Part IV

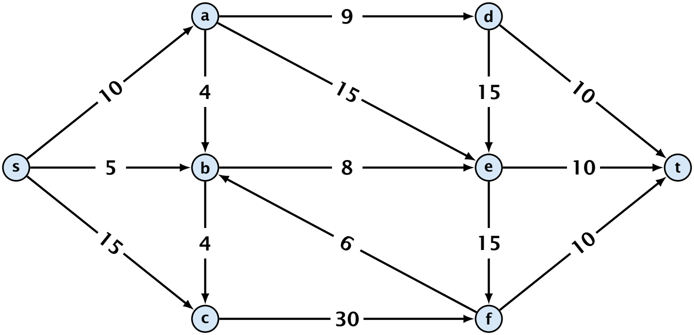
Flows and Cuts

The following slides are partially based on slides by Kevin Wayne.

6 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$



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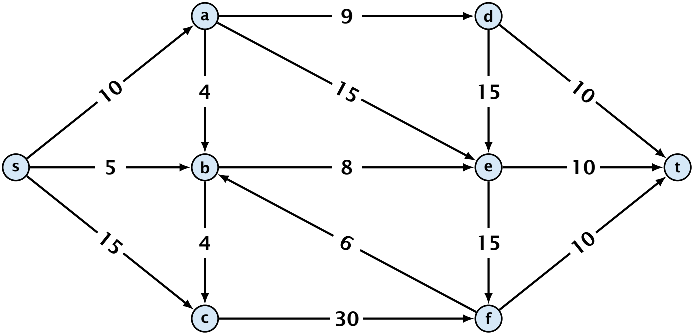

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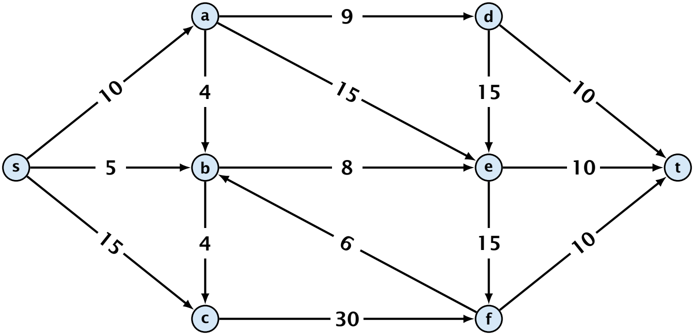
- two special nodes: source s; target t;
- no edges entering s or leaving t;



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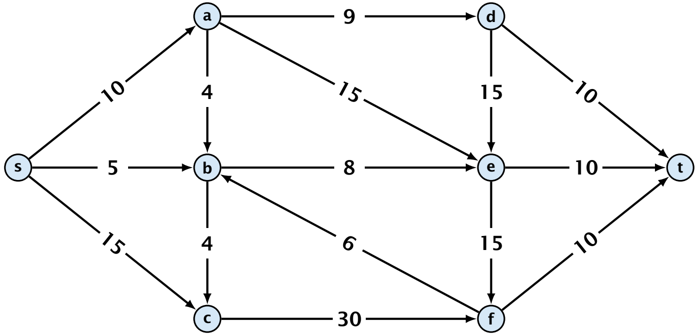
Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source s; target t;
- no edges entering s or leaving t;
- at least for now: no parallel edges;



Cuts

Definition 28
An (s, t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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The capacity of a cut A is defined as

$$\operatorname{cap}(A, V \setminus A) := \sum_{e \in \operatorname{out}(A)} c(e) \;,$$

where $\operatorname{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

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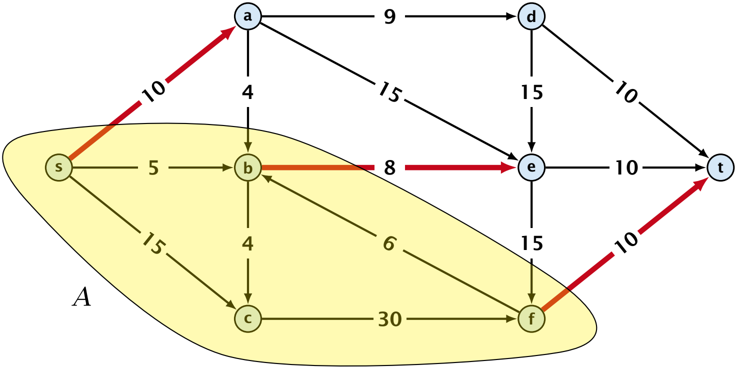
Minimum Cut Problem: Find an (s, t)-cut with minimum capacity.

Cuts

Example 30



The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$.

Flows

Definition 31

An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \le f(e) \le c(e) \enspace .$$

(capacity constraints)

Flows

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An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
$$0 \le f(e) \le c(e) \ .$$
(capacity constraints)
2. For each $v \in V \setminus \{s,t\}$
$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \ .$$
(flow conservation constraints)

Flows

Definition 32
The value of an (s,t)-flow f is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) \ .$$

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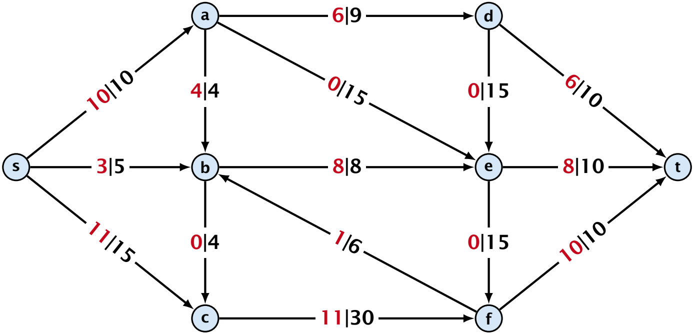
Maximum Flow Problem: Find an (s,t)-flow with maximum value.

Flows

Example 33



The value of the flow is $\mathrm{val}(f) = 24$.

Flows

Lemma 34 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \ .$$

Proof.

$$\mathrm{val}(f)$$

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$$
\begin{aligned}
\text{val}(f) &= \sum_{e\in\text{out}(s)} f(e) \\
&= \sum_{e\in\text{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}} \left(\sum_{e\in\text{out}(v)} f(e) - \sum_{e\in\text{in}(v)} f(e) \right)
\end{aligned}
$$

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\end{aligned}$$

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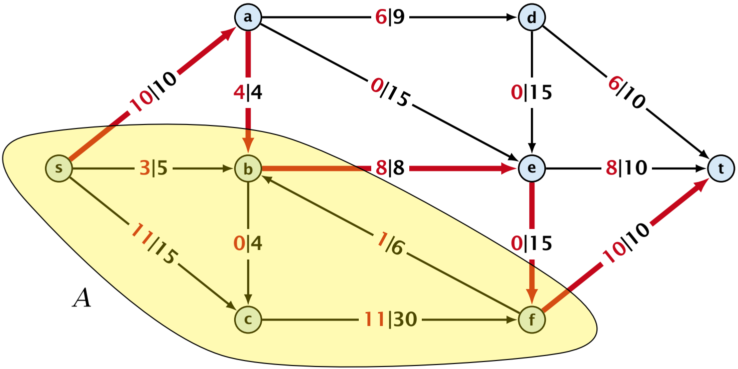
&= \sum_{e\in\mathrm{out}(A)} f(e) - \sum_{e\in\mathrm{into}(A)} f(e)
\end{aligned}$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. □

Example 35



The net-flow across the cut is $\text{val}(f) = 24$.

Corollary 36

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

$$\operatorname{val}(f) = \operatorname{cap}(A, V \setminus A).$$

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Suppose that there is a flow f' with larger value. Then

$$\mathrm{cap}(A, V \setminus A) < \mathrm{val}(f')$$

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Proof.

Suppose that there is a flow f' with larger value. Then

$$\begin{aligned}\mathrm{cap}(A, V \setminus A) &< \mathrm{val}(f') \\ &= \sum_{e \in \mathrm{out}(A)} f'(e) - \sum_{e \in \mathrm{into}(A)} f'(e)\end{aligned}$$

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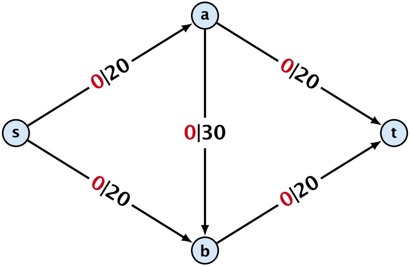
&\le \operatorname{cap}(A, V \setminus A)
\end{aligned}$$

□

7 Augmenting Path Algorithms

Greedy-algorithm:

- start with $f(e) = 0$ everywhere
- find an s-t path with $f(e) < c(e)$ on every edge
- augment flow along the path
- repeat as long as possible



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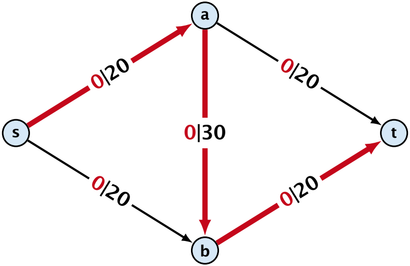

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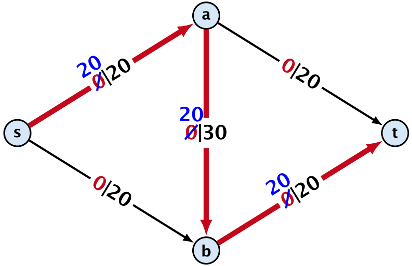

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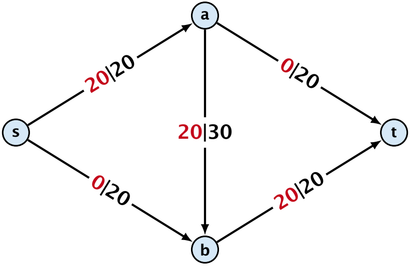

flow value: 20

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

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- Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v.
- G_f has edge e'_1 with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e'_2 with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

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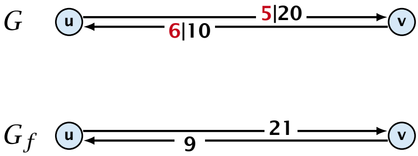
- Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v.
- G_f has edge e_1' with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e_2' with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.



Augmenting Path Algorithm

Definition 37

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

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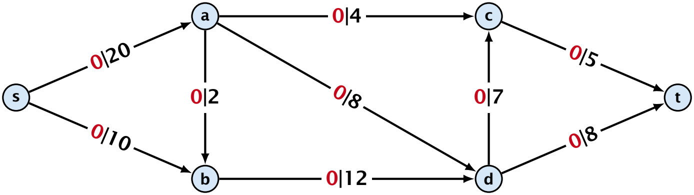
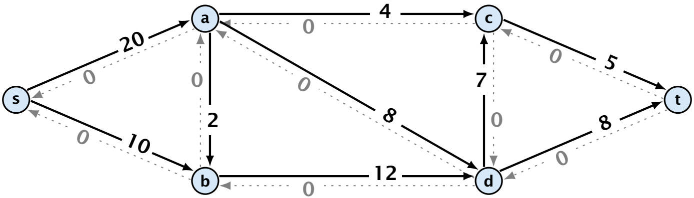
Algorithm 1 FordFulkerson$(G = (V, E, c))$

1: Initialize $f(e) \leftarrow 0$ for all edges.
2: **while** $\exists$ augmenting path p in G_f **do**
3: augment as much flow along p as possible.

