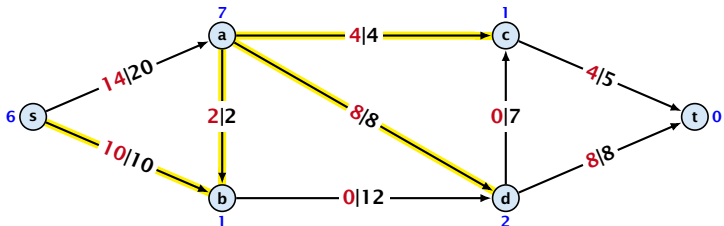
satürating and deactivating push



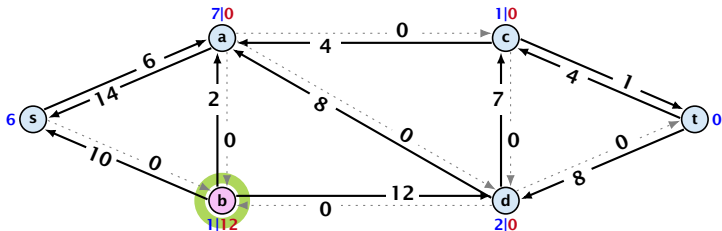
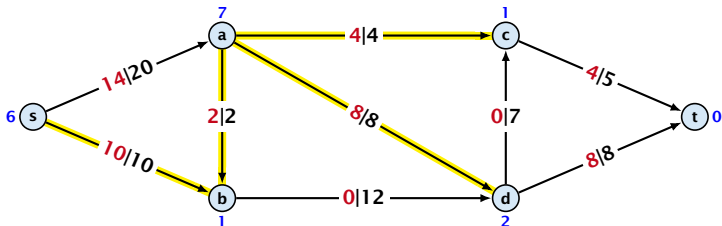
# Preflow Push



# Preflow Push

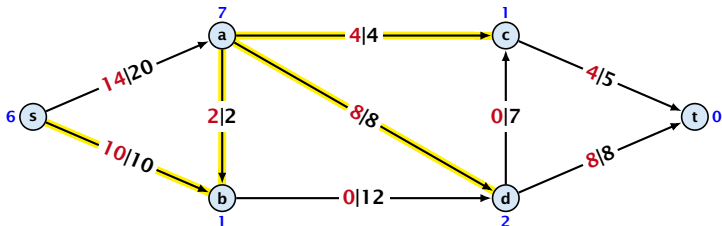


# Preflow Push

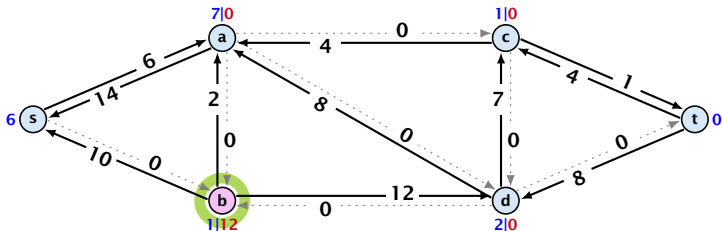




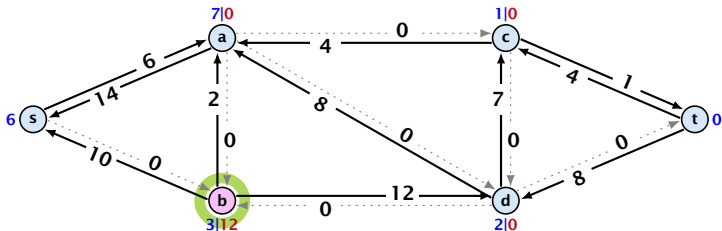
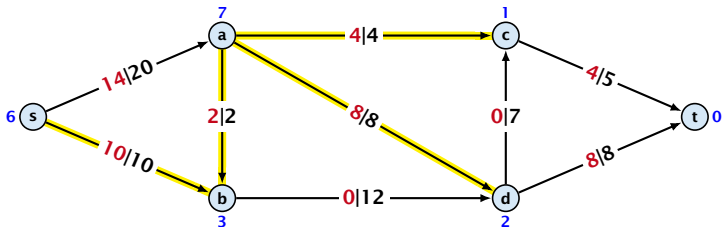
# Preflow Push



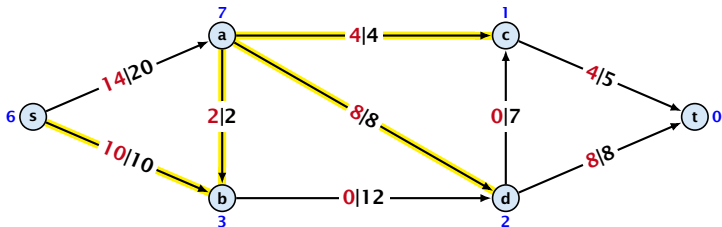
relabel to 3



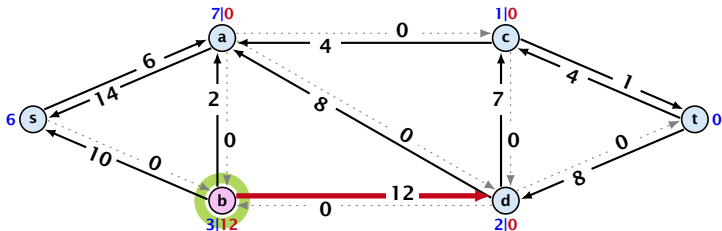
# Preflow Push



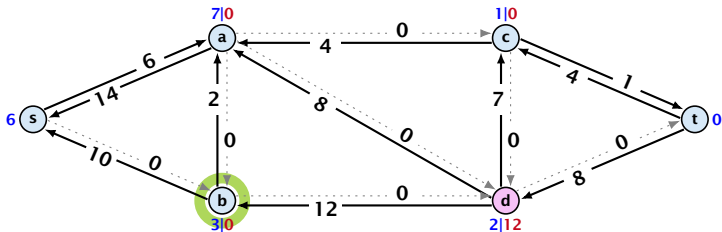
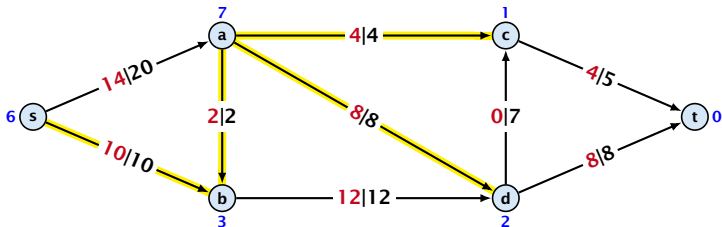
# Preflow Push



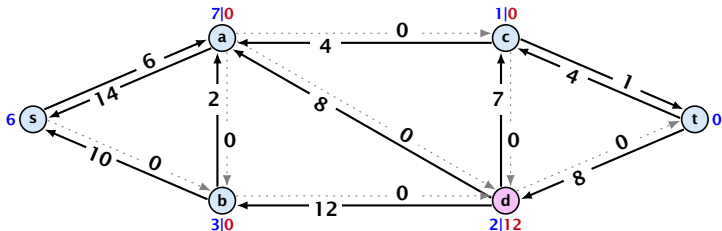
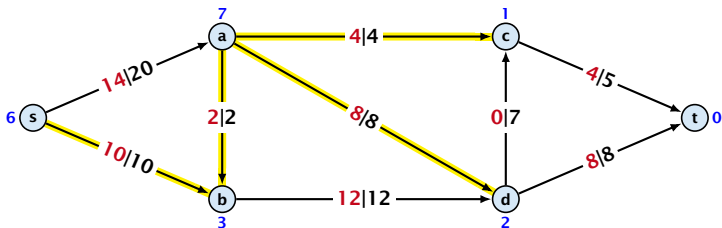
satürating and deactivating push



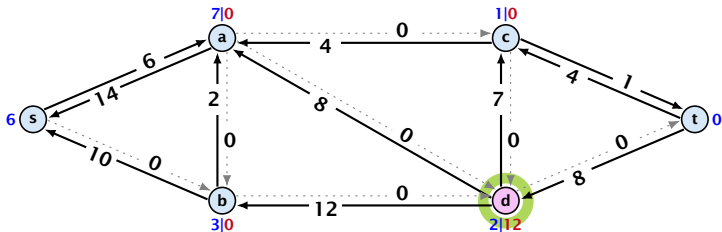
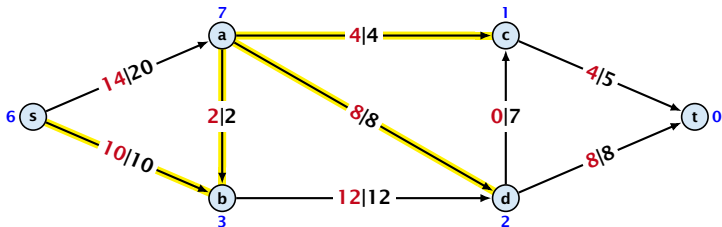
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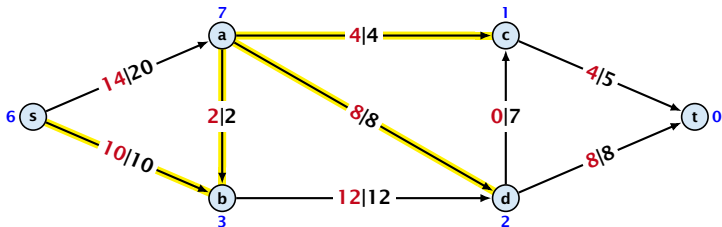
# Preflow Push



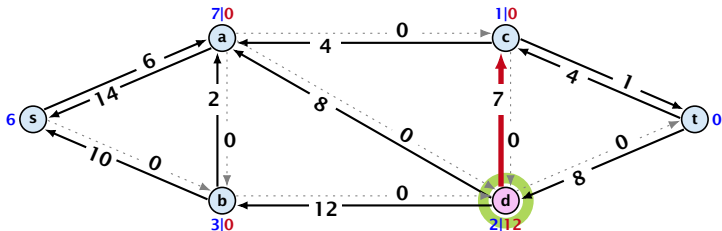
# Preflow Push



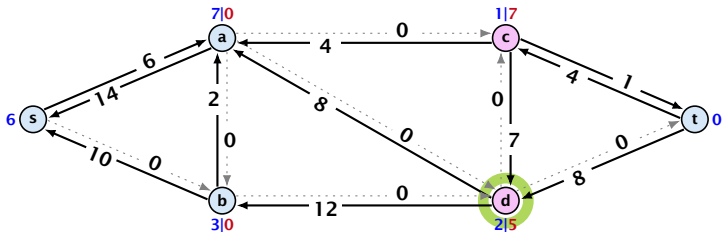
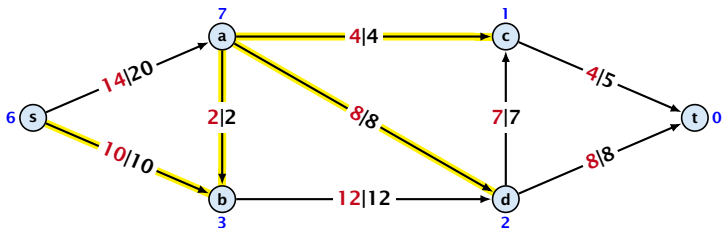
# Preflow Push



satürating push

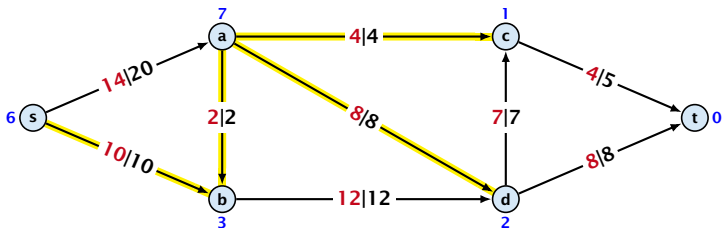


# Preflow Push

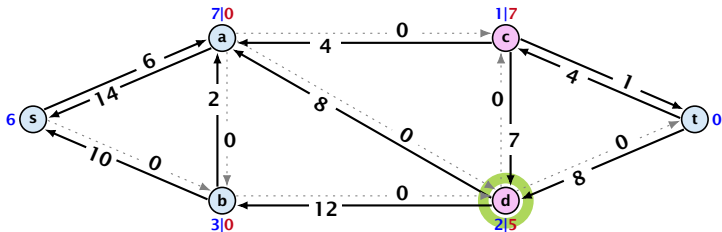




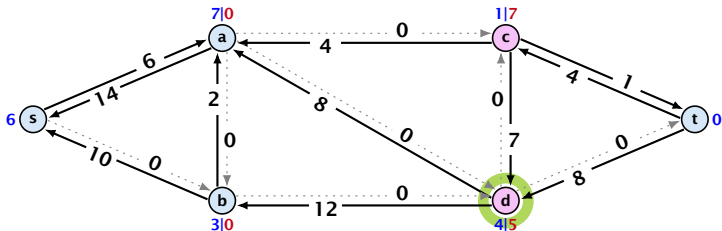
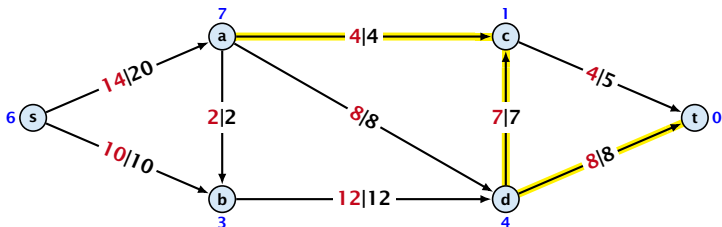
# Preflow Push



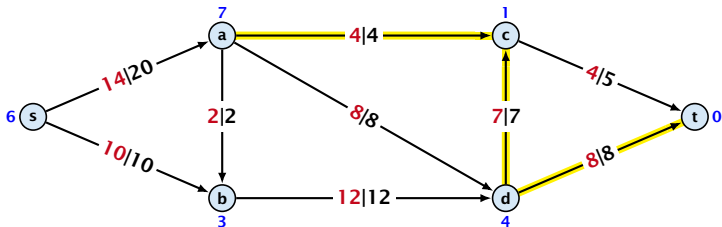
relabel to 4



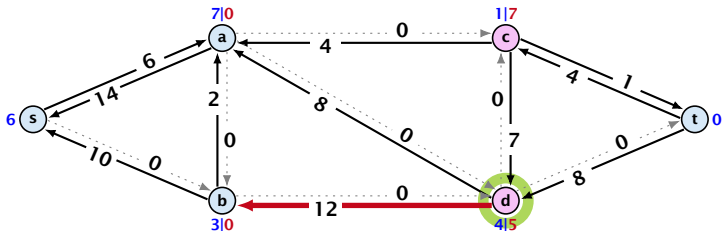
# Preflow Push



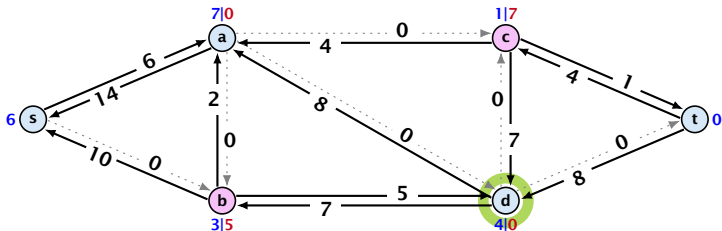
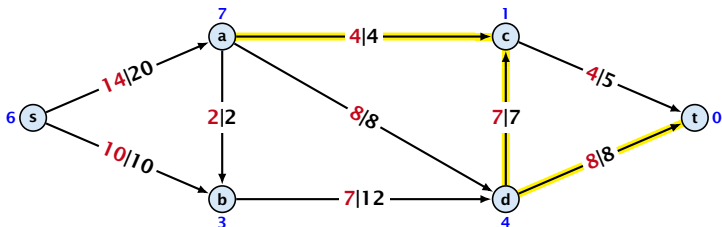
# Preflow Push



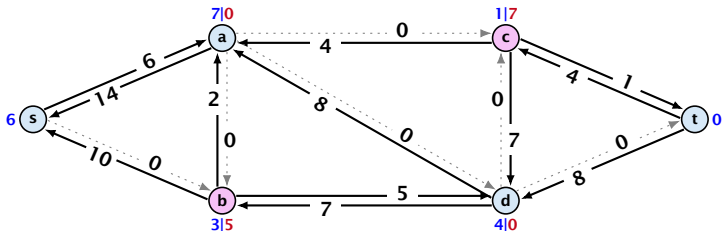
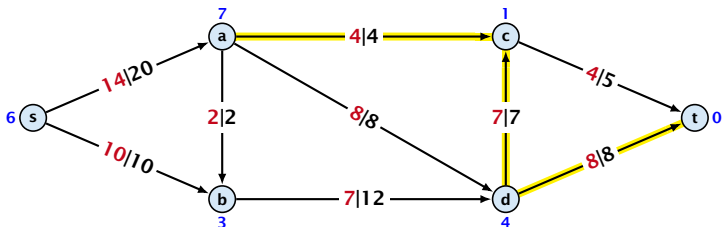
deactivating push



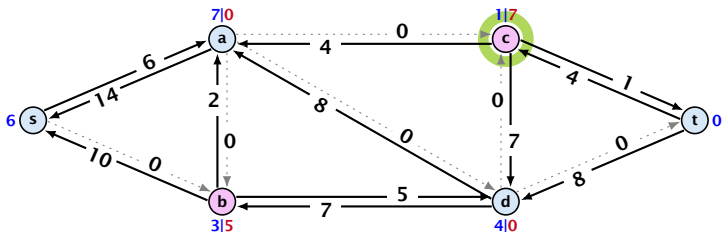
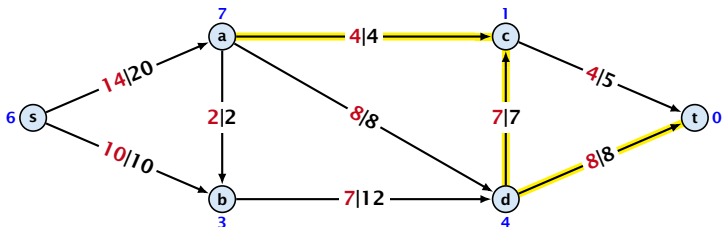
# Preflow Push



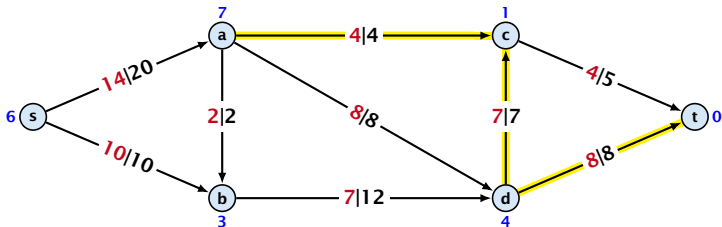
# Preflow Push



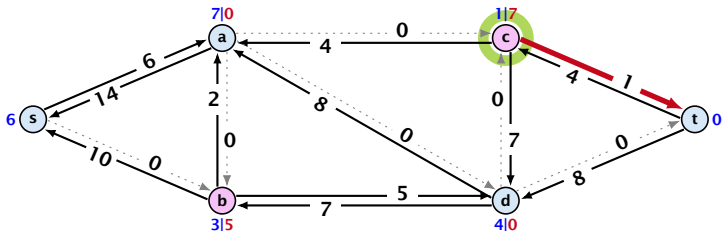
# Preflow Push



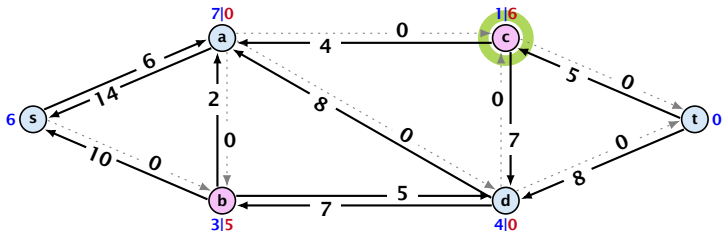
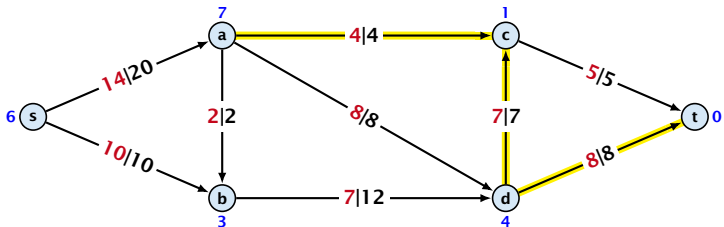
# Preflow Push



saturation push

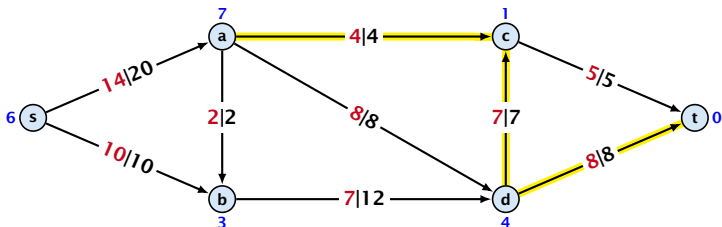


# Preflow Push

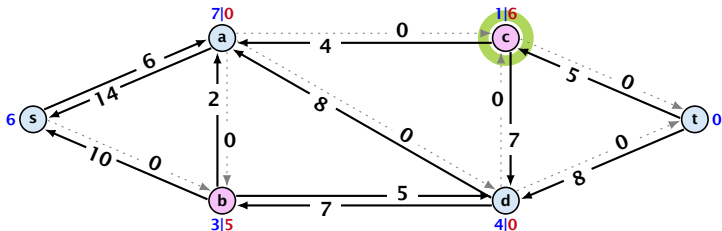




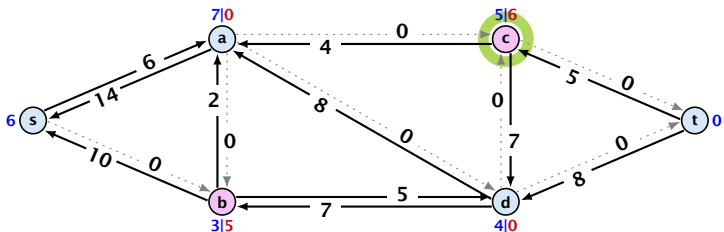
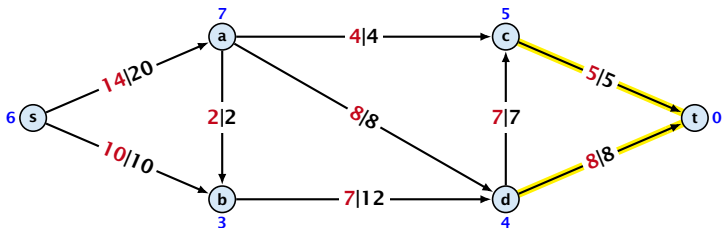
# Preflow Push



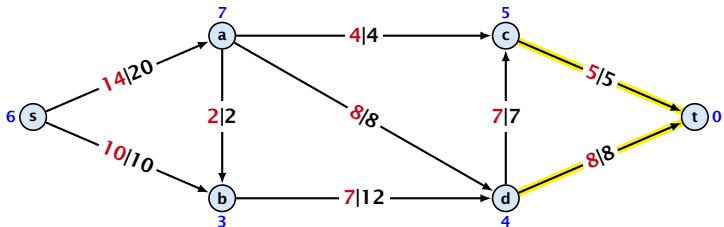
relabel to 5



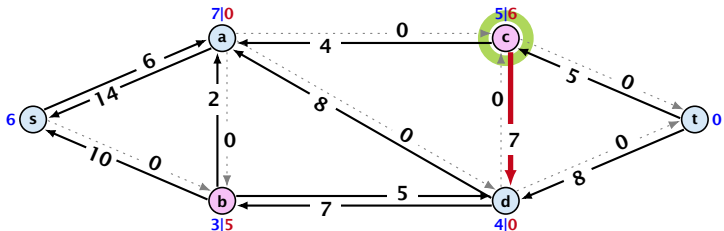
# Preflow Push



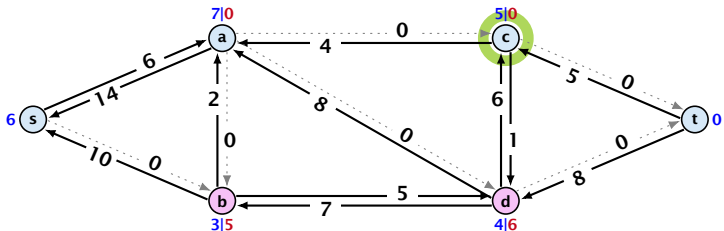
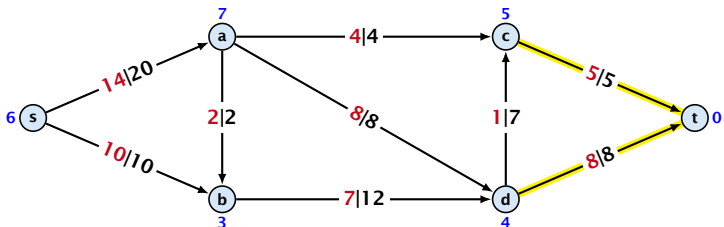
# Preflow Push



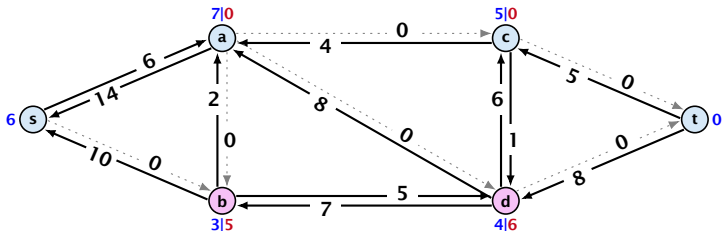
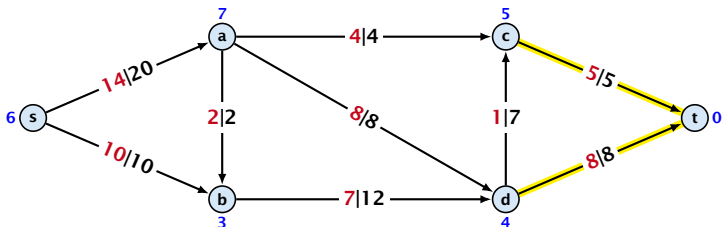
deactivating push



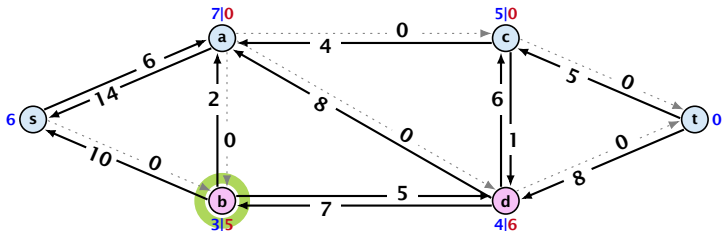
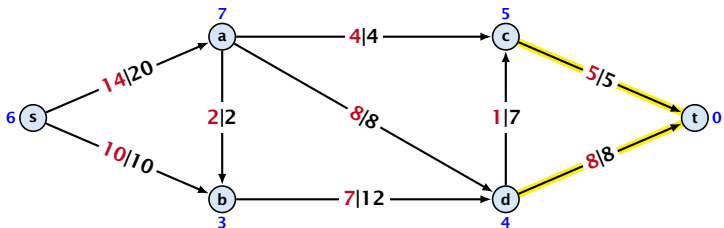
# Preflow Push



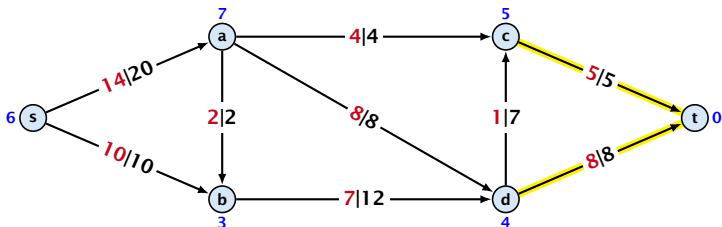
# Preflow Push



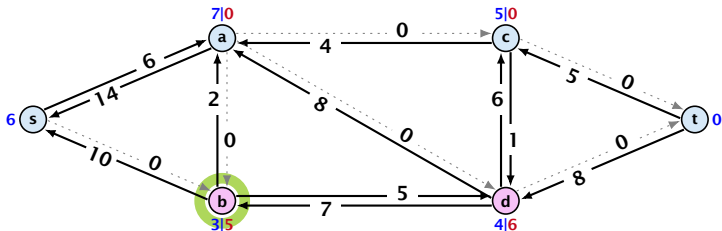
# Preflow Push



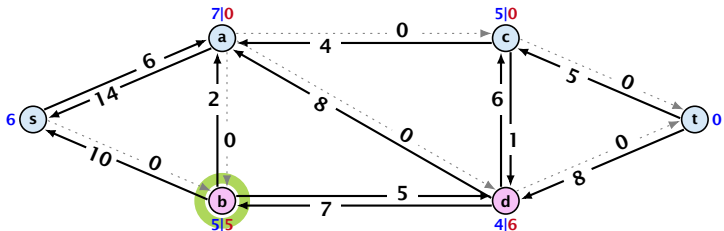
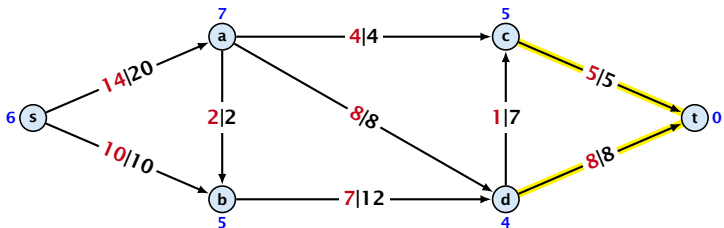
# Preflow Push



relabel to 5

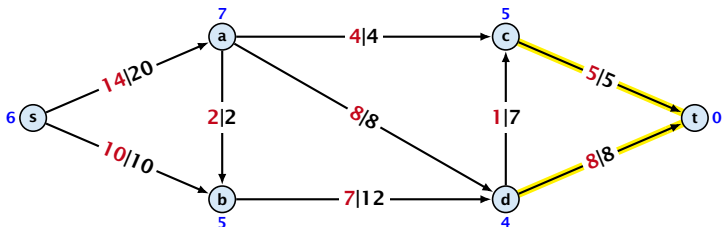


# Preflow Push

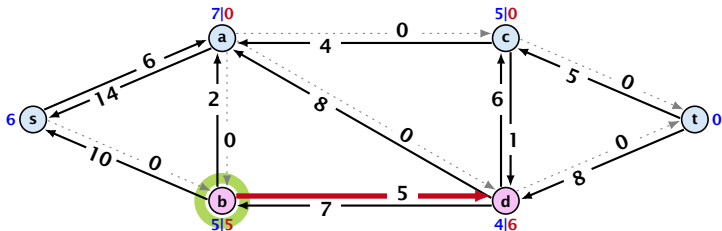




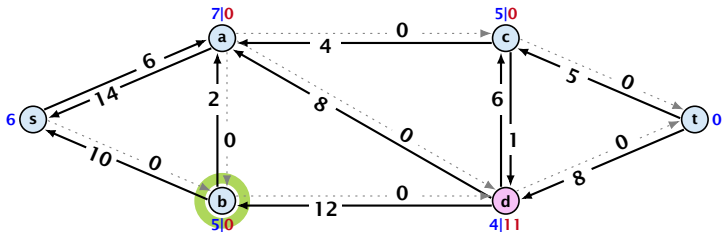
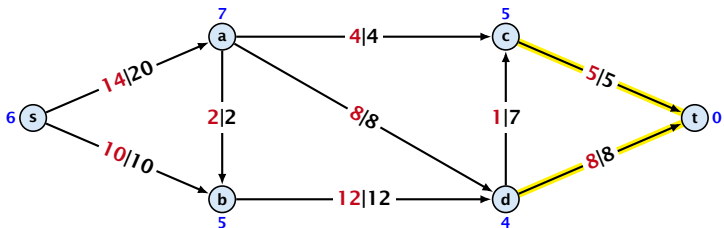
# Preflow Push



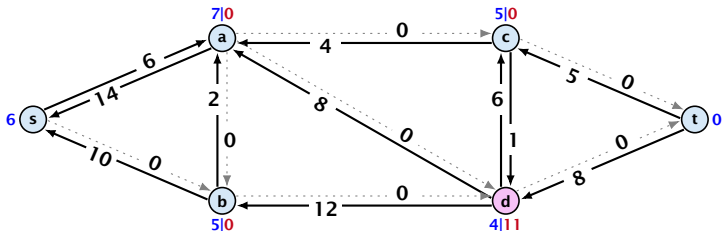
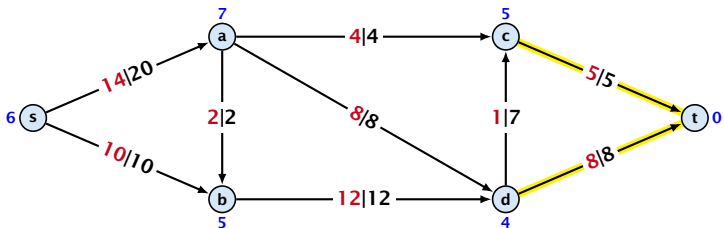
satürating and deactivating push



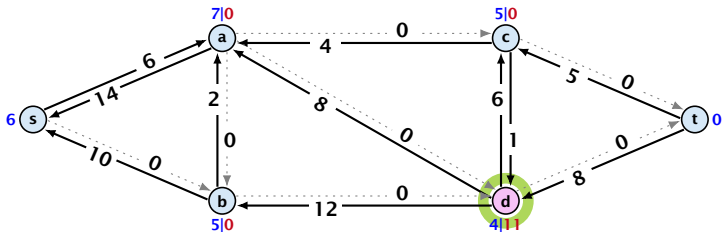
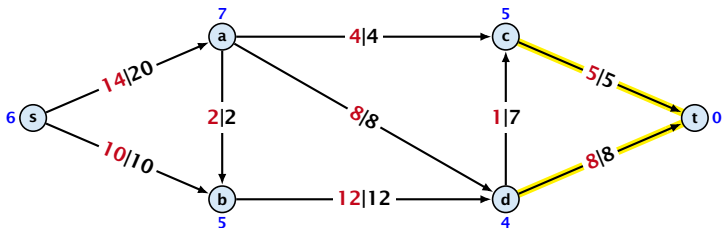
# Preflow Push



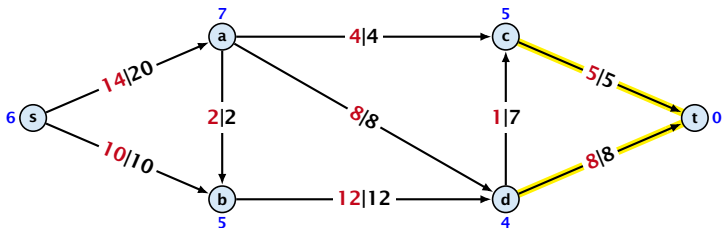
# Preflow Push



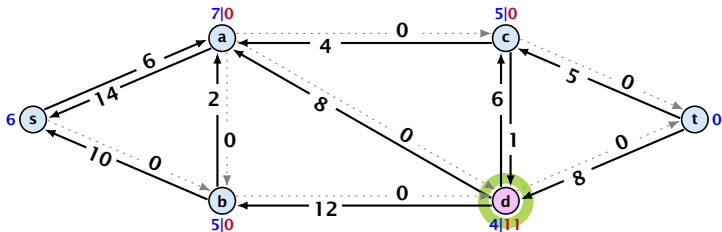
# Preflow Push



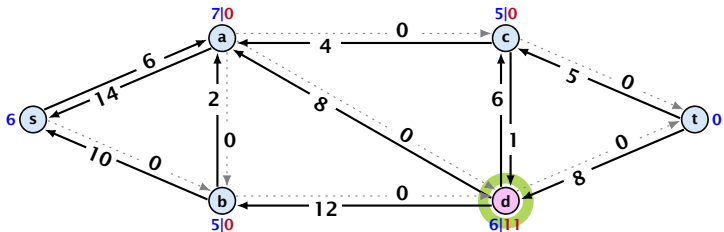
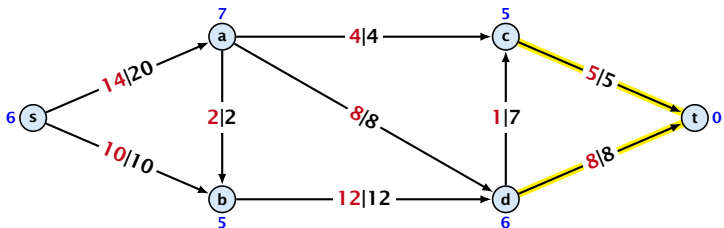
# Preflow Push



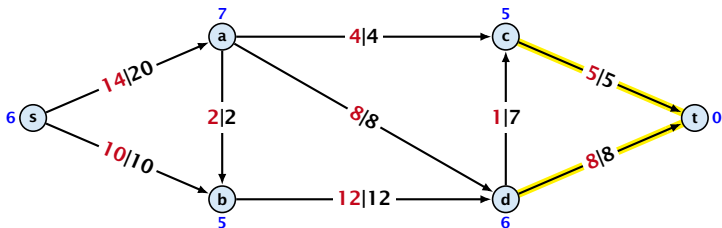
relabel to 6



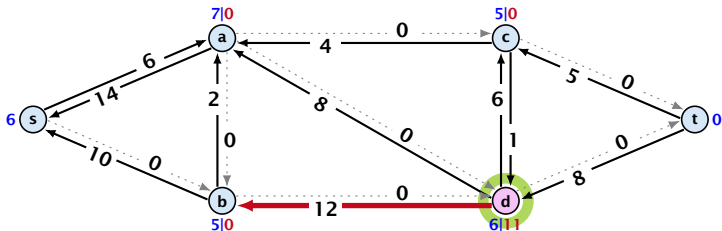
# Preflow Push



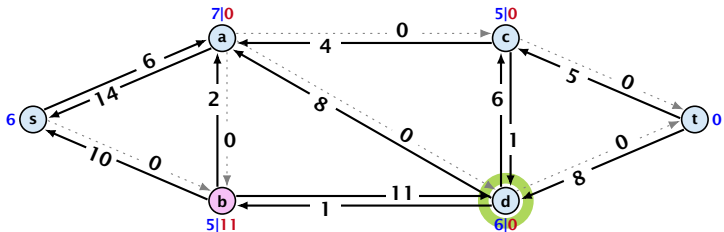
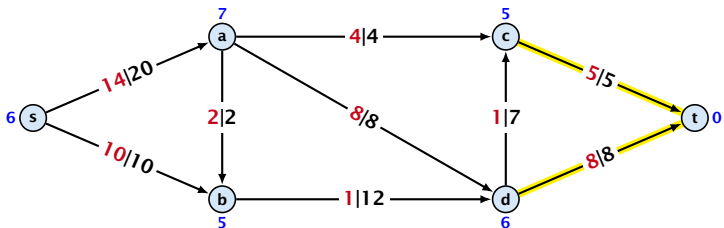
# Preflow Push



deactivating push

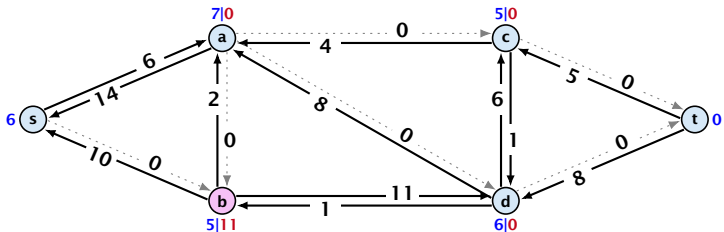
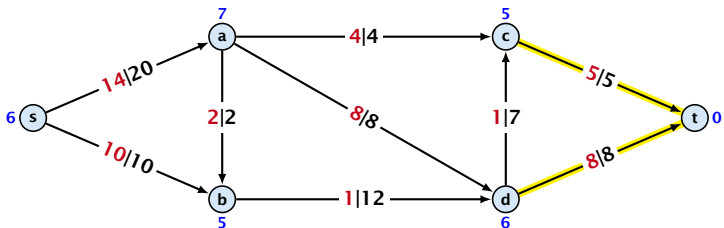


# Preflow Push

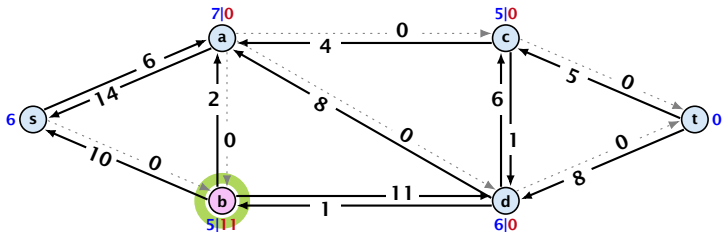
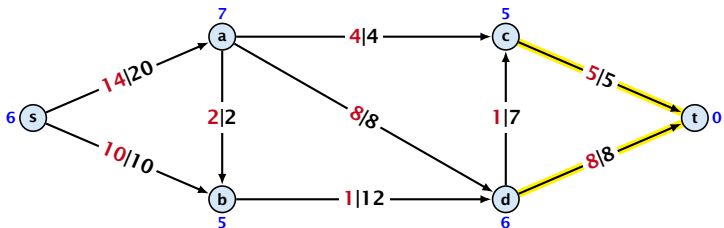