Augmenting Paths



flow value: 0



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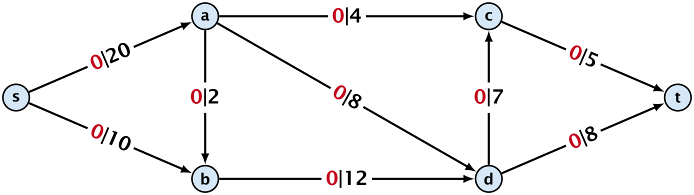

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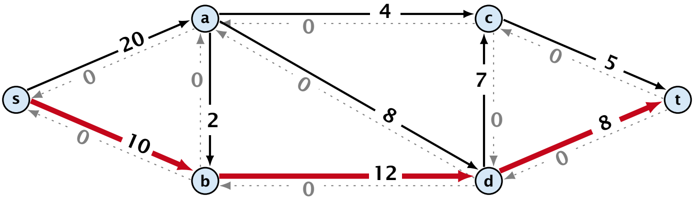

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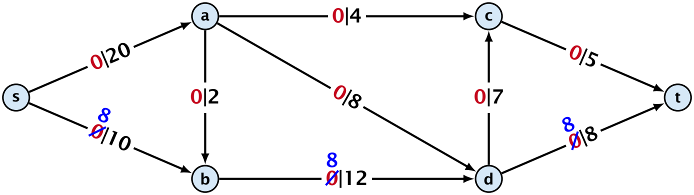

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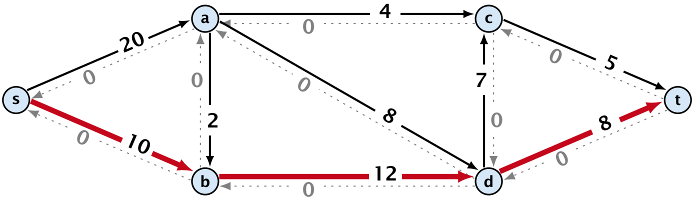

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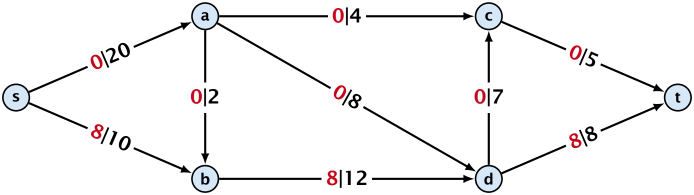
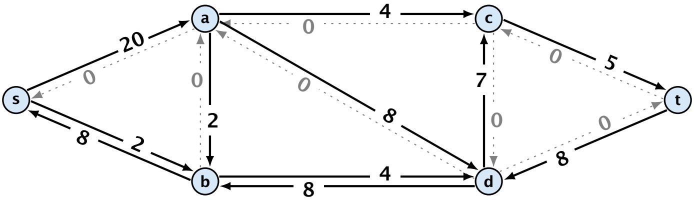


flow value: 8



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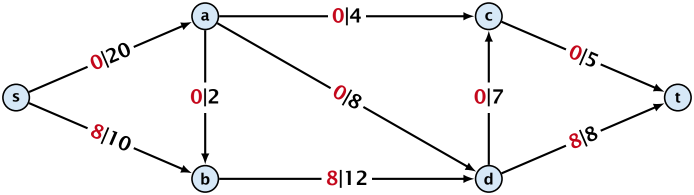

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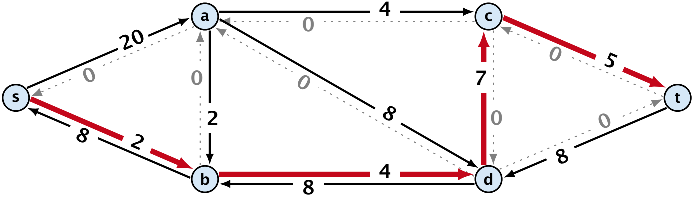

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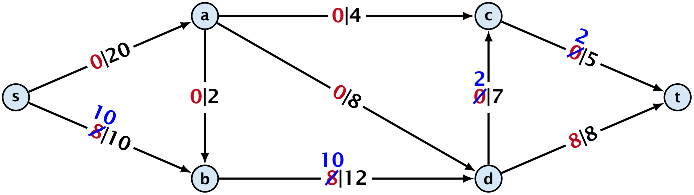

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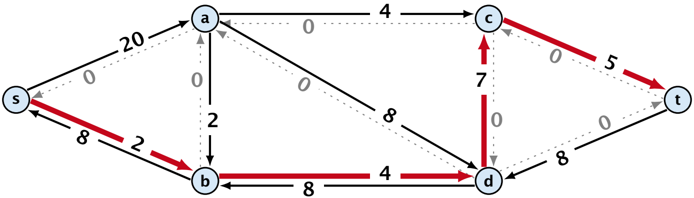

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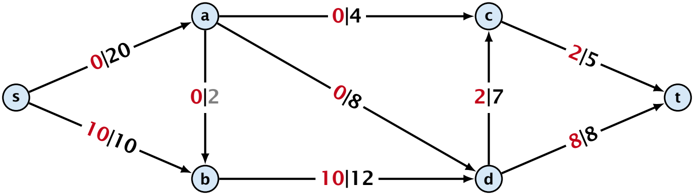

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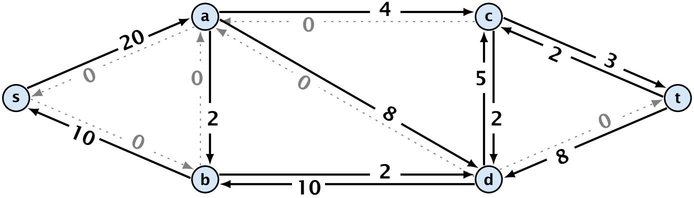

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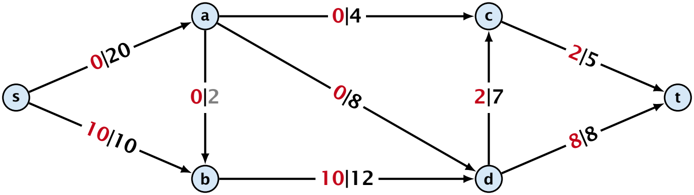

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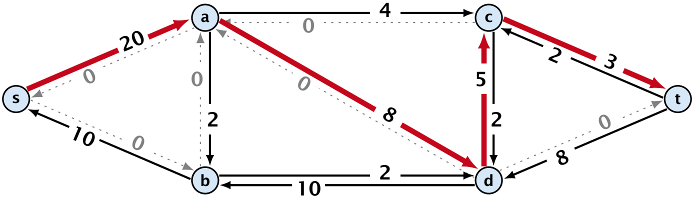

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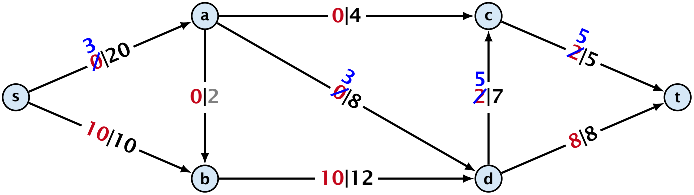

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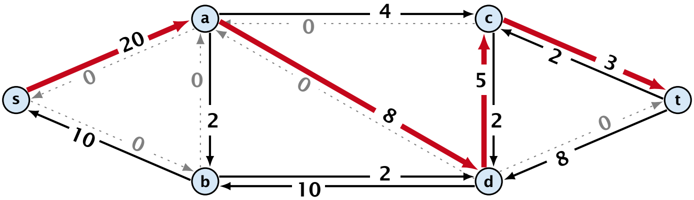

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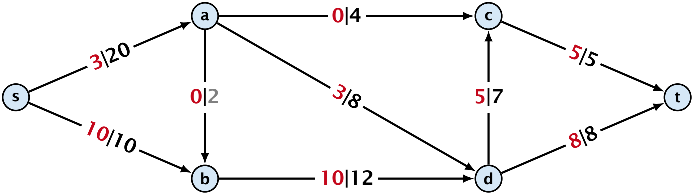
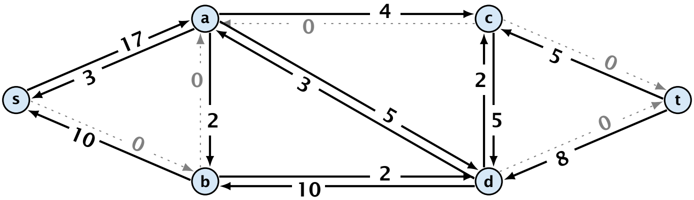


flow value: 13



Augmenting Path Algorithm

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Theorem 38
A flow f is a maximum flow ***iff*** *there are no augmenting paths.*

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Proof.

Let f be a flow. The following are equivalent:

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1. There exists a cut A such that $\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f.

□

Augmenting Path Algorithm

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$1. \implies 2.$

This we already showed.

Augmenting Path Algorithm

1. $\Longrightarrow$ 2.
This we already showed.

2. $\Longrightarrow$ 3.
If there were an augmenting path, we could improve the flow. Contradiction.

Augmenting Path Algorithm

1\. $\Longrightarrow$ 2.

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2\. $\Longrightarrow$ 3.

If there were an augmenting path, we could improve the flow. Contradiction.

3\. $\Longrightarrow$ 1.

- Let f be a flow with no augmenting paths.

Augmenting Path Algorithm

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If there were an augmenting path, we could improve the flow. Contradiction.

3\. $\Longrightarrow$ 1.

- Let f be a flow with no augmenting paths.
- Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.

Augmenting Path Algorithm

1\. $\Longrightarrow$ 2.

This we already showed.

2\. $\Longrightarrow$ 3.

If there were an augmenting path, we could improve the flow. Contradiction.

3\. $\Longrightarrow$ 1.

- Let f be a flow with no augmenting paths.
- Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$\mathrm{val}(f)$

Augmenting Path Algorithm

$$\operatorname{val}(f) = \sum_{e \in \operatorname{out}(A)} f(e) - \sum_{e \in \operatorname{into}(A)} f(e)$$

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$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(A)} f(e) - \sum_{e\in\mathrm{into}(A)} f(e)\\ &= \sum_{e\in\mathrm{out}(A)} c(e)\end{aligned}$$

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

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All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

Lemma 40

The algorithm terminates in at most $\mathrm{val}(f^*) \le nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$*.*

Lemma 40

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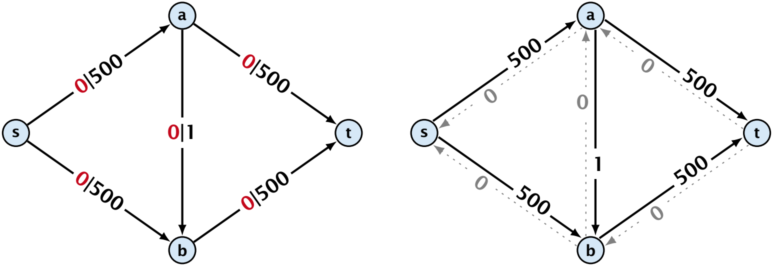
Theorem 41

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ *is integral.*

A Bad Input

Problem: The running time may not be polynomial



flow value: 0

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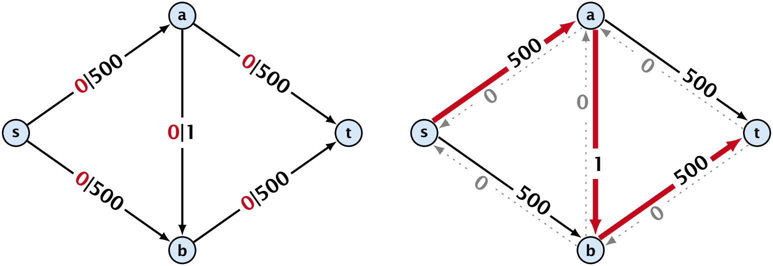

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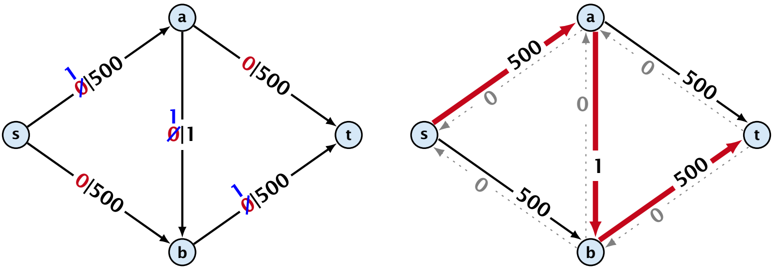