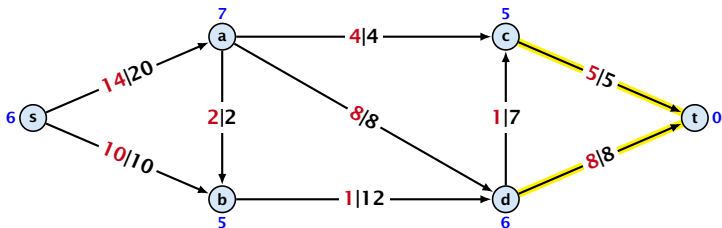
# Preflow Push



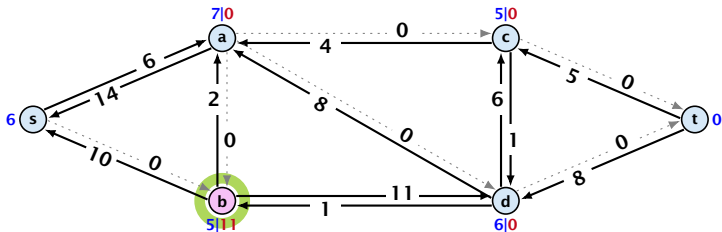
# Preflow Push



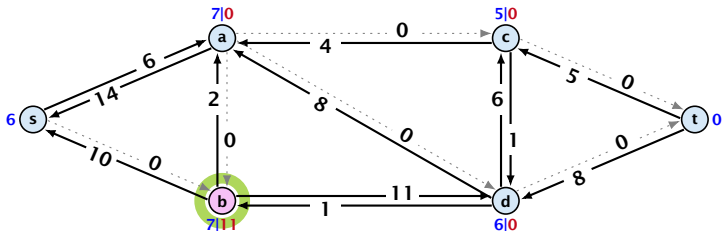
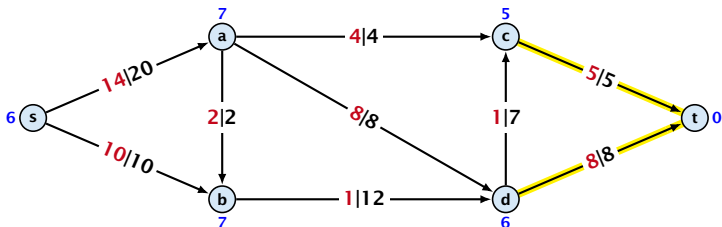
# Preflow Push



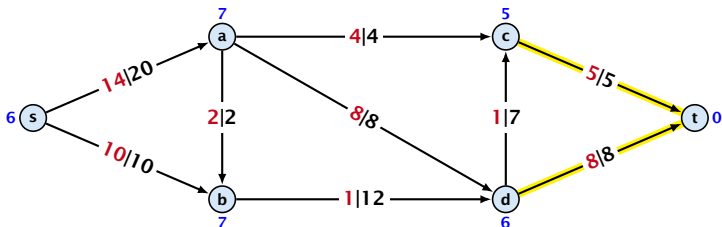
relabel to 7



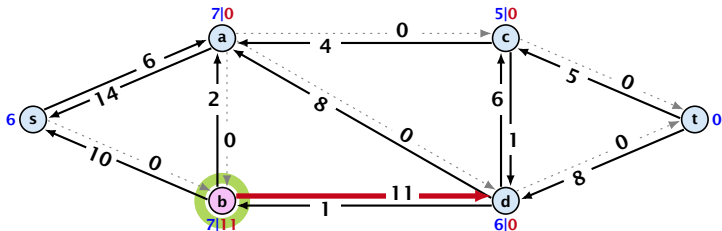
# Preflow Push



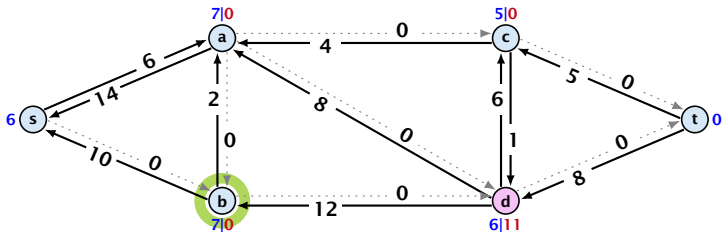
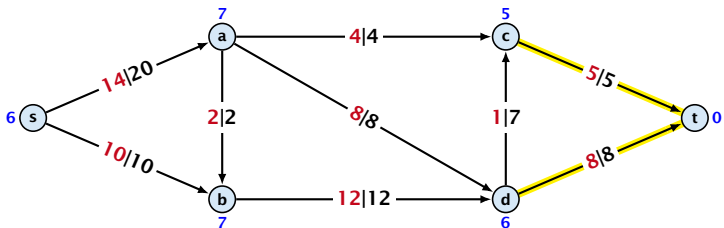
# Preflow Push



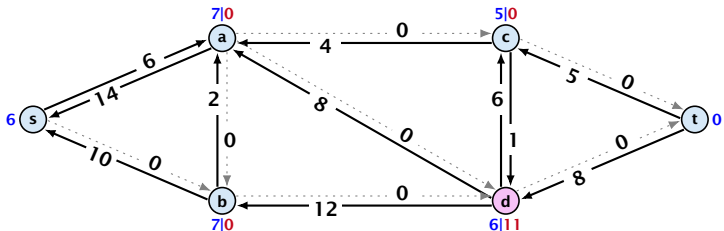
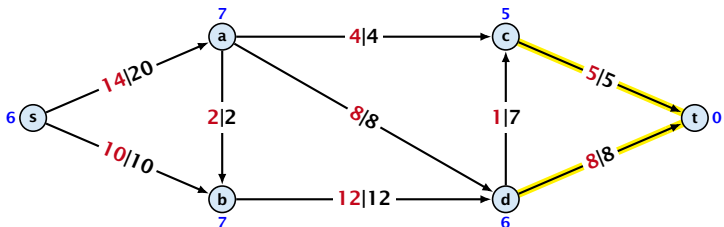
saturation and deactivating push



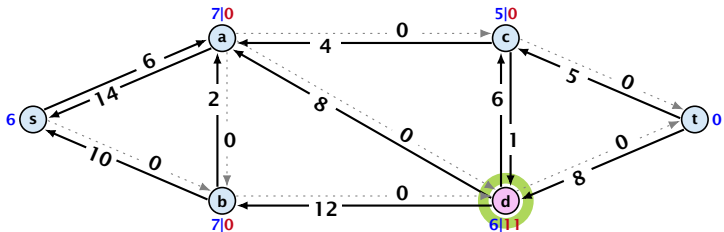
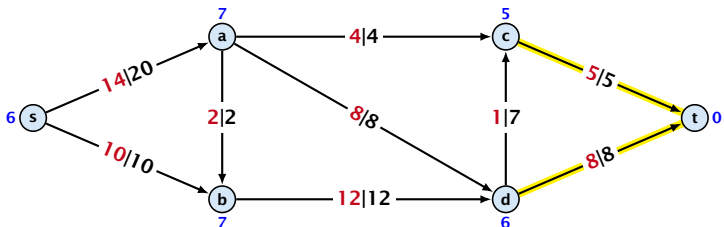
# Preflow Push



# Preflow Push

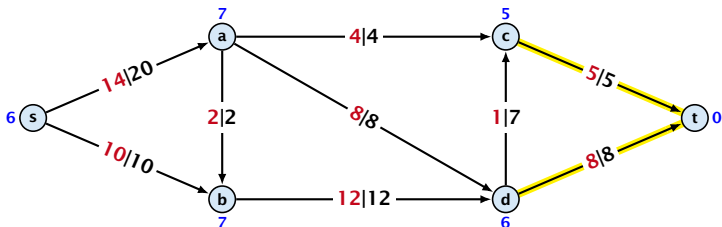


# Preflow Push

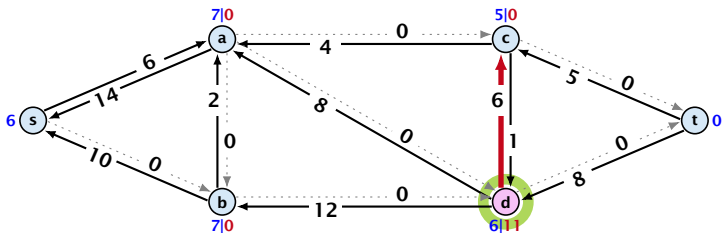




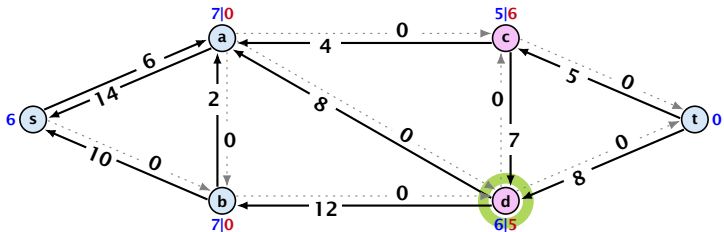
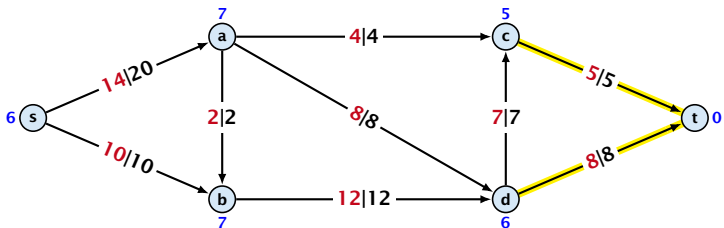
# Preflow Push



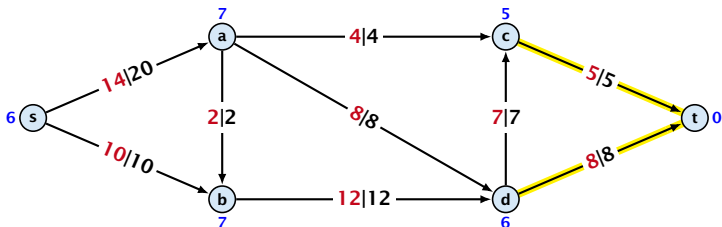
saturating push



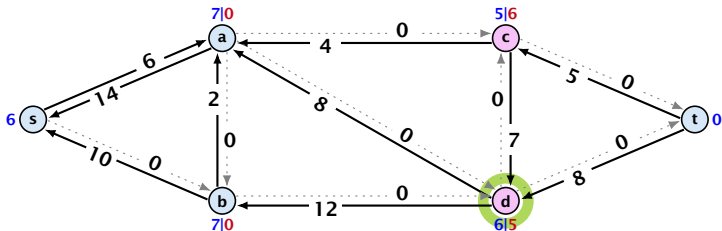
# Preflow Push



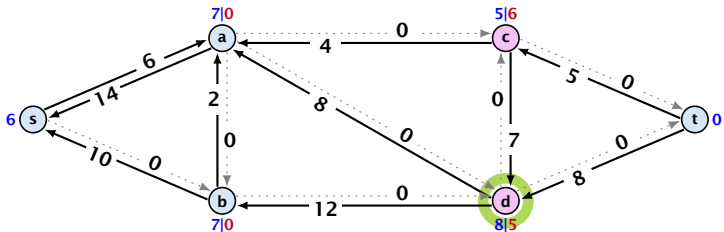
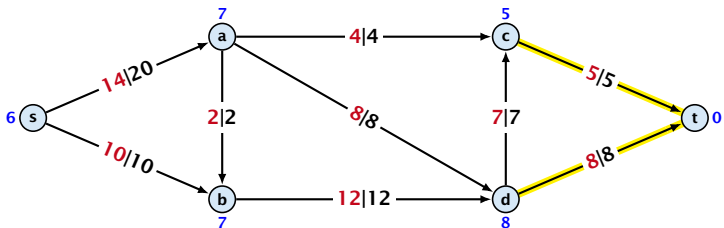
# Preflow Push



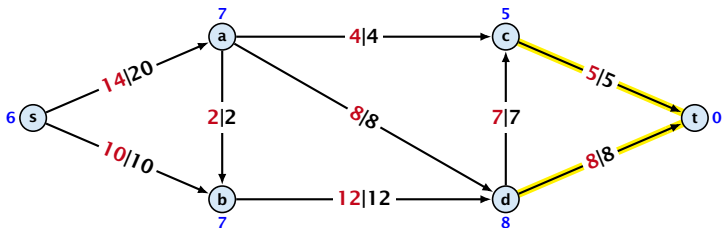
relabel to 8



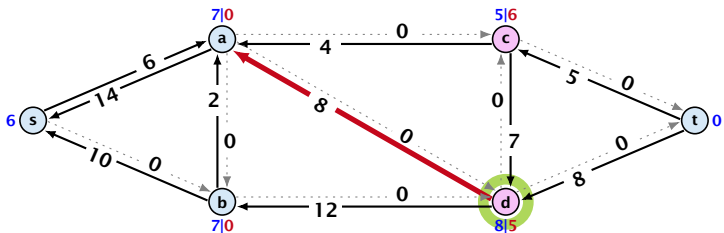
# Preflow Push



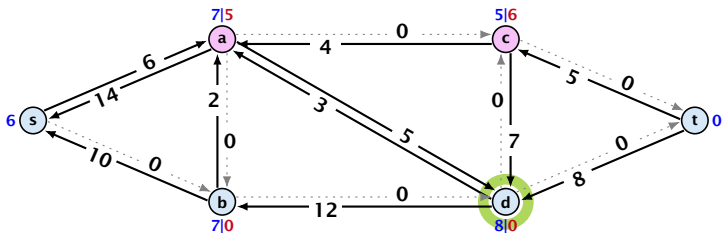
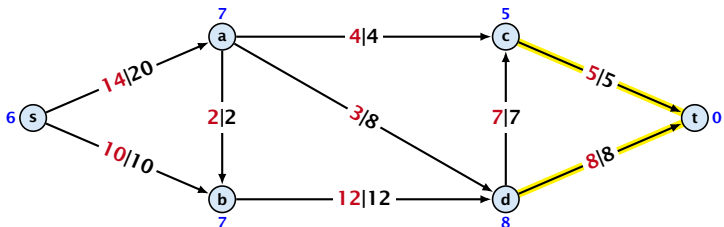
# Preflow Push



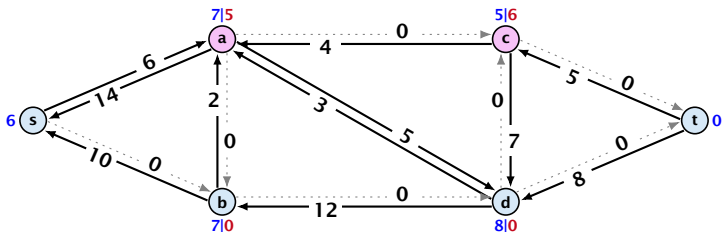
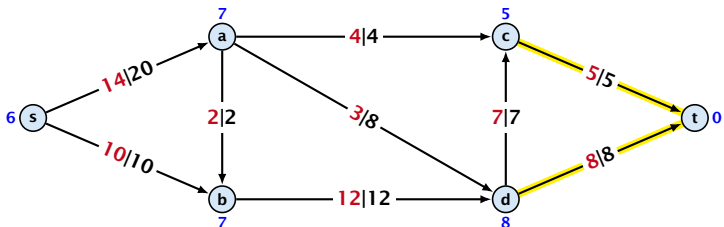
deactivating push



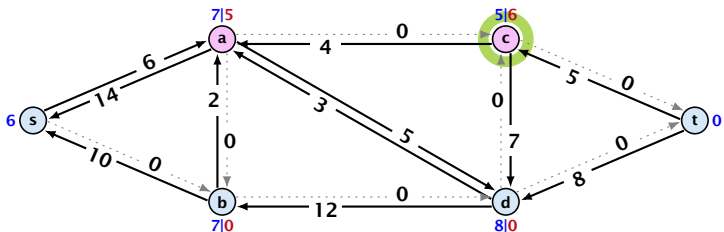
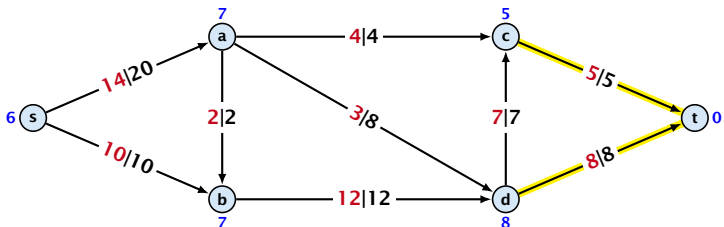
# Preflow Push



# Preflow Push

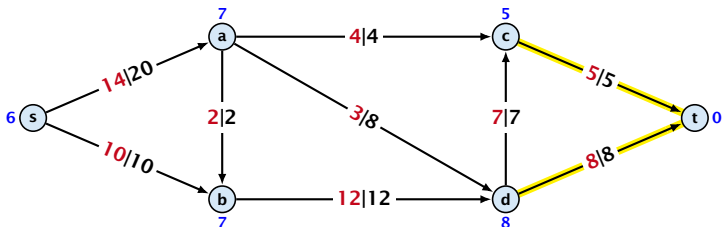


# Preflow Push

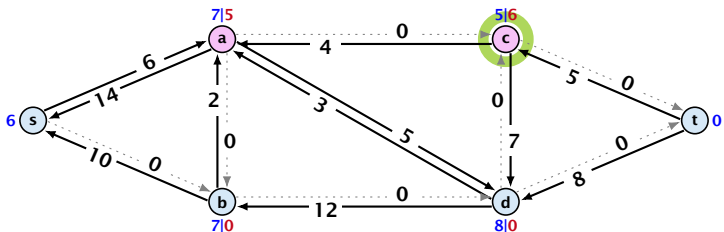




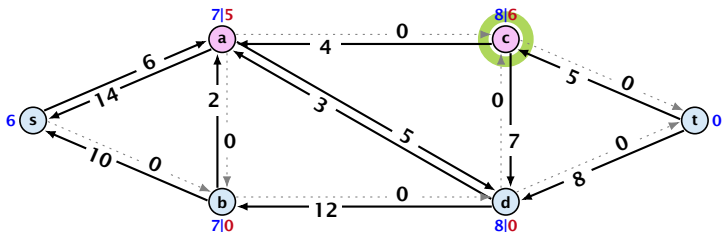
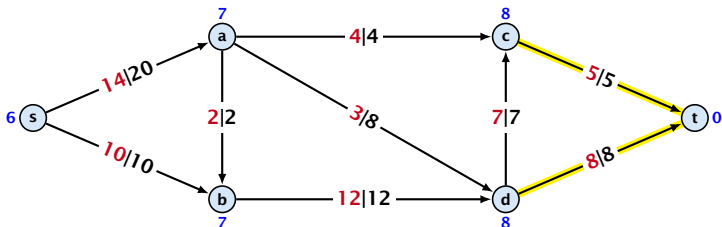
# Preflow Push



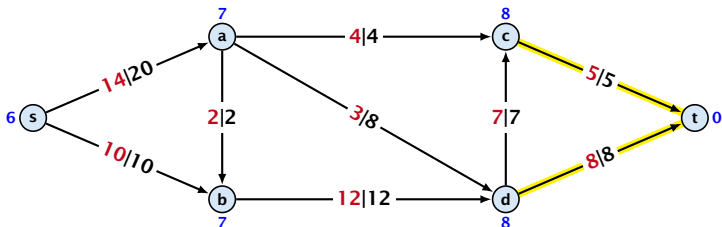
relabel to 8



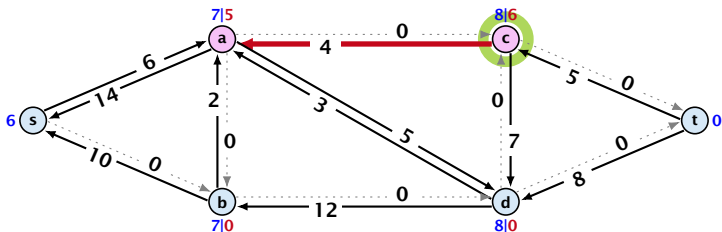
# Preflow Push



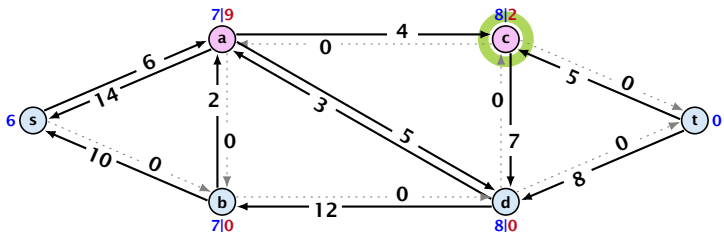
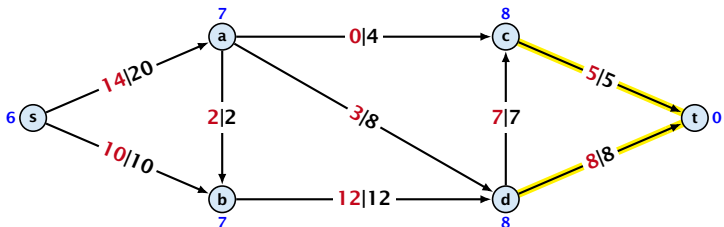
# Preflow Push



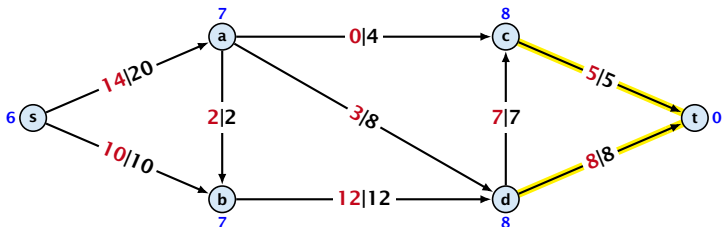
satürating push



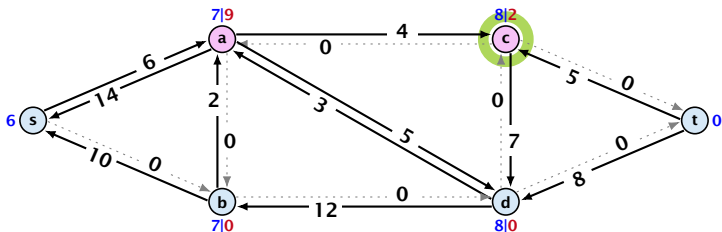
# Preflow Push



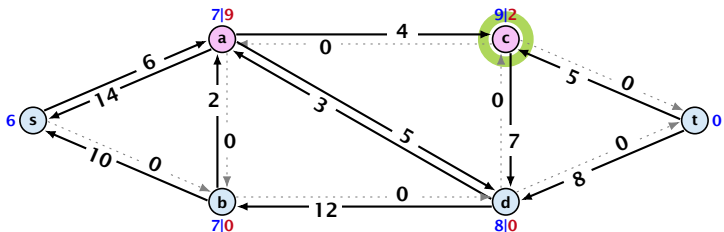
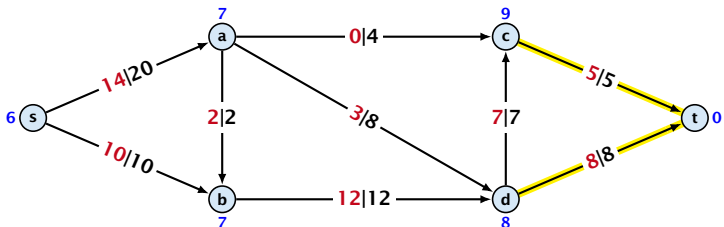
# Preflow Push



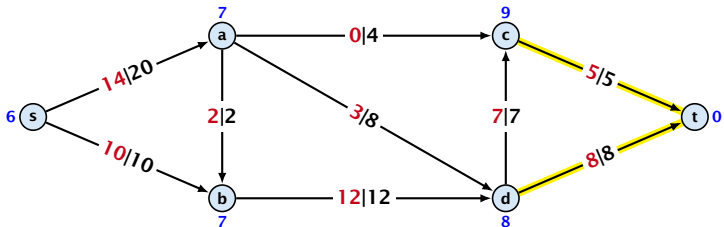
relabel to 9



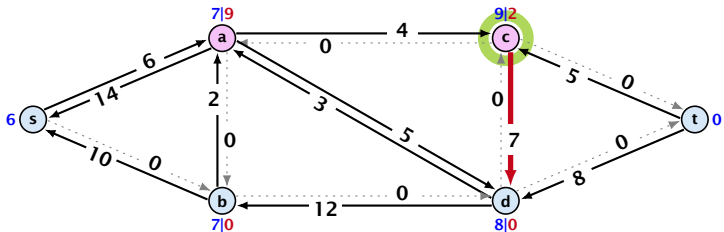
# Preflow Push



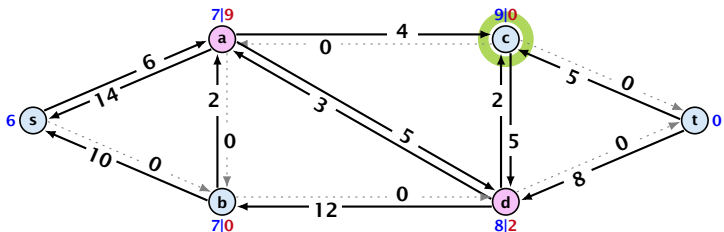
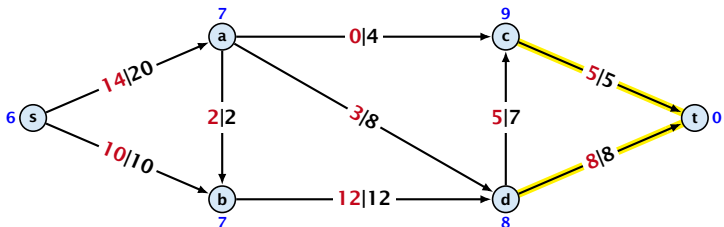
# Preflow Push



deactivating push

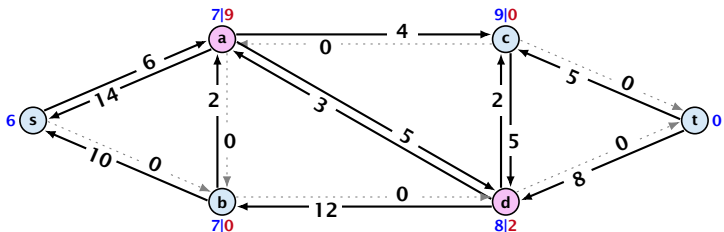
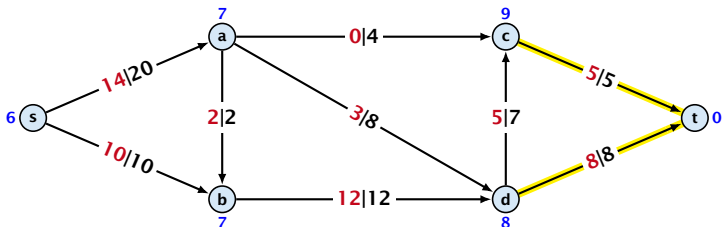


# Preflow Push

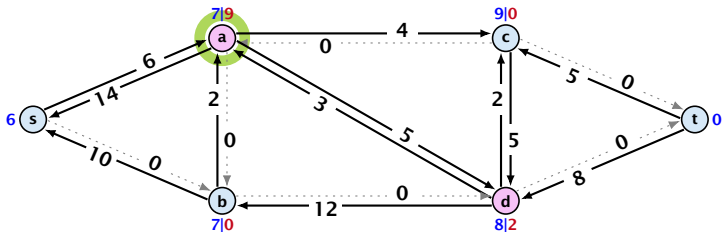
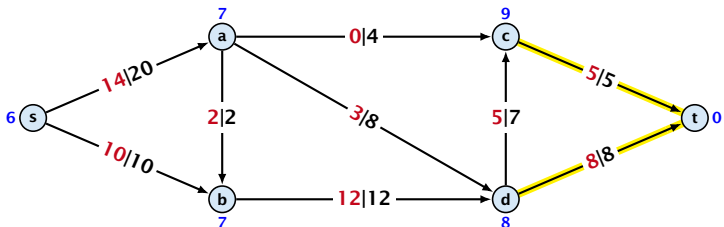




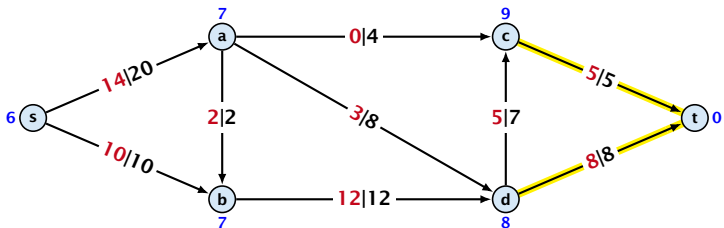
# Preflow Push



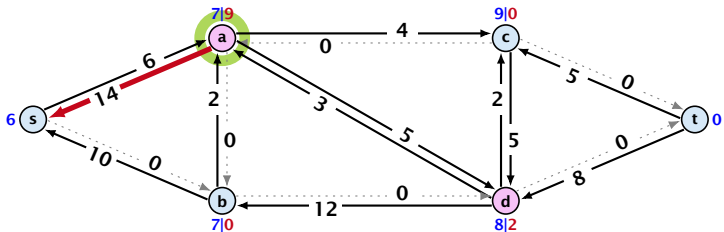
# Preflow Push



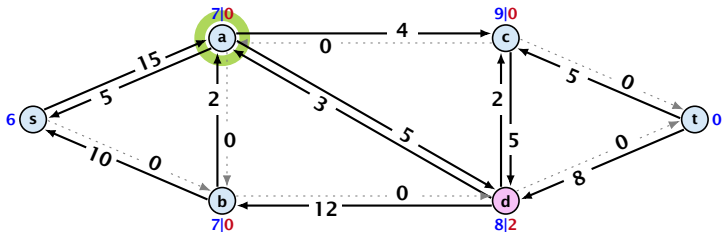
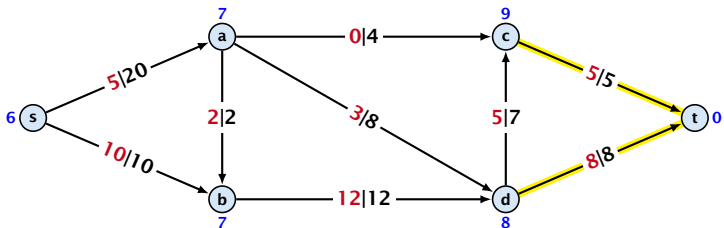
# Preflow Push



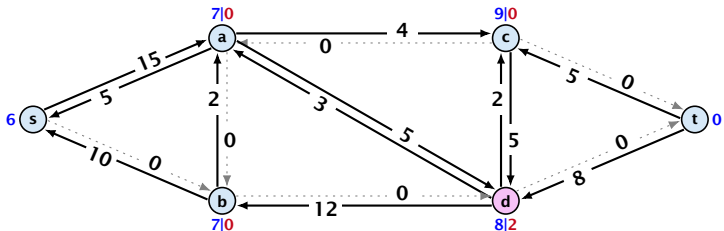
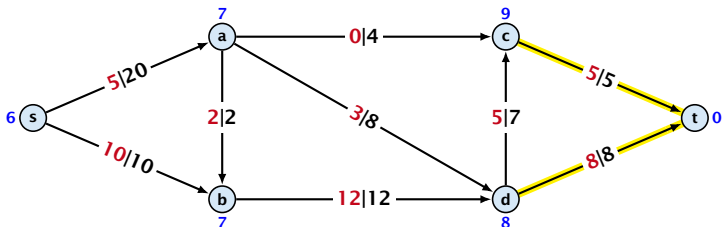
deactivating push



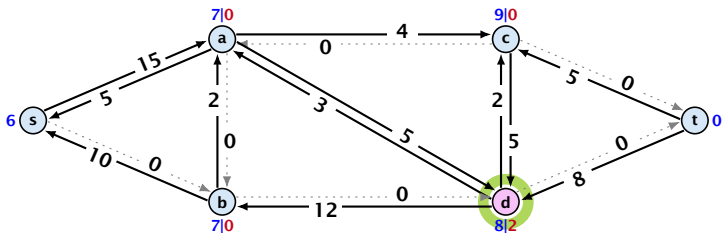
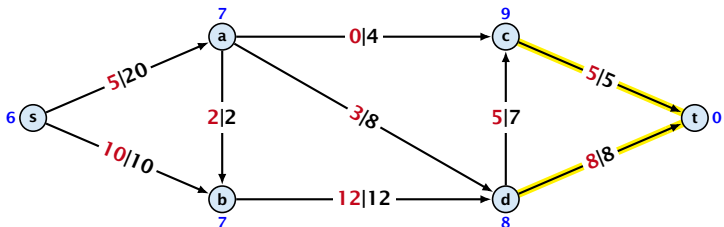
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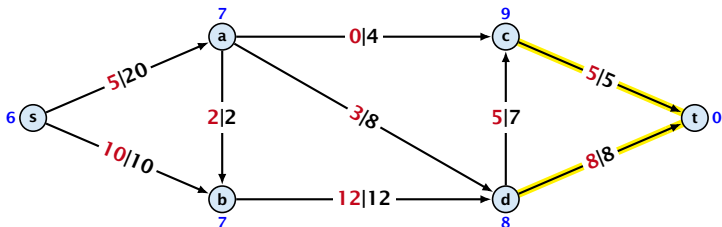
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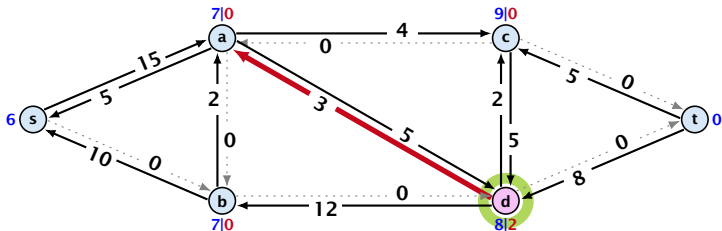
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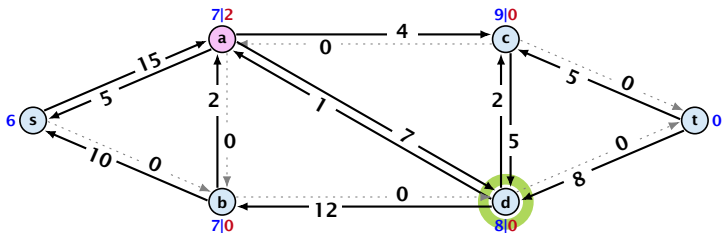
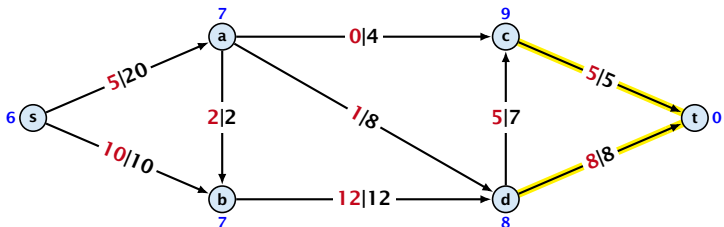
# Preflow Push



deactivating push

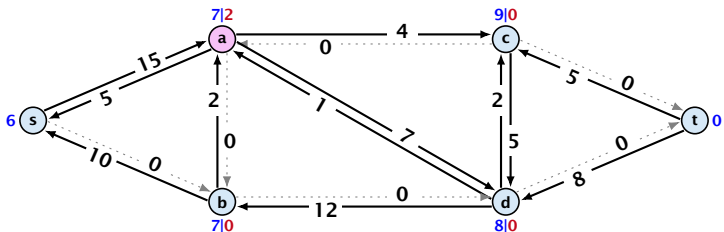
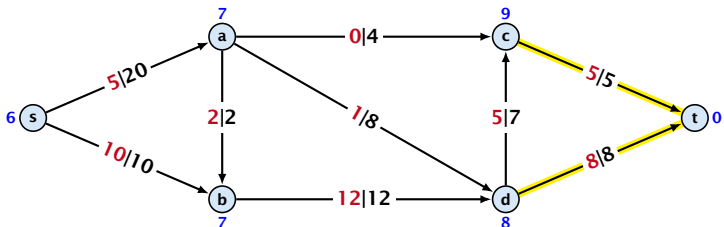


# Preflow Push

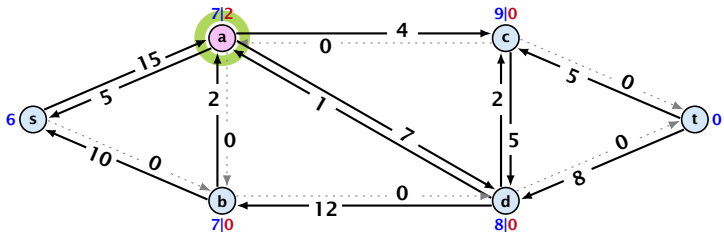
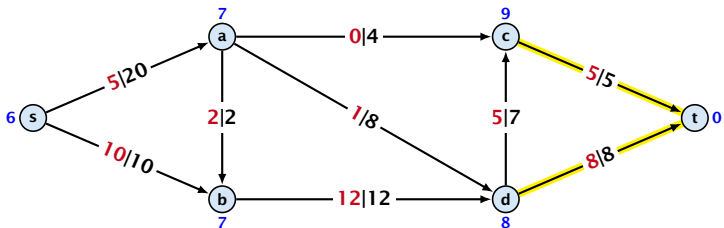




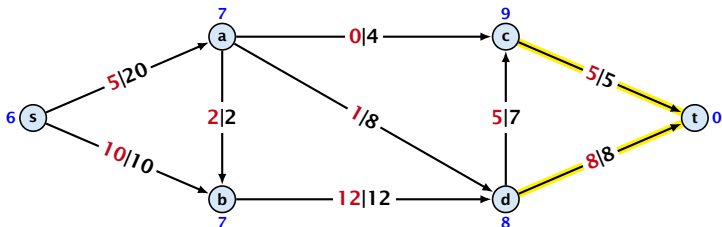
# Preflow Push



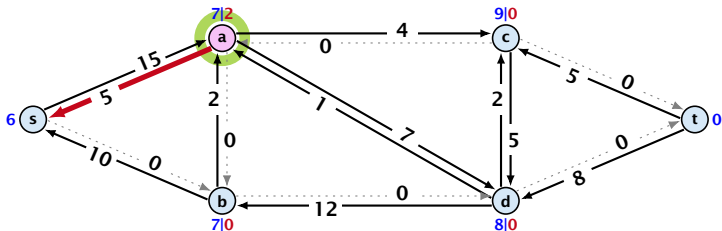
# Preflow Push



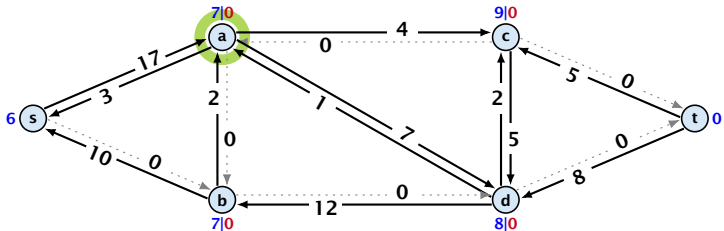
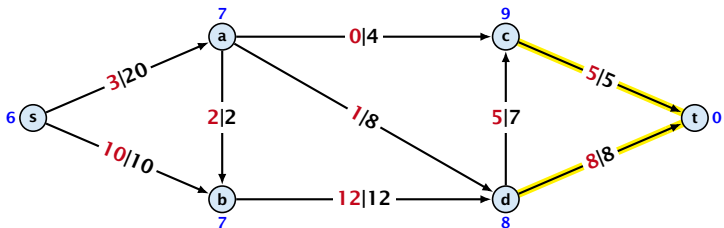
# Preflow Push



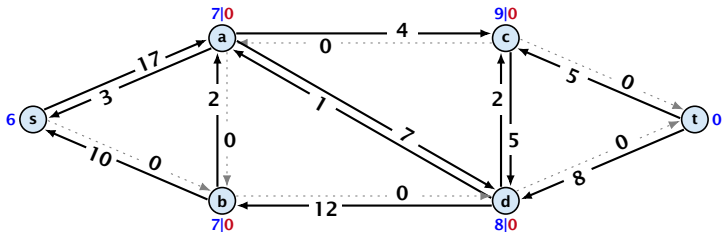
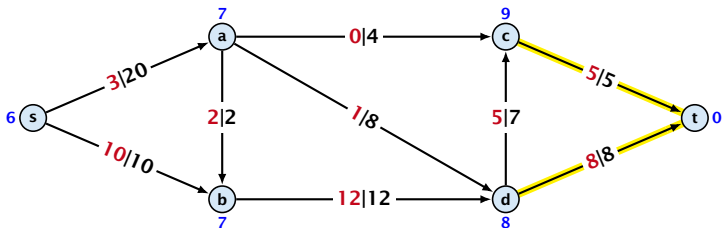
deactivating push



# Preflow Push



# Preflow Push



# Analysis

## Lemma 69

*An active node has a path to  $s$  in the residual graph.*

# Analysis

## Lemma 69

*An active node has a path to  $s$  in the residual graph.*

### Proof.

- ▶ Let  $A$  denote the set of nodes that can reach  $s$ , and let  $B$  denote the remaining nodes. Note that  $s \in A$ .

# Analysis

## Lemma 69

*An active node has a path to  $s$  in the residual graph.*

### Proof.

- ▶ Let  $A$  denote the set of nodes that can reach  $s$ , and let  $B$  denote the remaining nodes. Note that  $s \in A$ .
- ▶ In the following we show that a node  $b \in B$  has excess flow  $f(b) = 0$  which gives the lemma.



# Analysis

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*An active node has a path to  $s$  in the residual graph.*

### Proof.

- ▶ Let  $A$  denote the set of nodes that can reach  $s$ , and let  $B$  denote the remaining nodes. Note that  $s \in A$ .
- ▶ In the following we show that a node  $b \in B$  has excess flow  $f(b) = 0$  which gives the lemma.
- ▶ In the residual graph there are no edges into  $A$ , and, hence, no edges leaving  $A$ /entering  $B$  can carry any flow.

# Analysis

## Lemma 69

*An active node has a path to  $s$  in the residual graph.*

### Proof.

- ▶ Let  $A$  denote the set of nodes that can reach  $s$ , and let  $B$  denote the remaining nodes. Note that  $s \in A$ .
- ▶ In the following we show that a node  $b \in B$  has excess flow  $f(b) = 0$  which gives the lemma.
- ▶ In the residual graph there are no edges into  $A$ , and, hence, no edges leaving  $A$ /entering  $B$  can carry any flow.
- ▶ Let  $f(B) = \sum_{v \in B} f(v)$  be the excess flow of all nodes in  $B$ .

Let  $f : E \rightarrow \mathbb{R}_0^+$  be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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$$f(B)$$

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We have

$$\begin{aligned} f(B) &= \sum_{b \in B} f(b) \\ &= \sum_{b \in B} \left( \sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \end{aligned}$$

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Let  $f : E \rightarrow \mathbb{R}_0^+$  be a preflow. We introduce the notation

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Hence, the excess flow  $f(b)$  must be 0 for every node  $b \in B$ .



# Analysis

## Lemma 70

*The label of a node cannot become larger than  $2n - 1$ .*

# Analysis

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### Proof.

- ▶ When increasing the label at a node  $u$  there exists a path from  $u$  to  $s$  of length at most  $n - 1$ . Along each edge of the path the height/label can at most drop by  $1$ , and the label of the source is  $n$ .

# Analysis

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## Lemma 71

*There are only  $\mathcal{O}(n^2)$  relabel operations.*

# Analysis

## Lemma 72

The number of *saturating pushes* performed is at most  $O(mn)$ .

# Analysis

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### Proof.

- ▶ Suppose that we just made a saturating push along  $(u, v)$ .
- ▶ Hence, the edge  $(u, v)$  is deleted from the residual graph.

# Analysis

## Lemma 72

The number of *saturating pushes* performed is at most  $\mathcal{O}(mn)$ .

### Proof.

- ▶ Suppose that we just made a saturating push along  $(u, v)$ .
- ▶ Hence, the edge  $(u, v)$  is deleted from the residual graph.
- ▶ For the edge to appear again, a push from  $v$  to  $u$  is required.

# Analysis

## Lemma 72

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### Proof.

- ▶ Suppose that we just made a saturating push along  $(u, v)$ .
- ▶ Hence, the edge  $(u, v)$  is deleted from the residual graph.
- ▶ For the edge to appear again, a push from  $v$  to  $u$  is required.
- ▶ Currently,  $\ell(u) = \ell(v) + 1$ , as we only make pushes along admissible edges.



# Analysis

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### Proof.

- ▶ Suppose that we just made a saturating push along  $(u, v)$ .
- ▶ Hence, the edge  $(u, v)$  is deleted from the residual graph.
- ▶ For the edge to appear again, a push from  $v$  to  $u$  is required.
- ▶ Currently,  $\ell(u) = \ell(v) + 1$ , as we only make pushes along admissible edges.
- ▶ For a push from  $v$  to  $u$  the edge  $(v, u)$  must become admissible. The label of  $v$  must increase by at least 2.

# Analysis

## Lemma 72

The number of *saturating pushes* performed is at most  $\mathcal{O}(mn)$ .

### Proof.

- ▶ Suppose that we just made a saturating push along  $(u, v)$ .
- ▶ Hence, the edge  $(u, v)$  is deleted from the residual graph.
- ▶ For the edge to appear again, a push from  $v$  to  $u$  is required.
- ▶ Currently,  $\ell(u) = \ell(v) + 1$ , as we only make pushes along admissible edges.
- ▶ For a push from  $v$  to  $u$  the edge  $(v, u)$  must become admissible. The label of  $v$  must increase by at least 2.
- ▶ Since the label of  $v$  is at most  $2n - 1$ , there are at most  $n$  pushes along  $(u, v)$ .

### Lemma 73

The number of *deactivating pushes* performed is at most  $\mathcal{O}(n^2m)$ .

## Lemma 73

The number of *deactivating pushes* performed is at most  $\mathcal{O}(n^2m)$ .

### Proof.

- ▶ Define a potential function  $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$

## Lemma 73

The number of *deactivating pushes* performed is at most  $\mathcal{O}(n^2m)$ .

### Proof.

- ▶ Define a potential function  $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$
- ▶ A saturating push increases  $\Phi$  by  $\leq 2n$  (when the target node becomes active it may contribute at most  $2n$  to the sum).

## Lemma 73

The number of *deactivating pushes* performed is at most  $\mathcal{O}(n^2m)$ .

### Proof.

- ▶ Define a potential function  $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$
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- ▶ A relabel increases  $\Phi$  by at most 1.

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- ▶ Hence,

$$\begin{aligned} \#deactivating\_pushes &\leq \#relabels + 2n \cdot \#saturating\_pushes \\ &\leq \mathcal{O}(n^2m) . \end{aligned}$$



## Theorem 74

*There is an implementation of the generic push relabel algorithm with running time  $\mathcal{O}(n^2m)$ .*

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A relabel at a node  $u$  can be performed in time  $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible
- ▶ check for all incoming edges if they become non-admissible



## Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph  $G_f$ ). Then we use the discharge-operation:

### Algorithm 2 discharge( $u$ )

```
1: while  $u$  is active do  
2:    $v \leftarrow u.current\text{-neighbour}$   
3:   if  $v = \text{null}$  then  
4:     relabel( $u$ )  
5:      $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$   
6:   else  
7:     if  $(u, v)$  admissible then push( $u, v$ )  
8:     else  $u.current\text{-neighbour} \leftarrow v.next\text{-in-list}$ 
```

Note that  $u.current\text{-neighbour}$  is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

## Lemma 75

If  $v = \text{null}$  in Line 3, then there is no outgoing admissible edge from  $u$ .

### Proof.

- ▶ While pushing from  $u$  the current-neighbour pointer is only advanced if the current edge is not admissible.
- ▶ The only thing that could make the edge admissible again would be a relabel at  $u$ .
- ▶ If we reach the end of the list ( $v = \text{null}$ ) all edges are not admissible. □

This shows that  $\text{discharge}(u)$  is correct, and that we can perform a relabel in Line 4.

## 14.2 Relabel to Front

### Algorithm 1 relabel-to-front( $G, s, t$ )

```
1: initialize preflow
2: initialize node list  $L$  containing  $V \setminus \{s, t\}$  in any order
3: foreach  $u \in V \setminus \{s, t\}$  do
4:    $u.current-neighbour \leftarrow u.neighbour-list-head$ 
5:  $u \leftarrow L.head$ 
6: while  $u \neq null$  do
7:    $old-height \leftarrow \ell(u)$ 
8:   discharge( $u$ )
9:   if  $\ell(u) > old-height$  then // relabel happened
10:    move  $u$  to the front of  $L$ 
11:    $u \leftarrow u.next$ 
```

## 14.2 Relabel to Front

### Lemma 76 (Invariant)

*In Line 6 of the relabel-to-front algorithm the following invariant holds.*

- 1. The sequence  $L$  is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge  $(x, y)$  the node  $x$  appears before  $y$  in sequence  $L$ .*
- 2. No node before  $u$  in the list  $L$  is active.*

## Proof:

### ▶ Initialization:

1. In the beginning  $s$  has label  $n \geq 2$ , and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering  $L$  is permitted.
2. We start with  $u$  being the head of the list; hence no node before  $u$  can be active

### ▶ Maintenance:

1.
  - ▶ Pushes do not create any new admissible edges. Therefore, if `discharge()` does not relabel  $u$ ,  $L$  is still topologically sorted.
  - ▶ After relabeling,  $u$  cannot have admissible incoming edges as such an edge  $(x, u)$  would have had a difference  $\ell(x) - \ell(u) \geq 2$  before the re-labeling (such edges do not exist in the residual graph).  
Hence, moving  $u$  to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving  $u$  that were generated by the relabeling.

## 14.2 Relabel to Front

### Proof:

► Maintenance:

2. If we do a relabel there is nothing to prove because the only node before  $u'$  ( $u$  in the next iteration) will be the current  $u$ ; the discharge( $u$ ) operation only terminates when  $u$  is not active anymore.

For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of  $u$ .

Note that the invariant means that for  $u = \text{null}$  we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

## 14.2 Relabel to Front

### Lemma 77

*There are at most  $\mathcal{O}(n^3)$  calls to  $\text{discharge}(u)$ .*

Every discharge operation without a relabel advances  $u$  (the current node within list  $L$ ). Hence, if we have  $n$  discharge operations without a relabel we have  $u = \text{null}$  and the algorithm terminates.

Therefore, the number of calls to discharge is at most  $n(\#\text{relabels} + 1) = \mathcal{O}(n^3)$ .

## 14.2 Relabel to Front

### Lemma 78

*The cost for all relabel-operations is only  $\mathcal{O}(n^2)$ .*

A relabel-operation at a node is constant time (increasing the label and resetting  *$u$ .current-neighbour*). In total we have  $\mathcal{O}(n^2)$  relabel-operations.



## 14.2 Relabel to Front

Recall that a saturating push operation ( $\min\{c_f(e), f(u)\} = c_f(e)$ ) can also be a deactivating push operation ( $\min\{c_f(e), f(u)\} = f(u)$ ).

### Lemma 79

*The cost for all saturating push-operations that are **not** deactivating is only  $\mathcal{O}(mn)$ .*

Note that such a push-operation leaves the node  $u$  active but makes the edge  $e$  disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer  $u.current-neighbour$ .

This pointer can traverse the neighbour-list at most  $\mathcal{O}(n)$  times (upper bound on number of relabels) and the neighbour-list has only  $degree(u) + 1$  many entries (+1 for null-entry).

## 14.2 Relabel to Front

### Lemma 80

*The cost for all deactivating push-operations is only  $\mathcal{O}(n^3)$ .*

A deactivating push-operation takes constant time and ends the current call to `discharge()`. Hence, there are only  $\mathcal{O}(n^3)$  such operations.

### Theorem 81

*The push-relabel algorithm with the rule relabel-to-front takes time  $\mathcal{O}(n^3)$ .*

## 14.3 Highest Label

### Algorithm 1 highest-label( $G, s, t$ )

- 1: initialize preflow
- 2: **foreach**  $u \in V \setminus \{s, t\}$  **do**
- 3:      $u.current-neighbour \leftarrow u.neighbour-list-head$
- 4: **while**  $\exists$  active node  $u$  **do**
- 5:     select active node  $u$  with highest label
- 6:     discharge( $u$ )

## 14.3 Highest Label

### Lemma 82

*When using highest label the number of deactivating pushes is only  $\mathcal{O}(n^3)$ .*

A push from a node on level  $\ell$  can only “activate” nodes on levels strictly less than  $\ell$ .

This means, after a deactivating push from  $u$  a relabel is required to make  $u$  active again.

Hence, after  $n$  deactivating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of deactivating pushes is at most  $n(\#relabels + 1) = \mathcal{O}(n^3)$ .

## 14.3 Highest Label

Since a discharge-operation is terminated by a deactivating push this gives an upper bound of  $\mathcal{O}(n^3)$  on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

### Question:

How do we find the next node for a discharge operation?

## 14.3 Highest Label

Maintain lists  $L_i$ ,  $i \in \{0, \dots, 2n\}$ , where list  $L_i$  contains active nodes with label  $i$  (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node  $u$  with label  $k$ , traverse the lists  $L_k, L_{k-1}, \dots, L_0$ , (in that order) until you find a non-empty list.

Unless the last (deactivating) push was to  $s$  or  $t$  the list  $k-1$  must be non-empty (i.e., the search takes constant time).

## 14.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#deactivating-pushes-to-s-or-t)$$

### Lemma 83

*The number of deactivating pushes to  $s$  or  $t$  is at most  $\mathcal{O}(n^2)$ .*

With this lemma we get

### Theorem 84

*The push-relabel algorithm with the rule highest-label takes time  $\mathcal{O}(n^3)$ .*

## 14.3 Highest Label

### Proof of the Lemma.

- ▶ We only show that the number of pushes to the source is at most  $\mathcal{O}(n^2)$ . A similar argument holds for the target.
- ▶ After a node  $v$  (which must have  $\ell(v) = n + 1$ ) made a deactivating push to the source there needs to be another node whose label is increased from  $\leq n + 1$  to  $n + 2$  before  $v$  can become active again.
- ▶ This happens for every push that  $v$  makes to the source. Since, every node can pass the threshold  $n + 2$  at most once,  $v$  can make at most  $n$  pushes to the source.
- ▶ As this holds for every node the total number of pushes to the source is at most  $\mathcal{O}(n^2)$ .



## Problem Definition:

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

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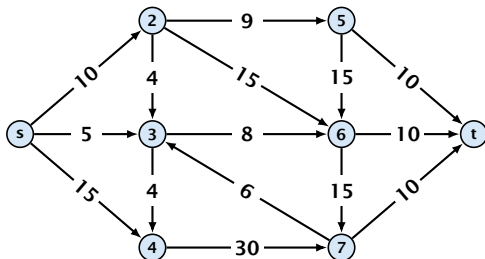
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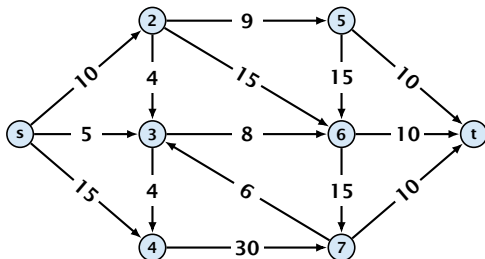
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(note that  $c(e)$  may be negative).
- ▶  $b : V \rightarrow \mathbb{R}, \sum_{v \in V} b(v) = 0$  is a **demand function**.

# Solve Maxflow Using Mincost Flow

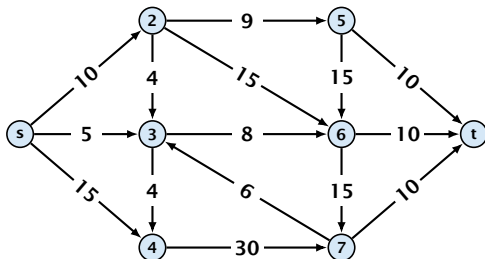


# Solve Maxflow Using Mincost Flow



- ▶ Given a flow network for a standard maxflow problem.

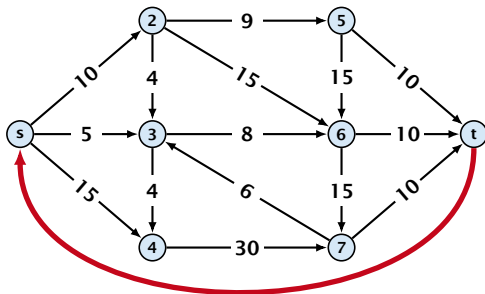
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- ▶ Given a flow network for a standard maxflow problem.
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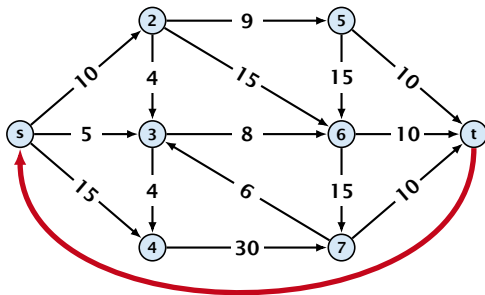


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- ▶ Then,  $\text{val}(f^*) = -\text{cost}(f_{\min})$ , where  $f^*$  is a maxflow, and  $f_{\min}$  is a mincost-flow.

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- ▶ Set edge-costs to zero, and keep the capacities.
- ▶ There exists a maxflow of value at least  $k$  if and only if the mincost-flow problem is feasible.

# Generalization

Our model:

$$\begin{aligned} \min \quad & \sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

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**A more general model?**

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where  $a: V \rightarrow \mathbb{R}$ ,  $b: V \rightarrow \mathbb{R}$ ;  $\ell: E \rightarrow \mathbb{R} \cup \{-\infty\}$ ,  $u: E \rightarrow \mathbb{R} \cup \{\infty\}$   
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## Differences

- ▶ Flow along an edge  $e$  may have non-zero lower bound  $\ell(e)$ .
- ▶ Flow along  $e$  may have negative upper bound  $u(e)$ .
- ▶ The demand at a node  $v$  may have lower bound  $a(v)$  and upper bound  $b(v)$  instead of just lower bound = upper bound =  $b(v)$ .

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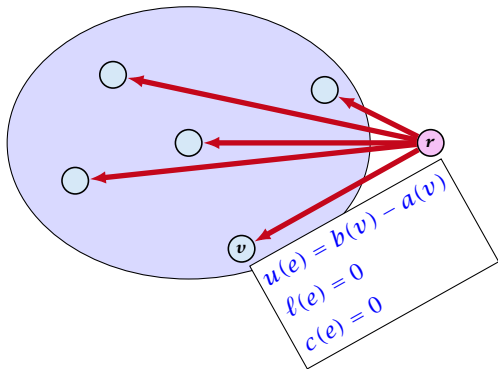
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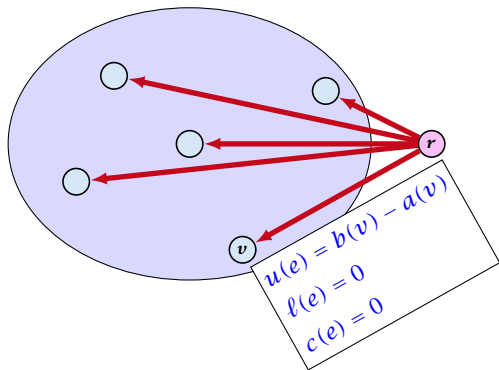


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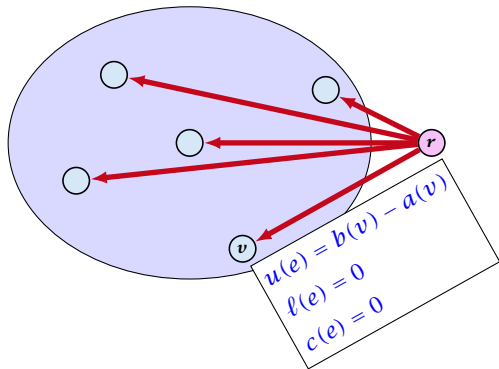
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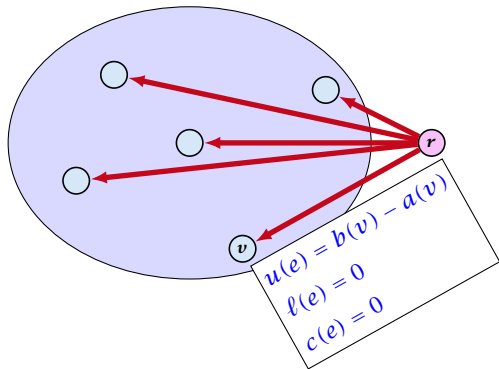
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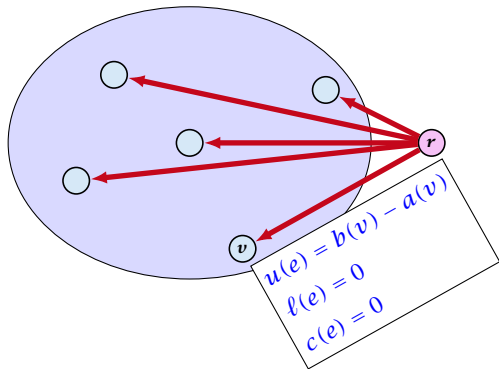
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Set  $u(e) = b(v) - a(v)$  for edge  $(r, v)$ .





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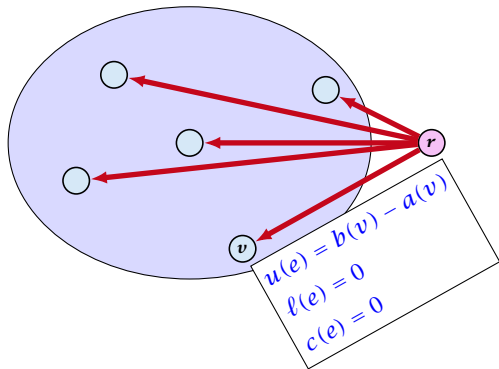
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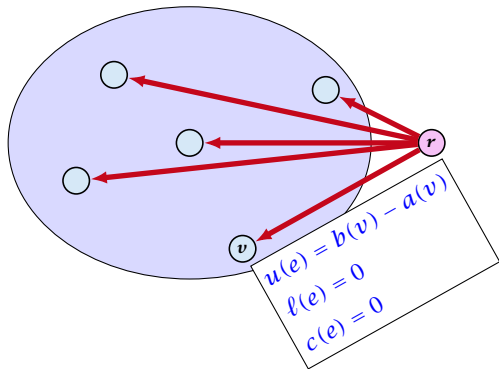
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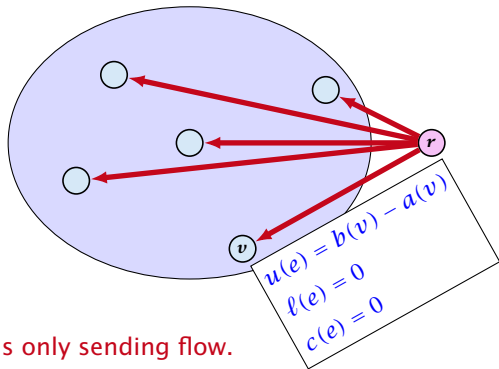
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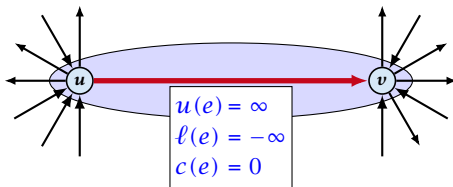
$-\sum_v b(v)$  is negative; hence  $r$  is only sending flow.



## Reduction II

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

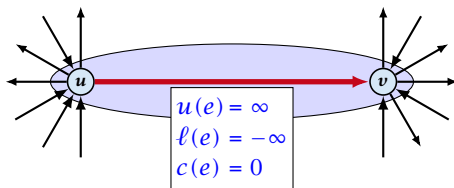
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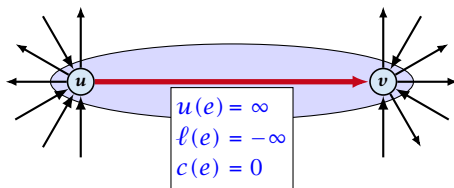


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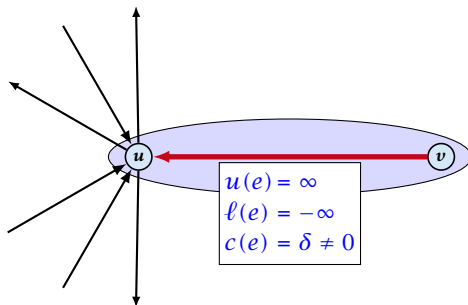


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If  $c(e) \neq 0$  we can transform the graph so that  $c(e) = 0$ .

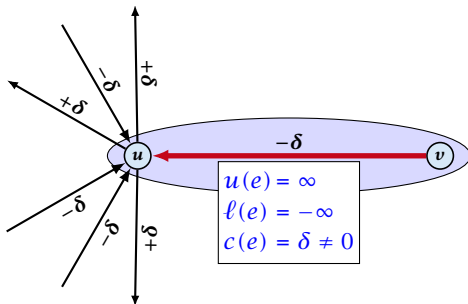
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We can transform any network so that a particular edge has cost  $c(e) = 0$ :



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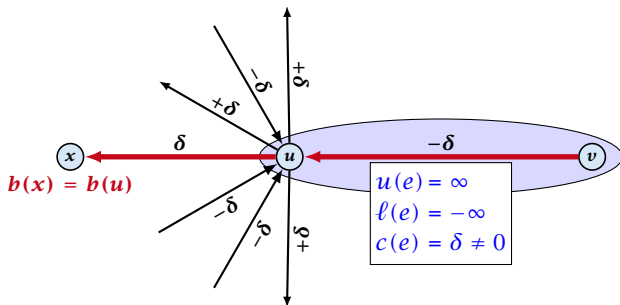
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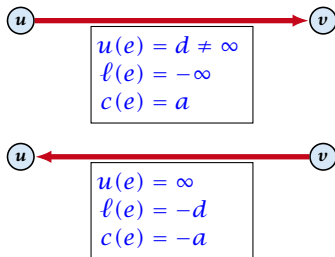


Additionally we set  $b(u) = 0$ .

## Reduction III

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

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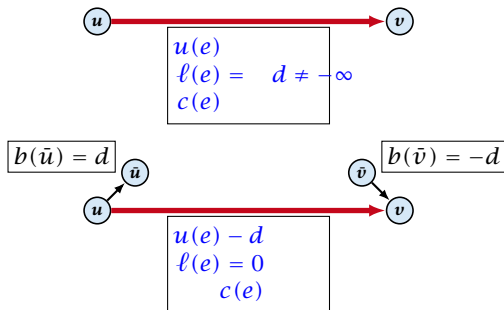


Replace the edge by an edge in opposite direction.

## Reduction IV

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that  $\ell(e) = 0$ :



The added edges have infinite capacity and cost  $c(e)/2$ .

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- ▶ She needs to supply  $r_i$  napkins on  $N$  successive days.

# Applications

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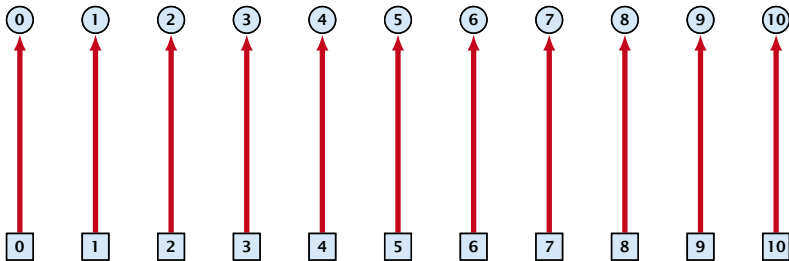
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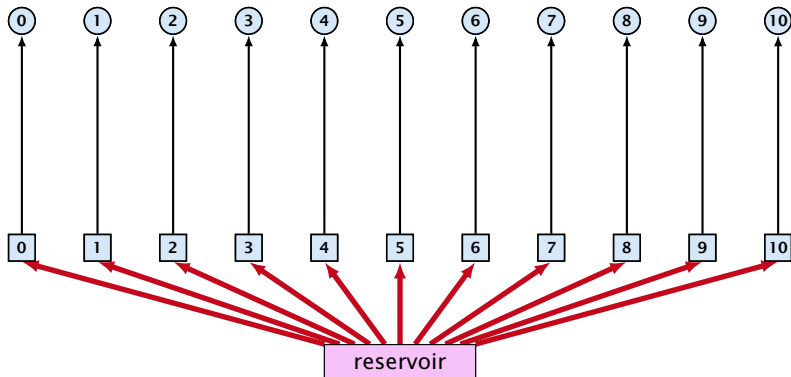
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- ▶ Minimize cost.





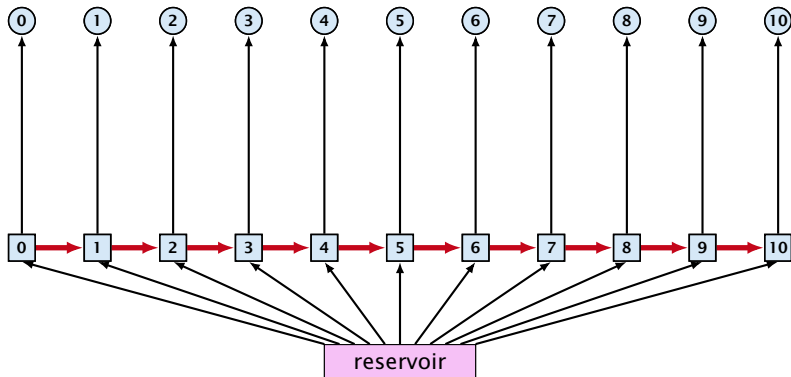
day edges:

upper bound:  $u(e_i) = \infty$ ;  
lower bound:  $\ell(e_i) = r_i$ ;  
cost:  $c(e) = 0$



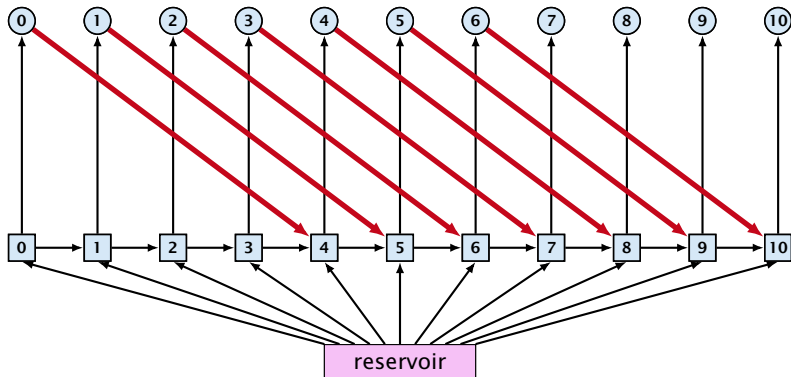
buy edges:

upper bound:  $u(e_i) = \infty$ ;  
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cost:  $c(e) = p$



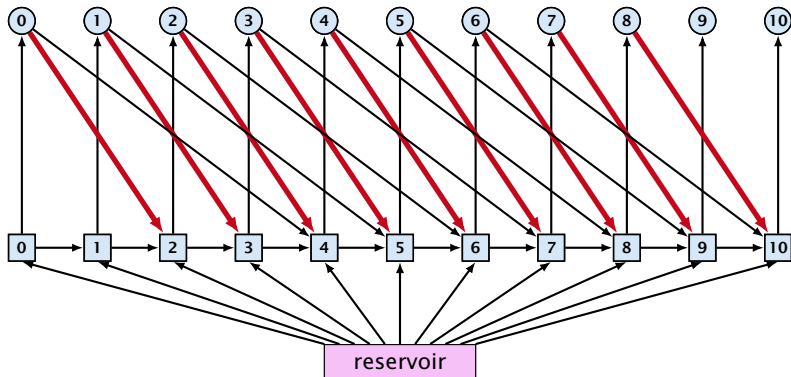
forward edges:

upper bound:  $u(e_i) = \infty$ ;  
lower bound:  $\ell(e_i) = 0$ ;  
cost:  $c(e) = 0$



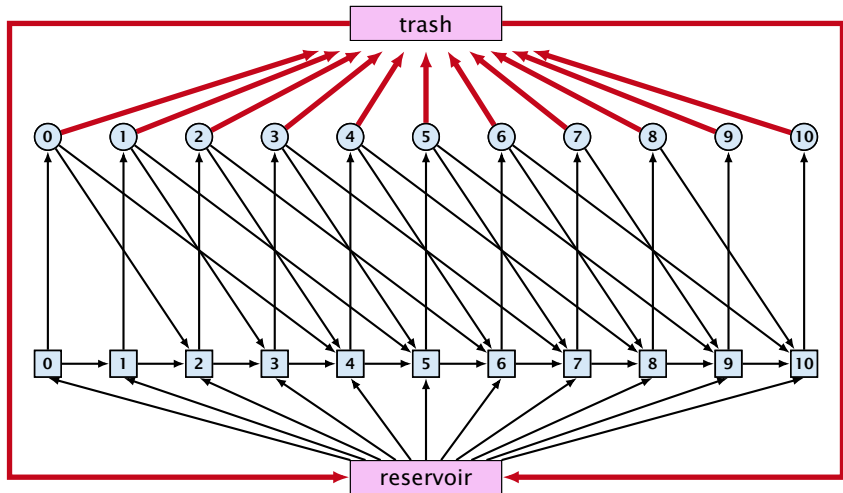
slow edges:

upper bound:  $u(e_i) = \infty$ ;  
 lower bound:  $\ell(e_i) = 0$ ;  
 cost:  $c(e) = s$



fast edges:

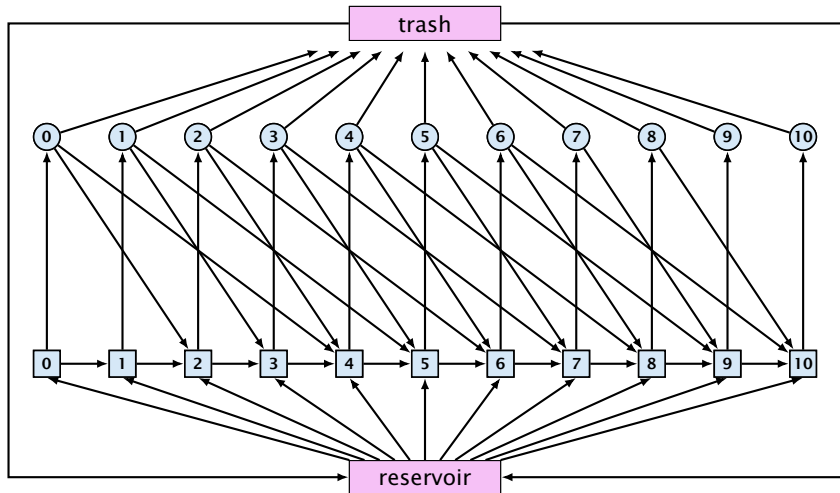
upper bound:  $u(e_i) = \infty$ ;  
 lower bound:  $\ell(e_i) = 0$ ;  
 cost:  $c(e) = f$



trash edges:

upper bound:  $u(e_i) = \infty$ ;  
 lower bound:  $\ell(e_i) = 0$ ;  
 cost:  $c(e) = 0$





# Residual Graph

## Version A:

The residual graph  $G'$  for a mincost flow is just a copy of the graph  $G$ .

If we send  $f(e)$  along an edge, the corresponding edge  $e'$  in the residual graph has its lower and upper bound changed to  $\ell(e') = \ell(e) - f(e)$  and  $u(e') = u(e) - f(e)$ .

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## Version B:

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of  $z$  from  $u$  to  $v$  the residual edge  $(v, u)$  has capacity  $z$  and a cost of  $-c((u, v))$ .

# 15 Mincost Flow

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A circulation is **feasible** if it fulfills capacity constraints, i.e.,  $f(e) \leq u(e)$  for every edge of  $G$ .

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Clearly  $f^* - f$  is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending  $-f$  in the residual graph (pushing all flow back) we arrive at the original graph; for this  $f^*$  is clearly feasible)

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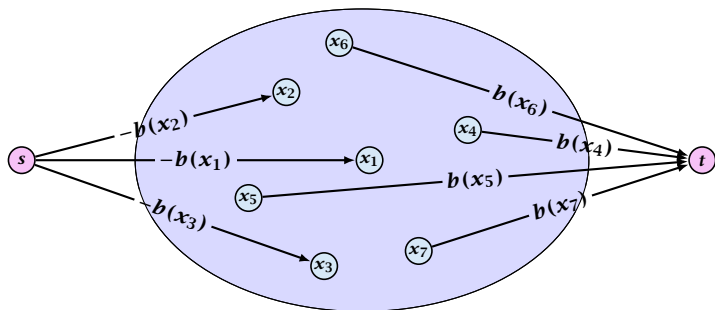
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- ▶ You still have a circulation with negative cost.
- ▶ Repeat.

# 15 Mincost Flow

## Algorithm 72 CycleCanceling( $G = (V, E), c, u, b$ )

- 1: establish a feasible flow  $f$  in  $G$
- 2: **while**  $G_f$  contains negative cycle **do**
- 3:     use Bellman-Ford to find a negative circuit  $Z$
- 4:      $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
- 5:     augment  $\delta$  units along  $Z$  and update  $G_f$

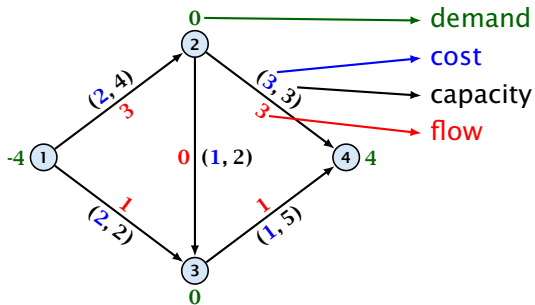
## How do we find the initial feasible flow?



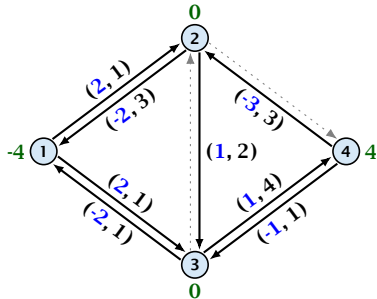
- ▶ Connect new node  $s$  to all nodes with negative  $b(v)$ -value.
- ▶ Connect nodes with positive  $b(v)$ -value to a new node  $t$ .
- ▶ There exist a feasible flow in the original graph iff in the resulting graph there exists an  $s$ - $t$  flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) .$$

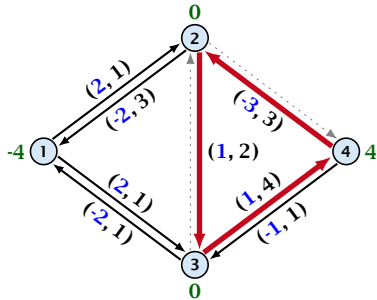
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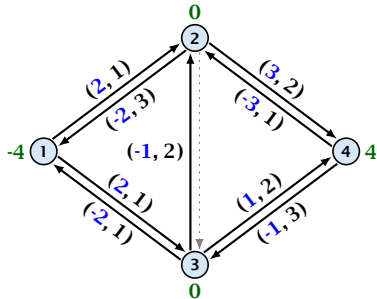
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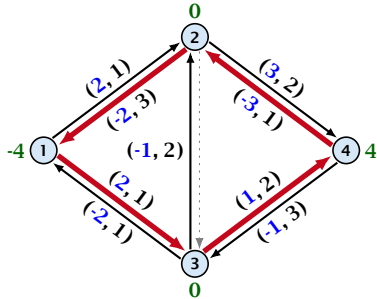
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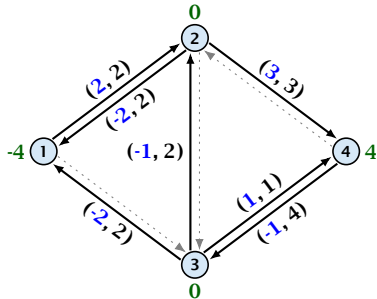


# 15 Mincost Flow





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## Lemma 87

The improving cycle algorithm runs in time  $\mathcal{O}(nm^2CU)$ , for integer capacities and costs, when for all edges  $e$ ,  $|c(e)| \leq C$  and  $|u(e)| \leq U$ .

- ▶ Running time of Bellman-Ford is  $\mathcal{O}(mn)$ .
- ▶ Pushing flow along the cycle can be done in time  $\mathcal{O}(n)$ .
- ▶ Each iteration decreases the total cost by at least 1.
- ▶ The true optimum cost must lie in the interval  $[-mCU, \dots, +mCU]$ .

Note that this lemma is weak since it does not allow for edges with infinite capacity.

# 15 Mincost Flow

A **general mincost flow problem** is of the following form:

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{array}$$

where  $a: V \rightarrow \mathbb{R}$ ,  $b: V \rightarrow \mathbb{R}$ ;  $\ell: E \rightarrow \mathbb{R} \cup \{-\infty\}$ ,  $u: E \rightarrow \mathbb{R} \cup \{\infty\}$   
 $c: E \rightarrow \mathbb{R}$ ;

## Lemma 88 (without proof)

*A general mincost flow problem can be solved in polynomial time.*

## 16 Gomory Hu Trees

Given an undirected, weighted graph  $G = (V, E, c)$  a **cut-tree**  $T = (V, F, w)$  is a tree with edge-set  $F$  and capacities  $w$  that fulfills the following properties.

- 1. Equivalent Flow Tree:** For any pair of vertices  $s, t \in V$ ,  $f(s, t)$  in  $G$  is equal to  $f_T(s, t)$ .
- 2. Cut Property:** A minimum  $s-t$  cut in  $T$  is also a minimum cut in  $G$ .

Here,  $f(s, t)$  is the value of a maximum  $s-t$  flow in  $G$ , and  $f_T(s, t)$  is the corresponding value in  $T$ .

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The algorithm maintains a partition of  $V$ , (sets  $S_1, \dots, S_t$ ), and a spanning tree  $T$  on the vertex set  $\{S_1, \dots, S_t\}$ .

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- ▶  $X$  and  $Y$  are connected by an edge, and the edges that before the split were incident to  $S_i$  are attached to either  $X$  or  $Y$ .

# Overview of the Algorithm

The algorithm maintains a partition of  $V$ , (sets  $S_1, \dots, S_t$ ), and a spanning tree  $T$  on the vertex set  $\{S_1, \dots, S_t\}$ .

Initially, there exists only the set  $S_1 = V$ .

Then the algorithm performs  $n - 1$  split-operations:

- ▶ In each such split-operation it chooses a set  $S_i$  with  $|S_i| \geq 2$  and splits this set into two non-empty parts  $X$  and  $Y$ .
- ▶  $S_i$  is then removed from  $T$  and replaced by  $X$  and  $Y$ .
- ▶  $X$  and  $Y$  are connected by an edge, and the edges that before the split were incident to  $S_i$  are attached to either  $X$  or  $Y$ .

**In the end this gives a tree on the vertex set  $V$ .**

## Details of the Split-operation

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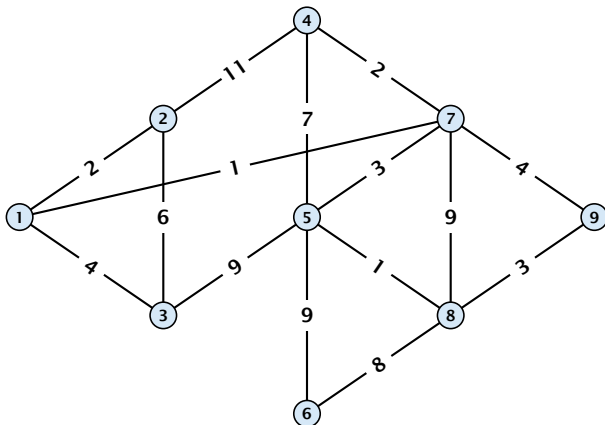
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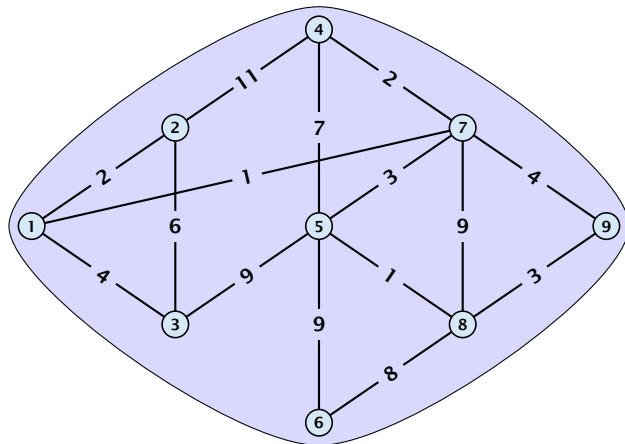
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- ▶ Replace an edge  $\{S_i, S_x\}$  by  $\{S_i^a, S_x\}$  if  $S_x \subset A$  and by  $\{S_i^b, S_x\}$  if  $S_x \subset B$ .