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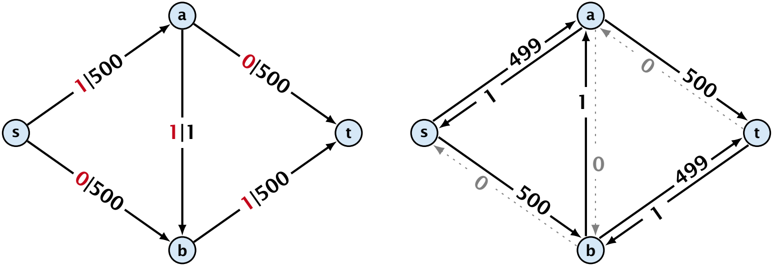

flow value: 1

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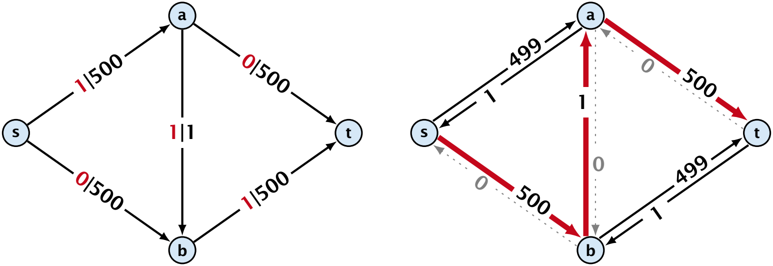

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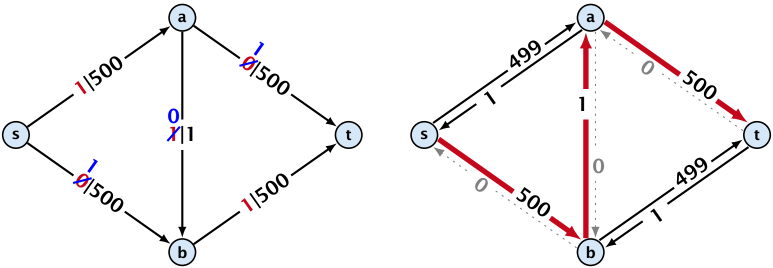

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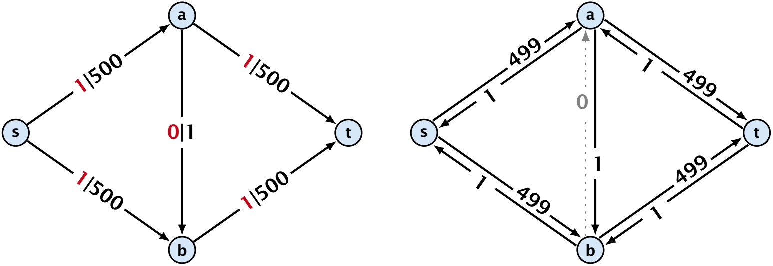
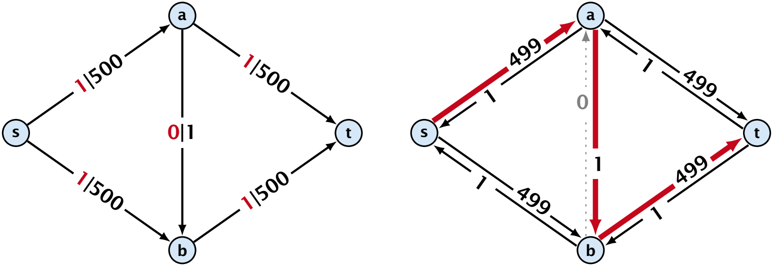


flow value: 2

A Bad Input

Problem: The running time may not be polynomial



flow value: 2

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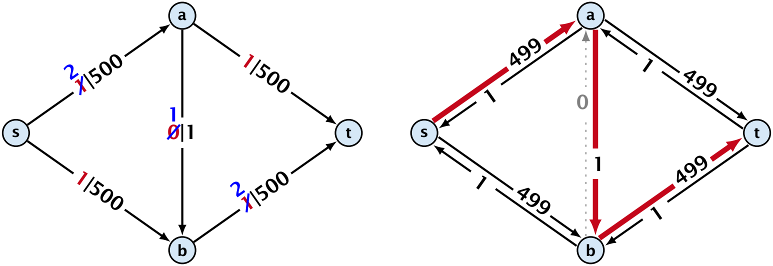

flow value: 2

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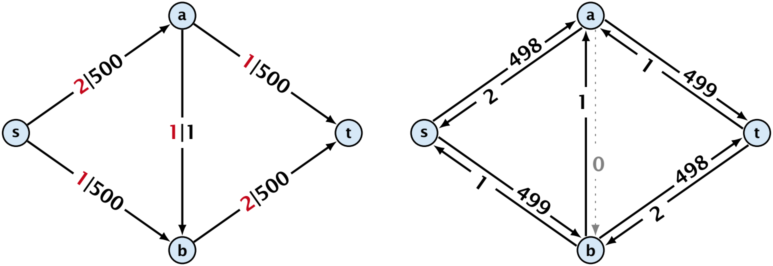
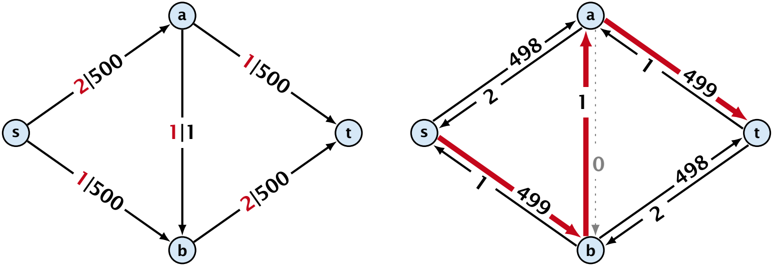


flow value: 3

A Bad Input

Problem: The running time may not be polynomial



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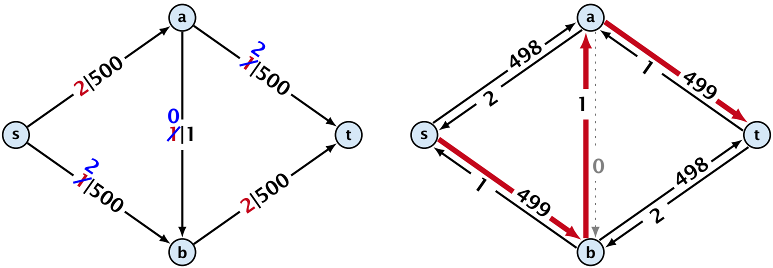

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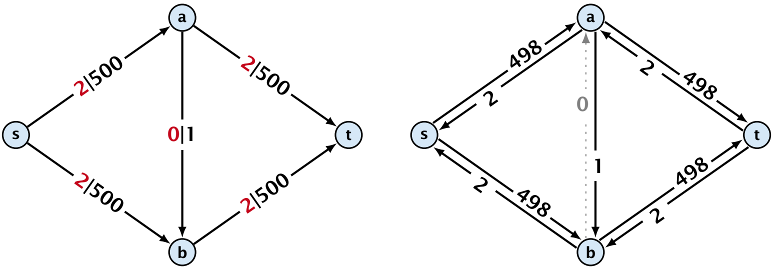
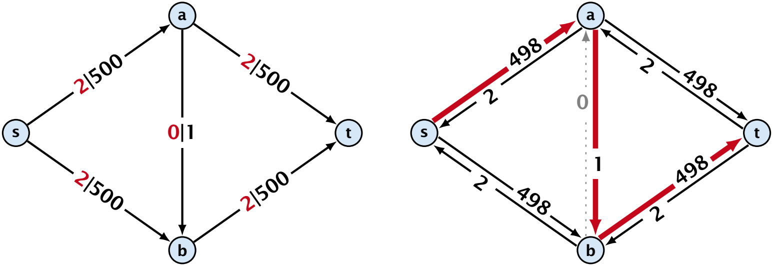


flow value: 4

A Bad Input

Problem: The running time may not be polynomial



flow value: 4

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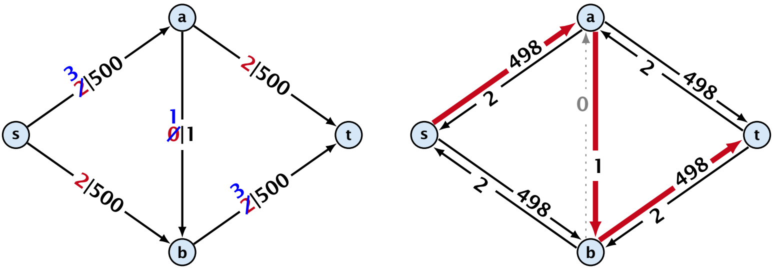
Problem: The running time may not be polynomial



flow value: 4

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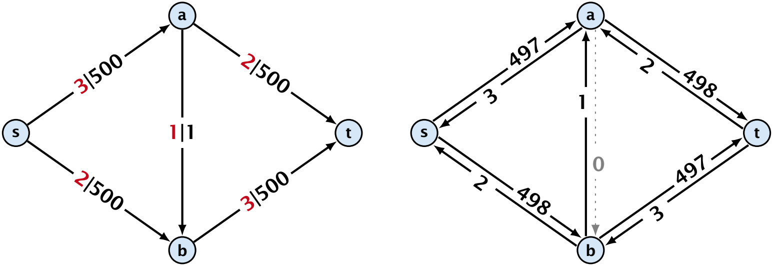
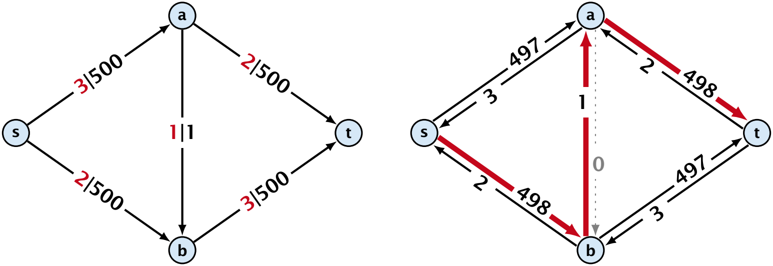


flow value: 5

A Bad Input

Problem: The running time may not be polynomial



flow value: 5

A Bad Input

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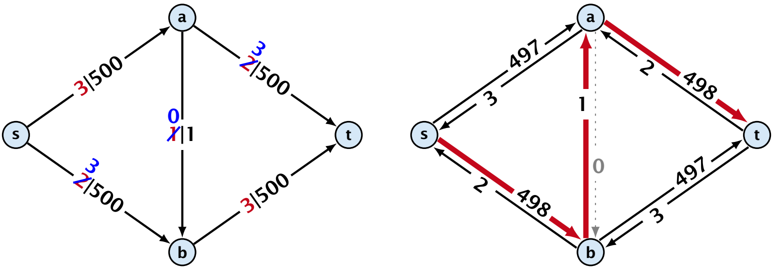

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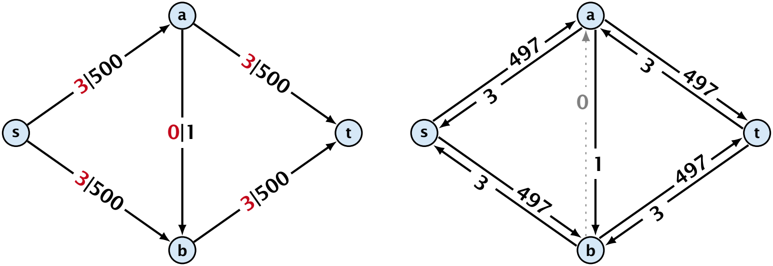
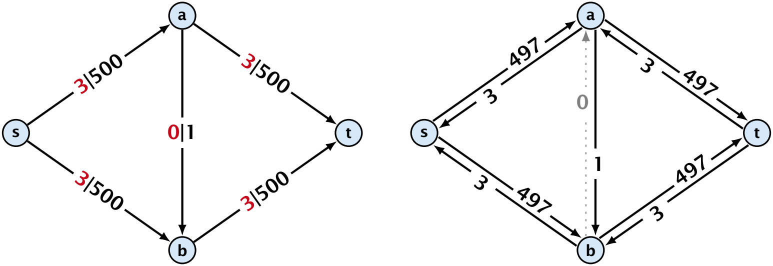


flow value: 6

A Bad Input

Problem: The running time may not be polynomial

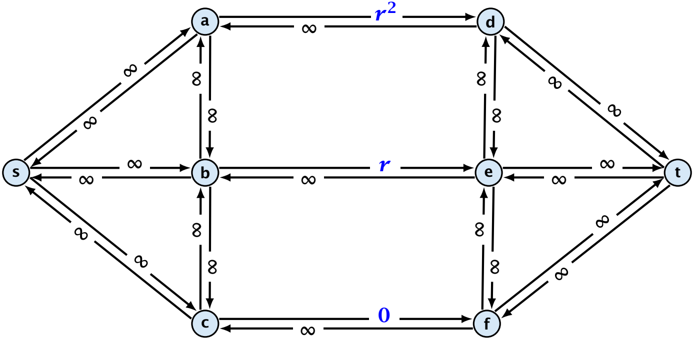


flow value: 6

Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



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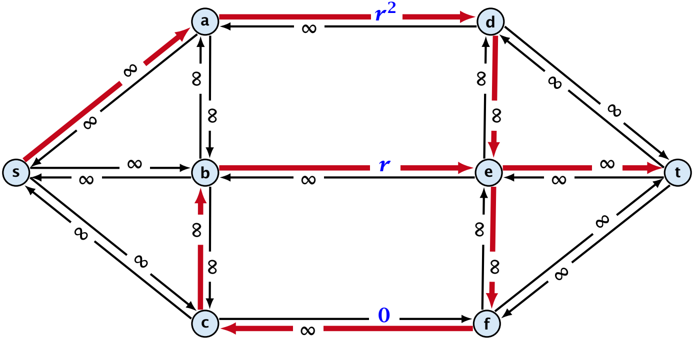

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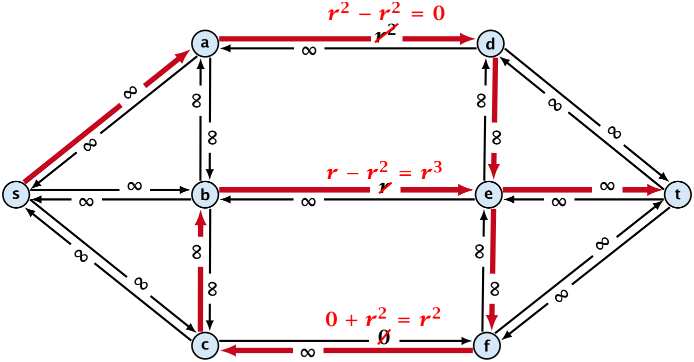

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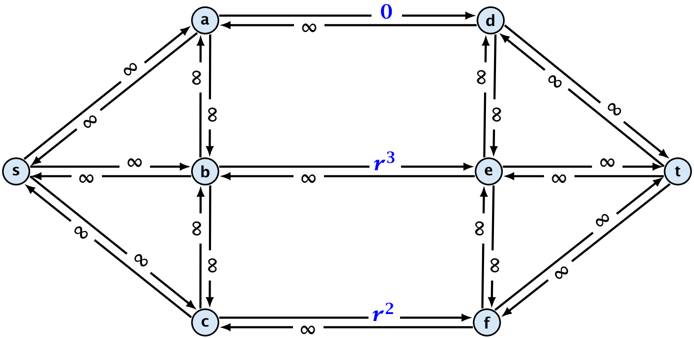

flow value: r^2

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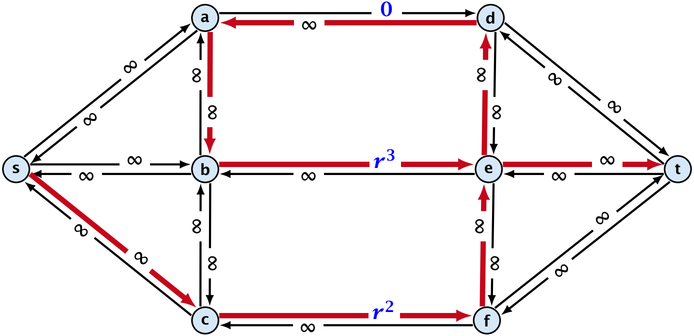

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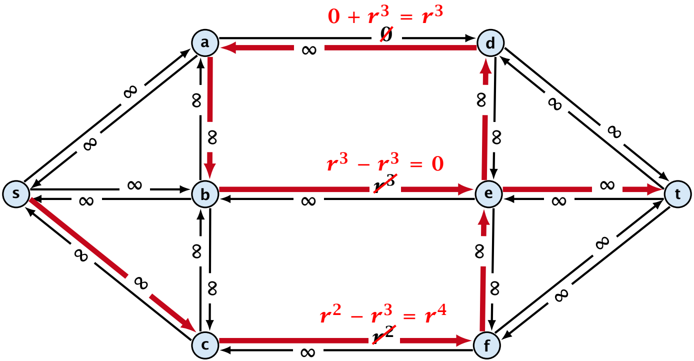

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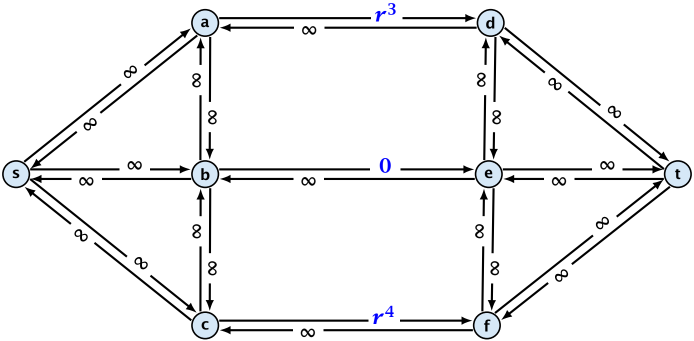

flow value: $r^2 + r^3$

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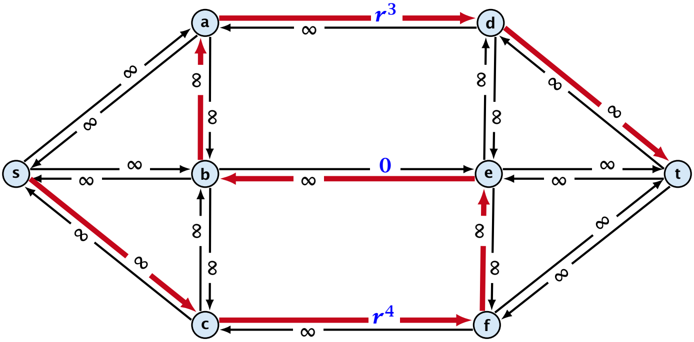

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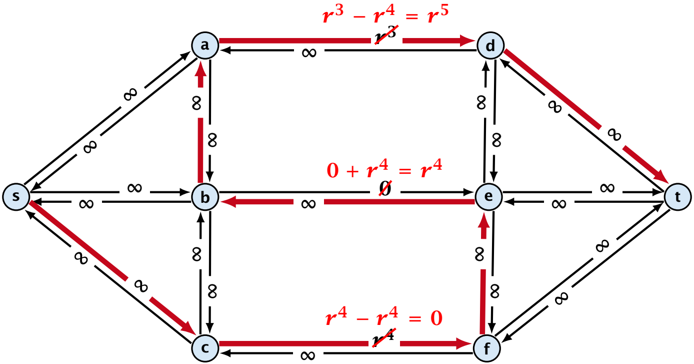

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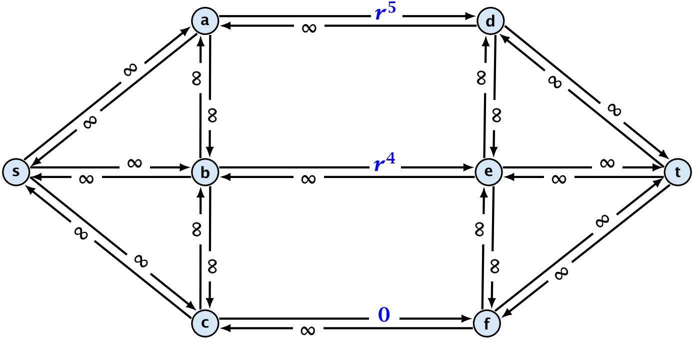

flow value: $r^2 + r^3 + r^4$

Running time may be infinite!!!

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Several possibilities:

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- Choose path with sufficiently large bottleneck capacity.
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Overview: Shortest Augmenting Paths

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Lemma 42

The length of the shortest augmenting path never decreases.

Overview: Shortest Augmenting Paths

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Lemma 43
After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

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Theorem 44
The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ *augmentations. This gives a running time of* $\mathcal{O}(m^2n)$.

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Proof.

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Proof.

- We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.

□

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Define the level $\ell(v)$ of a node as the length of the shortest s-v path in G_f (along non-zero edges).

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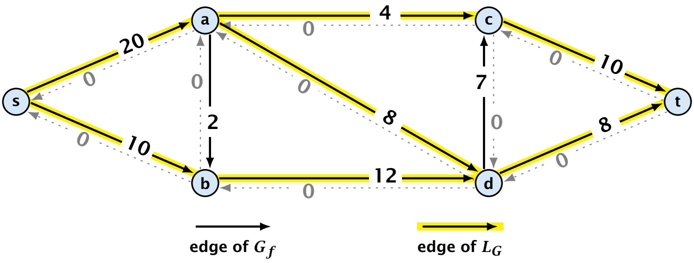
A path P is a shortest s-u path in G_f **iff** it is an s-u path in L_G.



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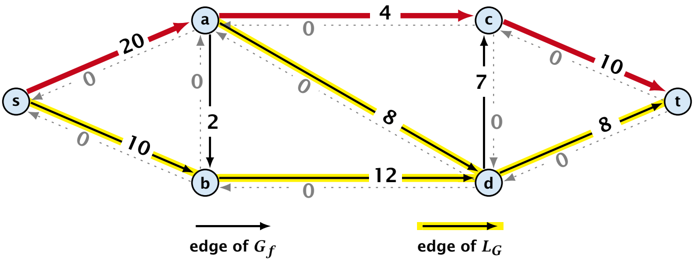

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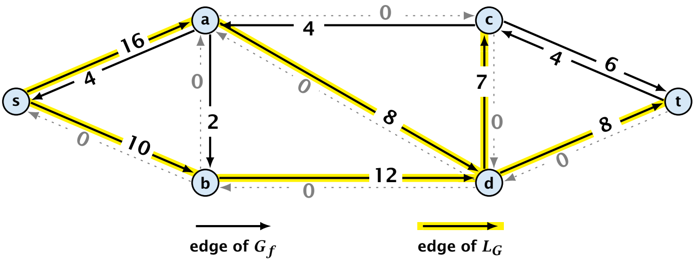

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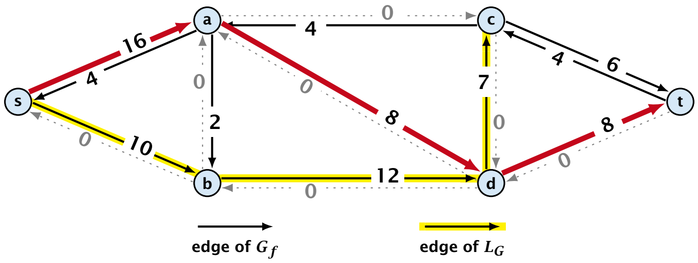