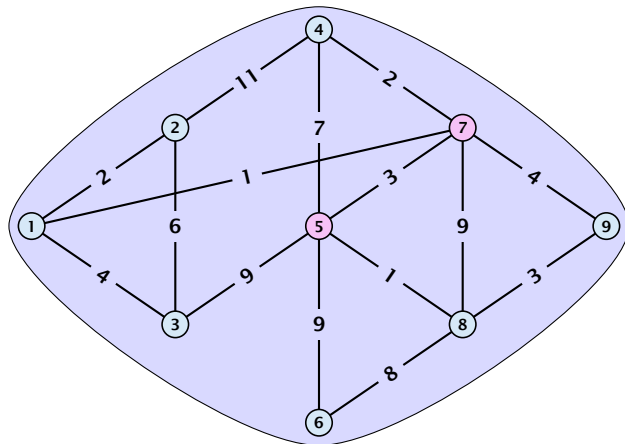
# Example: Gomory-Hu Construction



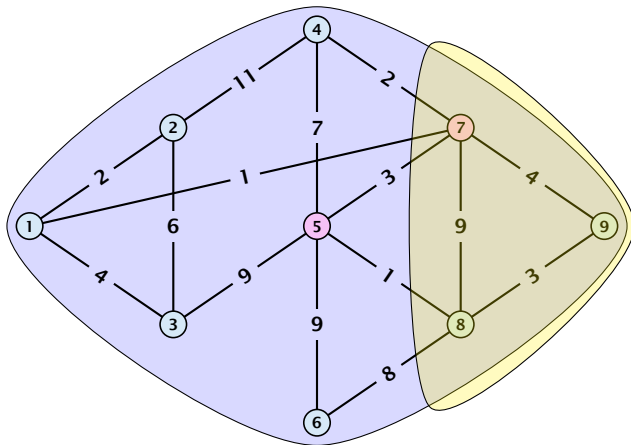
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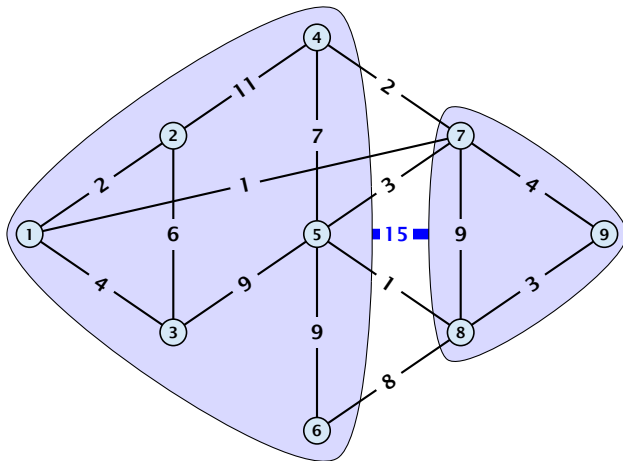
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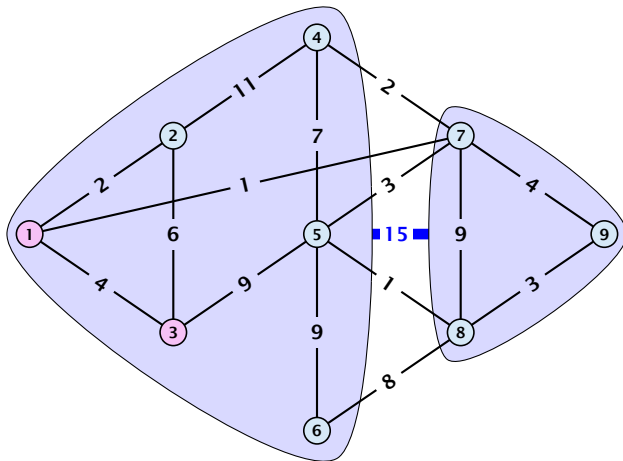
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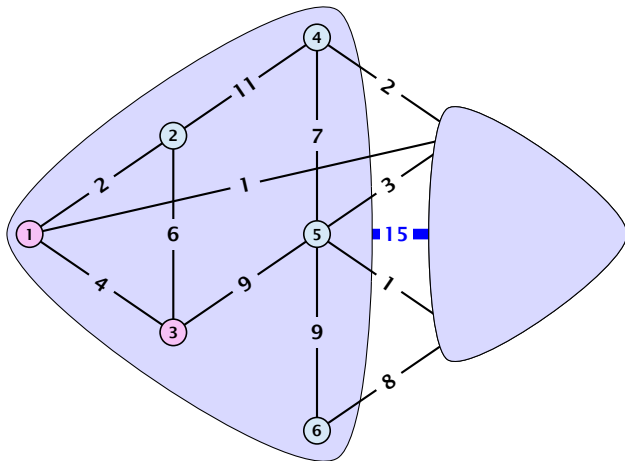
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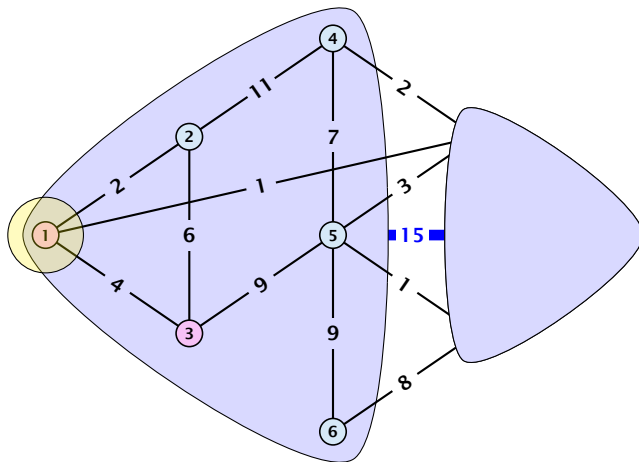


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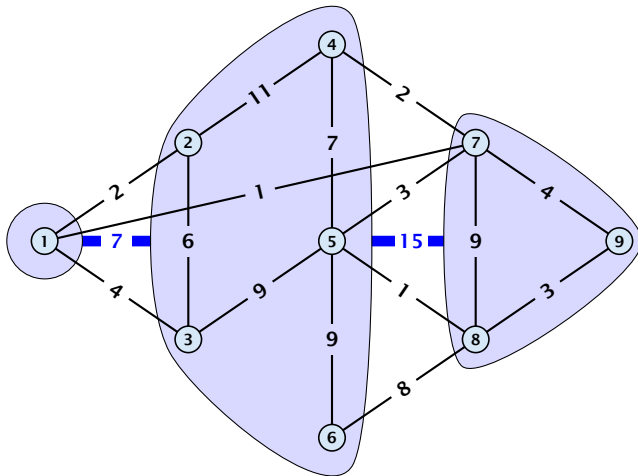




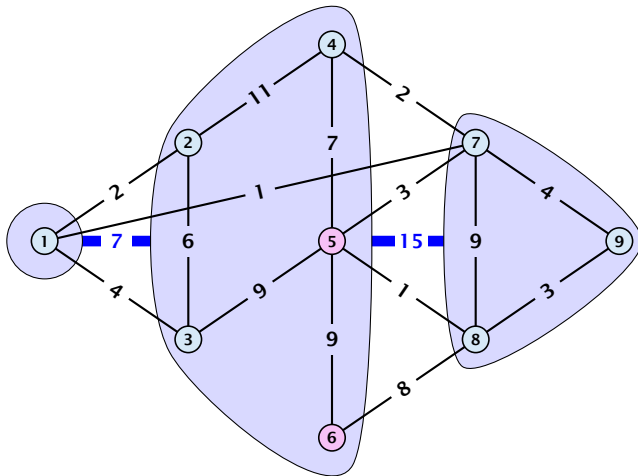
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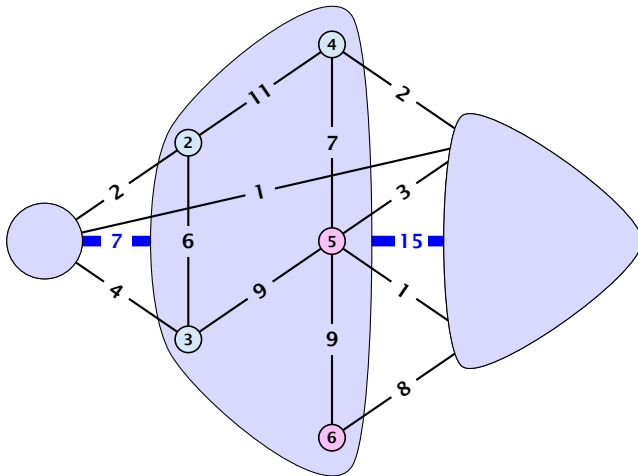
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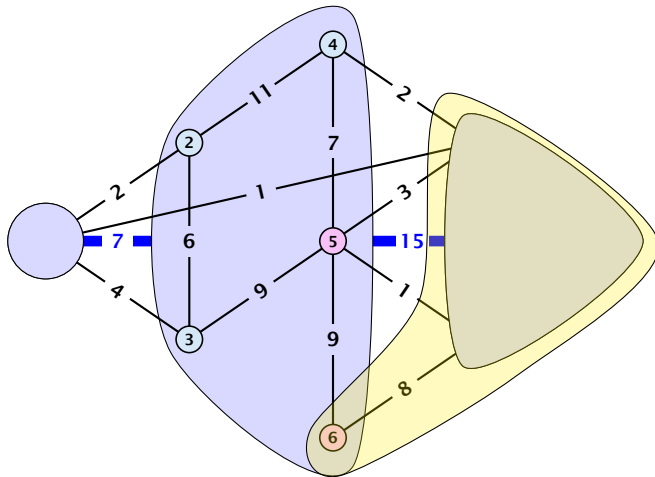
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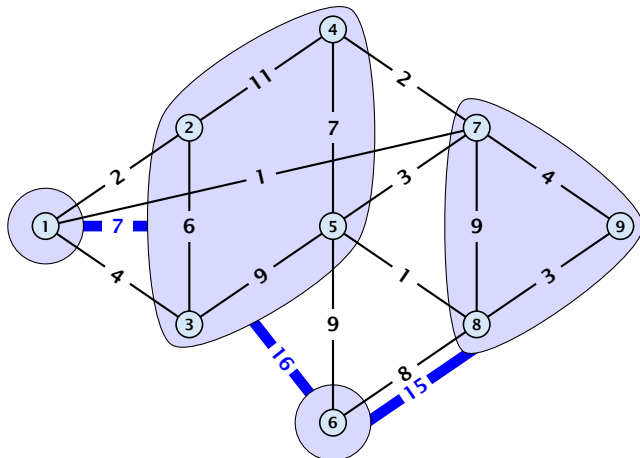
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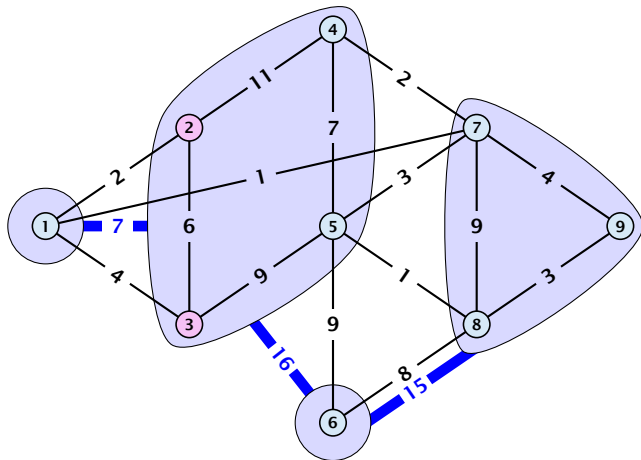
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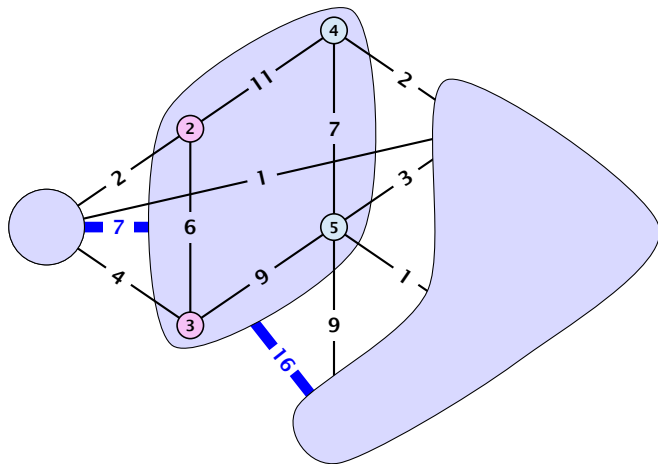
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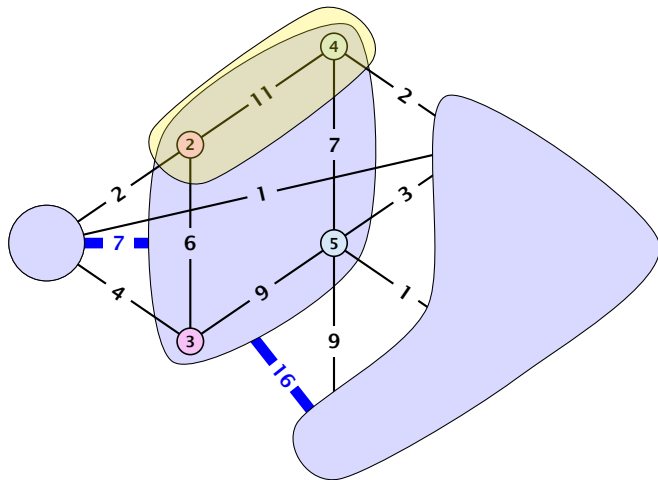


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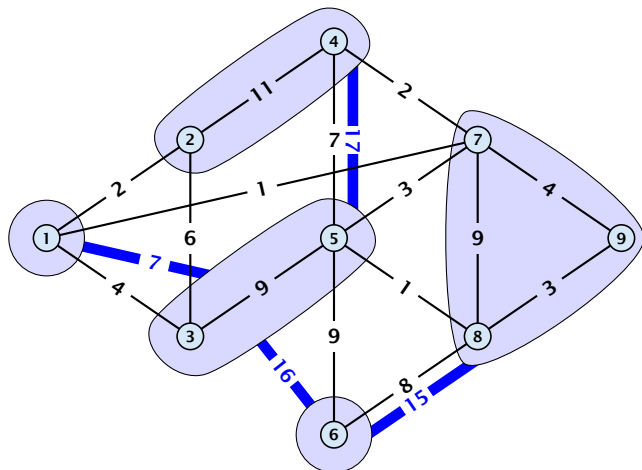




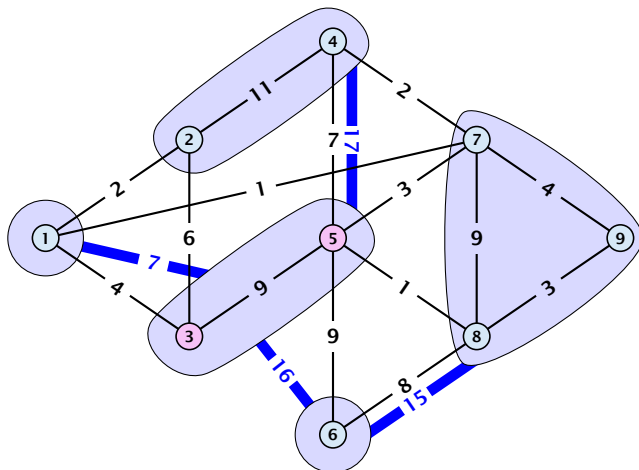
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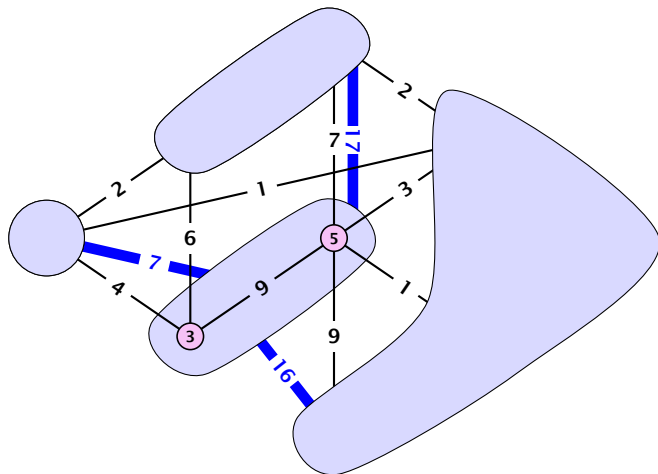
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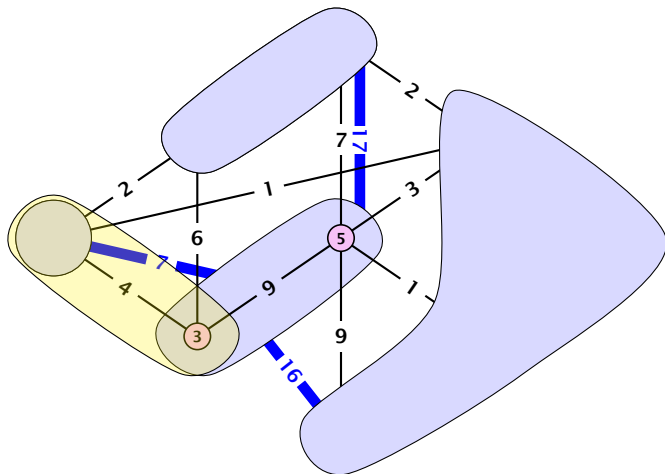
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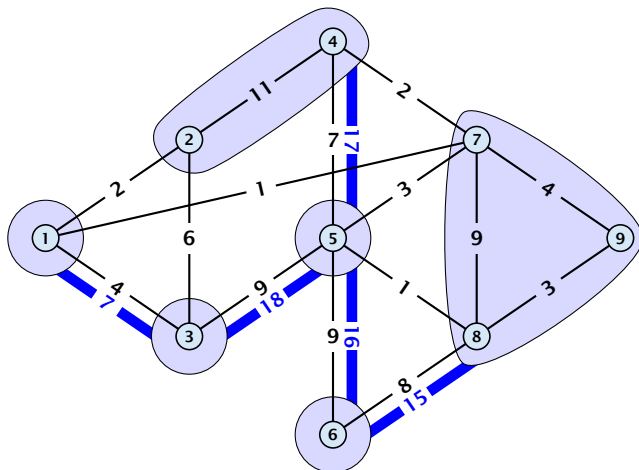
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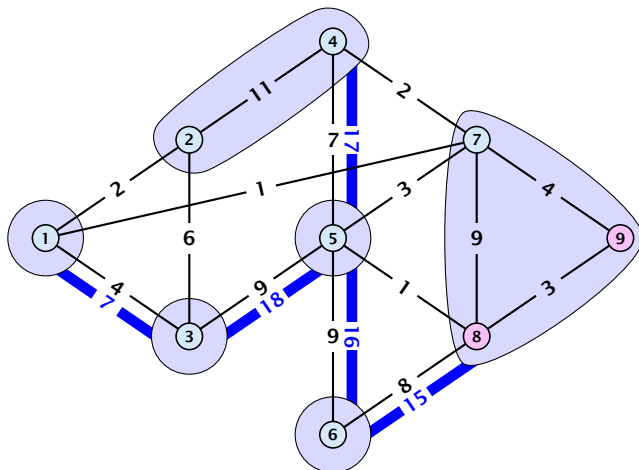
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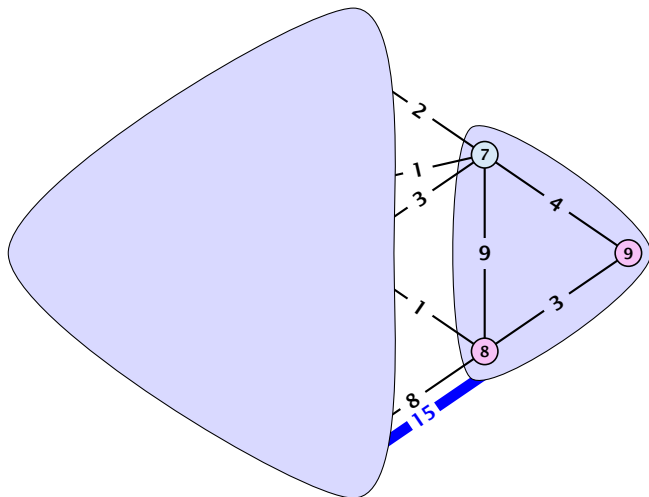
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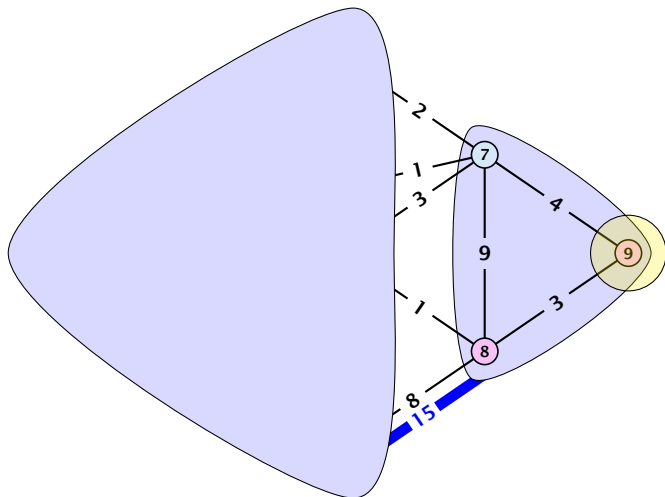


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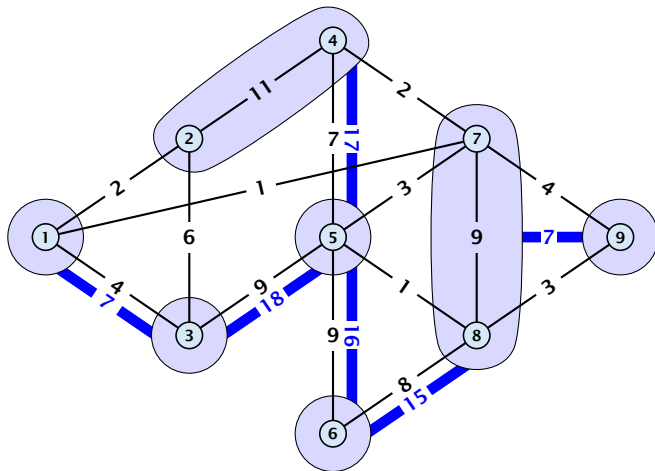




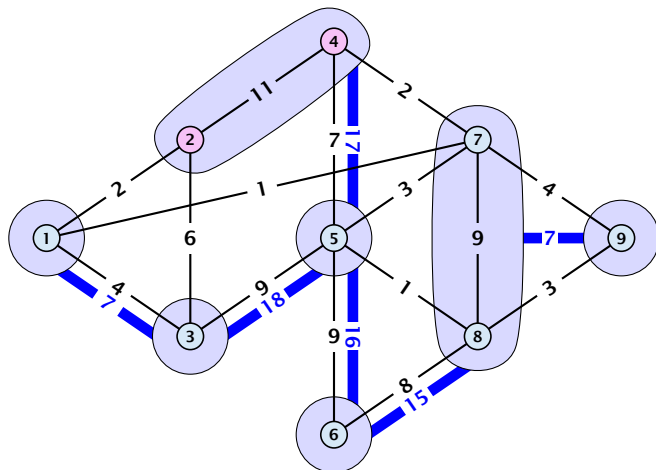
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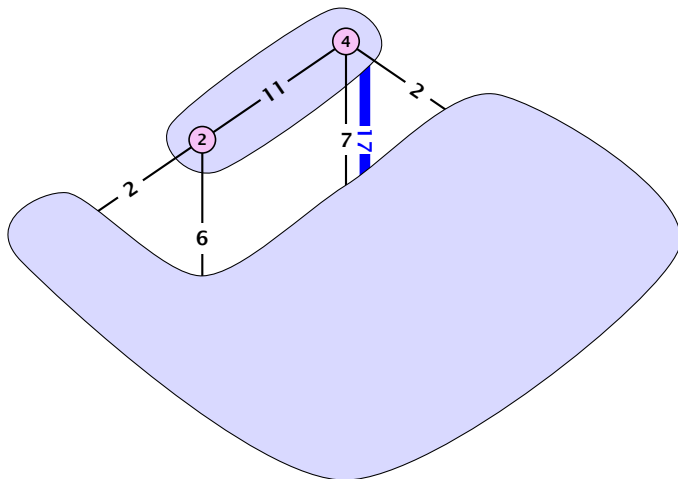
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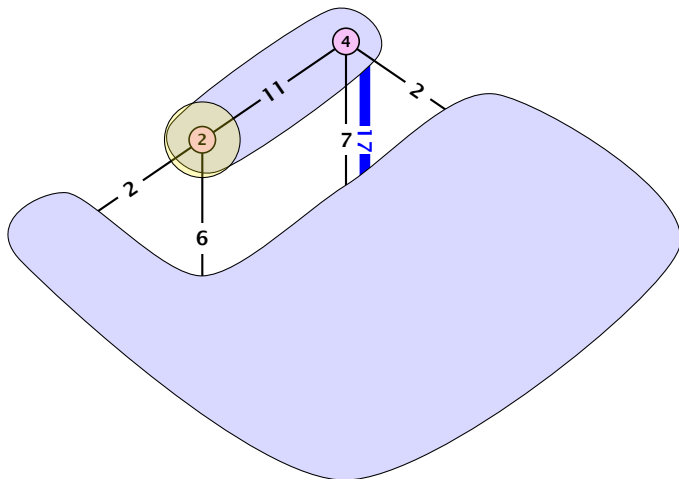
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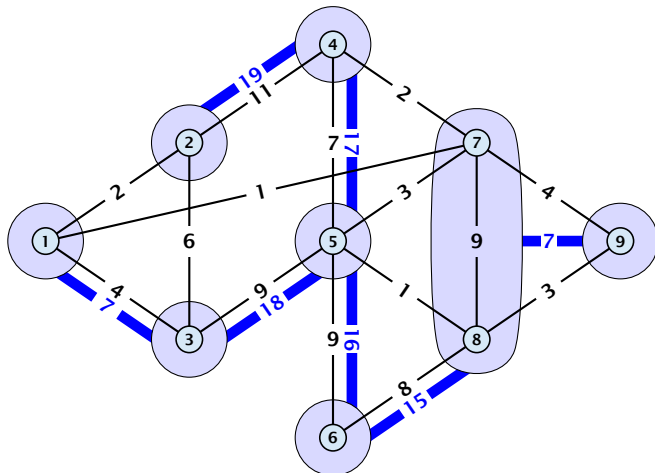
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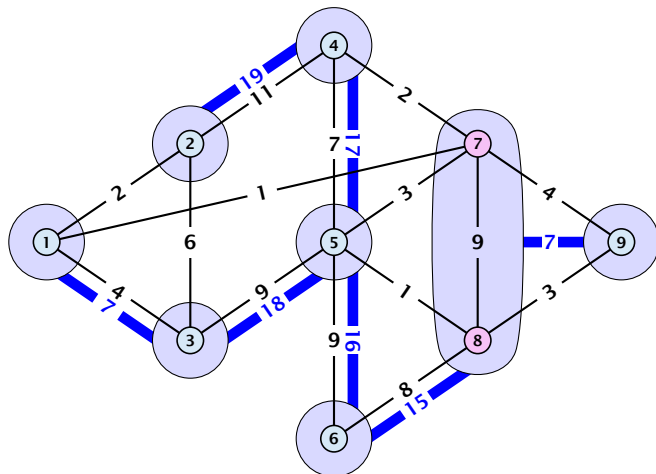
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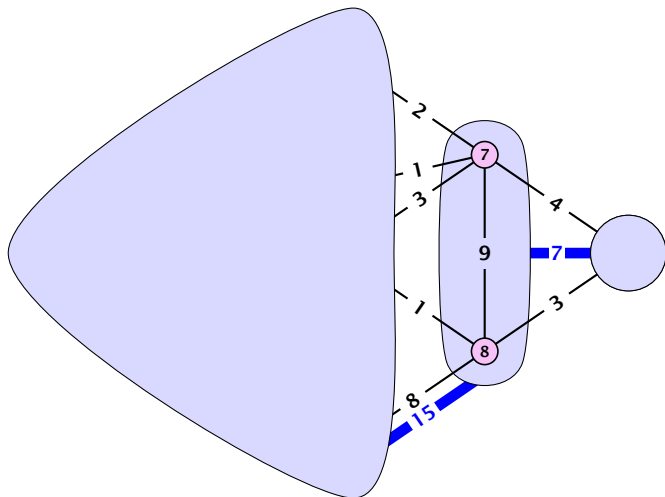
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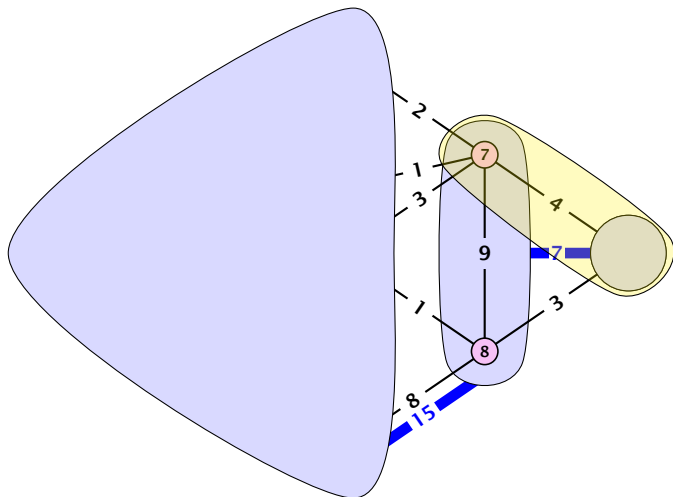


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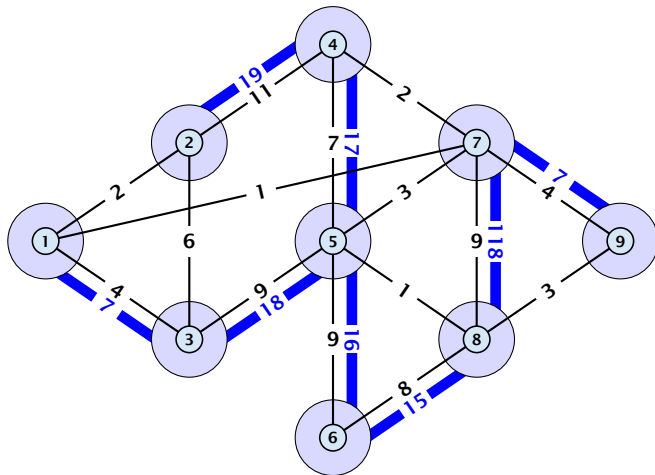




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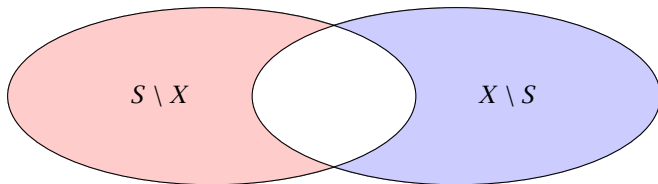
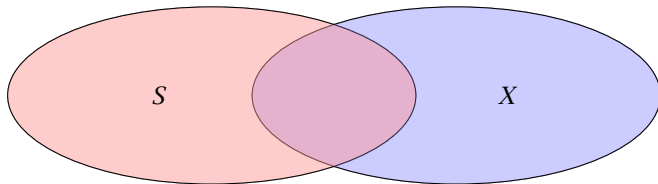
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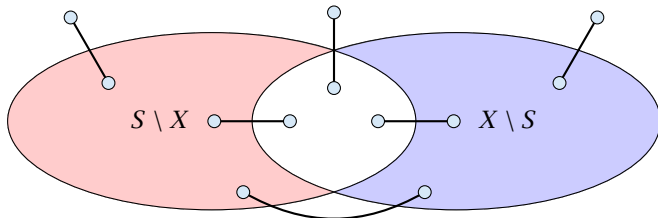
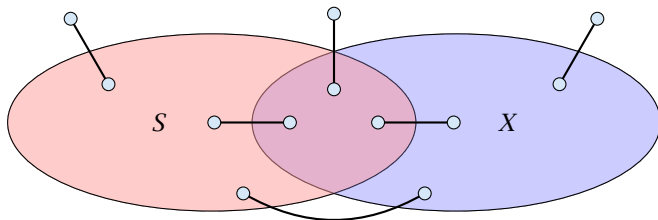
- ▶  $\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$ .
- ▶  $\text{cap}(X \cup S) \geq \text{cap}(S)$  because  $X \cup S$  is an  $r$ - $s$  cut.
- ▶ This gives  $\text{cap}(S \cap X) \leq \text{cap}(X)$ .



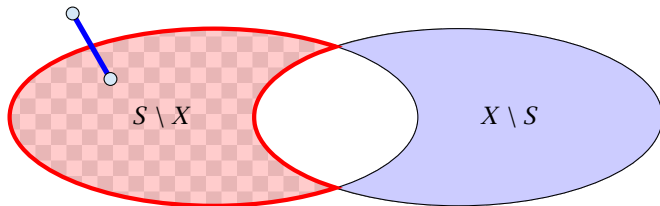
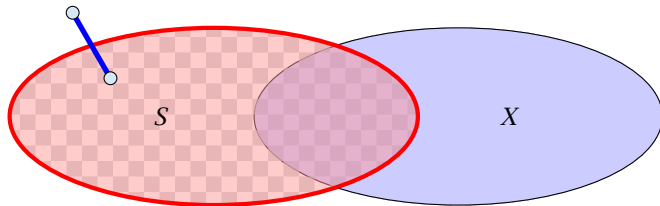
$$\text{cap}(S \setminus X) + \text{cap}(X \setminus S) \leq \text{cap}(S) + \text{cap}(X)$$



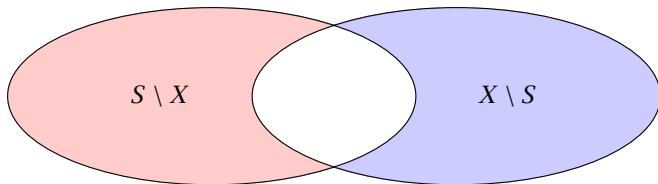
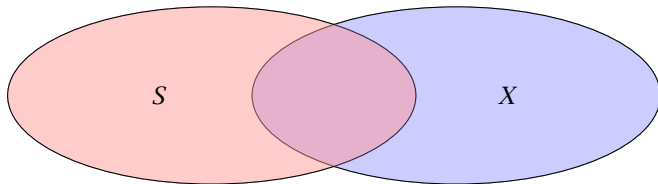
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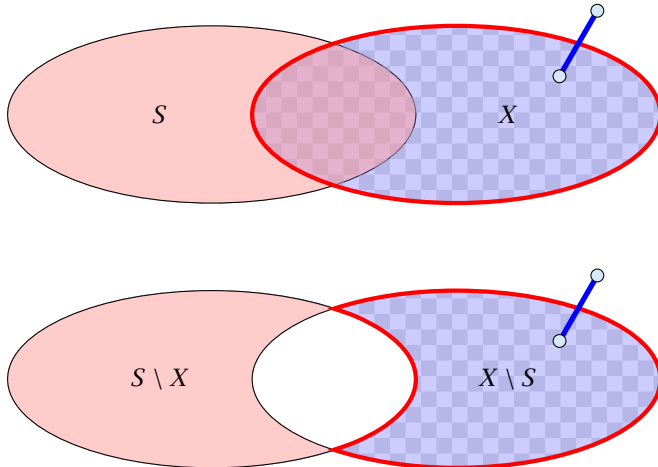
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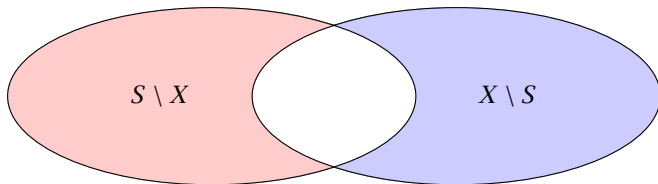
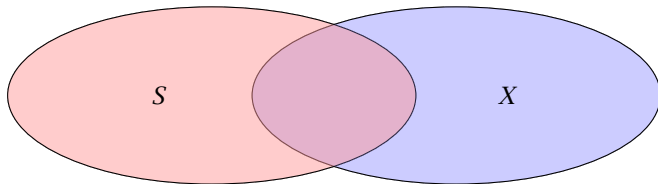
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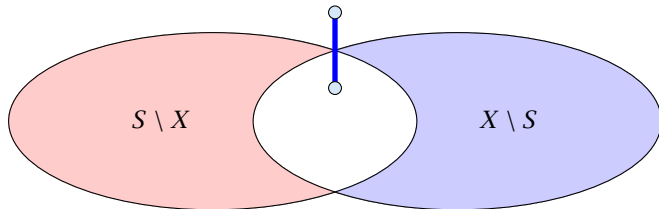
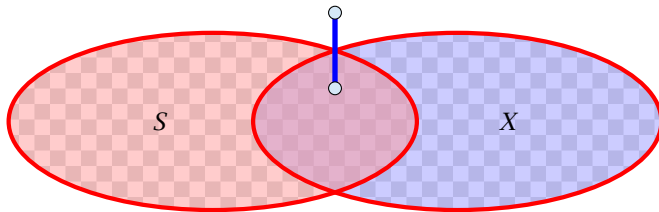
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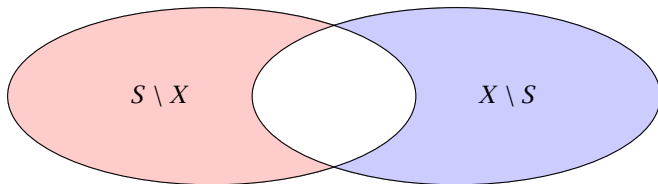
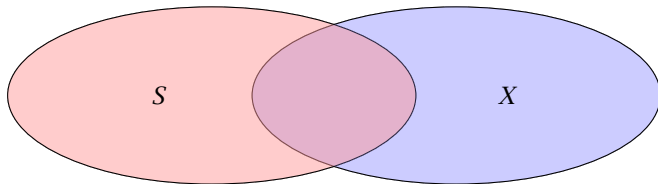
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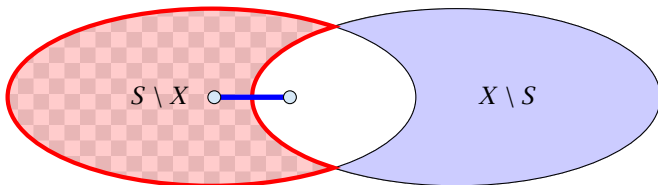
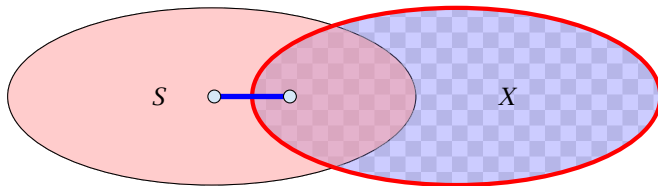


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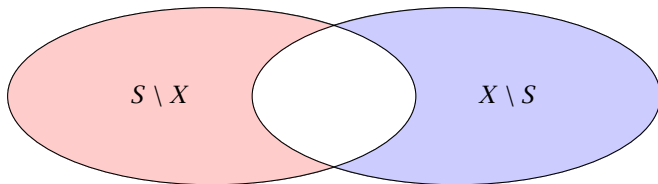
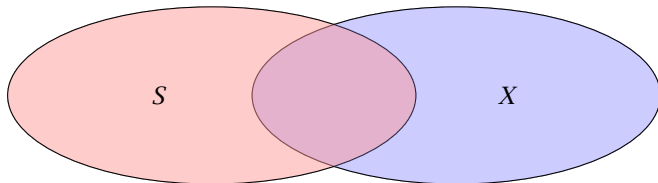