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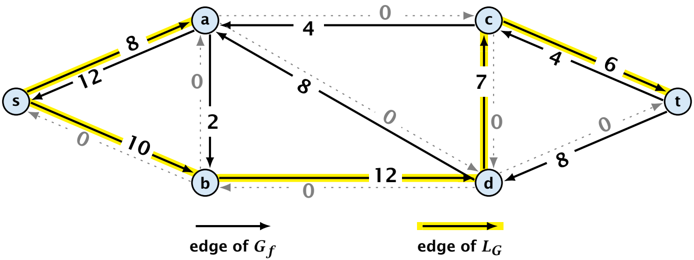

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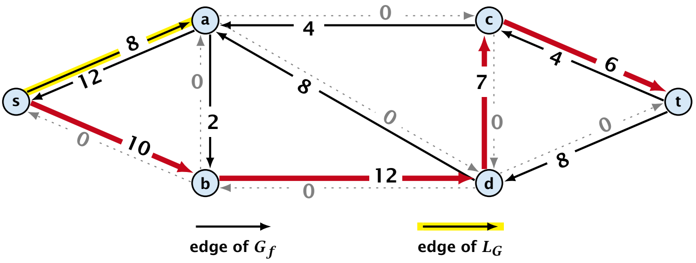

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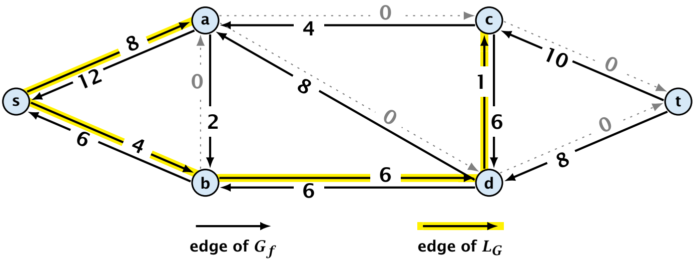

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

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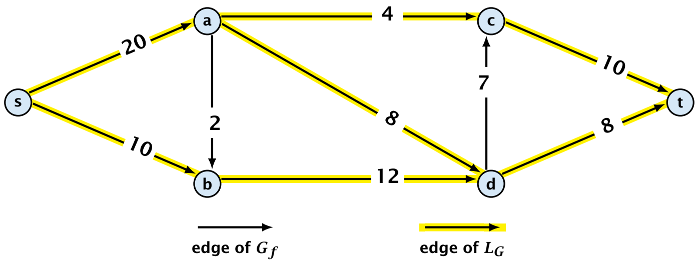

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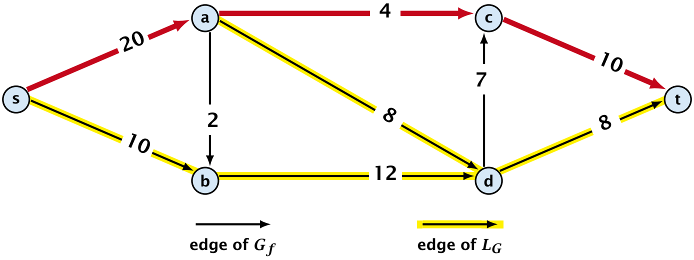

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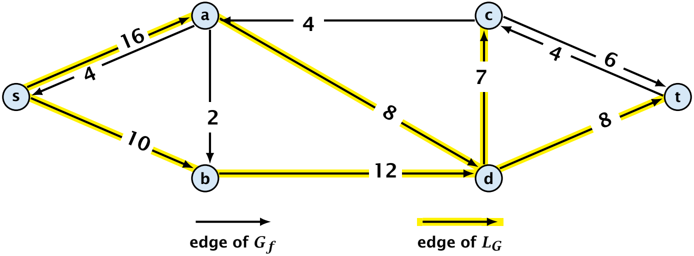

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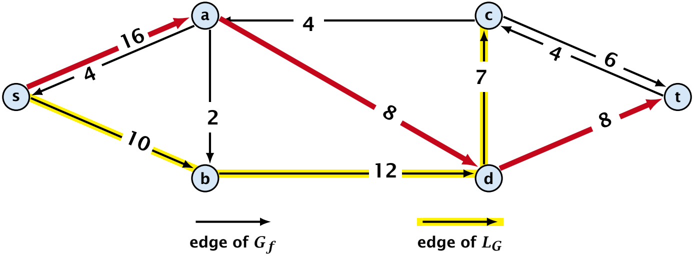

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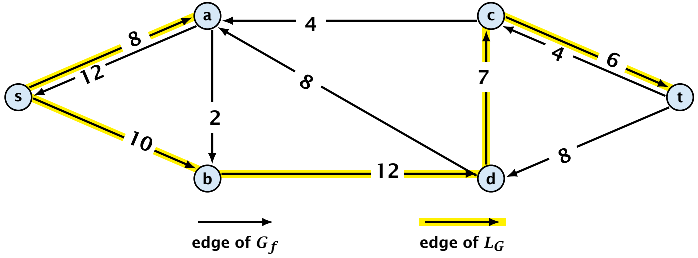

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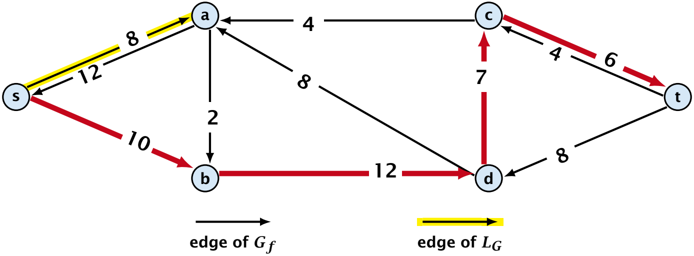

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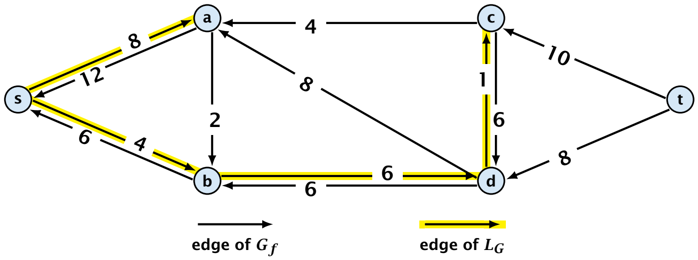

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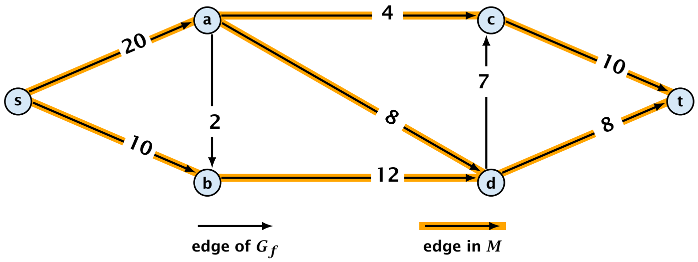

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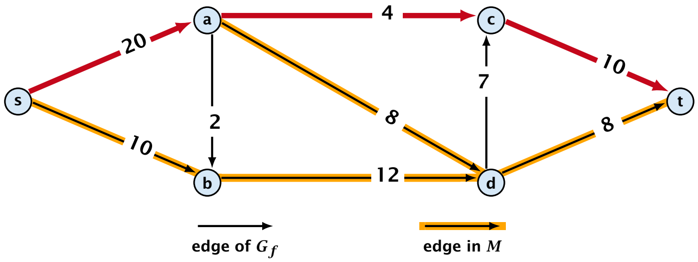

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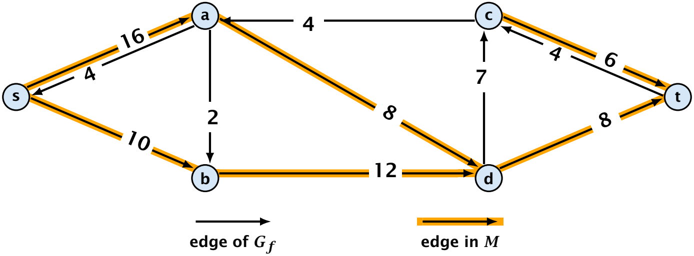

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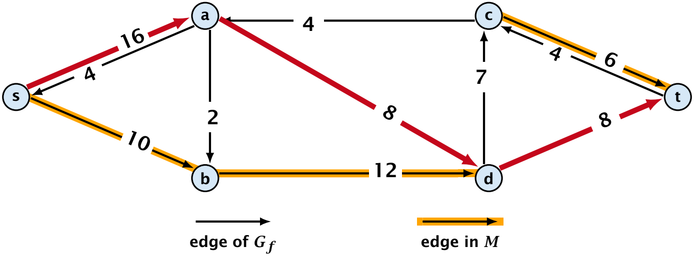

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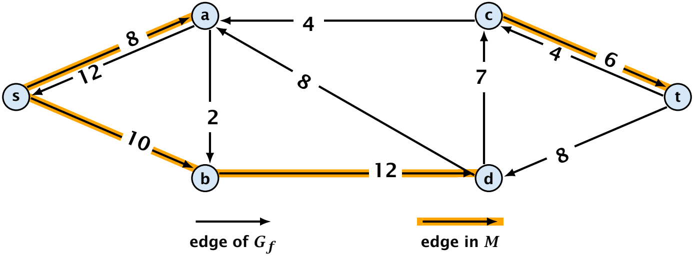

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Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

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When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

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However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

Shortest Augmenting Paths

We maintain a subset M of the edges of G_f with the guarantee that a shortest s-t path using only edges from M is a shortest augmenting path.

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Note that M is not the set of edges of the level graph but a subset of level-graph edges.

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You can delete incoming edges of v from M.

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There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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Several possibilities:

- Choose path with maximum bottleneck capacity.
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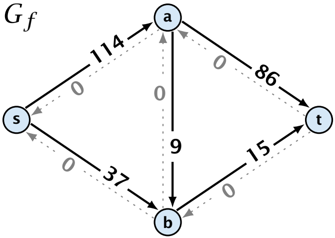

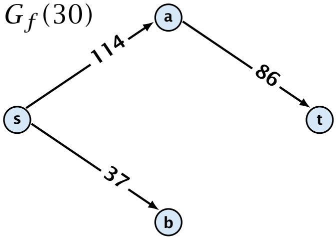

Capacity Scaling

Algorithm 1 maxflow(G, s, t, c)

foreach $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ **do**
    $G_f(\Delta) \leftarrow \Delta$-residual graph
    while there is augmenting path P in $G_f(\Delta)$ **do**
        $f \leftarrow$ augment(f, c, P)
        update$(G_f(\Delta))$
    $\Delta \leftarrow \Delta/2$
return f

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- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

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- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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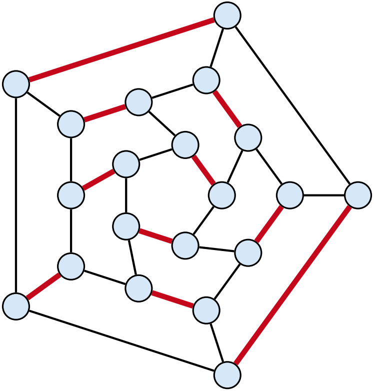
Theorem 50

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
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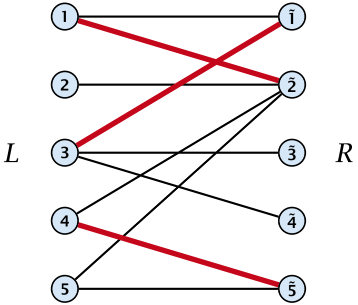

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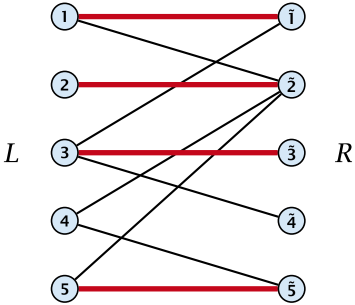
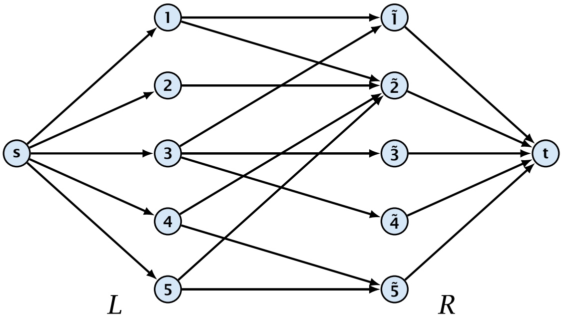


Maxflow Formulation

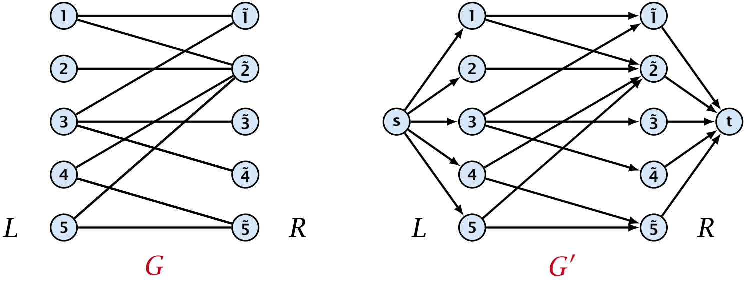
- Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s,t\}, E')$.
- Direct all edges from L to R.
- Add source s and connect it to all nodes on the left.
- Add t and connect all nodes on the right to t.
- All edges have unit capacity.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.



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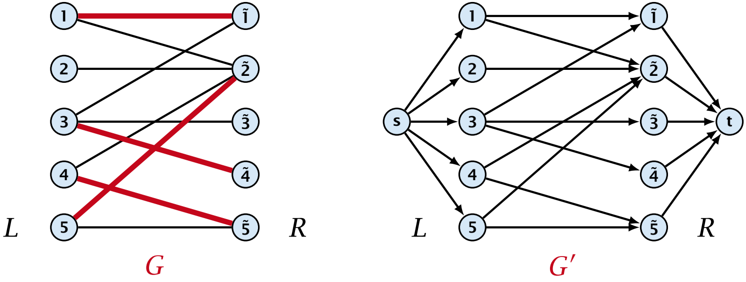

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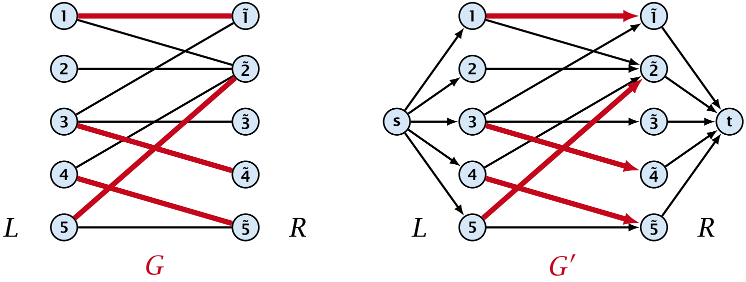

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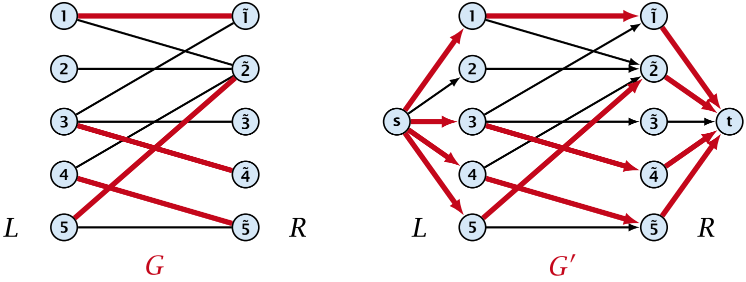

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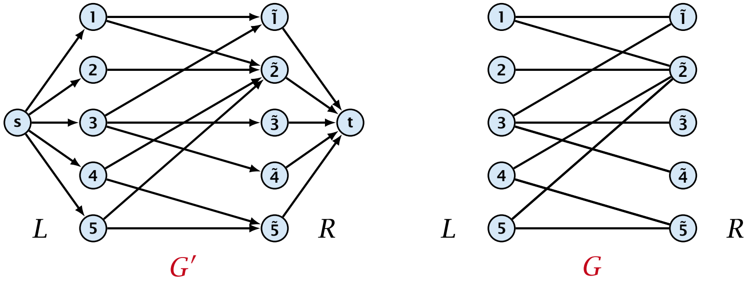
- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow k$ integral; we can assume f is 0/1.
- Consider M= set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.



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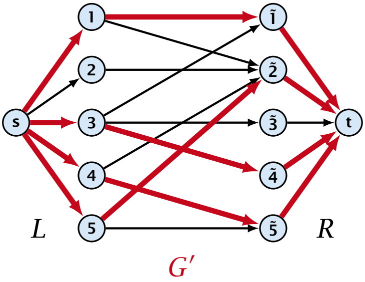

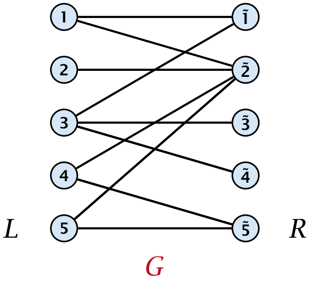

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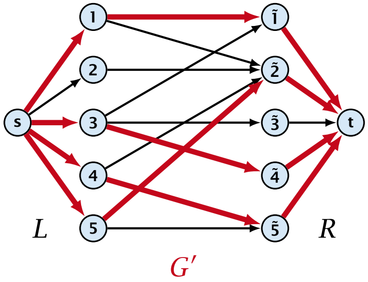


8.1 Matching

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- Shortest augmenting path: $\mathcal{O}(mn^2)$.

For unit capacity simple graphs shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

Baseball Elimination

team i	wins w_i	losses ℓ_i	remaining games Atl	Phi	NY	Mon
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- But also Philadelphia is eliminated. Why?

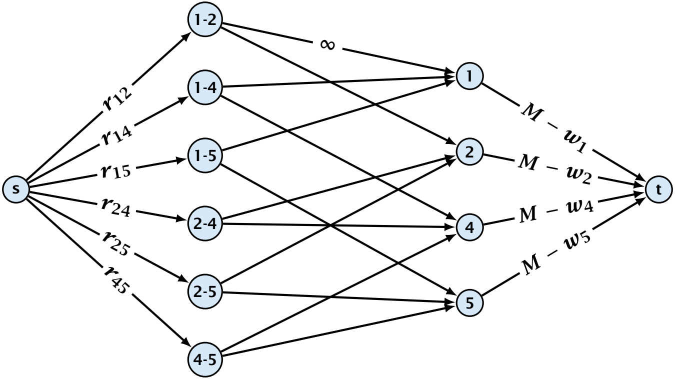
Baseball Elimination

Formal definition of the problem:

- Given a set S of teams, and one specific team $z \in S$.
- Team x has already won w_x games.
- Team x still has to play team y, r_{xy} times.
- Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.



Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i<j} r_{ij}$$

wins of teams in T

remaining games among teams in T

If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

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A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

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- If for node x-y not both team-nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.
- We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\})$$

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$$r(S \setminus \{z\}) > \text{cap}(A, V \setminus A)$$

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A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

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$$\begin{aligned} r(S\setminus\{z\}) &> \mathrm{cap}(A, V\setminus A) \\ &\geq \sum_{i<j:\, i\notin T \vee j\notin T} r_{ij} + \sum_{i\in T}(M - w_i) \end{aligned}$$

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&\geq r(S\setminus\{z\}) - r(T) + |T|M - w(T)
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- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

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- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

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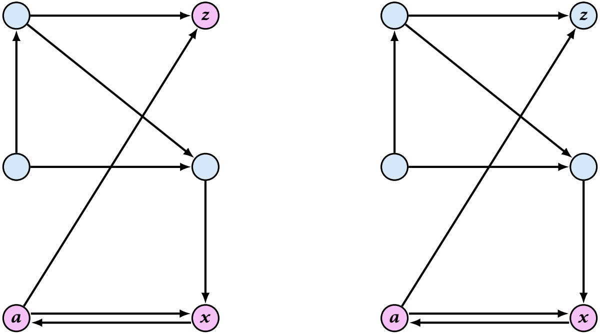
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

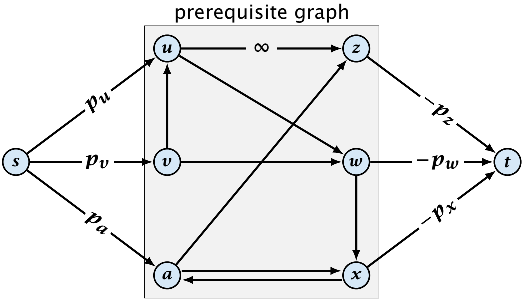
- $\{x, a, z\}$ is a feasible subset.
- $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- Edges in the prerequisite graph get infinite capacity.
- Add edge (s, v) with capacity p_v for nodes v with positive profit.
- Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



Theorem 52

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

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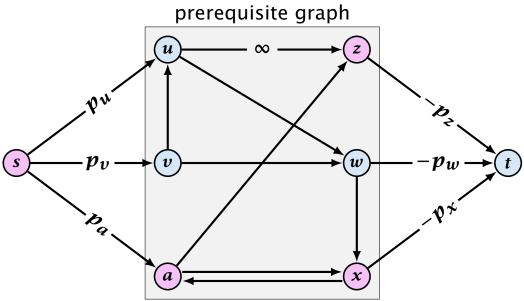
Proof.

- A is feasible because of capacity infinity edges.



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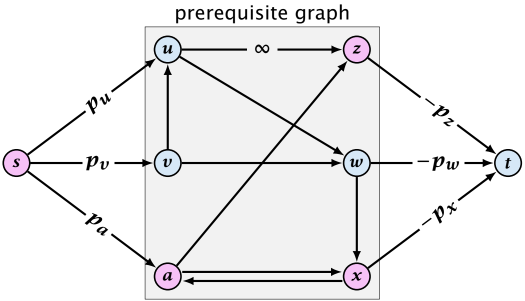
- A is feasible because of capacity infinity edges.
- $\text{cap}(A, V \setminus A)$



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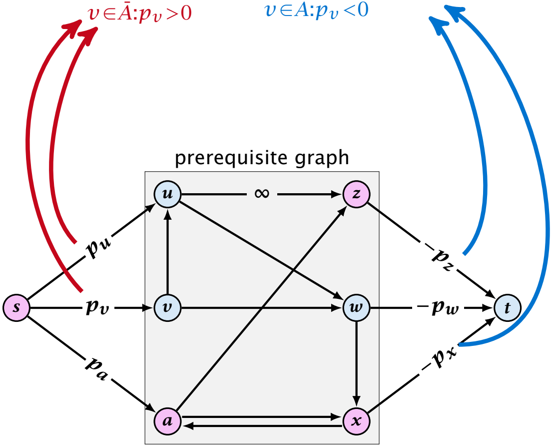
- A is feasible because of capacity infinity edges.
- $\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)$



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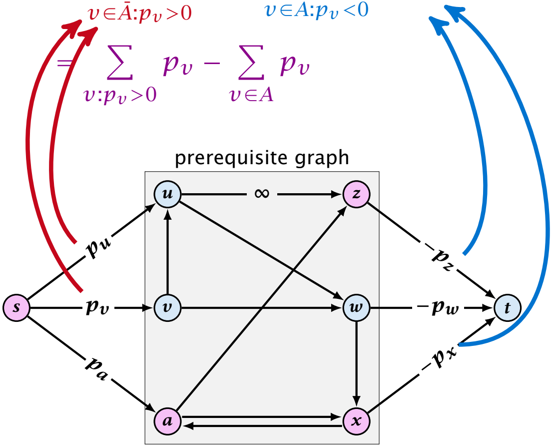
  $$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$



Preflows

Preflows

Definition 53
An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \le f(e) \le c(e) \enspace .$$