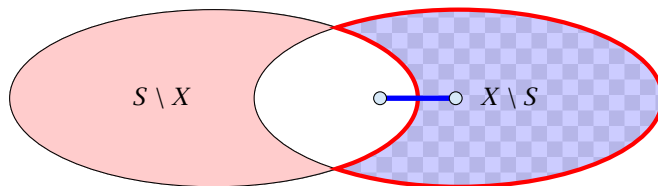
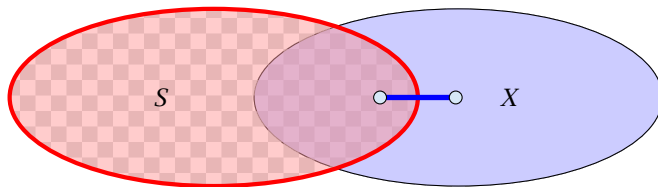
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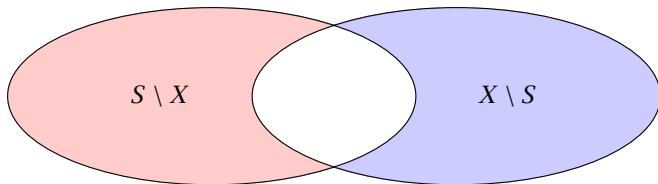
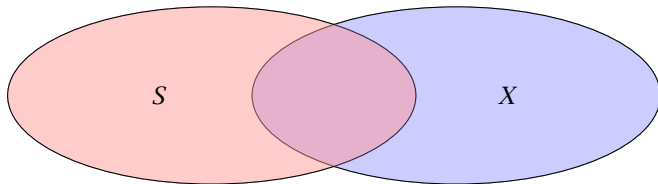
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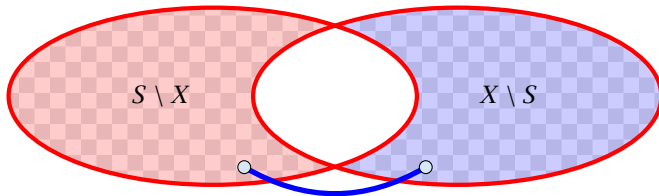
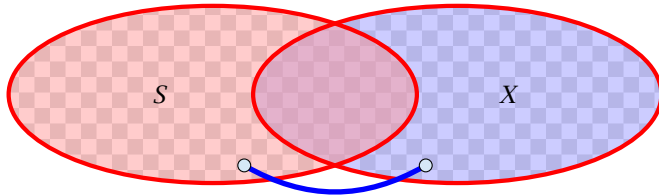
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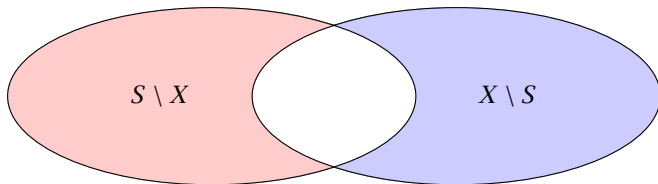
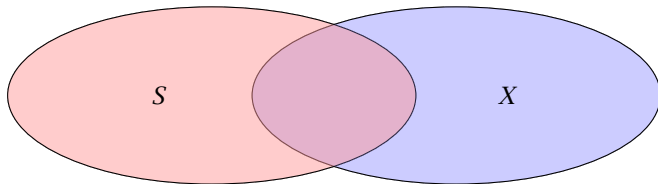
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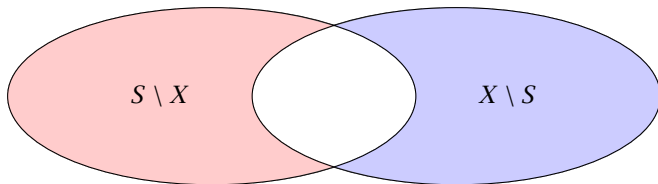
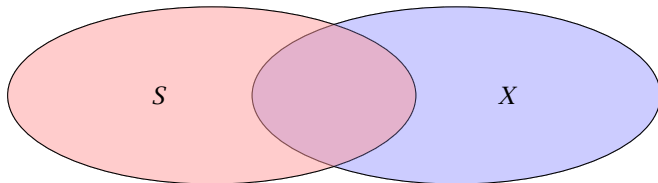
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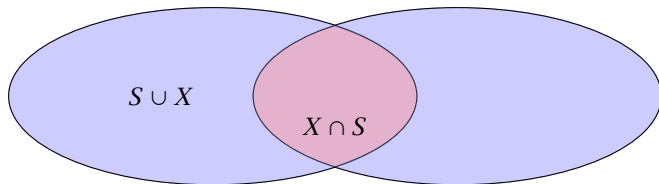
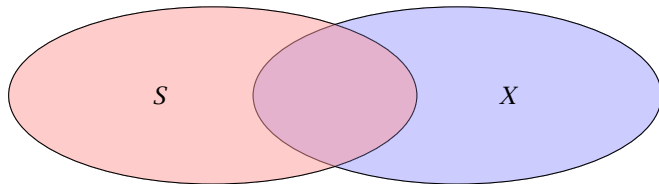
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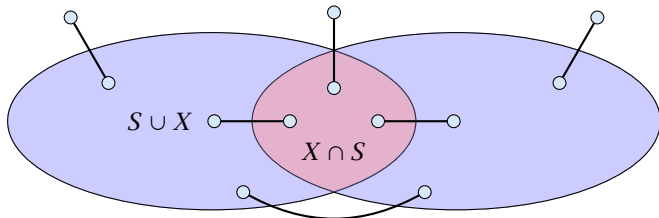
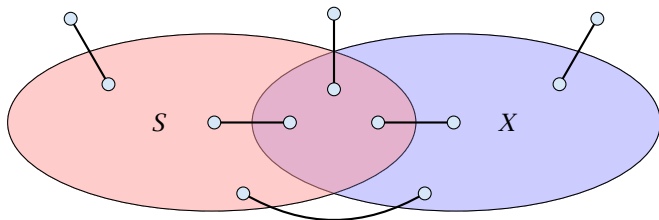


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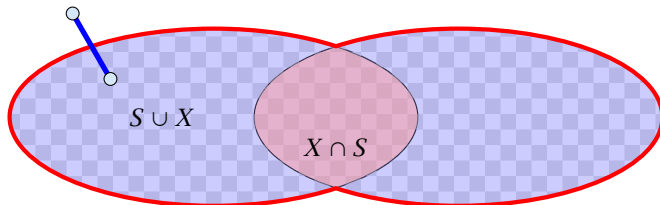
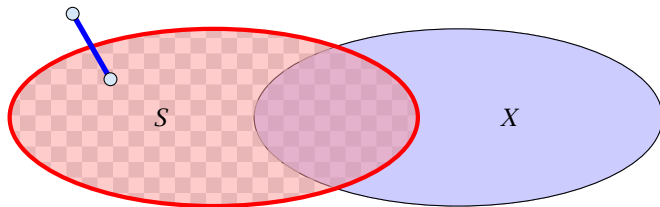




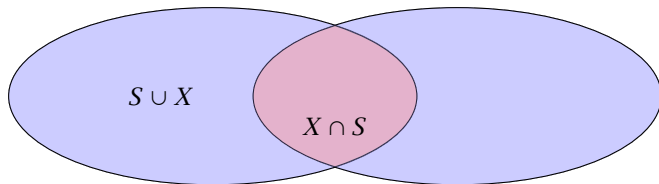
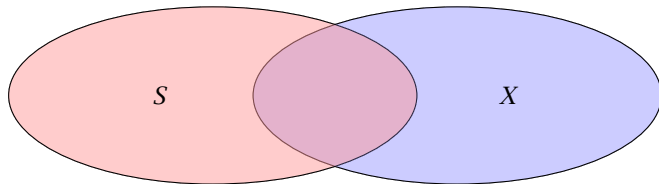
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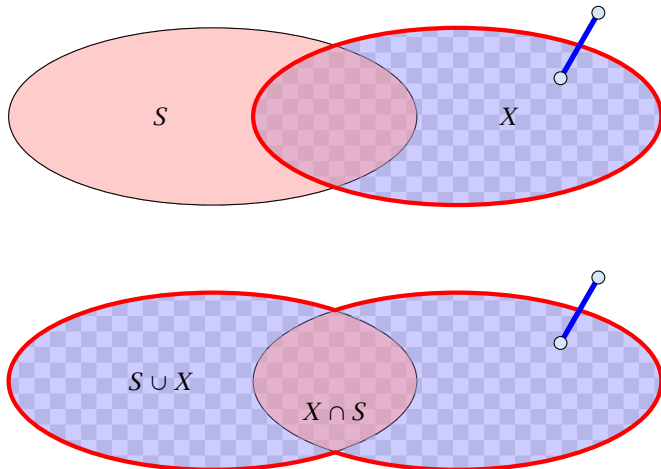
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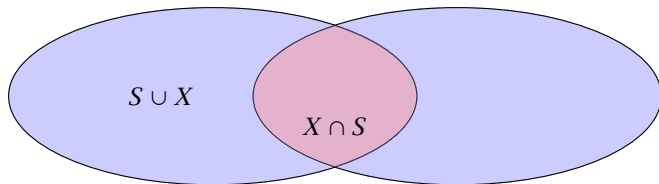
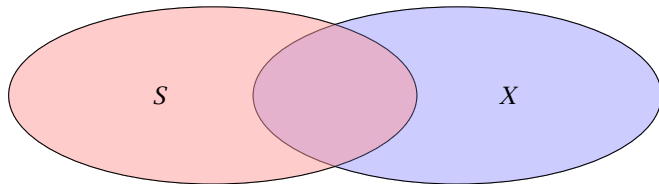
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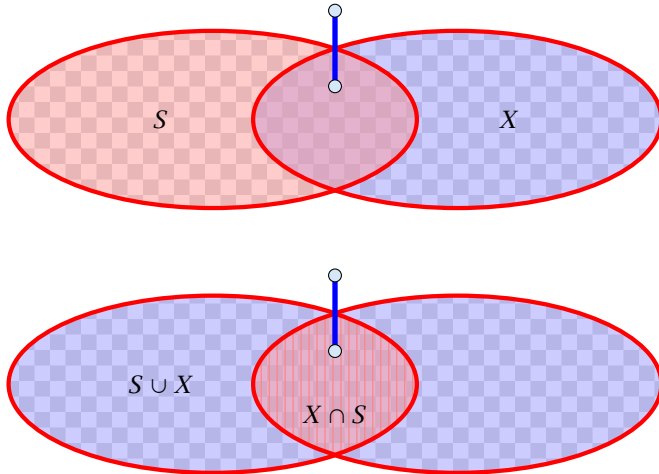
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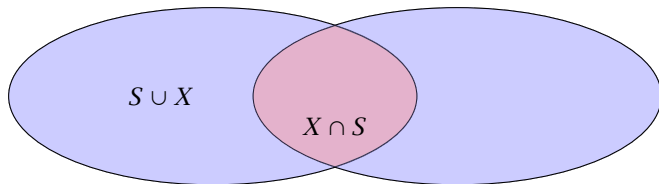
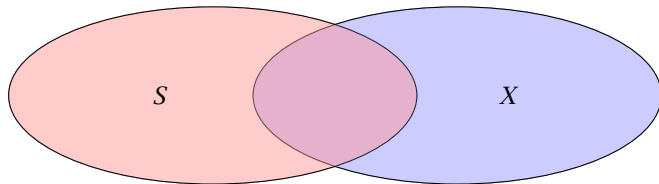
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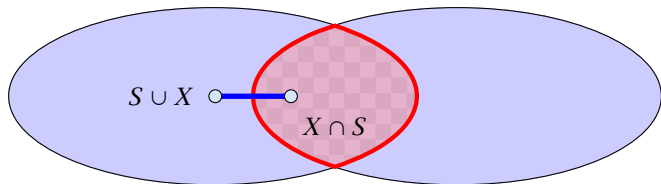
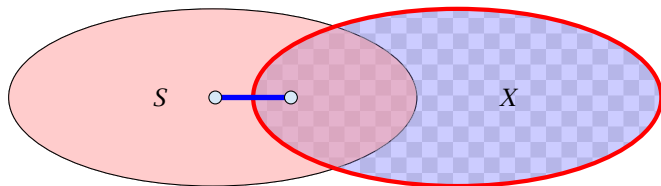
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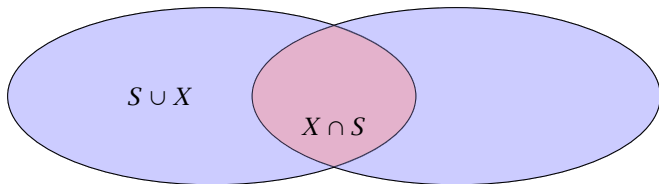
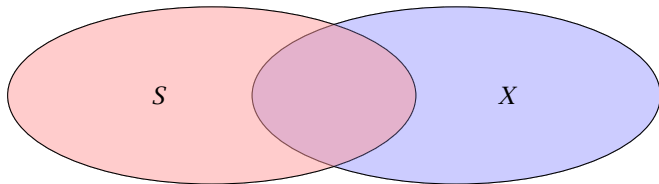


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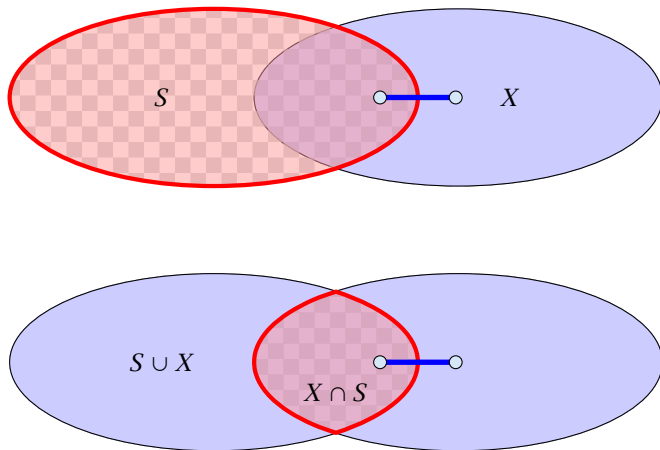




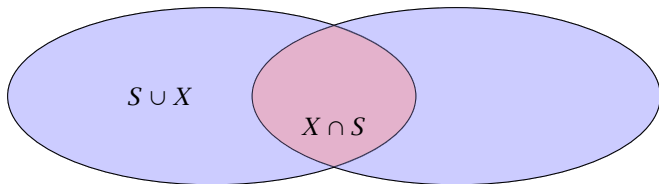
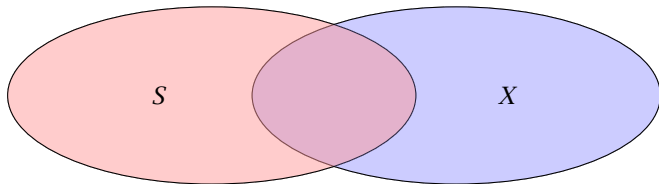
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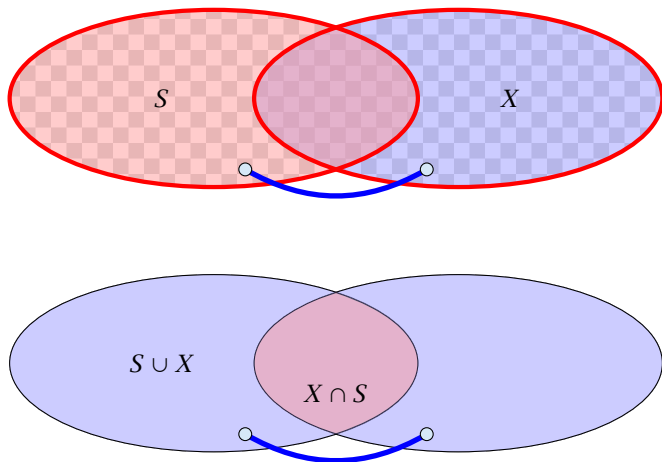
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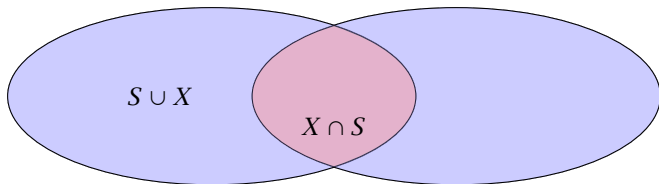
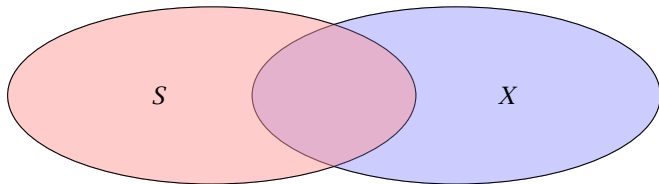
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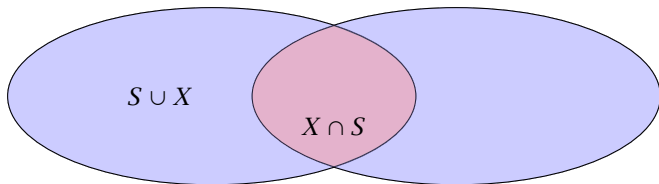
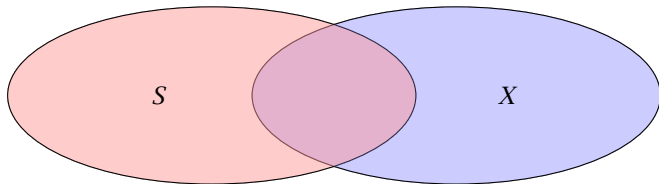
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# Analysis

Lemma 91 tells us that if we have a graph  $G = (V, E)$  and we contract a subset  $X \subset V$  that corresponds to some mincut, then the value of  $f(s, t)$  does not change for two nodes  $s, t \notin X$ .

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence,  $f_H(s, t) = f(s, t)$ , where  $f_H(s, t)$  is the value of a minimum  $s$ - $t$  mincut in graph  $H$ .

## Invariant [existence of representatives]:

For any edge  $\{S_i, S_j\}$  in  $T$ , there are vertices  $a \in S_i$  and  $b \in S_j$  such that  $w(S_i, S_j) = f(a, b)$  and the cut defined by edge  $\{S_i, S_j\}$  is a minimum  $a$ - $b$  cut in  $G$ .



## Analysis

We first show that the invariant implies that at the end of the algorithm  $T$  is indeed a cut-tree.

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$$f_T(s, t)$$

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$$\begin{aligned} f_T(s, t) &= \min_{i \in \{0, \dots, k-1\}} \{w(x_i, x_{i+1})\} \\ &= \min_{i \in \{0, \dots, k-1\}} \{f(x_i, x_{i+1})\} \end{aligned}$$

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- ▶ Let  $\{x_j, x_{j+1}\}$  be the edge with minimum weight on the path.
- ▶ Since by the invariant this edge induces an  $s$ - $t$  cut with capacity  $f(x_j, x_{j+1})$  we get  $f(s, t) \leq f(x_j, x_{j+1}) = f_T(s, t)$ .



# Analysis

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- ▶ By invariant, it forms a cut with capacity  $f(x_j, x_{j+1})$  in  $G$  (which separates  $s$  and  $t$ ).

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- ▶ The edge  $\{x_j, x_{j+1}\}$  is a mincut between  $s$  and  $t$  in  $T$ .
- ▶ By invariant, it forms a cut with capacity  $f(x_j, x_{j+1})$  in  $G$  (which separates  $s$  and  $t$ ).
- ▶ Since, we can send a flow of value  $f(x_j, x_{j+1})$  btw.  $s$  and  $t$ , this is an  $s$ - $t$  mincut (cut property).

# Proof of Invariant

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The invariant obviously holds at the beginning of the algorithm.

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Now, we show that it holds after a split-operation provided that it was true before the operation.

Let  $S_i$  denote our selected cluster with nodes  $a$  and  $b$ . Because of the invariant all edges leaving  $\{S_i\}$  in  $T$  correspond to some mincuts.



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After the split we have to choose representatives for all edges. For the new edge  $\{S_i^a, S_i^b\}$  with capacity  $w(S_i^a, S_i^b) = f_H(a, b)$  we can simply choose  $a$  and  $b$  as representatives.

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If  $s \in S_i^a$  we can keep  $x$  and  $s$  as representatives.

Otherwise, we choose  $x$  and  $a$  as representatives. We need to show that  $f(x, a) = f(x, s)$ .

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Because the invariant was true before the split we know that the edge  $\{X, S_i\}$  induces a cut in  $G$  of capacity  $f(x, s)$ . Since,  $x$  and  $a$  are on opposite sides of this cut, we know that  $f(x, a) \leq f(x, s)$ .

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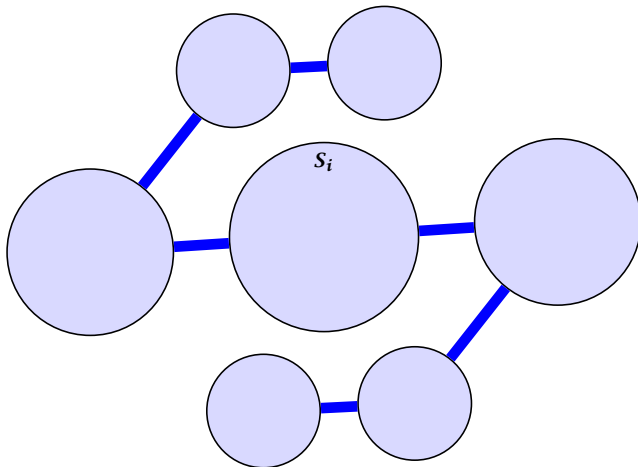
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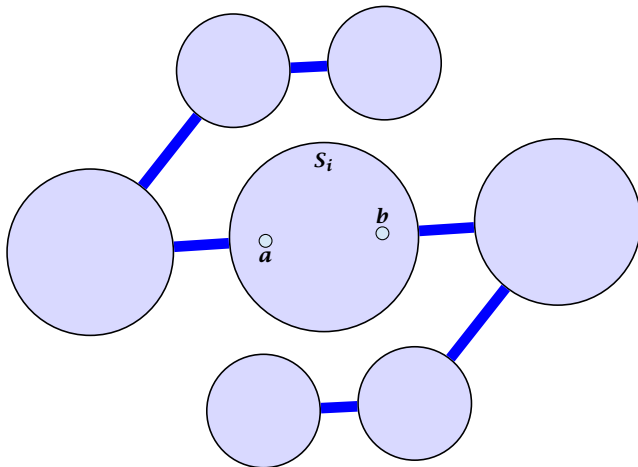
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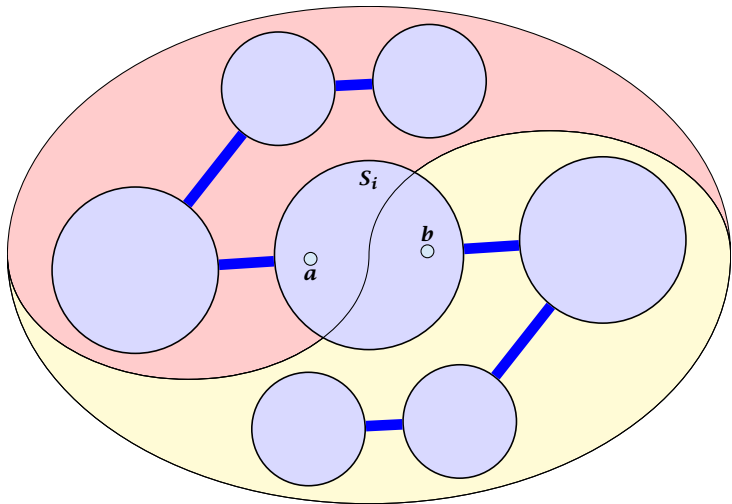
Also,  $f'(a, v_B) \geq f(a, b) \geq f(x, s)$  since the  $a$ - $b$  cut that splits  $S_i$  into  $S_i^a$  and  $S_i^b$  also separates  $s$  and  $x$ .



# Analysis

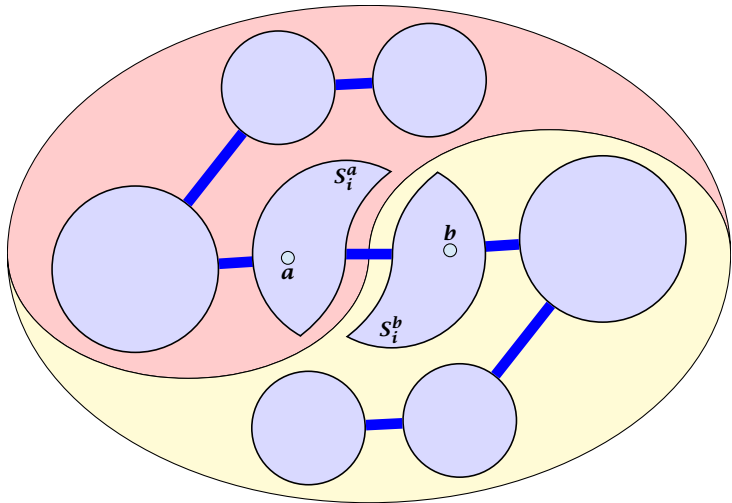


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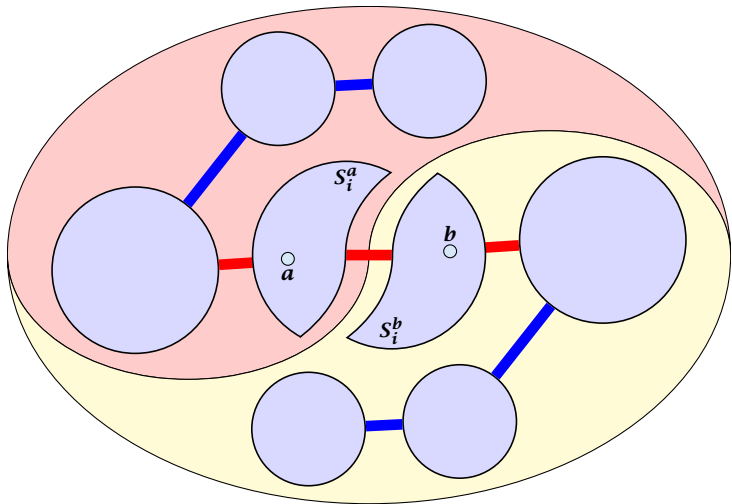




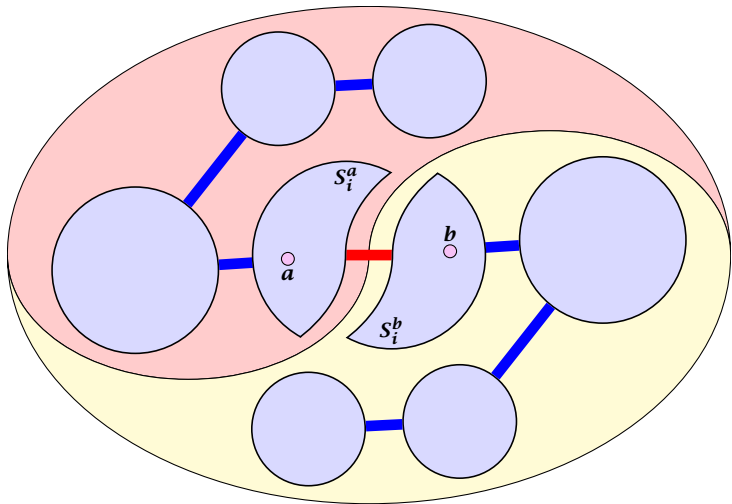
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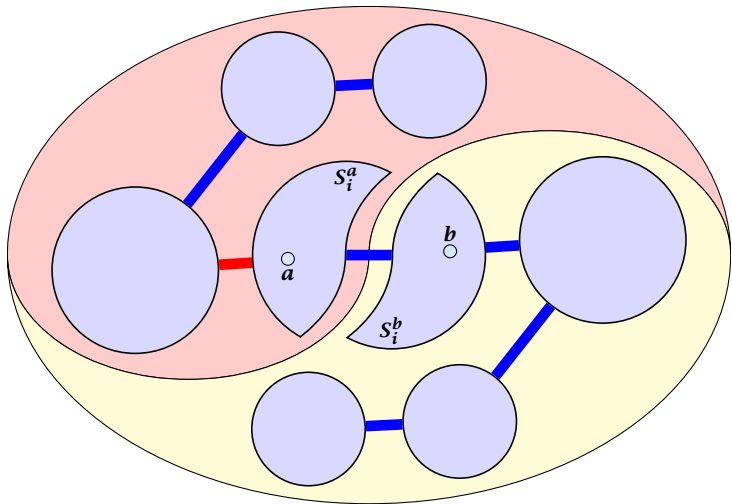
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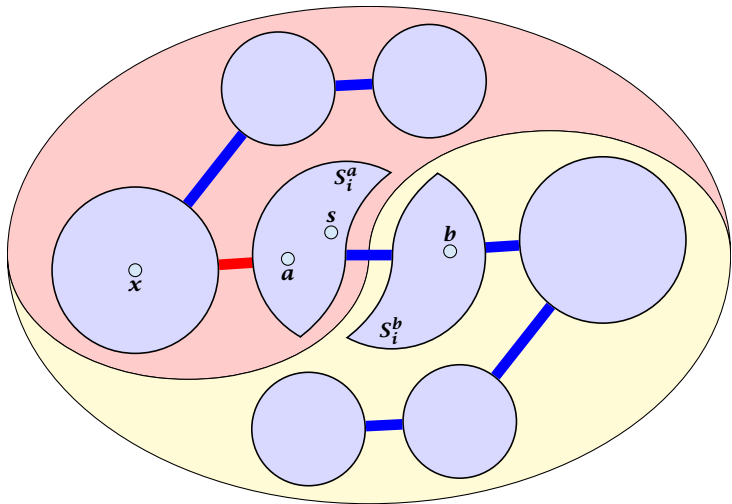
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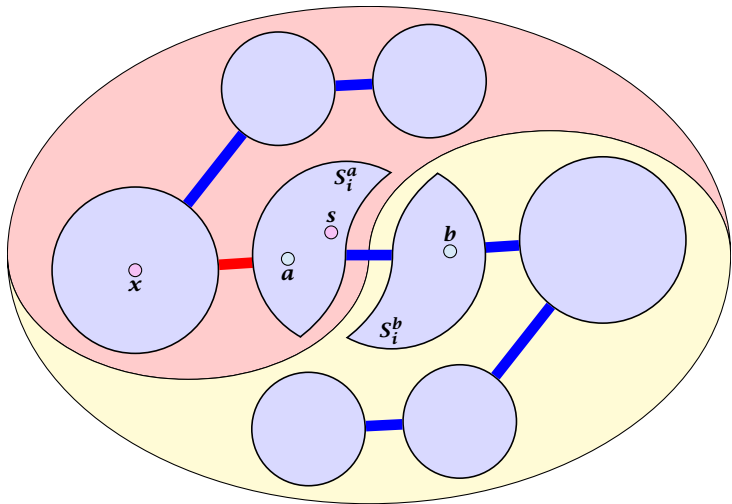
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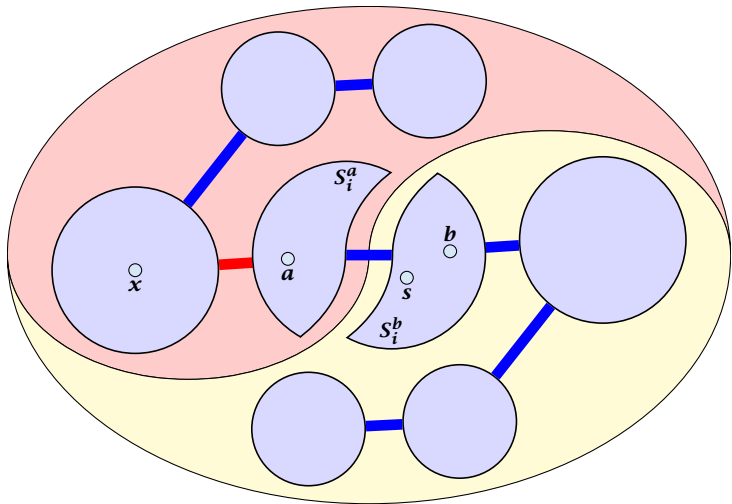
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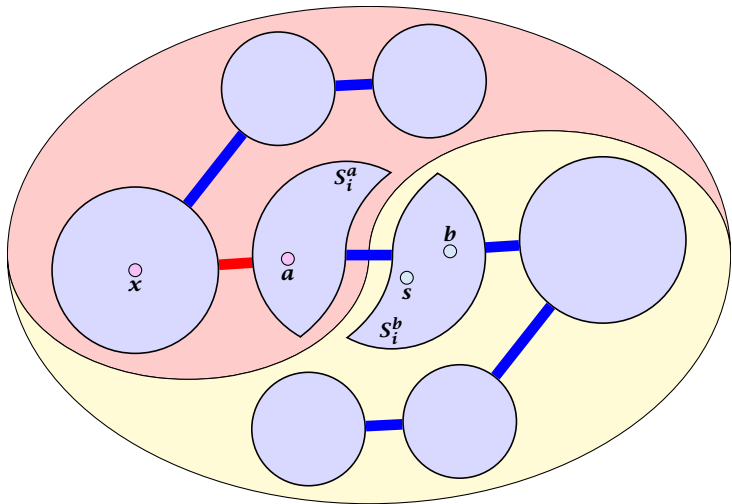
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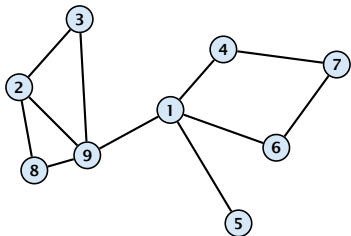
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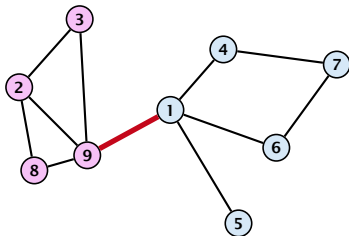
# 17 Global Mincut

Given an **undirected, capacitated graph**  $G = (V, E, c)$  find a partition of  $V$  into two non-empty sets  $S, V \setminus S$  s.t. the capacity of edges between both sets is minimized.



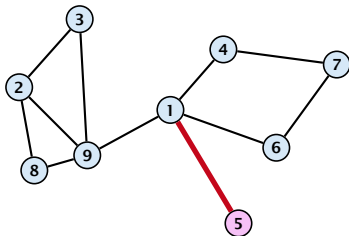
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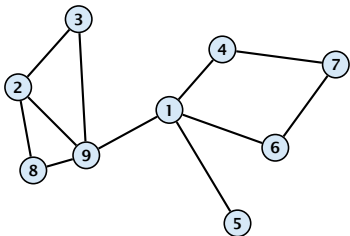
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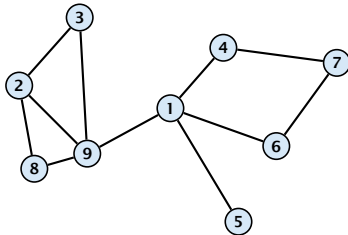
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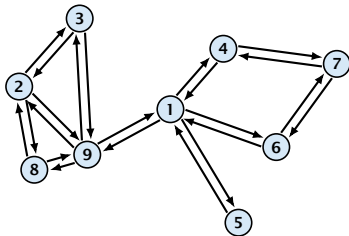
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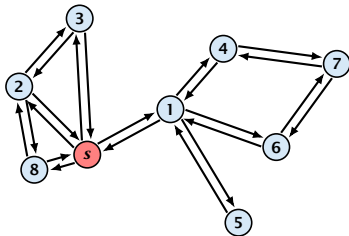
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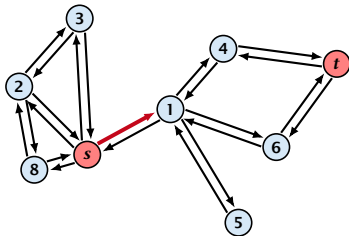
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- ▶ Let  $(S, V \setminus S)$  be a minimum global mincut. The above algorithm will output a cut of capacity  $\text{cap}(S, V \setminus S)$  whenever  $|\{s, t\} \cap S| = 1$ .





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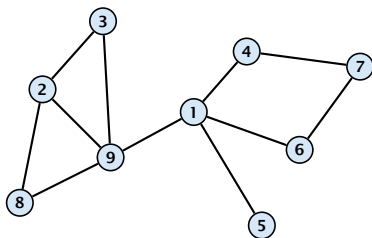
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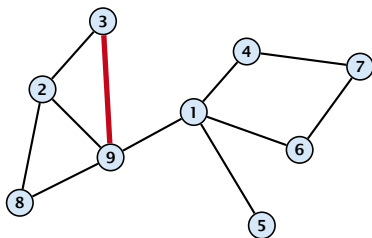
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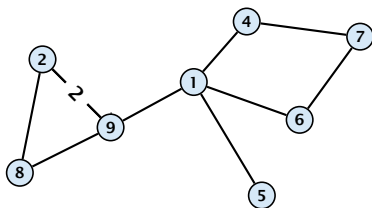
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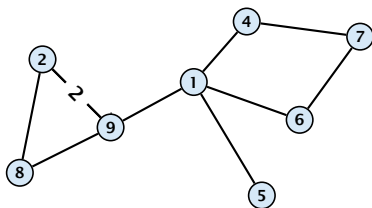
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- ▶ Edge-contractions do not decrease the size of the mincut.



# Edge Contractions

We can perform an edge-contraction in time  $\mathcal{O}(n)$ .

# Randomized Mincut Algorithm

**Algorithm 1** KargerMincut( $G = (V, E, c)$ )

- 1: **for**  $i = 1 \rightarrow n - 2$  **do**
- 2:     choose  $e \in E$  randomly with probability  $c(e)/c(E)$
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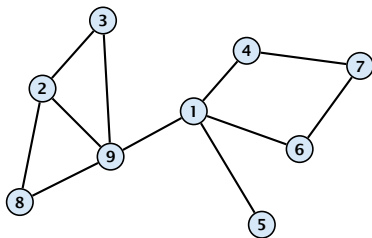
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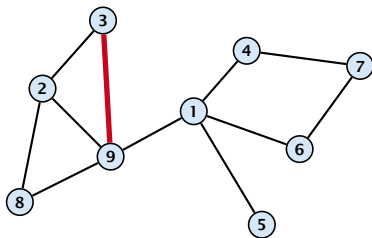
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- ▶ What is the probability that this algorithm returns a mincut?

# Example: Randomized Mincut Algorithm

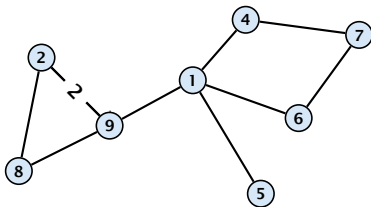


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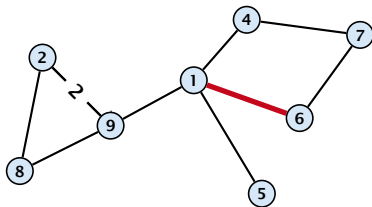




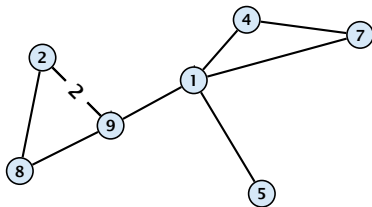
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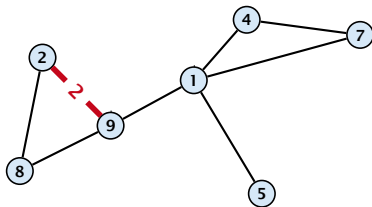
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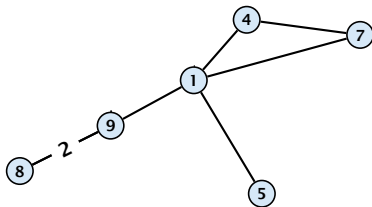
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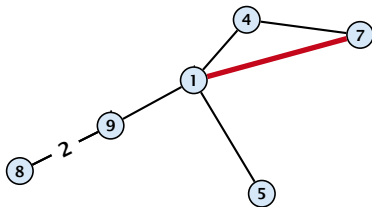
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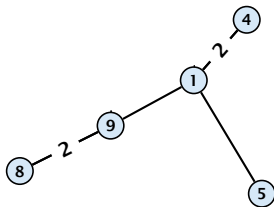
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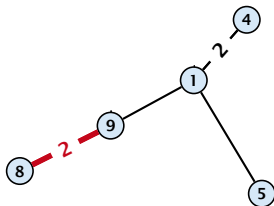
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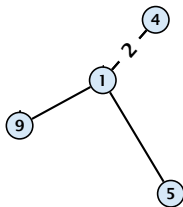


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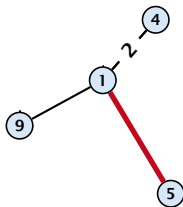




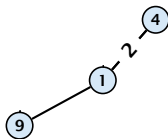
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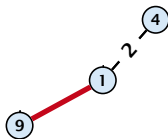
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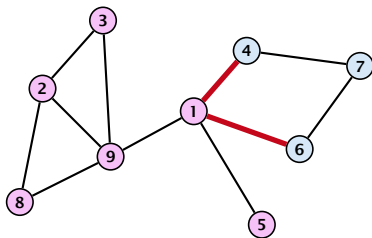
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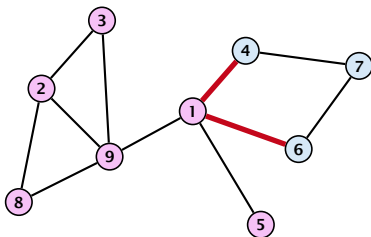
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**What is the probability that this algorithm returns a mincut?**



**What is the probability that a given mincut  $A$  is still possible after round  $i$ ?**

- ▶ It is still possible to obtain cut  $A$  in the end if so far **no** edge in  $(A, V \setminus A)$  has been contracted.

# Analysis

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- ▶ Hence, the probability of choosing an edge from the cut is at most  $\min / c(E) \leq 2 / (n - i + 1)$ .

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The probability that we do **not** choose an edge from the cut in iteration  $i$  is

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Choosing  $t = 2$  gives that with probability  $1/\binom{n}{2}$  the algorithm computes a mincut.