   (capacity constraints)

Preflows

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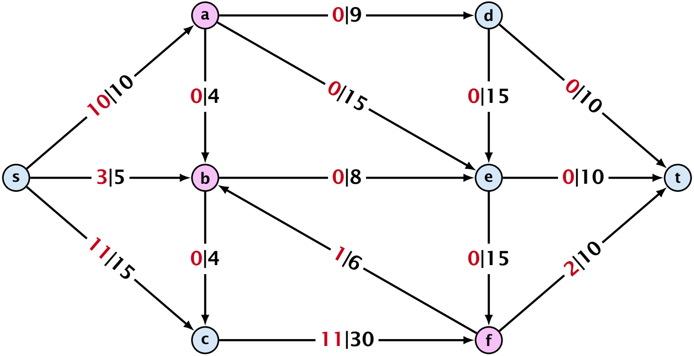
1. For each edge e
$$0 \leq f(e) \leq c(e) \ .$$
(capacity constraints)
2. For each $v \in V \setminus \{s, t\}$
$$\sum_{e \in \text{out}(v)} f(e) \leq \sum_{e \in \text{into}(v)} f(e) \ .$$

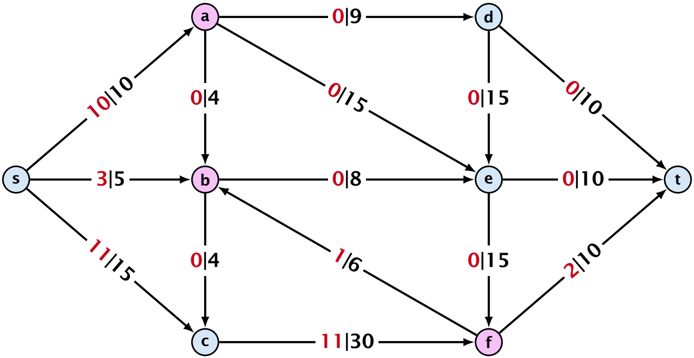
Preflows

Example 54



Preflows

Example 54



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

Preflows

Definition:
A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \leq \ell(v) + 1$ for all edges (u, v) in the residual graph G_f (only non-zero capacity edges!!!)

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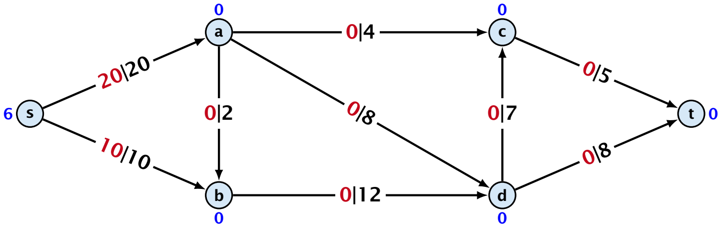
- $\ell(s) = n$
- $\ell(t) = 0$

Intuition:
The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows



0
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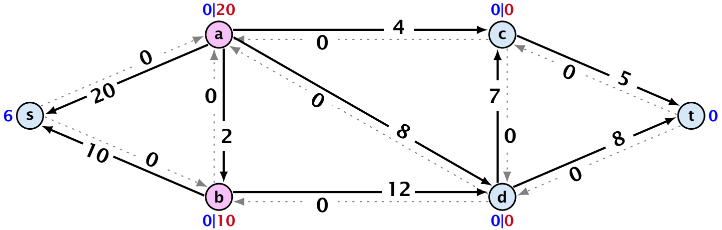

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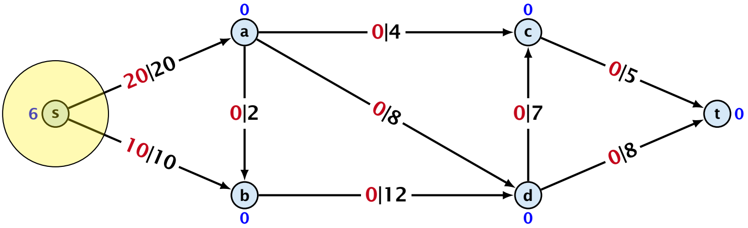
Preflows



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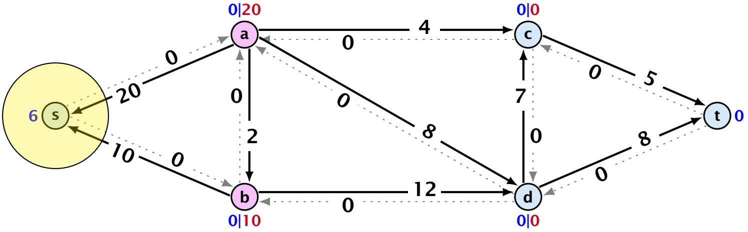

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Lemma 55

A preflow that has a valid labelling saturates a cut.

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Lemma 56

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

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Idea:

- start with some preflow and some valid labelling

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- start with some preflow and some valid labelling
- successively change the preflow while maintaining a valid labelling

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- start with some preflow and some valid labelling
- successively change the preflow while maintaining a valid labelling
- stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

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An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissible if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

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The push operation
Consider an active node u with excess flow $f(u) = \sum_{e\in\text{into}(u)} f(e) - \sum_{e\in\text{out}(u)} f(e)$ and suppose $e = (u,v)$ is an admissible arc with residual capacity $c_f(e)$.

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We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

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  the arc e is deleted from the residual graph
- deactivating push: $\min\{f(u), c_f(e)\} = f(u)$
  the node u becomes inactive

Push Relabel Algorithms

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Push Relabel Algorithms

The relabel operation

Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- Edges (w,u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.
- An outgoing edge (u,w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:
We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- Such a node is called active.
- A labelling is valid if for every edge (u, v) in the residual graph $\ell(u) \le \ell(v) + 1$.
- An arc (u, v) in residual graph is admissible if $\ell(u) = \ell(v) + 1$.
- A saturating push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).
- A deactivating push along $e = (u, v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 1 maxflow(G, s, t, c)

```
find initial preflow f
while there is active node u do
    if there is admiss. arc e out of u then
        push(G, e, f, c)
    else
        relabel(u)
return f
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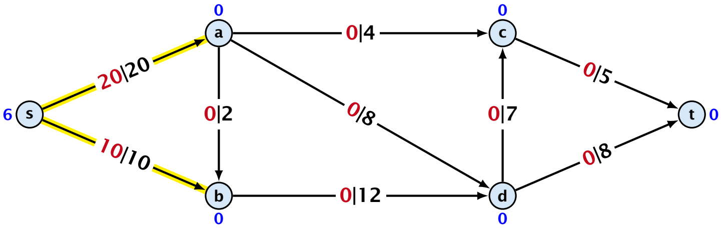
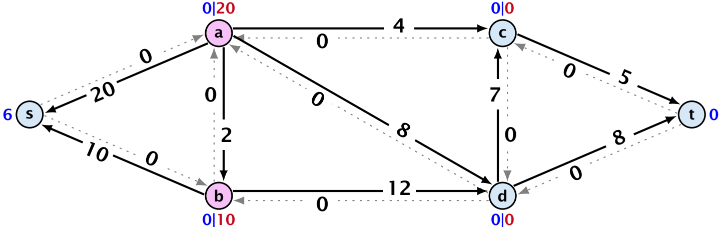
```

In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push





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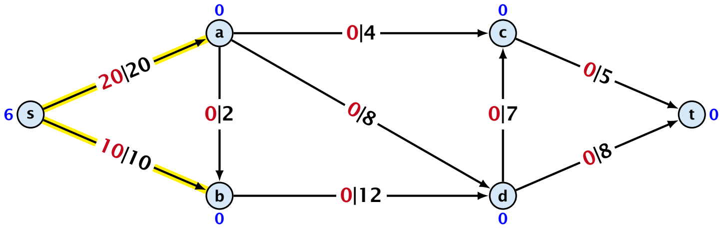
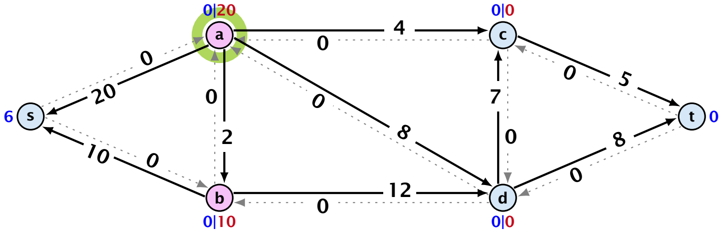
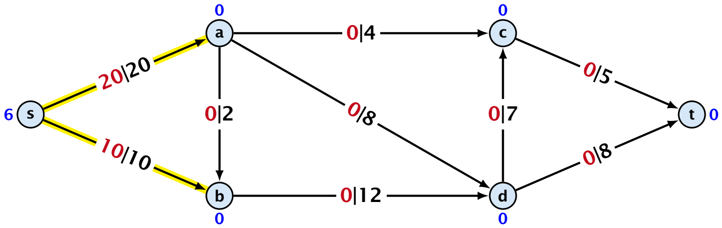
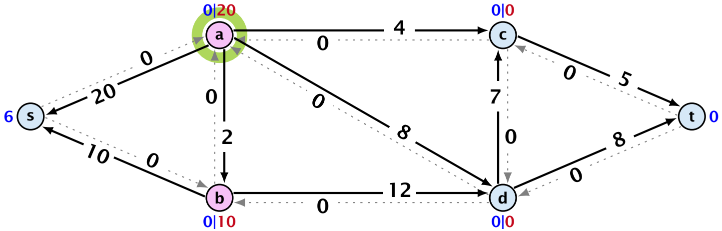




Preflow Push



relabel to 1



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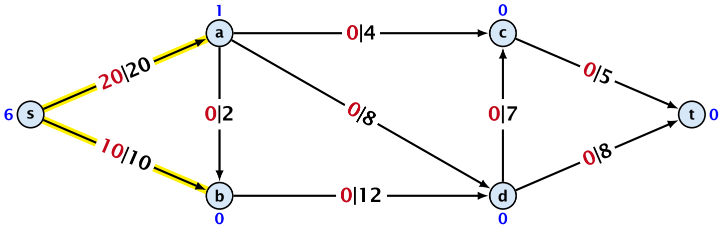
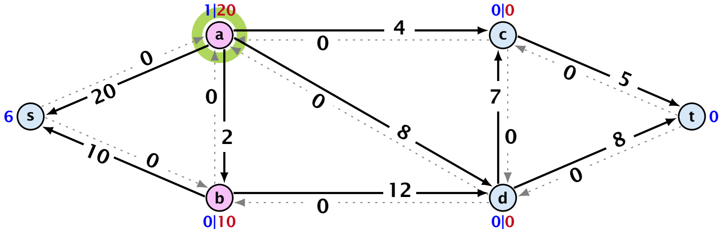
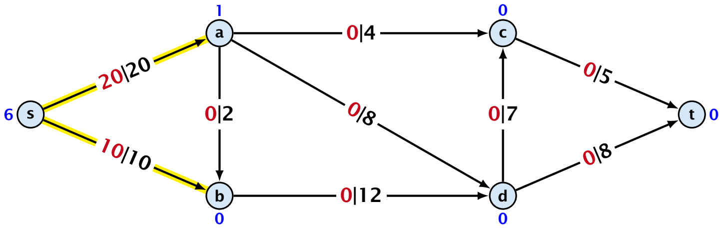
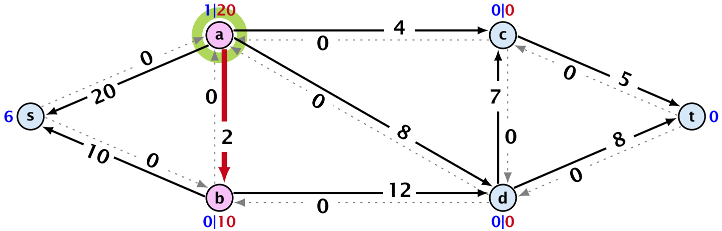