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## Theorem 93

*The randomized mincut algorithm computes an optimal cut with high probability. The total running time is  $\mathcal{O}(n^4 \log n)$ .*



# Improved Algorithm

**Algorithm 2** RecursiveMincut( $G = (V, E, c)$ )

```
1: for  $i = 1 \rightarrow n - n/\sqrt{2}$  do  
2:   choose  $e \in E$  randomly with probability  $c(e)/c(E)$   
3:    $G \leftarrow G/e$   
4: if  $|V| = 2$  return cut-value;  
5:  $cuta \leftarrow$  RecursiveMincut( $G$ );  
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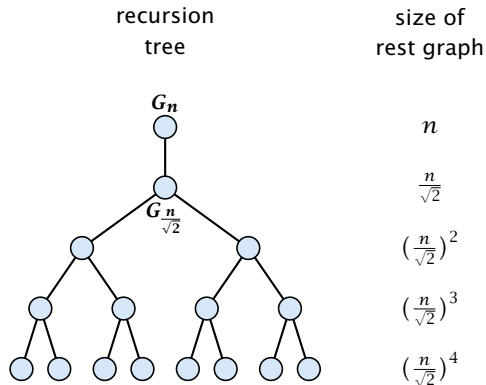
# Probability of Success

The probability of not contracting an edge from the mincut during one iteration through the for-loop is at least

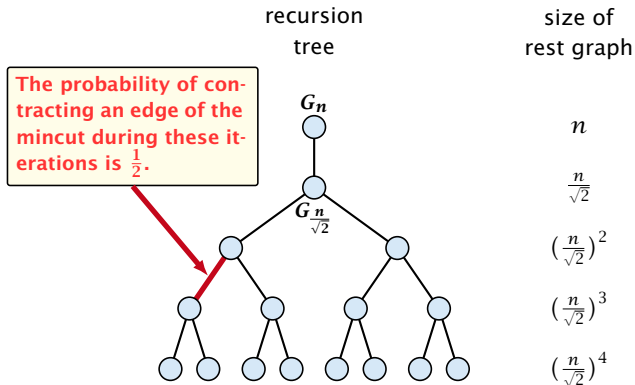
$$\frac{t(t-1)}{n(n-1)} \geq \frac{t^2}{n^2} = \frac{1}{2} ,$$

as  $t = \frac{n}{\sqrt{2}}$ .

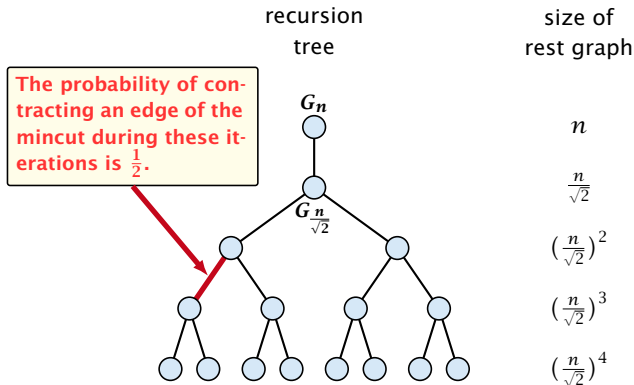
# Probability of Success



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# Probability of Success



We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability  $\frac{1}{2}$ . If in the end you have a path from the root to **at least one** leaf node you are successful.

# Probability of Success

Let for an edge  $e$  in the recursion tree,  $h(e)$  denote the height (distance to leaf level) of the parent-node of  $e$  (end-point that is higher up in the tree). Let  $h$  denote the height of the root node.



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## Lemma 94

*The probability that an edge  $e$  is alive is at least  $\frac{1}{h(e)+1}$ .*

# Probability of Success

## Proof.

- ▶ An edge  $e$  with  $h(e) = 1$  is alive if and only if it is not deleted.  
Hence, it is alive with probability at least  $\frac{1}{2}$ .

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- ▶ An edge  $e$  with  $h(e) = 1$  is alive if and only if it is not deleted. Hence, it is alive with probability at least  $\frac{1}{2}$ .
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$$p_d$$

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# 17 Global Mincut

## Lemma 95

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*Doing  $\Theta(\log^2 n)$  runs gives that the algorithm succeeds with high probability. The total running time is  $\mathcal{O}(n^2 \log^3 n)$ .*

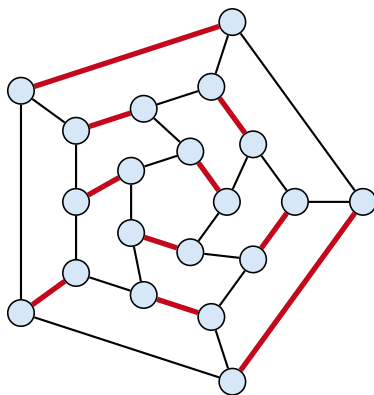
# Part V

## Matchings



# Matching

- ▶ Input: undirected graph  $G = (V, E)$ .
- ▶  $M \subseteq E$  is a **matching** if each node appears in at most one edge in  $M$ .
- ▶ Maximum Matching: find a matching of maximum cardinality



# 19 Bipartite Matching via Flows

## Which flow algorithm to use?

- ▶ Generic augmenting path:  $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$ .
- ▶ Capacity scaling:  $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$ .
- ▶ Shortest augmenting path:  $\mathcal{O}(mn^2)$ .

For **unit capacity simple graphs** shortest augmenting path can be implemented in time  $\mathcal{O}(m\sqrt{n})$ .

## 20 Augmenting Paths for Matchings

### Definitions.

- ▶ Given a matching  $M$  in a graph  $G$ , a vertex that is not incident to any edge of  $M$  is called a **free vertex** w. r. .t.  $M$ .

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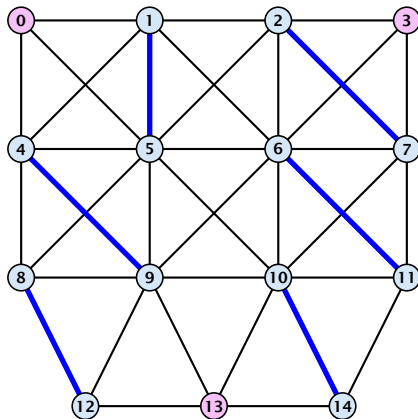
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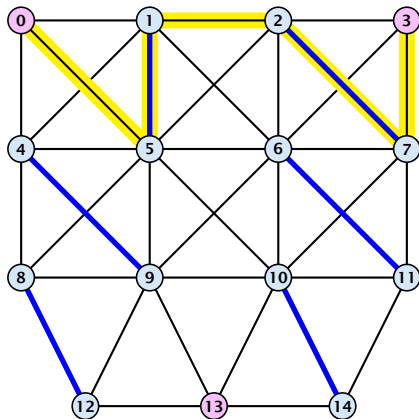
### Theorem 96

*A matching  $M$  is a maximum matching if and only if there is no augmenting path w. r. t.  $M$ .*

# Augmenting Paths in Action

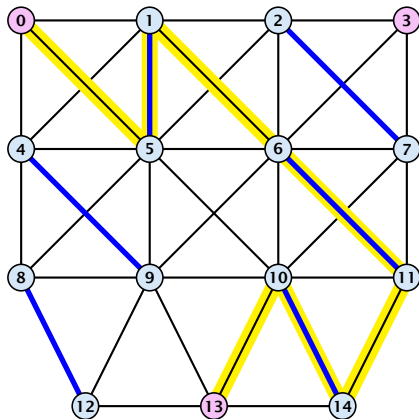


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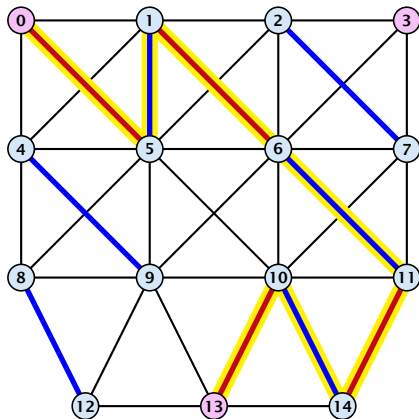




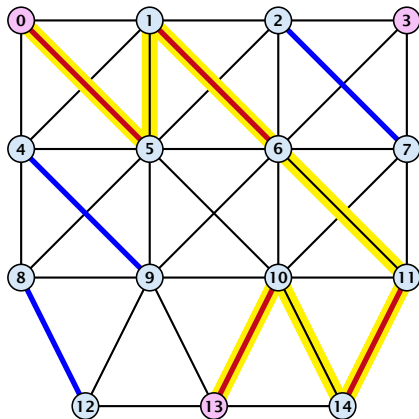
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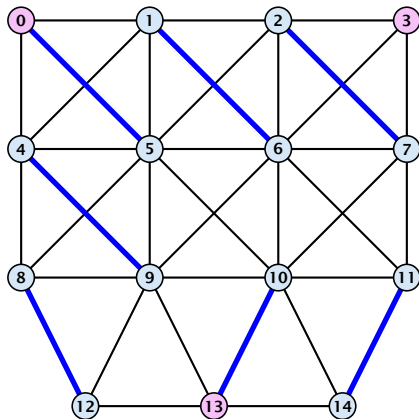
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## 20 Augmenting Paths for Matchings

### Proof.

- ⇒ If  $M$  is maximum there is no augmenting path  $P$ , because we could switch matching and non-matching edges along  $P$ . This gives matching  $M' = M \oplus P$  with larger cardinality.

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- ⇐ Suppose there is a matching  $M'$  with larger cardinality. Consider the graph  $H$  with edge-set  $M' \oplus M$  (i.e., only edges that are in either  $M$  or  $M'$  but not in both).

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As  $|M'| > |M|$  there is one connected component that is a path  $P$  for which both endpoints are incident to edges from  $M'$ .  $P$  is an augmenting path.



## 20 Augmenting Paths for Matchings

### **Algorithmic idea:**

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

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### Theorem 97

Let  $G$  be a graph,  $M$  a matching in  $G$ , and let  $u$  be a free vertex w.r.t.  $M$ . Further let  $P$  denote an augmenting path w.r.t.  $M$  and let  $M' = M \oplus P$  denote the matching resulting from augmenting  $M$  with  $P$ . If there was no augmenting path starting at  $u$  in  $M$  then there is no augmenting path starting at  $u$  in  $M'$ .

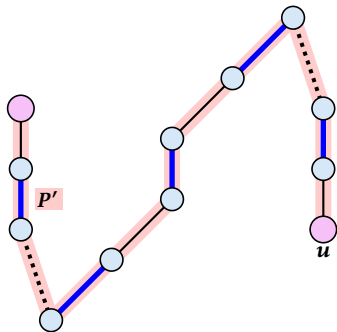
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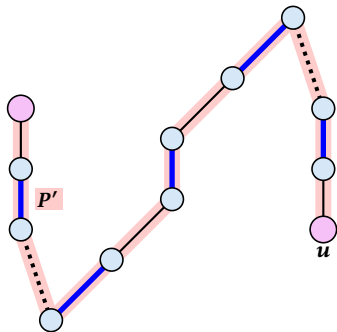
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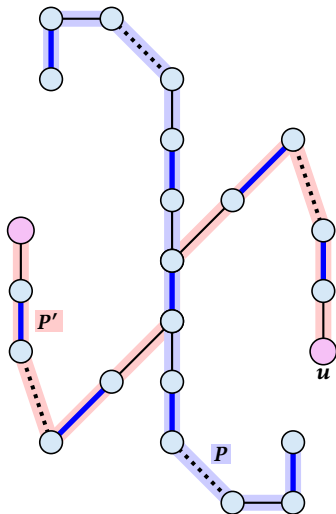
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- ▶ If  $P'$  and  $P$  are node-disjoint,  $P'$  is also augmenting path w.r.t.  $M$  ( $\neq$ ).



## 20 Augmenting Paths for Matchings

### Proof

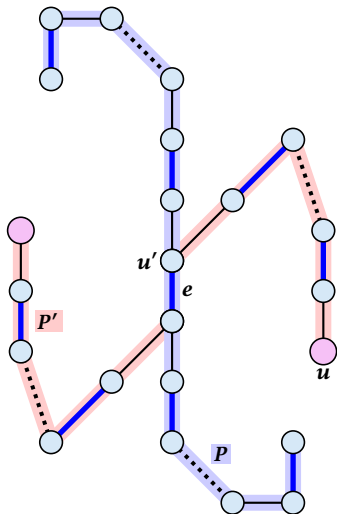
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# 20 Augmenting Paths for Matchings

## Proof

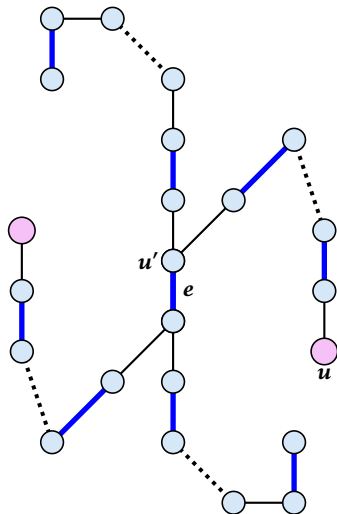
- ▶ Assume there is an augmenting path  $P'$  w.r.t.  $M'$  starting at  $u$ .
- ▶ If  $P'$  and  $P$  are node-disjoint,  $P'$  is also augmenting path w.r.t.  $M$  ( $\neq$ ).
- ▶ Let  $u'$  be the **first** node on  $P'$  that is in  $P$ , and let  $e$  be the matching edge from  $M'$  incident to  $u'$ .



# 20 Augmenting Paths for Matchings

## Proof

- ▶ Assume there is an augmenting path  $P'$  w.r.t.  $M'$  starting at  $u$ .
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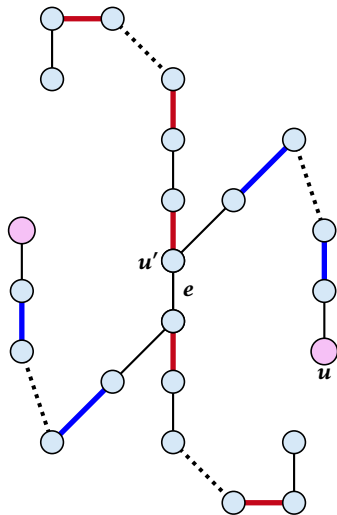




# 20 Augmenting Paths for Matchings

## Proof

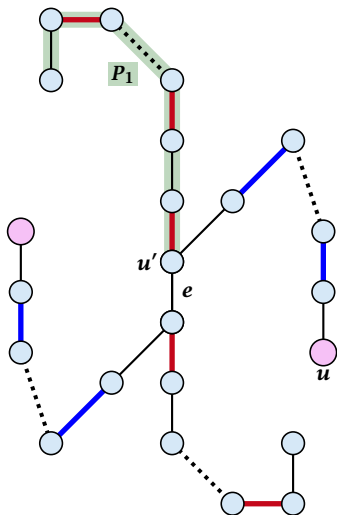
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- ▶ Let  $u'$  be the **first** node on  $P'$  that is in  $P$ , and let  $e$  be the matching edge from  $M'$  incident to  $u'$ .



## 20 Augmenting Paths for Matchings

### Proof

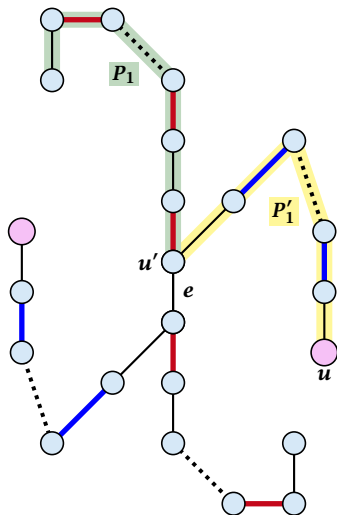
- ▶ Assume there is an augmenting path  $P'$  w.r.t.  $M'$  starting at  $u$ .
- ▶ If  $P'$  and  $P$  are node-disjoint,  $P'$  is also augmenting path w.r.t.  $M$  ( $\neq$ ).
- ▶ Let  $u'$  be the **first** node on  $P'$  that is in  $P$ , and let  $e$  be the matching edge from  $M'$  incident to  $u'$ .
- ▶  $u'$  splits  $P$  into two parts one of which does not contain  $e$ . Call this part  $P_1$ . Denote the sub-path of  $P'$  from  $u$  to  $u'$  with  $P'_1$ .



## 20 Augmenting Paths for Matchings

### Proof

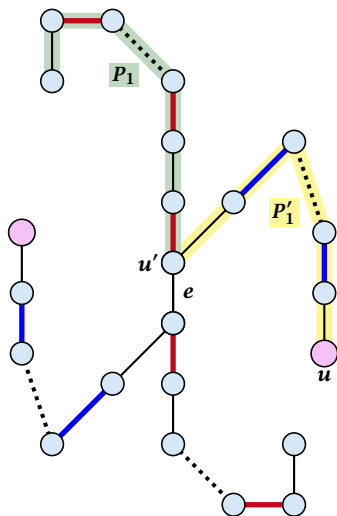
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- ▶ If  $P'$  and  $P$  are node-disjoint,  $P'$  is also augmenting path w.r.t.  $M$  ( $\neq$ ).
- ▶ Let  $u'$  be the **first** node on  $P'$  that is in  $P$ , and let  $e$  be the matching edge from  $M'$  incident to  $u'$ .
- ▶  $u'$  splits  $P$  into two parts one of which does not contain  $e$ . Call this part  $P_1$ . Denote the sub-path of  $P'$  from  $u$  to  $u'$  with  $P'_1$ .



## 20 Augmenting Paths for Matchings

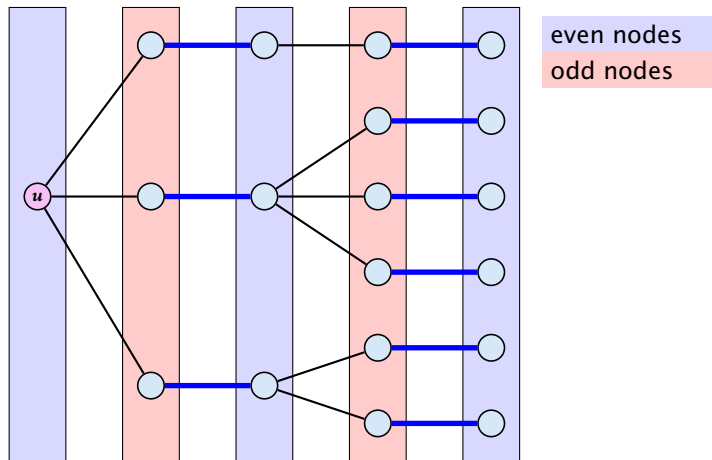
### Proof

- ▶ Assume there is an augmenting path  $P'$  w.r.t.  $M'$  starting at  $u$ .
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- ▶ Let  $u'$  be the **first** node on  $P'$  that is in  $P$ , and let  $e$  be the matching edge from  $M'$  incident to  $u'$ .
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- ▶  $P_1 \circ P'_1$  is augmenting path in  $M$  ( $\cancel{\neq}$ ).



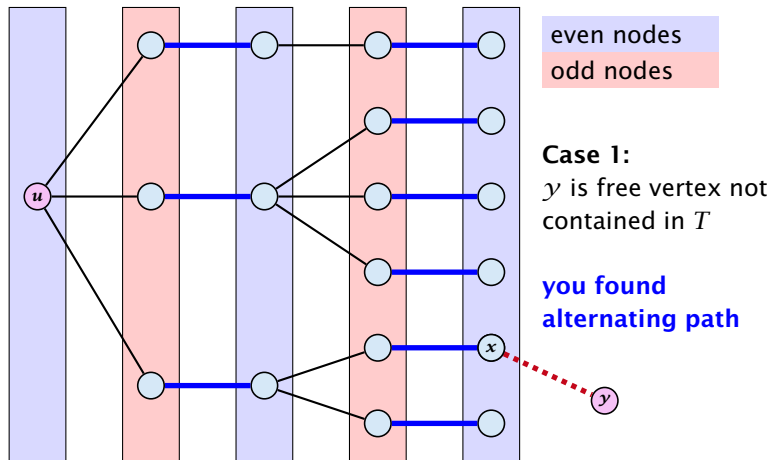
# How to find an augmenting path?

Construct an alternating tree.



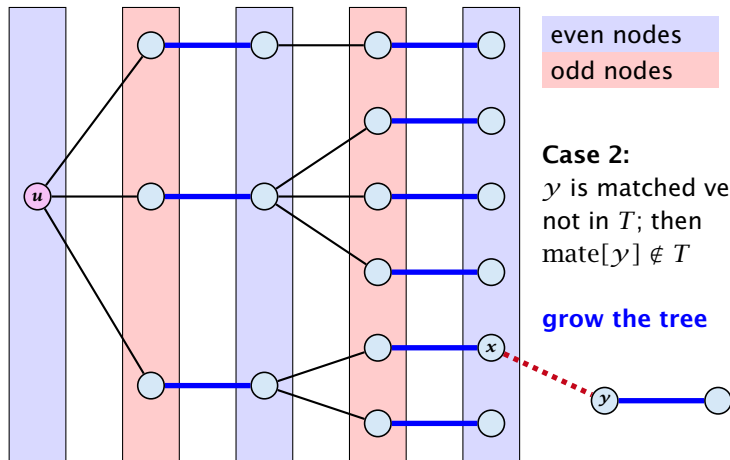
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Construct an alternating tree.



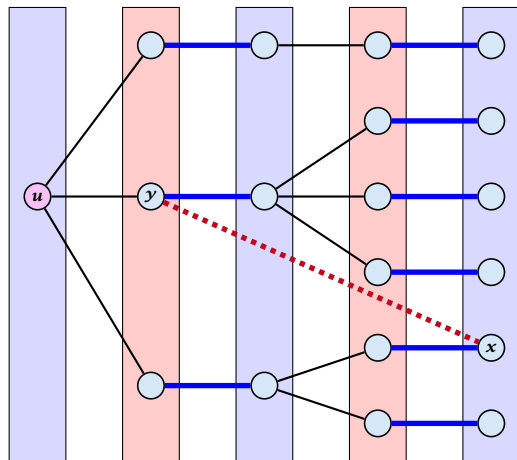
# How to find an augmenting path?

Construct an alternating tree.



# How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

**Case 3:**

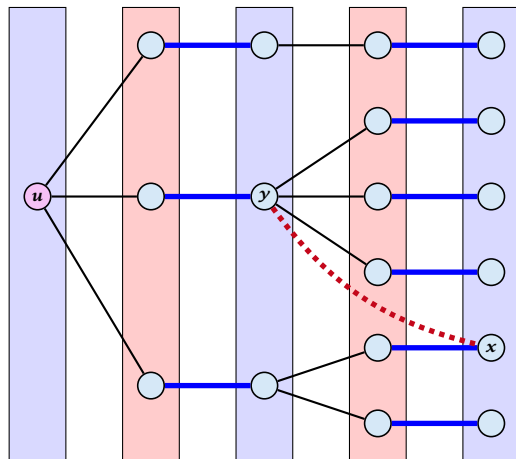
$y$  is already contained  
in  $T$  as an odd vertex

**ignore successor  $y$**



# How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

**Case 4:**

$y$  is already contained  
in  $T$  as an even vertex

can't ignore  $y$

does not happen in  
bipartite graphs

### Algorithm 75 BiMatch( $G, match$ )

```
1: for  $x \in V$  do  $mate[x] \leftarrow 0$ ;  
2:  $r \leftarrow 0$ ;  $free \leftarrow n$ ;  
3: while  $free \geq 1$  and  $r < n$  do  
4:    $r \leftarrow r + 1$   
5:   if  $mate[r] = 0$  then  
6:     for  $i = 1$  to  $n$  do  $parent[i'] \leftarrow 0$   
7:      $Q \leftarrow \emptyset$ ;  $Q.append(r)$ ;  $aug \leftarrow false$ ;  
8:     while  $aug = false$  and  $Q \neq \emptyset$  do  
9:        $x \leftarrow Q.dequeue()$ ;  
10:      for  $y \in A_x$  do  
11:        if  $mate[y] = 0$  then  
12:           $augm(mate, parent, y)$ ;  
13:           $aug \leftarrow true$ ;  
14:           $free \leftarrow free - 1$ ;  
15:        else  
16:          if  $parent[y] = 0$  then  
17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

graph  $G = (S \cup S', E)$

$S = \{1, \dots, n\}$

$S' = \{1', \dots, n'\}$

### Algorithm 75 BiMatch( $G, match$ )

```
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```

start with an  
empty matching

### Algorithm 75 BiMatch( $G, match$ )

```
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16:        if  $parent[y] = 0$  then  
17:           $parent[y] \leftarrow x$ ;  
18:           $Q.enqueue(mate[y])$ ;
```

*free*: number of  
unmatched nodes in  $S$

*r*: root of current tree

### Algorithm 75 BiMatch( $G, match$ )

```
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17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

as long as there are  
unmatched nodes and  
we did not yet try to  
grow from all nodes we  
continue

### Algorithm 75 BiMatch( $G, match$ )

```
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18:             $Q.enqueue(mate[y])$ ;
```

$r$  is the new node that we grow from.

### Algorithm 75 BiMatch( $G, match$ )

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18:             $Q.enqueue(mate[y])$ ;
```

If  $r$  is free start tree construction

### Algorithm 75 BiMatch( $G, match$ )

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16:          if  $parent[y] = 0$  then  
17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

Initialize an empty tree.  
Note that only nodes  $i'$   
have parent pointers.



### Algorithm 75 BiMatch( $G, match$ )

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18:           $Q.enqueue(mate[y])$ ;
```

$Q$  is a queue (BFS!!!).

$aug$  is a Boolean that stores whether we already found an augmenting path.

### Algorithm 75 BiMatch( $G, match$ )

```
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16:          if  $parent[y] = 0$  then  
17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

as long as we did not augment and there are still unexamined leaves continue...

### Algorithm 75 BiMatch( $G, match$ )

```
1: for  $x \in V$  do  $mate[x] \leftarrow 0$ ;  
2:  $r \leftarrow 0$ ;  $free \leftarrow n$ ;  
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16:          if  $parent[y] = 0$  then  
17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

take next unexamined  
leaf

### Algorithm 75 BiMatch( $G, match$ )

```
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15:        else  
16:          if  $parent[y] = 0$  then  
17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

if  $x$  has unmatched neighbour we found an augmenting path (note that  $y \neq r$  because we are in a bipartite graph)

### Algorithm 75 BiMatch( $G, match$ )

```
1: for  $x \in V$  do  $mate[x] \leftarrow 0$ ;  
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16:          if  $parent[y] = 0$  then  
17:             $parent[y] \leftarrow x$ ;  
18:             $Q.enqueue(mate[y])$ ;
```

do an augmentation...

### Algorithm 75 BiMatch( $G, match$ )

```
1: for  $x \in V$  do  $mate[x] \leftarrow 0$ ;  
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17:           $parent[y] \leftarrow x$ ;  
18:           $Q.enqueue(mate[y])$ ;
```

setting  $aug = true$   
ensures that the tree  
construction will not  
continue

### Algorithm 75 BiMatch( $G, match$ )

```
1: for  $x \in V$  do  $mate[x] \leftarrow 0$ ;  
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17:           $parent[y] \leftarrow x$ ;  
18:           $Q.enqueue(mate[y])$ ;
```

reduce number of free  
nodes

### Algorithm 75 BiMatch( $G, match$ )

```
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6:     for  $i = 1$  to  $n$  do  $parent[i'] \leftarrow 0$   
7:      $Q \leftarrow \emptyset$ ;  $Q.append(r)$ ;  $aug \leftarrow false$ ;  
8:     while  $aug = false$  and  $Q \neq \emptyset$  do  
9:        $x \leftarrow Q.dequeue()$ ;  
10:      for  $y \in A_x$  do  
11:        if  $mate[y] = 0$  then  
12:           $augm(mate, parent, y)$ ;  
13:           $aug \leftarrow true$ ;  
14:           $free \leftarrow free - 1$ ;  
15:      else  
16:        if  $parent[y] = 0$  then  
17:           $parent[y] \leftarrow x$ ;  
18:           $Q.enqueue(mate[y])$ ;
```

if  $y$  is not in the tree yet



### Algorithm 75 BiMatch( $G, match$ )

```
1: for  $x \in V$  do  $mate[x] \leftarrow 0$ ;  
2:  $r \leftarrow 0$ ;  $free \leftarrow n$ ;  
3: while  $free \geq 1$  and  $r < n$  do  
4:    $r \leftarrow r + 1$   
5:   if  $mate[r] = 0$  then  
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...put it into the tree

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add its buddy to the set  
of unexamined leaves

## 21 Weighted Bipartite Matching

### Weighted Bipartite Matching/Assignment

- ▶ Input: undirected, bipartite graph  $G = L \cup R, E$ .
- ▶ an edge  $e = (\ell, r)$  has weight  $w_e \geq 0$
- ▶ find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

### Simplifying Assumptions (wlog [why?]):

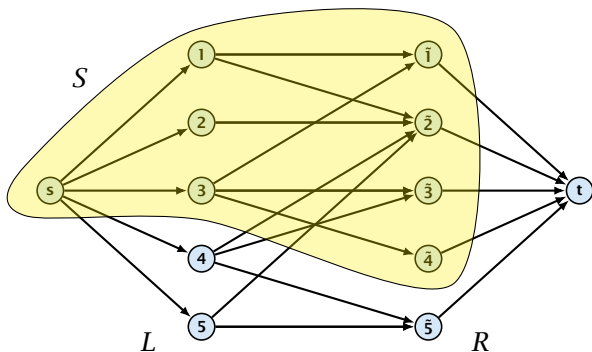
- ▶ assume that  $|L| = |R| = n$
- ▶ assume that there is an edge between every pair of nodes  $(\ell, r) \in V \times V$
- ▶ can assume goal is to construct maximum weight **perfect** matching

# Weighted Bipartite Matching

## Theorem 98 (Halls Theorem)

A bipartite graph  $G = (L \cup R, E)$  has a perfect matching if and only if for all sets  $S \subseteq L$ ,  $|\Gamma(S)| \geq |S|$ , where  $\Gamma(S)$  denotes the set of nodes in  $R$  that have a neighbour in  $S$ .

## 21 Weighted Bipartite Matching



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  - ▶ Let  $S$  denote a minimum cut and let  $L_S \stackrel{\text{def}}{=} L \cap S$  and  $R_S \stackrel{\text{def}}{=} R \cap S$  denote the portion of  $S$  inside  $L$  and  $R$ , respectively.



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  - ▶ The size of the cut is  $|L| - |L_S| + |R_S|$ .
  - ▶ Using the fact that  $|\Gamma(L_S)| \geq |L_S|$  gives that this is at least  $|L|$ .

# Algorithm Outline

## Idea:

We introduce a node weighting  $\vec{x}$ . Let for a node  $v \in V$ ,  $x_v \in \mathbb{R}$  denote the weight of node  $v$ .

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- ▶ Try to compute a perfect matching in the subgraph  $H(\vec{x})$ . If you are successful you found an optimal matching.



# Algorithm Outline

## Reason:

- ▶ The weight of your matching  $M^*$  is

$$\sum_{(u,v) \in M^*} w(u,v) = \sum_{(u,v) \in M^*} (x_u + x_v) = \sum_v x_v .$$

- ▶ Any other perfect matching  $M$  (in  $G$ , not necessarily in  $H(\vec{x})$ ) has

$$\sum_{(u,v) \in M} w(u,v) \leq \sum_{(u,v) \in M} (x_u + x_v) = \sum_v x_v .$$

# Algorithm Outline

## What if you don't find a perfect matching?

Then, Hall's theorem guarantees you that there is a set  $S \subseteq L$ , with  $|\Gamma(S)| < |S|$ , where  $\Gamma$  denotes the neighbourhood w.r.t. the subgraph  $H(\vec{x})$ .

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**Idea:** reweight such that:

- ▶ the total weight assigned to nodes decreases
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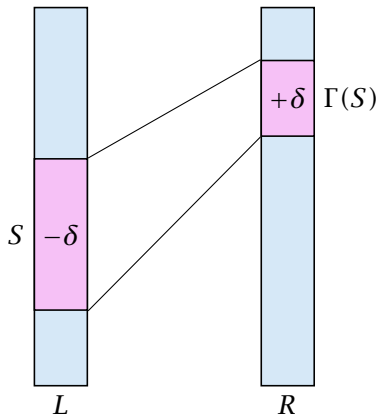
**Idea:** reweight such that:

- ▶ the total weight assigned to nodes decreases
- ▶ the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

# Changing Node Weights

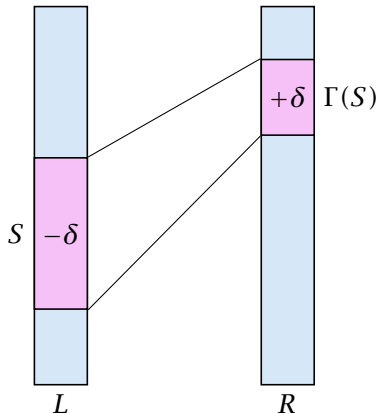
Increase node-weights in  $\Gamma(S)$  by  $+\delta$ , and decrease the node-weights in  $S$  by  $-\delta$ .



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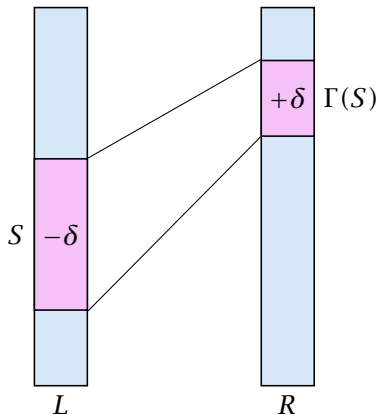
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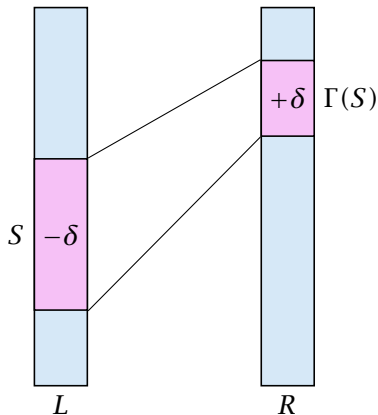
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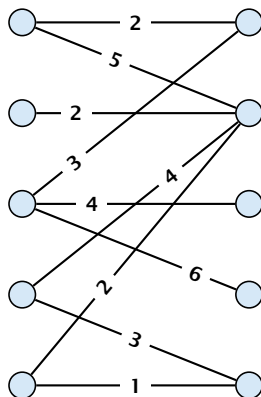
- ▶ Total node-weight decreases.
- ▶ Only edges from  $S$  to  $R - \Gamma(S)$  decrease in their weight.
- ▶ Since, none of these edges is tight (otw. the edge would be contained in  $H(\vec{x})$ , and hence would go between  $S$  and  $\Gamma(S)$ ) we can do this decrement for small enough  $\delta > 0$  until a new edge gets tight.





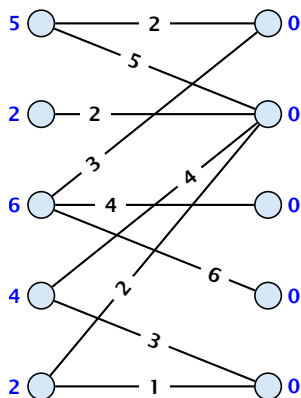
# Weighted Bipartite Matching

Edges not drawn have weight 0.



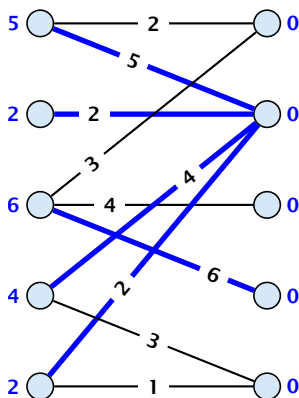
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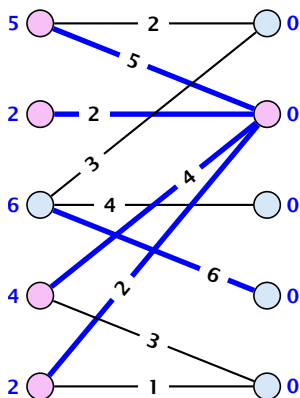
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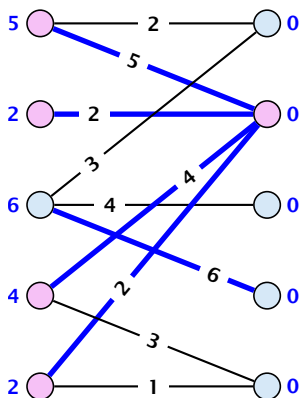
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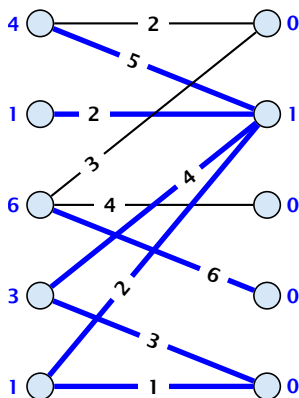
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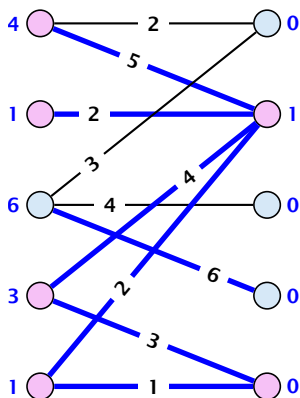
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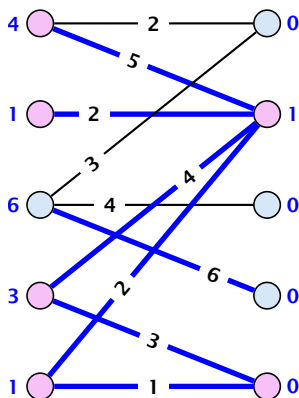
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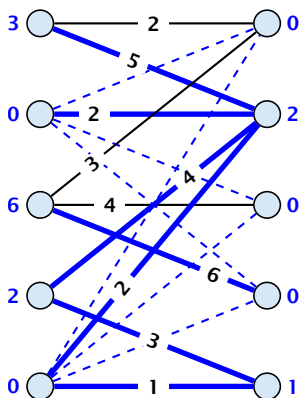
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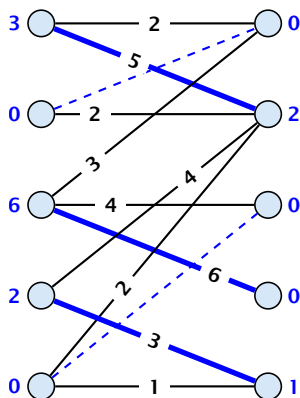
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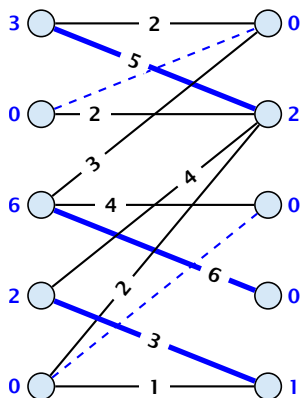
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- ▶ This matching is still contained in the new graph, because all its edges either go between  $\Gamma(S)$  and  $S$  or between  $L - S$  and  $R - \Gamma(S)$ .
- ▶ Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

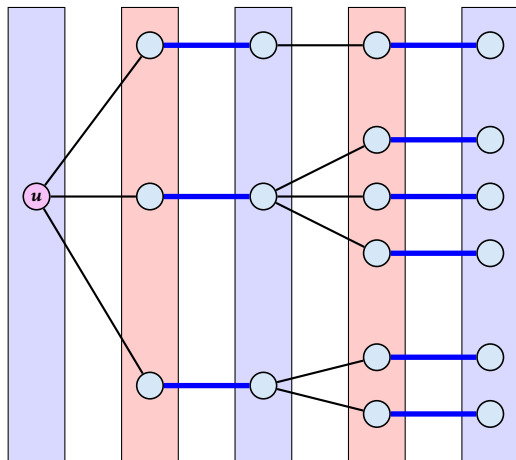
# Analysis

- ▶ We will show that after at most  $n$  reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- ▶ This gives a polynomial running time.



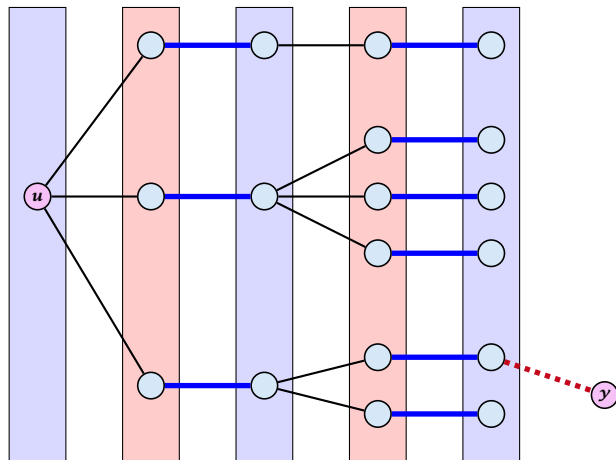
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- ▶ All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex  $u$ . Hence,  $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$ , and all odd vertices are saturated in the current matching.

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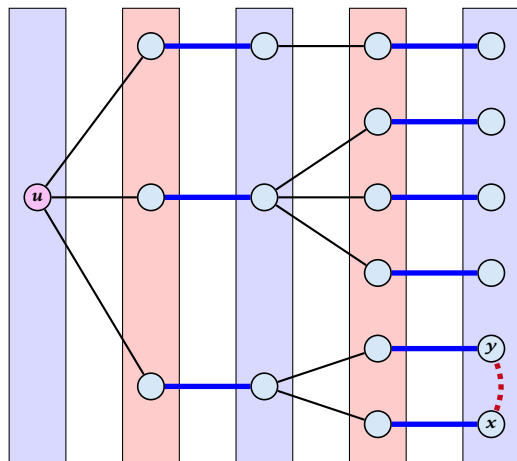
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- ▶ A more careful implementation of the algorithm obtains a running time of  $\mathcal{O}(n^3)$ .

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even nodes

odd nodes

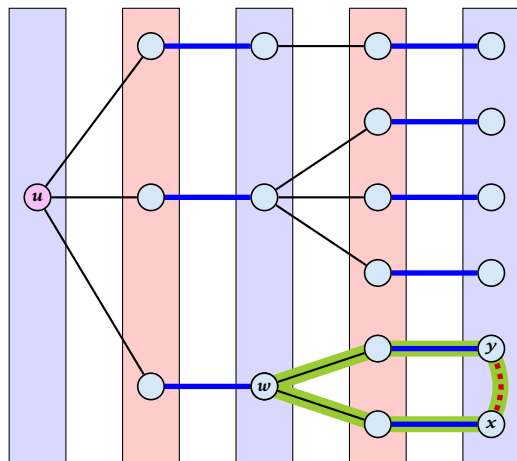
**Case 4:**

$y$  is already contained  
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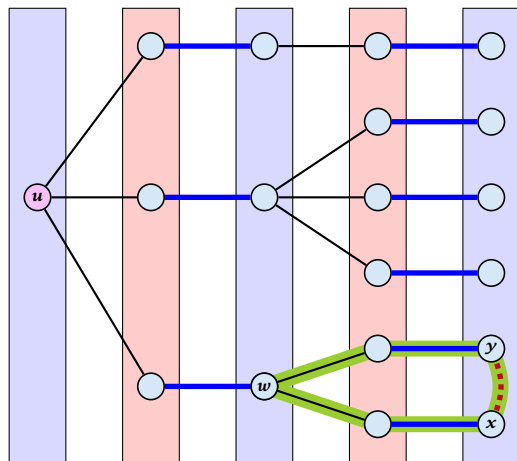
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The cycle  $w \leftrightarrow y - x \leftrightarrow w$   
is called a **blossom**.  
 $w$  is called the **base** of the  
blossom (even node!!!).  
The path  $u-w$  is called the  
**stem** of the blossom.

# Flowers and Blossoms

## Definition 99

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- ▶ A **stem** is an even length alternating path that starts at the root node  $r$  and terminates at some node  $w$ . We permit the possibility that  $r = w$  (empty stem).

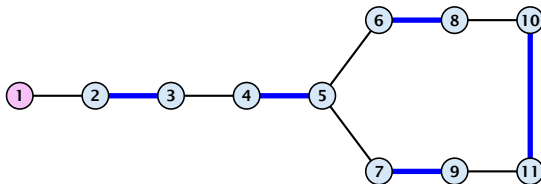
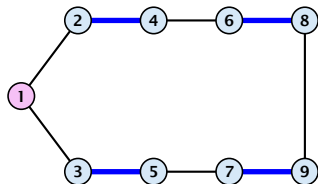
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- ▶ A **blossom** is an odd length alternating cycle that starts and terminates at the terminal node  $w$  of a stem and has no other node in common with the stem.  $w$  is called the **base** of the blossom.

# Flowers and Blossoms



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3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at  $r$ ).

# Flowers and Blossoms

## Properties:

4. Every node  $x$  in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.

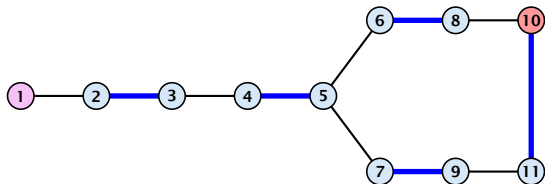
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## Properties:

4. Every node  $x$  in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
5. The even alternating path to  $x$  terminates with a matched edge and the odd path with an unmatched edge.



# Flowers and Blossoms



# Shrinking Blossoms

When during the alternating tree construction we discover a blossom  $B$  we replace the graph  $G$  by  $G' = G/B$ , which is obtained from  $G$  by contracting the blossom  $B$ .

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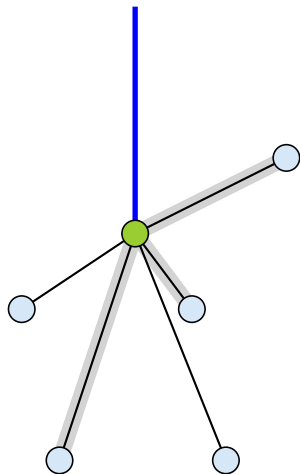
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- ▶ Delete all vertices in  $B$  (and its incident edges) from  $G$ .
- ▶ Add a new (pseudo-)vertex  $b$ . The new vertex  $b$  is connected to all vertices in  $V \setminus B$  that had at least one edge to a vertex from  $B$ .

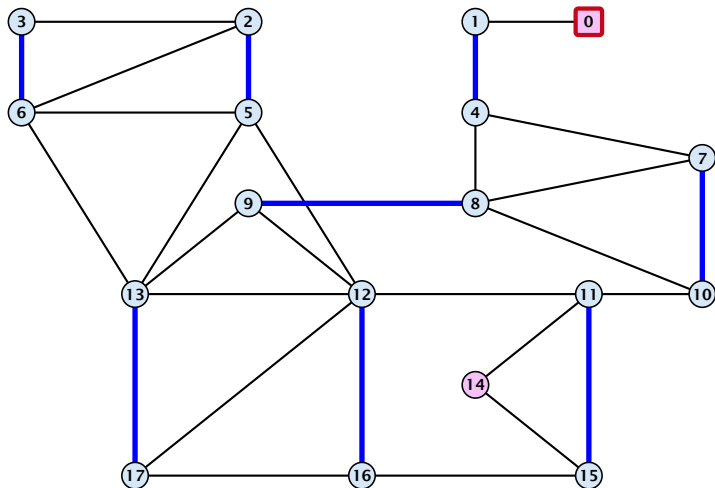


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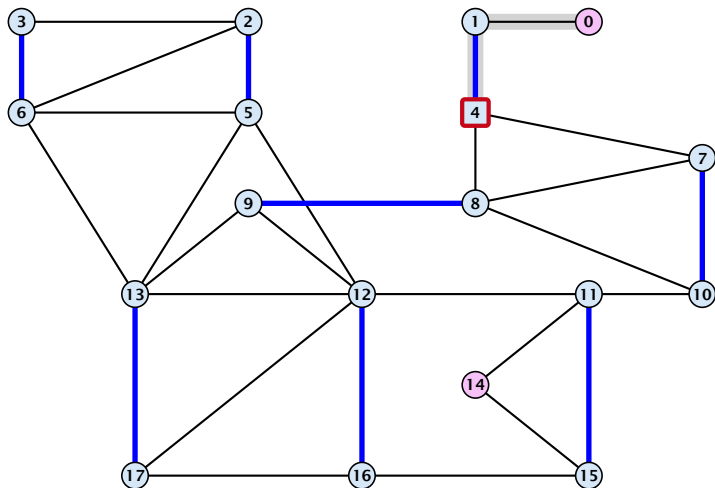
- ▶ Edges of  $T$  that connect a node  $u$  not in  $B$  to a node in  $B$  become tree edges in  $T'$  connecting  $u$  to  $b$ .
- ▶ Matching edges (there is at most one) that connect a node  $u$  not in  $B$  to a node in  $B$  become matching edges in  $M'$ .
- ▶ Nodes that are connected in  $G$  to at least one node in  $B$  become connected to  $b$  in  $G'$ .



# Example: Blossom Algorithm

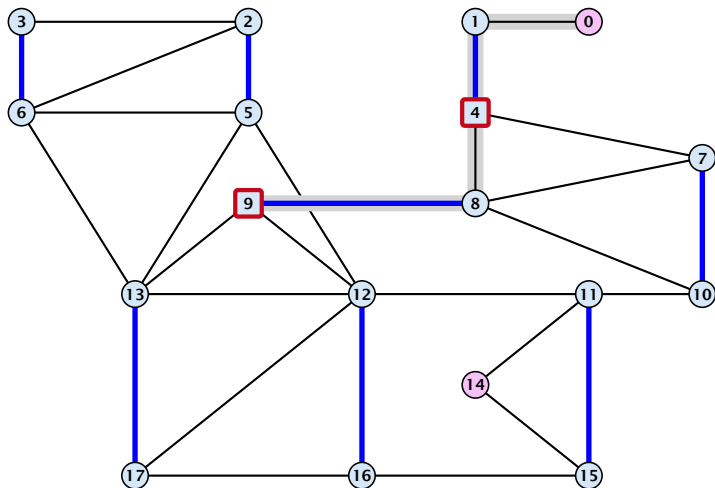


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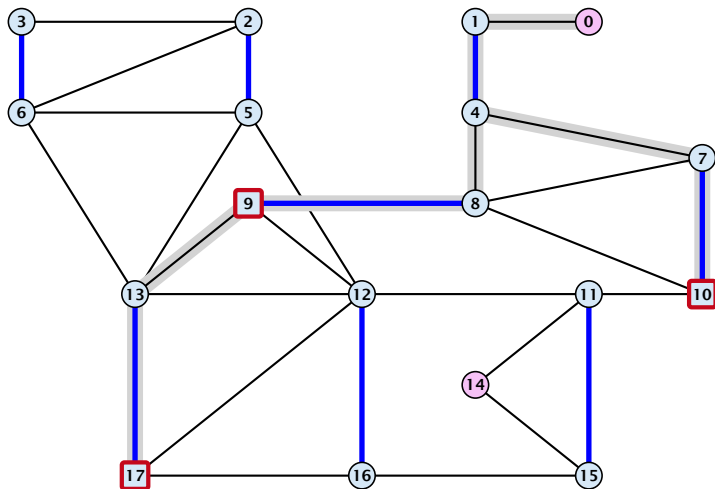


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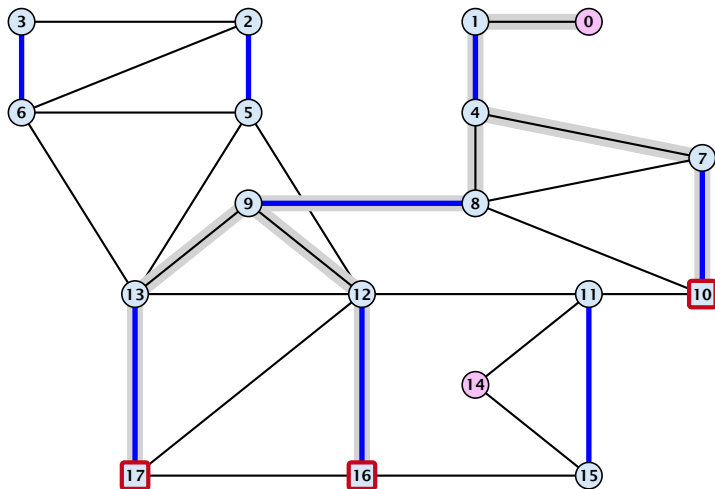




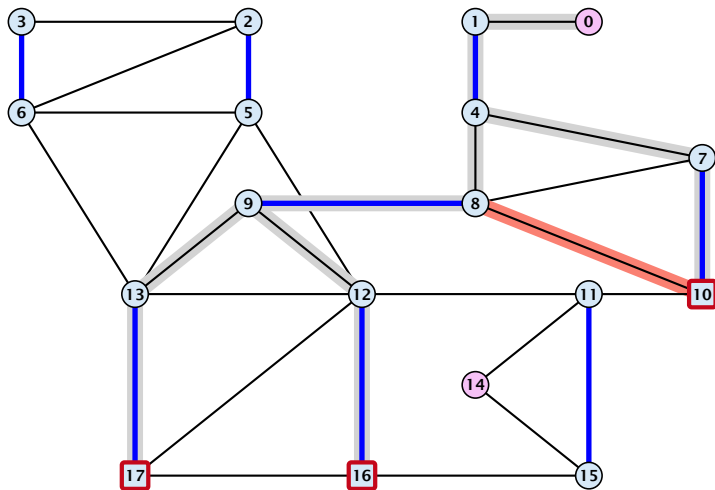
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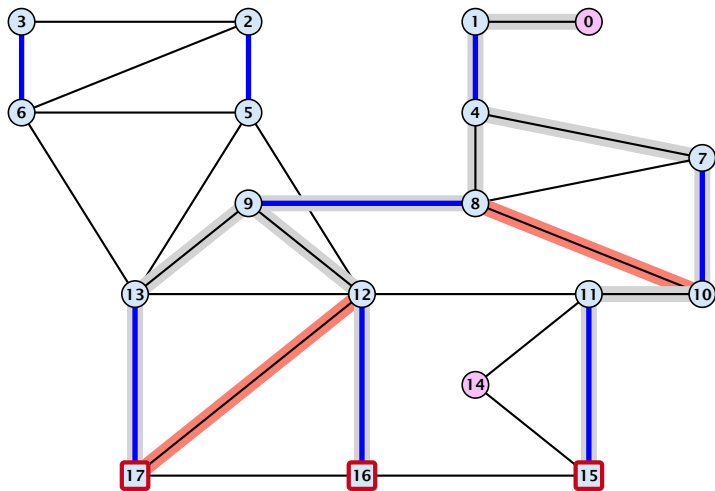


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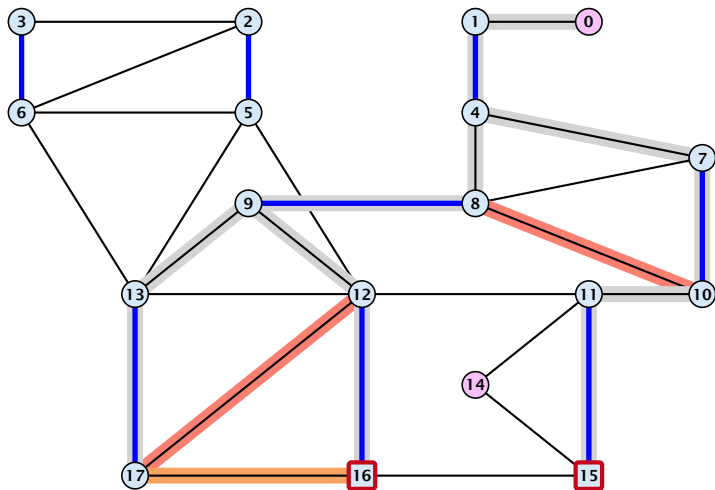




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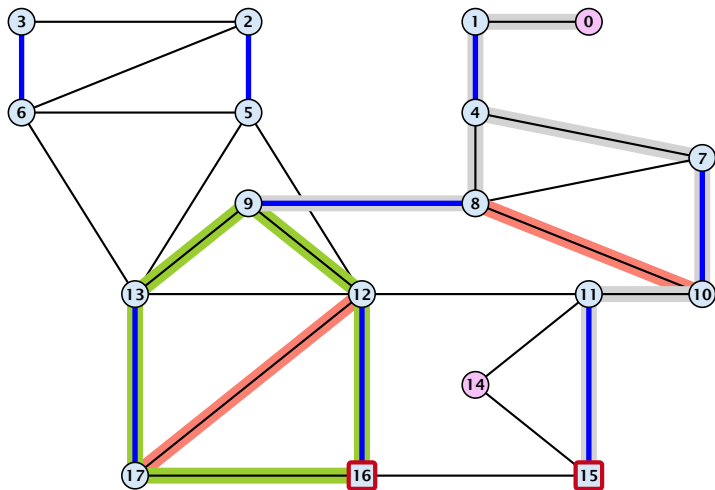


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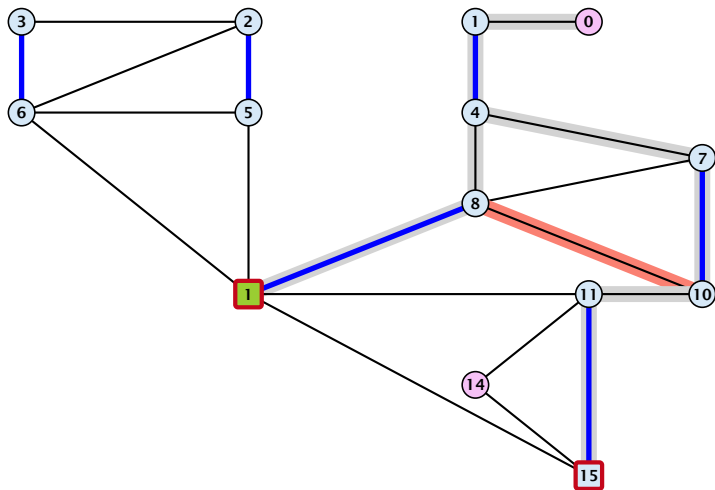




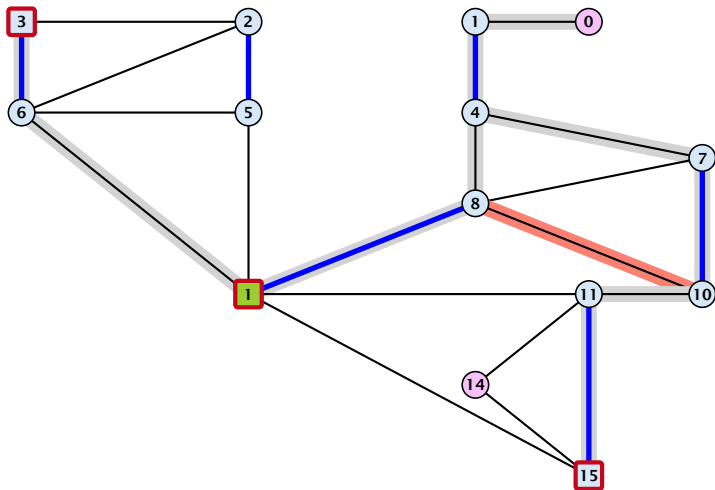
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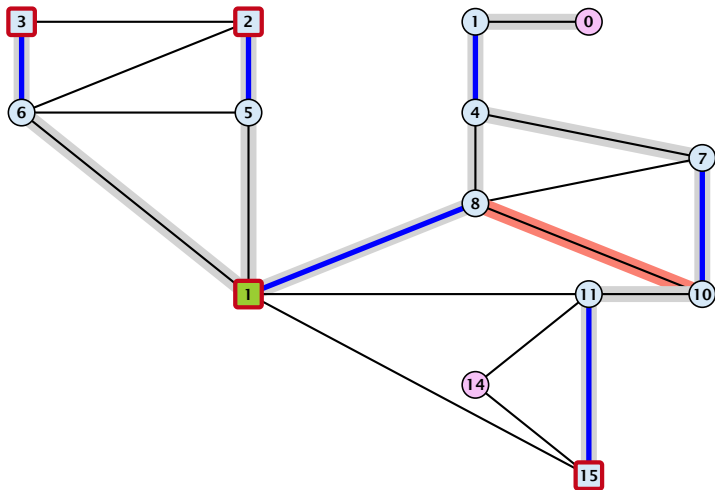
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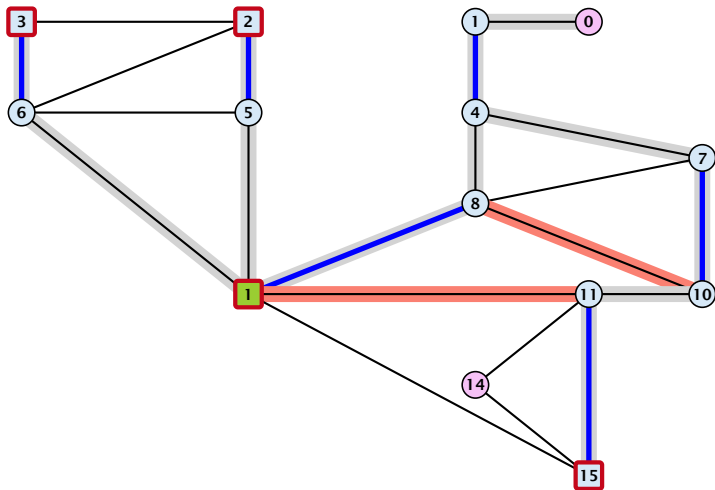
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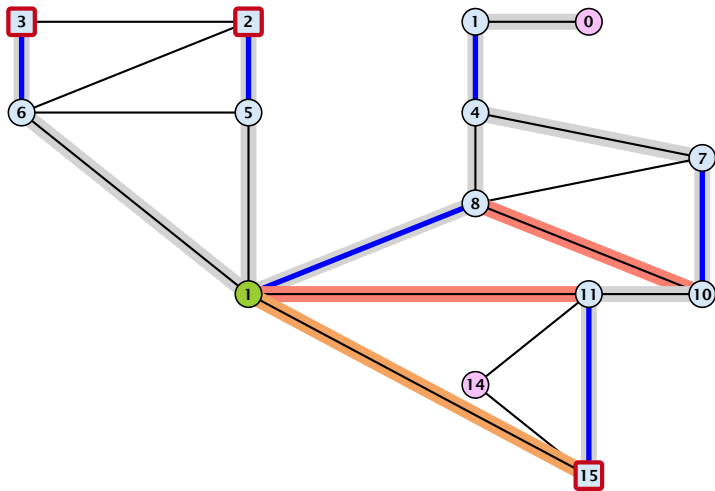
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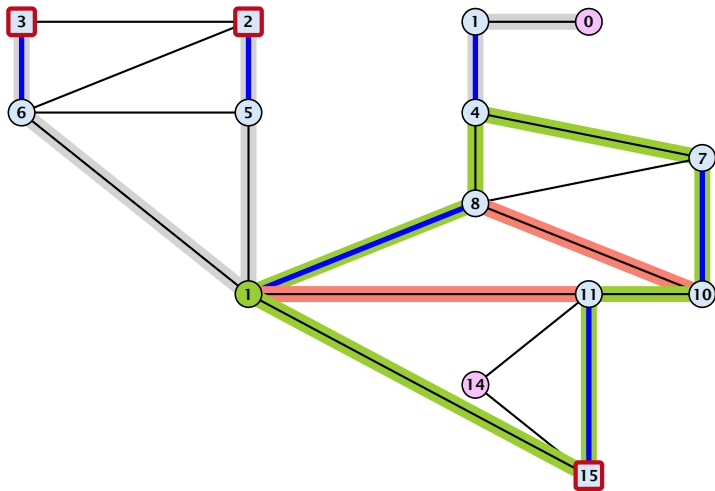
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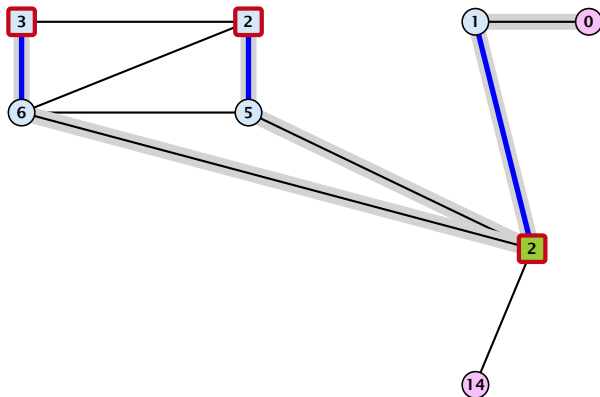
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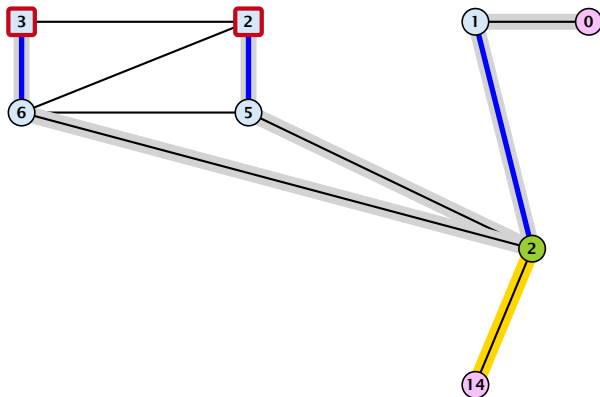


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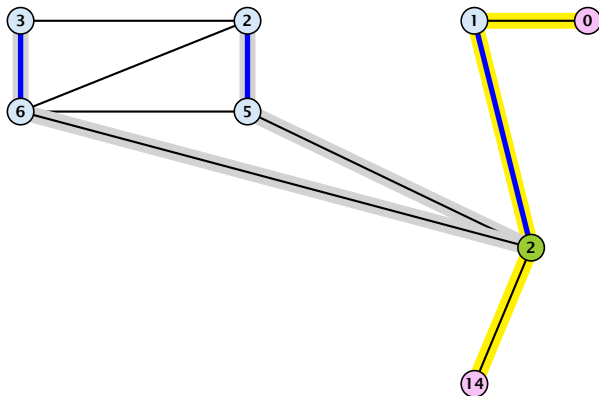




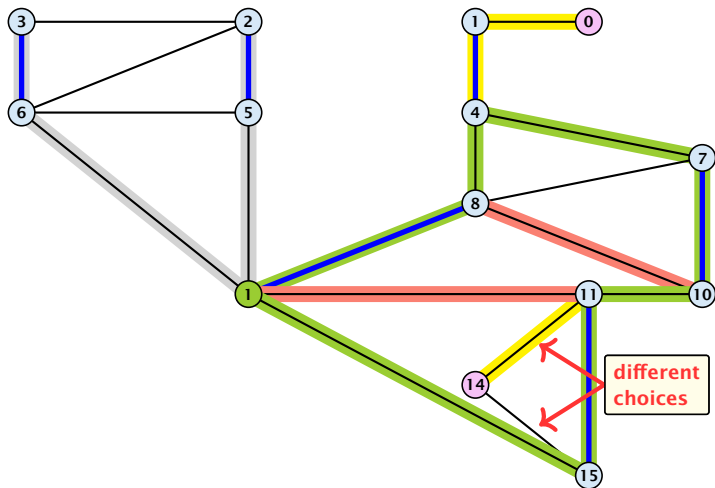
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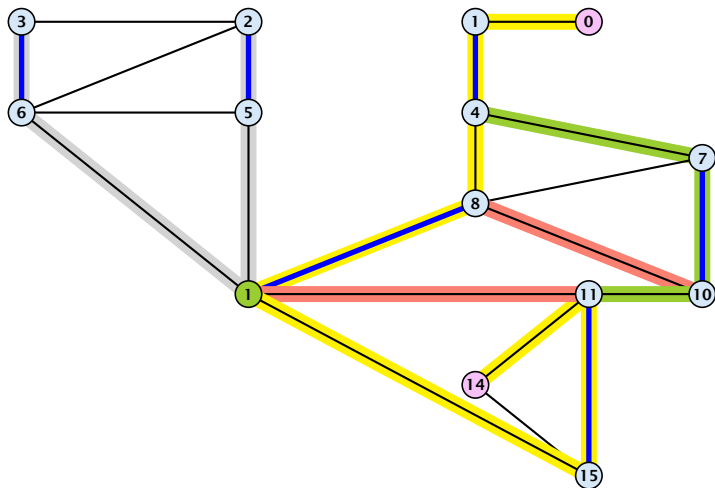
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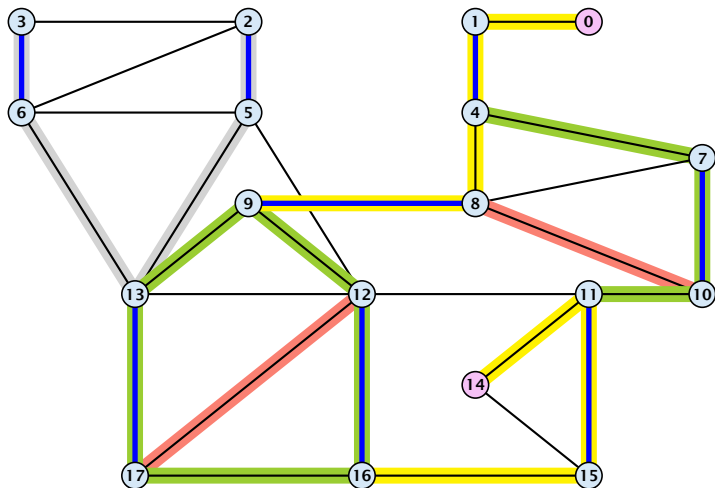
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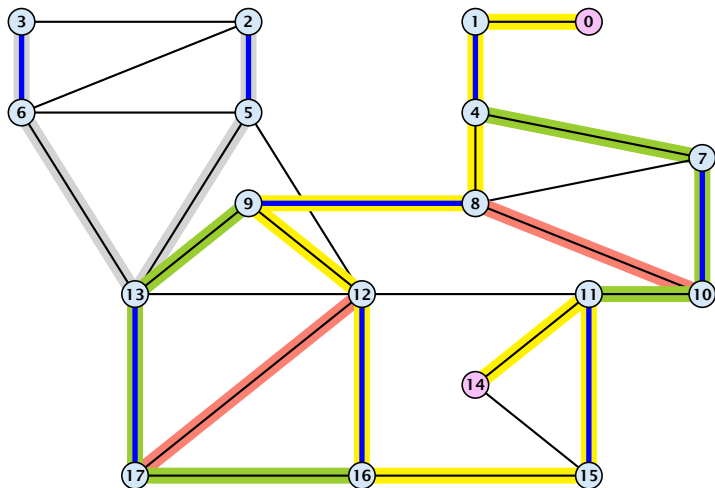
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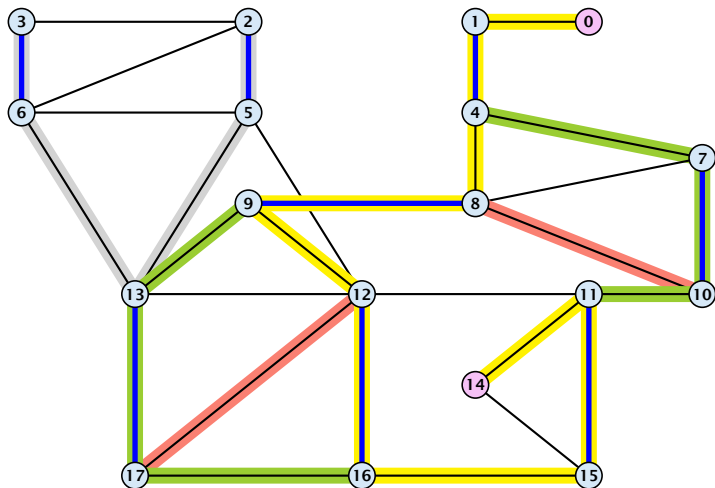
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# Correctness

Assume that in  $G$  we have a flower w.r.t. matching  $M$ . Let  $r$  be the root,  $B$  the blossom, and  $w$  the base. Let graph  $G' = G/B$  with pseudonode  $b$ . Let  $M'$  be the matching in the contracted graph.



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## Lemma 100

*If  $G'$  contains an augmenting path  $P'$  starting at  $r$  (or the pseudo-node containing  $r$ ) w.r.t. the matching  $M'$  then  $G$  contains an augmenting path starting at  $r$  w.r.t. matching  $M$ .*

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**Proof.**

If  $P'$  does not contain  $b$  it is also an augmenting path in  $G$ .

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## Case 1: non-empty stem

- ▶ Next suppose that the stem is non-empty.

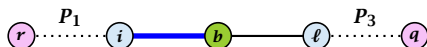
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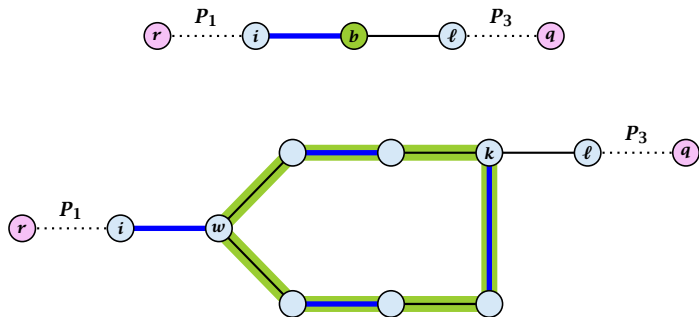
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# Correctness

- ▶ After the expansion  $\ell$  must be incident to some node in the blossom. Let this node be  $k$ .
- ▶ If  $k \neq w$  there is an alternating path  $P_2$  from  $w$  to  $k$  that ends in a matching edge.
- ▶  $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$  is an alternating path.
- ▶ If  $k = w$  then  $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$  is an alternating path.

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**Case 2: empty stem**

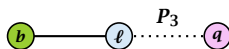
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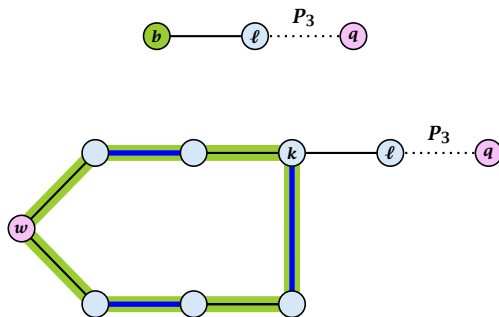


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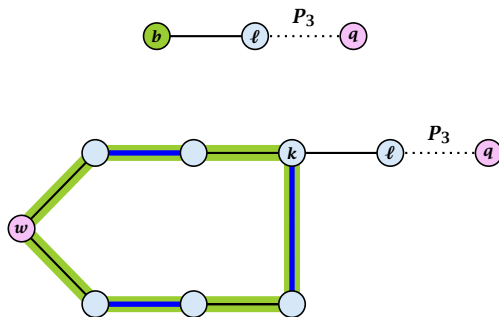


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- ▶ The path  $r \circ P_2 \circ (k, l) \circ P_3$  is an alternating path.

## Lemma 101

*If  $G$  contains an augmenting path  $P$  from  $r$  to  $q$  w.r.t. matching  $M$  then  $G'$  contains an augmenting path from  $r$  (or the pseudo-node containing  $r$ ) to  $q$  w.r.t.  $M'$ .*

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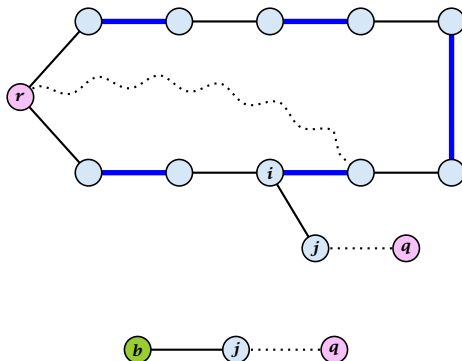
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$(b, j) \circ P_2$  is an augmenting path in the contracted network.

# Correctness

## Illustration for Case 1:



# Correctness

**Case 2: non-empty stem**

## Correctness

### Case 2: non-empty stem

Let  $P_3$  be alternating path from  $r$  to  $w$ ; this exists because  $r$  and  $w$  are root and base of a blossom. Define  $M_+ = M \oplus P_3$ .

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**Algorithm 76**  $\text{search}(r, \text{found})$

- 1: set  $\bar{A}(i) \leftarrow A(i)$  for all nodes  $i$
- 2:  $\text{found} \leftarrow \text{false}$
- 3: unlabel all nodes;
- 4: give an even label to  $r$  and initialize  $\text{list} \leftarrow \{r\}$
- 5: **while**  $\text{list} \neq \emptyset$  **do**
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- 7:      $\text{examine}(i, \text{found})$
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Search for an augmenting path  
starting at  $r$ .

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$A(i)$  contains neighbours of node  $i$ .

We create a copy  $\bar{A}(i)$  so that we later  
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*found* is just a Boolean that allows  
to abort the search process...

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In the beginning no node is in the tree.

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- 6:     delete a node  $i$  from  $\text{list}$
- 7:     examine( $i, \text{found}$ )
- 8:     **if**  $\text{found} = \text{true}$  **then return**

Put the root in the tree.

*list* could also be a set or a stack.

**Algorithm 76**  $\text{search}(r, \text{found})$

- 1: set  $\bar{A}(i) \leftarrow A(i)$  for all nodes  $i$
- 2:  $\text{found} \leftarrow \text{false}$
- 3: unlabel all nodes;
- 4: give an even label to  $r$  and initialize  $\text{list} \leftarrow \{r\}$
- 5: **while**  $\text{list} \neq \emptyset$  **do**
- 6:     delete a node  $i$  from  $\text{list}$
- 7:      $\text{examine}(i, \text{found})$
- 8:     **if**  $\text{found} = \text{true}$  **then return**

As long as there are nodes with  
unexamined neighbours...



**Algorithm 76**  $\text{search}(r, \text{found})$

- 1: set  $\bar{A}(i) \leftarrow A(i)$  for all nodes  $i$
- 2:  $\text{found} \leftarrow \text{false}$
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- 4: give an even label to  $r$  and initialize  $\text{list} \leftarrow \{r\}$
- 5: **while**  $\text{list} \neq \emptyset$  **do**
- 6:     delete a node  $i$  from  $\text{list}$
- 7:      $\text{examine}(i, \text{found})$
- 8:     **if**  $\text{found} = \text{true}$  **then return**

...examine the next one

**Algorithm 76**  $\text{search}(r, \text{found})$

- 1: set  $\bar{A}(i) \leftarrow A(i)$  for all nodes  $i$
- 2:  $\text{found} \leftarrow \text{false}$
- 3: unlabel all nodes;
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- 5: **while**  $\text{list} \neq \emptyset$  **do**
- 6:     delete a node  $i$  from  $\text{list}$
- 7:     examine( $i, \text{found}$ )
- 8:     **if**  $\text{found} = \text{true}$  **then return**

If you found augmenting path  
abort and start from next root.

**Algorithm 77** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i, j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to list
```

Examine the neighbours of a node  $i$

**Algorithm 77** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i, j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to list
```

For all neighbours  $j$  do...

**Algorithm 77** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do
2:   if  $j$  is even then contract( $i, j$ ) and return
3:   if  $j$  is unmatched then
4:      $q \leftarrow j$ ;
5:     pred( $q$ )  $\leftarrow i$ ;
6:      $found \leftarrow \text{true}$ ;
7:     return
8:   if  $j$  is matched and unlabeled then
9:     pred( $j$ )  $\leftarrow i$ ;
10:    pred(mate( $j$ ))  $\leftarrow j$ ;
11:    add mate( $j$ ) to list
```

You have found a blossom...

**Algorithm 77** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i, j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to list
```

You have found a free node which gives you an augmenting path.

**Algorithm 77** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i, j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to list
```

If you find a matched node that is not  
in the tree you grow...

**Algorithm 77** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i, j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to list
```

*mate*( $j$ ) is a new node from  
which you can grow further.



### Algorithm 78 contract( $i, j$ )

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label  $b$  even and add to *list*
- 4: update  $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$  for each  $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Contract blossom identified by  
nodes  $i$  and  $j$

### Algorithm 78 contract( $i, j$ )

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label  $b$  even and add to *list*
- 4: update  $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$  for each  $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Get all nodes of the blossom.

Time:  $\mathcal{O}(m)$

### Algorithm 78 contract( $i, j$ )

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label  $b$  even and add to *list*
- 4: update  $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$  for each  $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Identify all neighbours of  $b$ .

Time:  $\mathcal{O}(m)$  (how?)

### Algorithm 78 $\text{contract}(i, j)$

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label  $b$  even and add to *list*
- 4: update  $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$  for each  $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

$b$  will be an even node, and it has unexamined neighbours.

### Algorithm 78 $\text{contract}(i, j)$

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label  $b$  even and add to *list*
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- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Every node that was adjacent to a node  
in  $B$  is now adjacent to  $b$

### Algorithm 78 contract( $i, j$ )

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
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- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Only for making a blossom expansion easier.

### Algorithm 78 $\text{contract}(i, j)$

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label  $b$  even and add to *list*
- 4: update  $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$  for each  $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Only delete links from nodes not in  $B$  to  $B$ .  
When expanding the blossom again we can  
recreate these links in time  $\mathcal{O}(m)$ .

## Analysis

- ▶ A contraction operation can be performed in time  $\mathcal{O}(m)$ .  
Note, that any graph created will have at most  $m$  edges.



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# Analysis

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- ▶ The expansion can trivially be done in the same time as needed for all contractions.

# Analysis

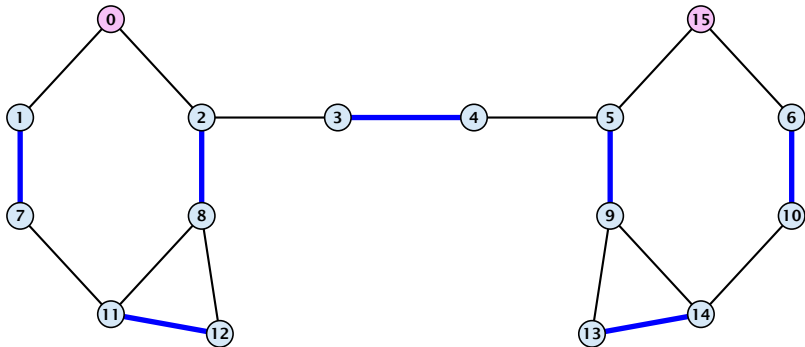
- ▶ A contraction operation can be performed in time  $\mathcal{O}(m)$ . Note, that any graph created will have at most  $m$  edges.
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- ▶ The expansion can trivially be done in the same time as needed for all contractions.
- ▶ An augmentation requires time  $\mathcal{O}(n)$ . There are at most  $n$  of them.

# Analysis

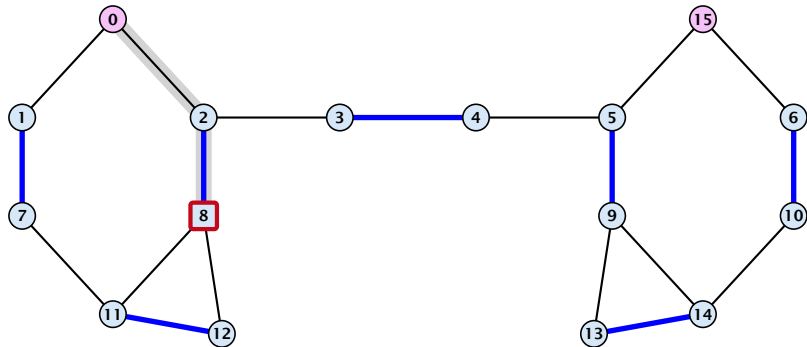
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- ▶ There are at most  $n$  contractions as each contraction reduces the number of vertices.
- ▶ The expansion can trivially be done in the same time as needed for all contractions.
- ▶ An augmentation requires time  $\mathcal{O}(n)$ . There are at most  $n$  of them.
- ▶ In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) .$$

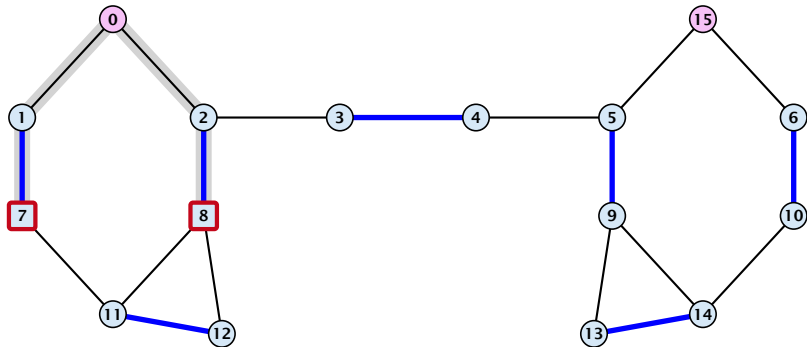
# Example: Blossom Algorithm



# Example: Blossom Algorithm

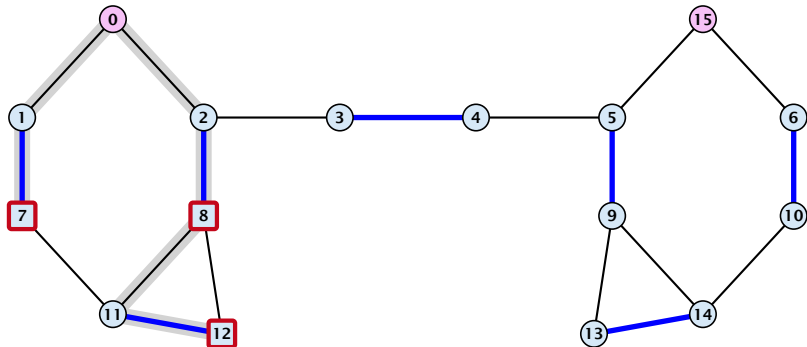


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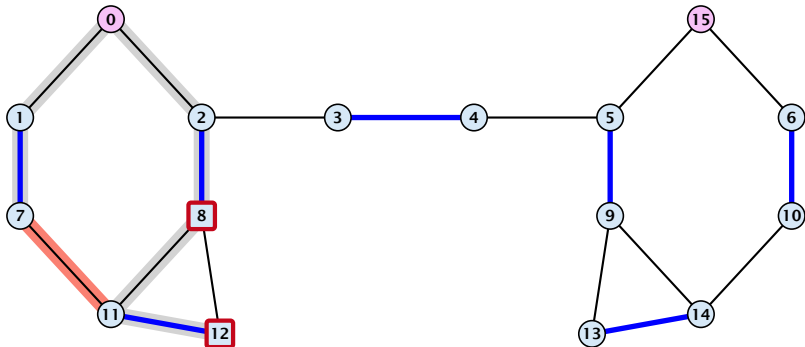




# Example: Blossom Algorithm

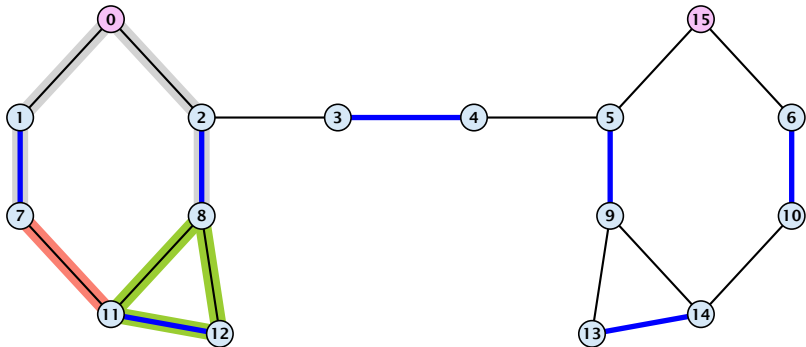


# Example: Blossom Algorithm



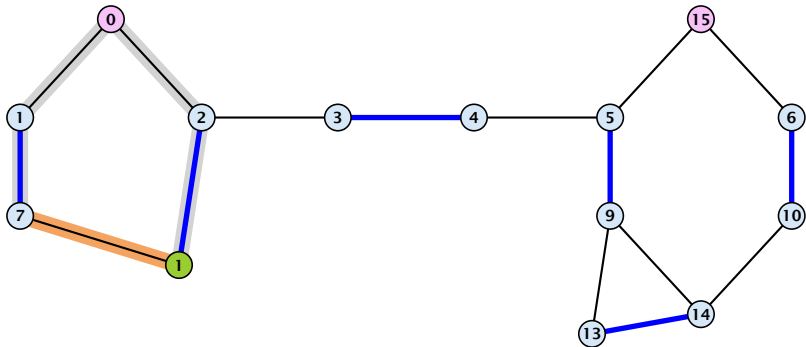


# Example: Blossom Algorithm

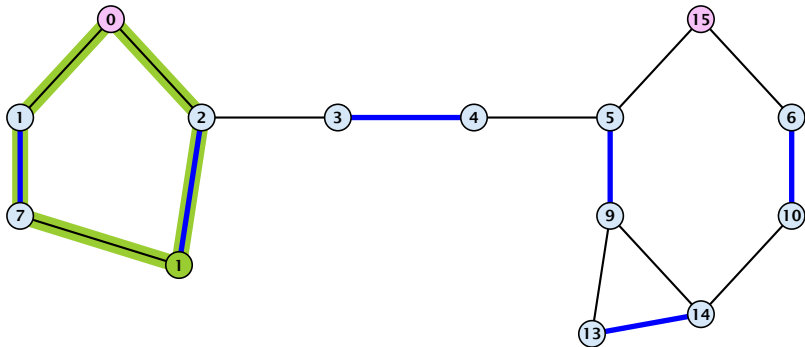




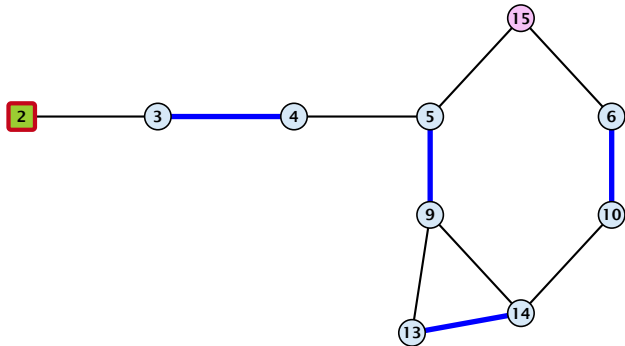
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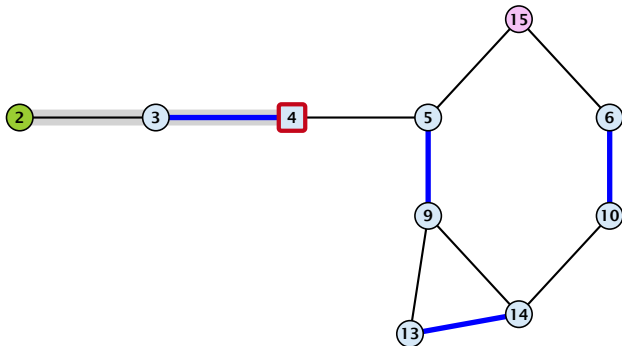


# Example: Blossom Algorithm



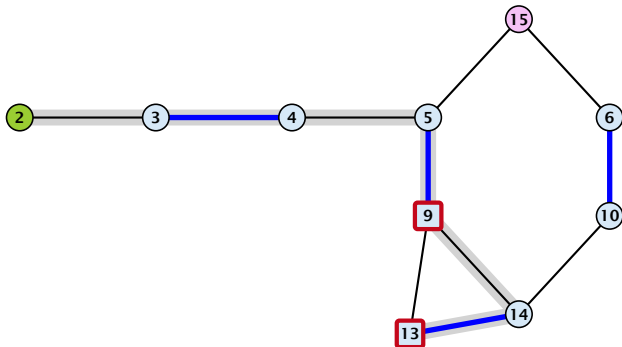


# Example: Blossom Algorithm

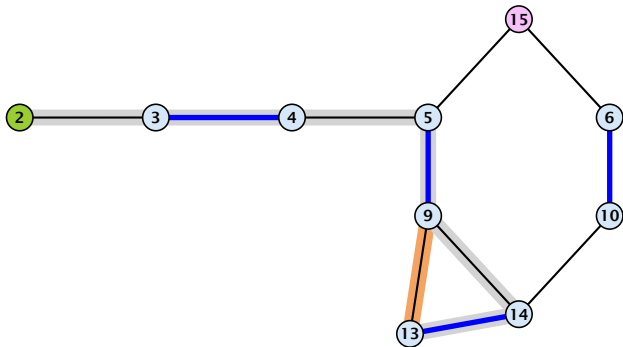




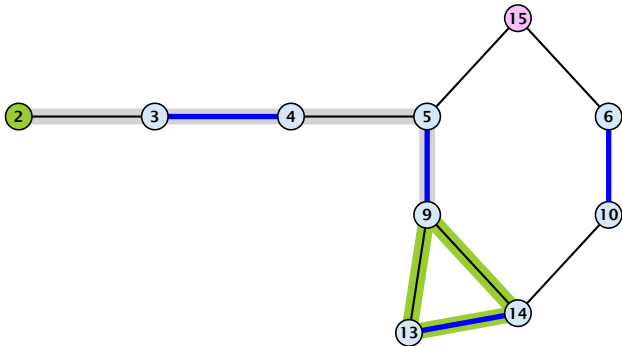
# Example: Blossom Algorithm



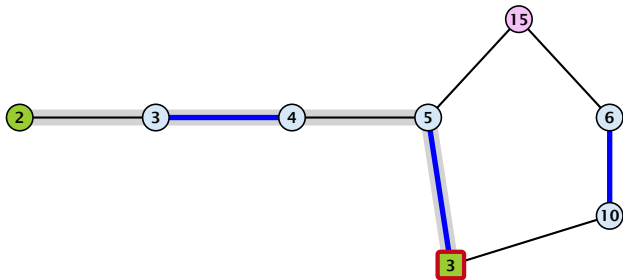
# Example: Blossom Algorithm



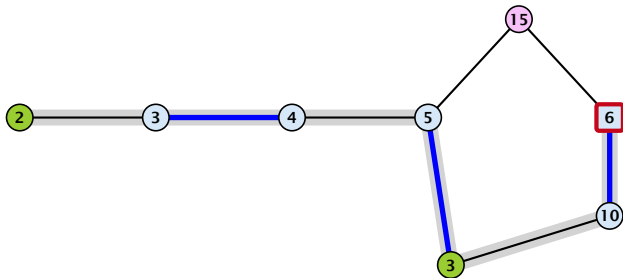
# Example: Blossom Algorithm



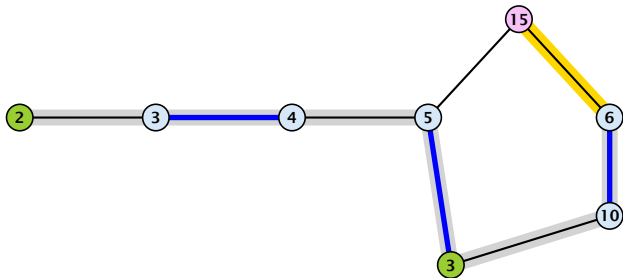
# Example: Blossom Algorithm



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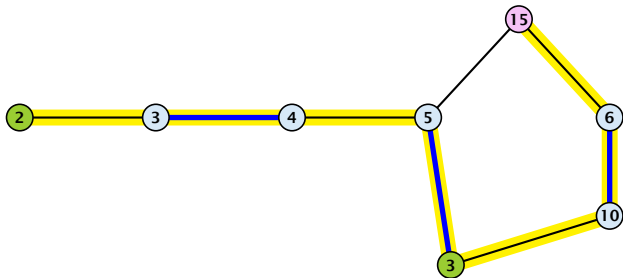


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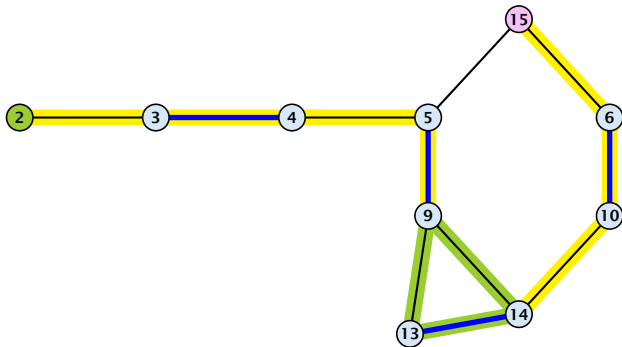




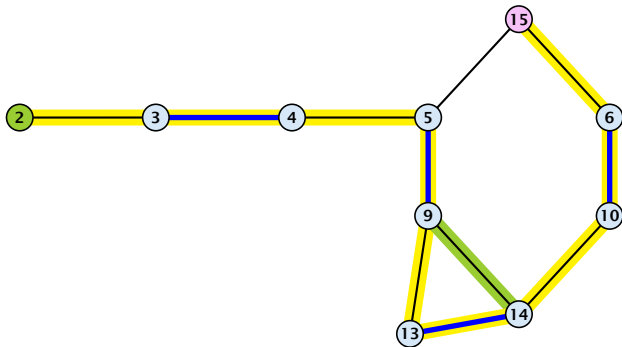
# Example: Blossom Algorithm



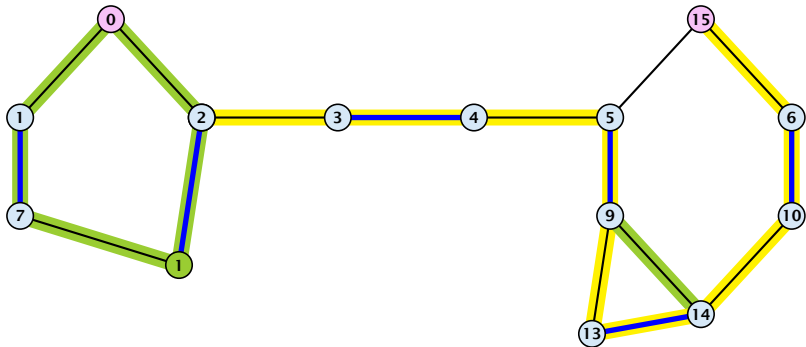
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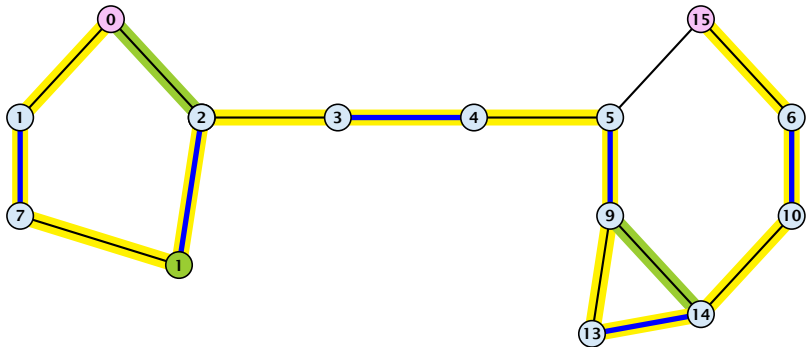
# Example: Blossom Algorithm



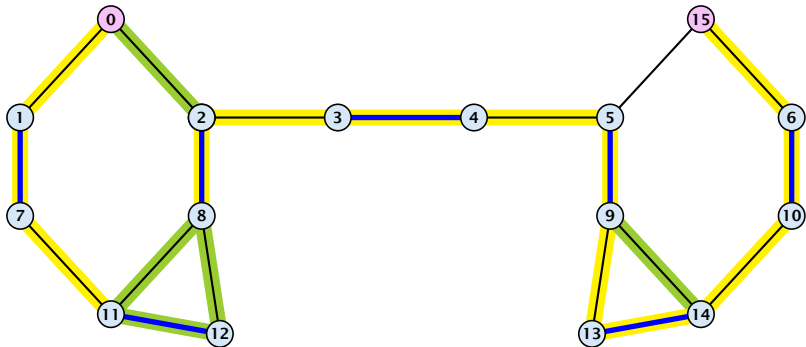
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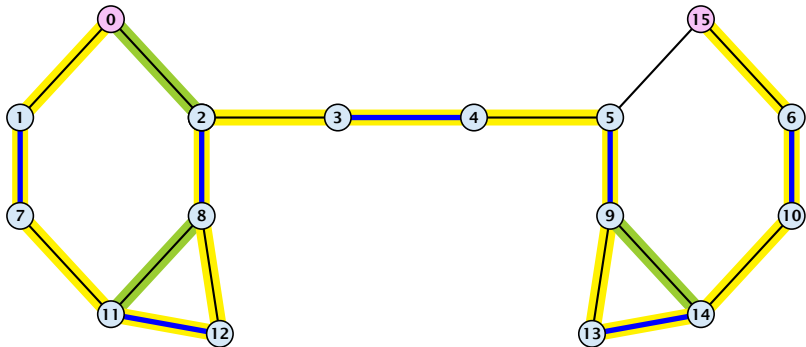
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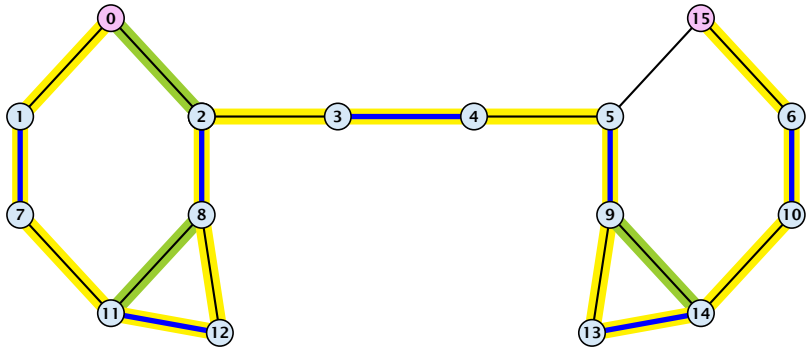
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# A Fast Matching Algorithm

## Algorithm 79 Bimatch-Hopcroft-Karp( $G$ )

```
1:  $M \leftarrow \emptyset$ 
2: repeat
3:   let  $\mathcal{P} = \{P_1, \dots, P_k\}$  be maximal set of
4:   vertex-disjoint, shortest augmenting path w.r.t.  $M$ .
5:    $M \leftarrow M \oplus (P_1 \cup \dots \cup P_k)$ 
6: until  $\mathcal{P} = \emptyset$ 
7: return  $M$ 
```

We call one iteration of the repeat-loop a **phase** of the algorithm.

# Analysis Hopcroft-Karp

## Lemma 102

Given a matching  $M$  and a matching  $M^*$  with  $|M^*| - |M| \geq 0$ .

There exist  $|M^*| - |M|$  *vertex-disjoint* augmenting path w.r.t.  $M$ .

# Analysis Hopcroft-Karp

## Lemma 102

Given a matching  $M$  and a matching  $M^*$  with  $|M^*| - |M| \geq 0$ .  
There exist  $|M^*| - |M|$  *vertex-disjoint* augmenting path w.r.t.  $M$ .

### Proof:

- ▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.

# Analysis Hopcroft-Karp

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### Proof:

- ▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting path.
- ▶ Consider the graph  $G = (V, M \oplus M^*)$ , and mark edges in this graph blue if they are in  $M$  and red if they are in  $M^*$ .

# Analysis Hopcroft-Karp

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- ▶ Consider the graph  $G = (V, M \oplus M^*)$ , and mark edges in this graph blue if they are in  $M$  and red if they are in  $M^*$ .
- ▶ The connected components of  $G$  are cycles and paths.

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- ▶ The graph contains  $k \stackrel{\text{def}}{=} |M^*| - |M|$  more red edges than blue edges.

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### Proof:

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- ▶ Consider the graph  $G = (V, M \oplus M^*)$ , and mark edges in this graph blue if they are in  $M$  and red if they are in  $M^*$ .
- ▶ The connected components of  $G$  are cycles and paths.
- ▶ The graph contains  $k \stackrel{\text{def}}{=} |M^*| - |M|$  more red edges than blue edges.
- ▶ Hence, there are at least  $k$  components that form a path starting and ending with a red edge. These are augmenting paths w.r.t.  $M$ .

# Analysis Hopcroft-Karp

- ▶ Let  $P_1, \dots, P_k$  be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t.  $M$  (let  $\ell = |P_i|$ ).



# Analysis Hopcroft-Karp

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- ▶  $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \dots \cup P_k) = M \oplus P_1 \oplus \dots \oplus P_k$ .

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- ▶ Let  $P$  be an augmenting path in  $M'$ .

# Analysis Hopcroft-Karp

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- ▶ Let  $P$  be an augmenting path in  $M'$ .

## Lemma 103

The set  $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \dots \cup P_k) \oplus P$  contains at least  $(k + 1)\ell$  edges.

# Analysis Hopcroft-Karp

## Proof.

- ▶ The set describes exactly the symmetric difference between matchings  $M$  and  $M' \oplus P$ .

# Analysis Hopcroft-Karp

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- ▶ Each of these paths is of length at least  $\ell$ .

# Analysis Hopcroft-Karp

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# Analysis Hopcroft-Karp

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- ▶ Hence,  $|A| \leq k\ell + |P| - 1$ .
- ▶ The lower bound on  $|A|$  gives  $(k + 1)\ell \leq |A| \leq k\ell + |P| - 1$ , and hence  $|P| \geq \ell + 1$ .

# Analysis Hopcroft-Karp

If the shortest augmenting path w.r.t. a matching  $M$  has  $\ell$  edges then the cardinality of the maximum matching is of size at most  $|M| + \frac{|V|}{\ell+1}$ .

# Analysis Hopcroft-Karp

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## Proof.

The symmetric difference between  $M$  and  $M^*$  contains  $|M^*| - |M|$  vertex-disjoint augmenting paths. Each of these paths contains at least  $\ell + 1$  vertices. Hence, there can be at most  $\frac{|V|}{\ell+1}$  of them.

# Analysis Hopcroft-Karp

## Lemma 105

*The Hopcroft-Karp algorithm requires at most  $2\sqrt{|V|}$  phases.*

# Analysis Hopcroft-Karp

## Lemma 105

The Hopcroft-Karp algorithm requires at most  $2\sqrt{|V|}$  phases.

### Proof.

- ▶ After iteration  $\lfloor \sqrt{|V|} \rfloor$  the length of a shortest augmenting path must be at least  $\lfloor \sqrt{|V|} \rfloor + 1 \geq \sqrt{|V|}$ .
- ▶ Hence, there can be at most  $|V| / (\sqrt{|V|} + 1) \leq \sqrt{|V|}$  additional augmentations.



# Analysis Hopcroft-Karp

## Lemma 106

*One phase of the Hopcroft-Karp algorithm can be implemented in time  $\mathcal{O}(m)$ .*

construct a “level graph”  $G'$ :

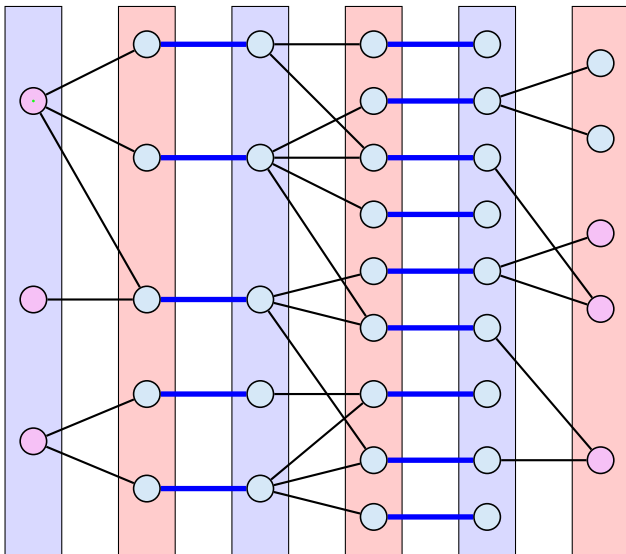
- ▶ construct Level 0 that includes all free vertices on left side  $L$
- ▶ construct Level 1 containing all neighbors of Level 0
- ▶ construct Level 2 containing **matching** neighbors of Level 1
- ▶ construct Level 3 containing all neighbors of Level 2
- ▶ ...
- ▶ stop when a level (apart from Level 0) contains a free vertex

can be done in time  $\mathcal{O}(m)$  by a modified BFS

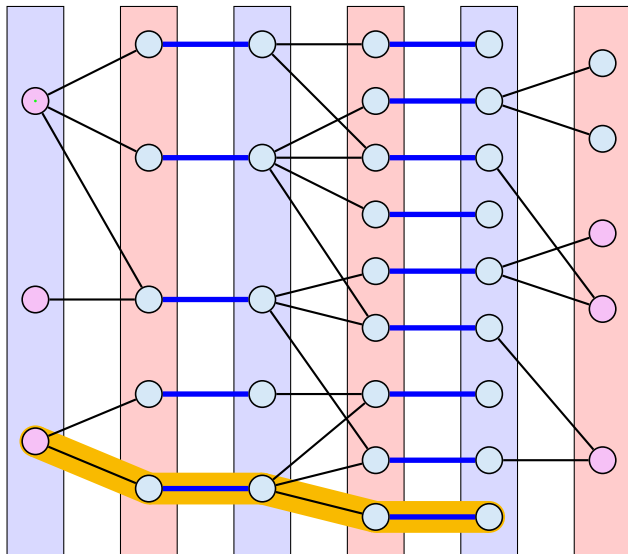
# Analysis Hopcroft-Karp

- ▶ a shortest augmenting path **must** go from Level 0 to the last layer constructed
- ▶ it can only use edges between layers
- ▶ construct a maximal set of vertex disjoint augmenting path connecting the layers
- ▶ for this, go forward until you either reach a free vertex or you reach a “dead end”  $v$
- ▶ if you reach a free vertex delete the augmenting path and all incident edges from the graph
- ▶ if you reach a dead end backtrack and delete  $v$  together with its incident edges

# Analysis Hopcroft-Karp

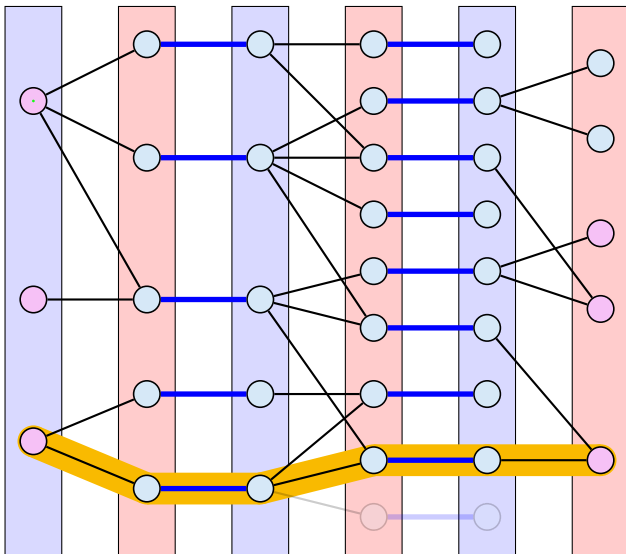


# Analysis Hopcroft-Karp

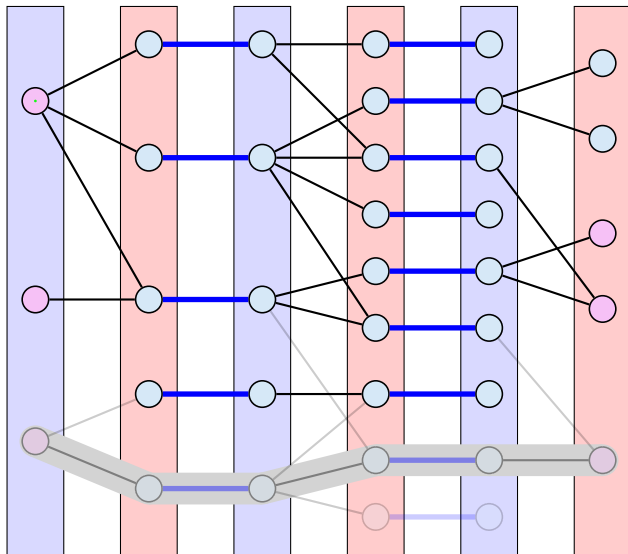




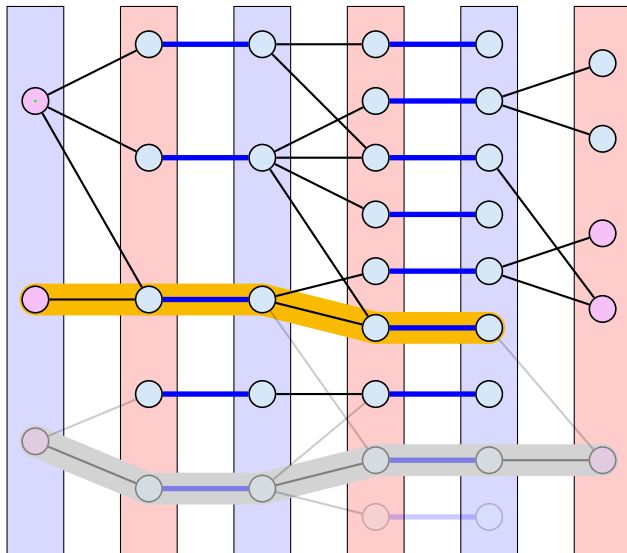
# Analysis Hopcroft-Karp



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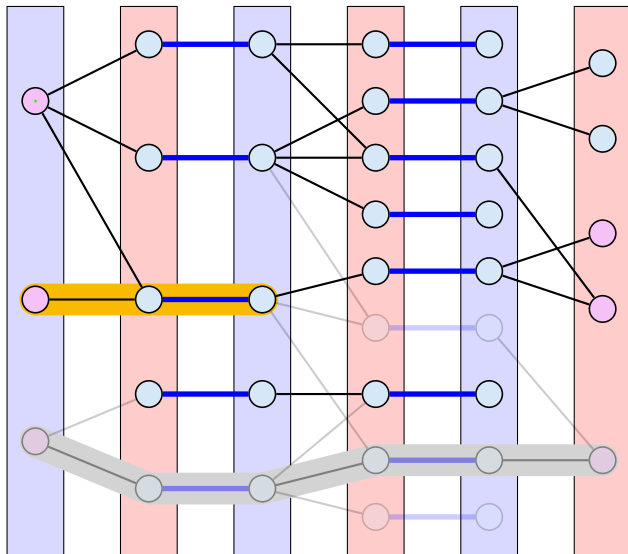


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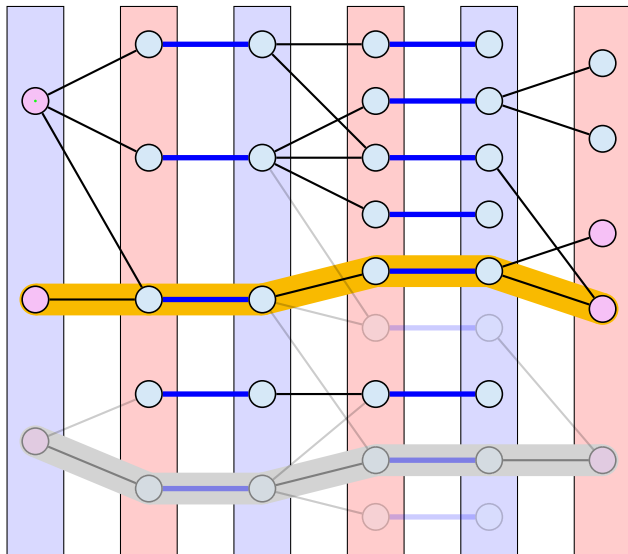




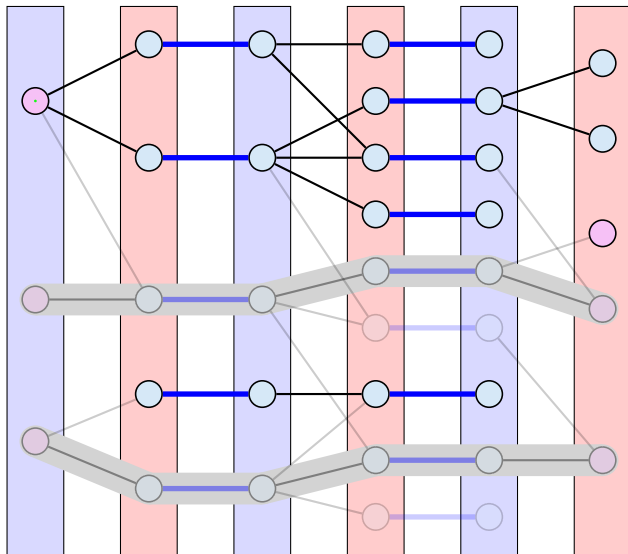
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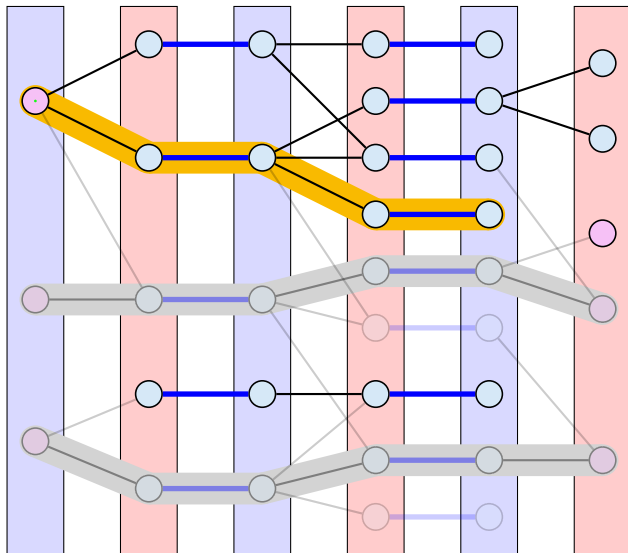
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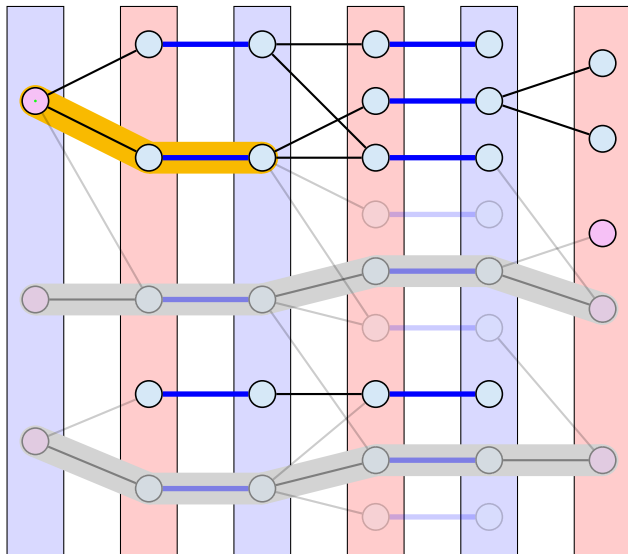
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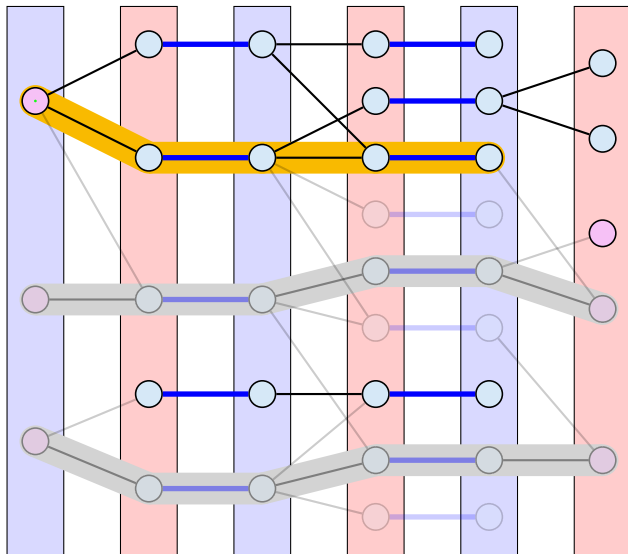
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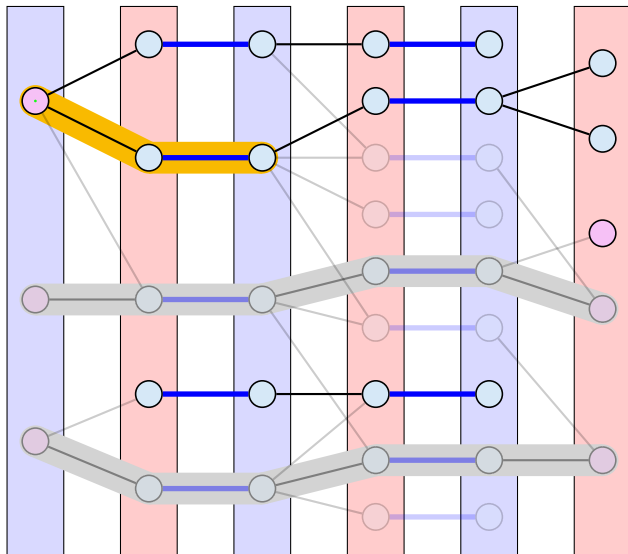
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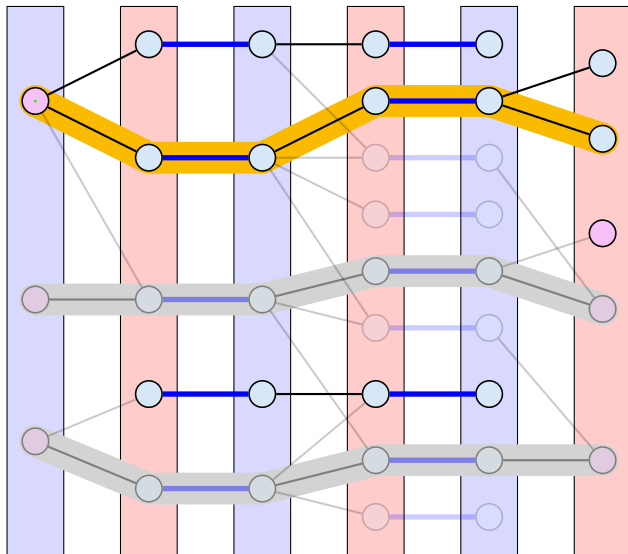
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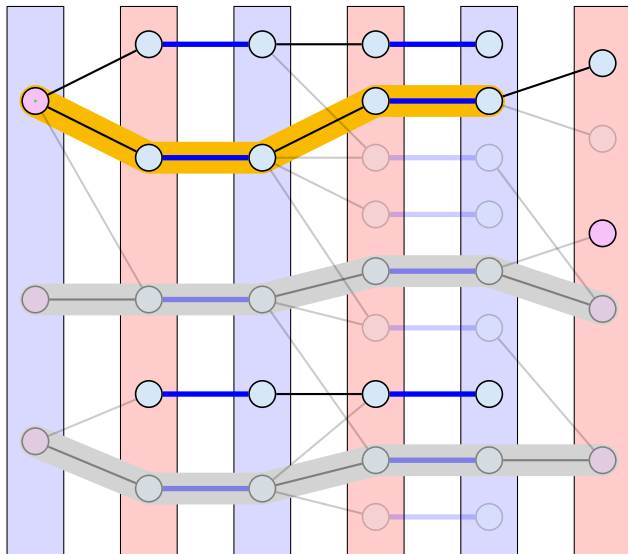


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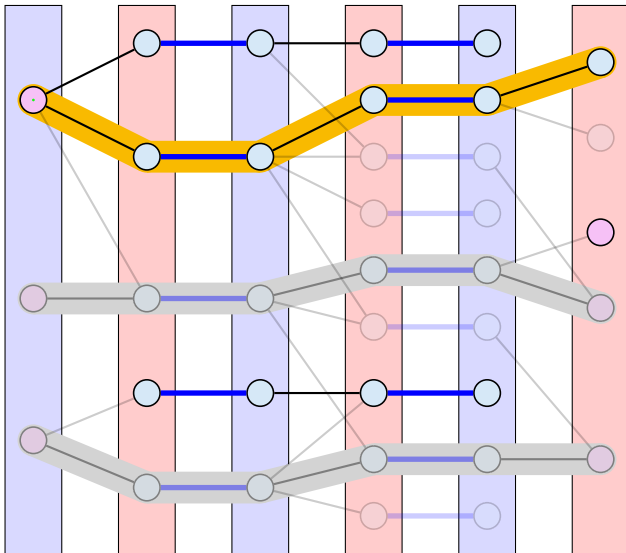




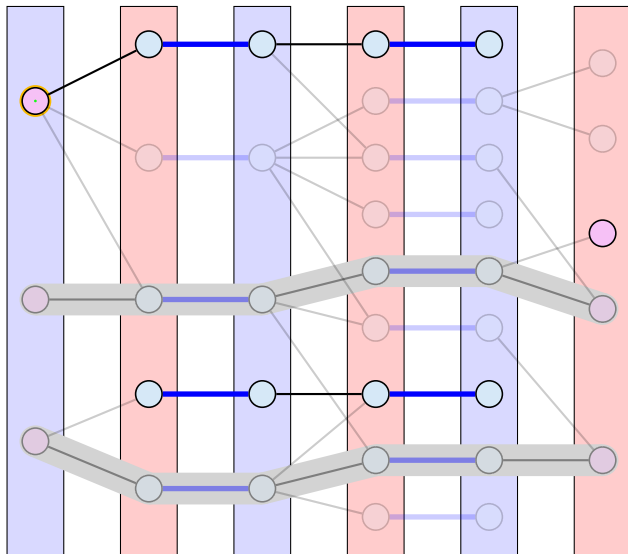
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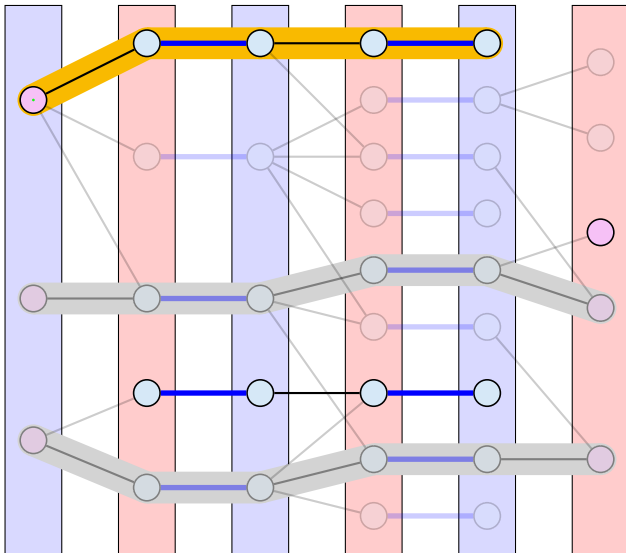
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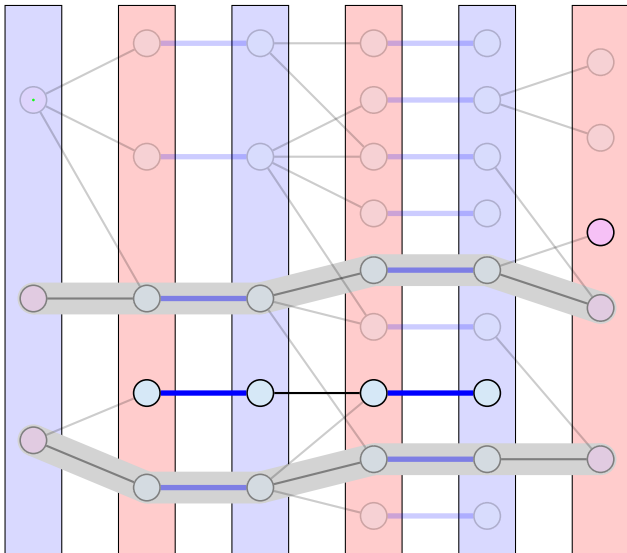
# Analysis Hopcroft-Karp



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# Analysis: Shortest Augmenting Path for Flows

**cost for searches during a phase is  $\mathcal{O}(mn)$**

- ▶ a search (successful or unsuccessful) takes time  $\mathcal{O}(n)$
- ▶ a search deletes at least one edge from the level graph

**there are at most  $n$  phases**

Time:  $\mathcal{O}(mn^2)$ .

# Analysis for Unit-capacity Simple Networks

**cost for searches during a phase is  $\mathcal{O}(m)$**

- ▶ an edge/vertex is traversed at most twice

**need at most  $\mathcal{O}(\sqrt{n})$  phases**

- ▶ after  $\sqrt{n}$  phases there is a cut of size at most  $\sqrt{n}$  in the residual graph
- ▶ hence at most  $\sqrt{n}$  additional augmentations required

Time:  $\mathcal{O}(m\sqrt{n})$ .