Preflow Push



saturating push



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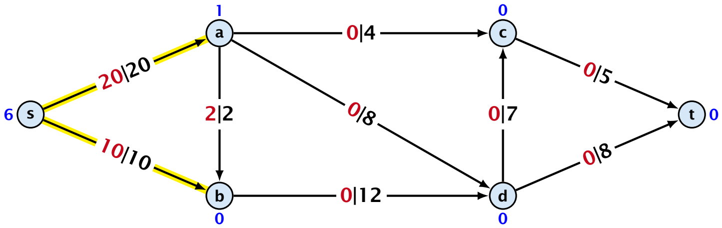

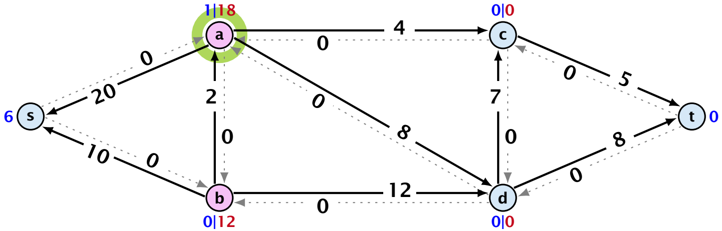

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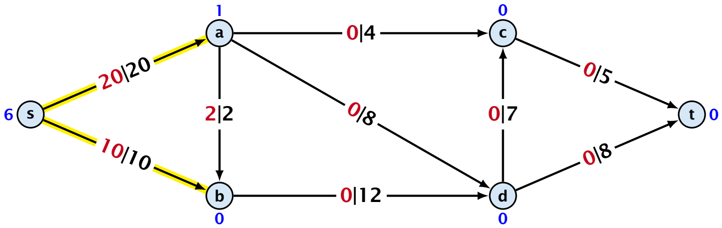

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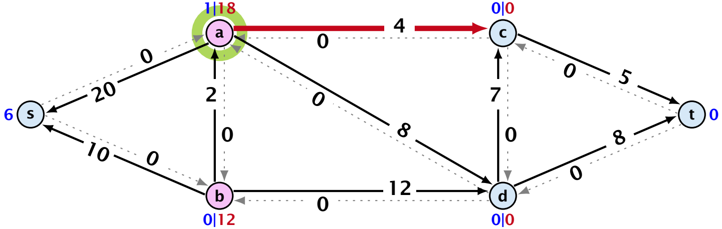

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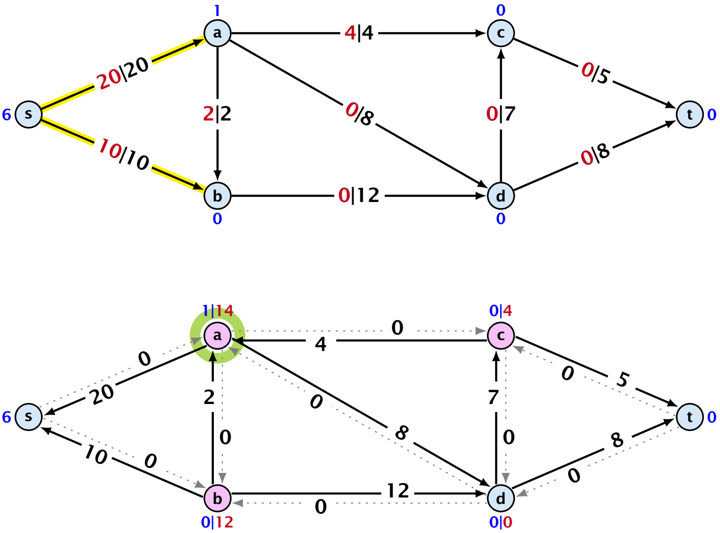

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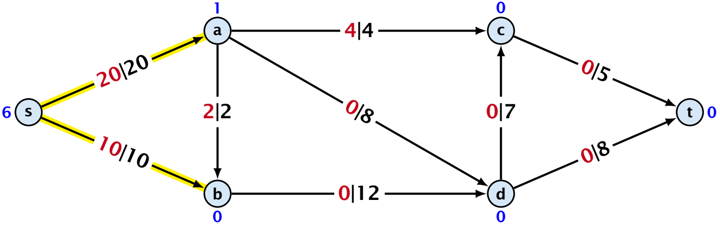

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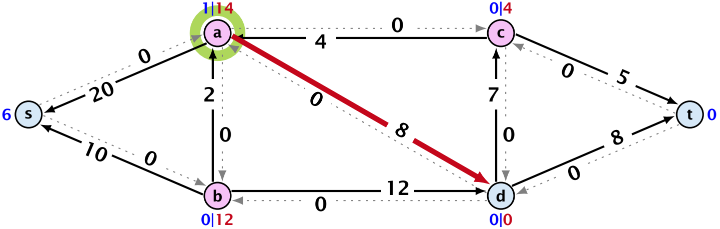

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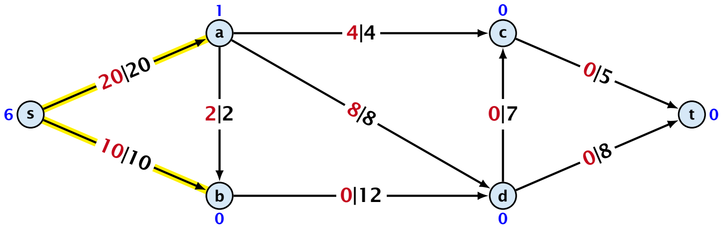

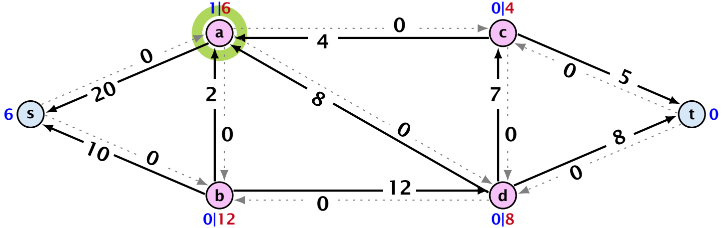

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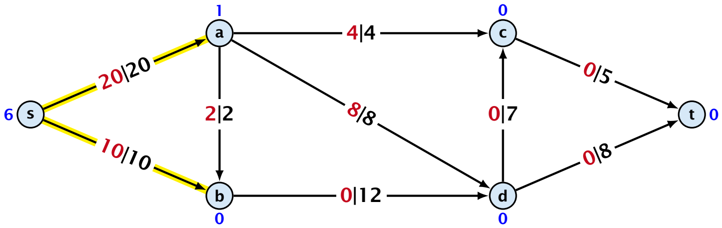

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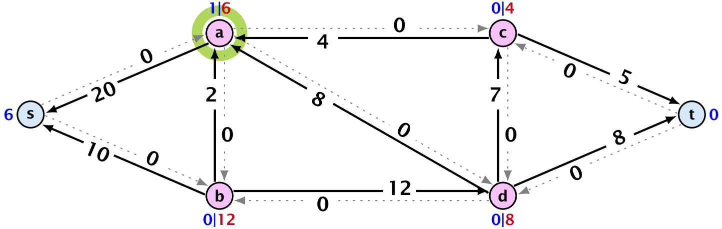

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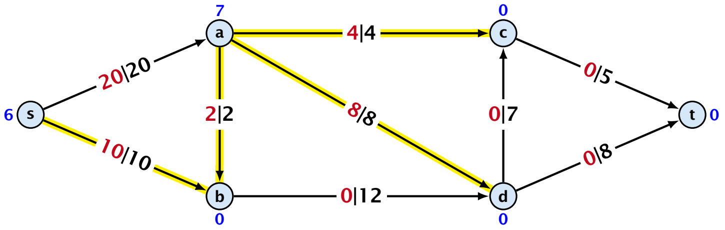

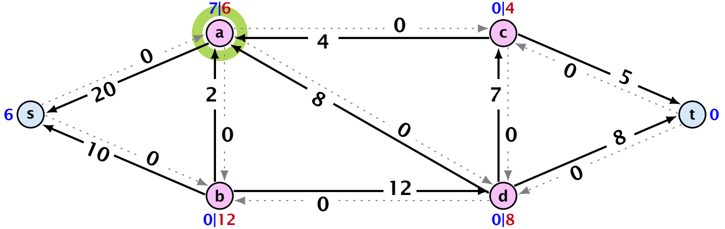

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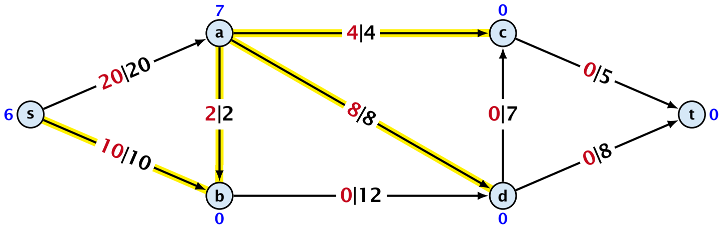
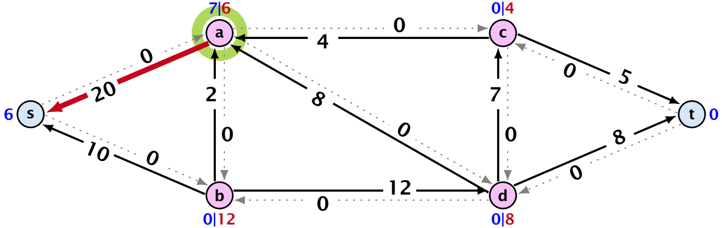


deactivating push



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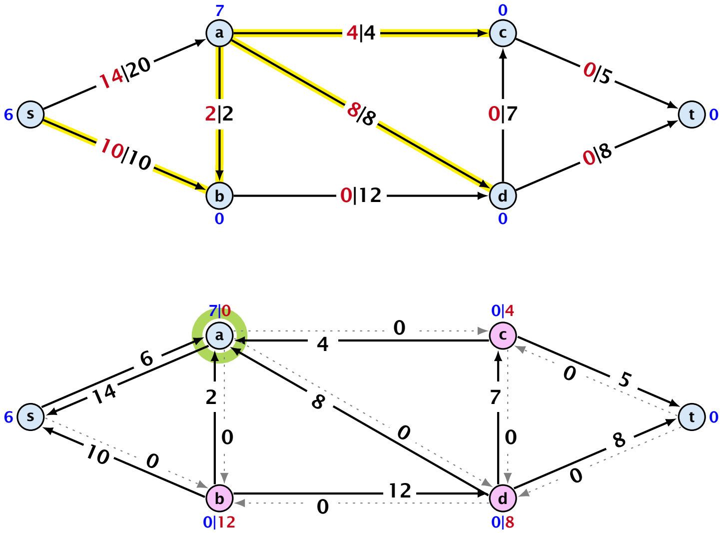

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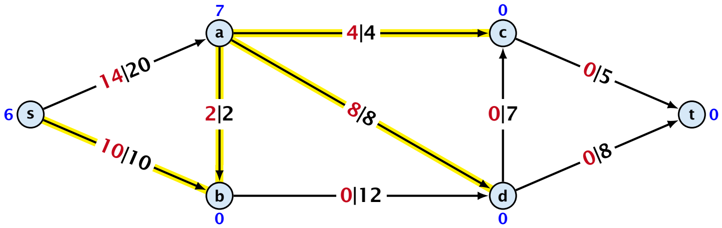

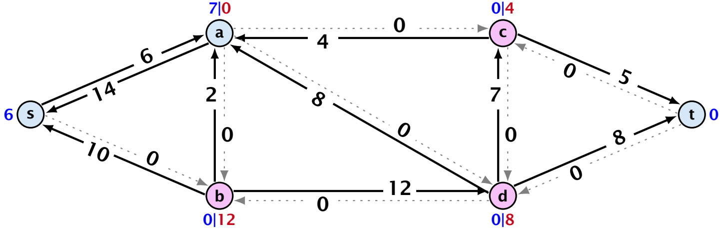

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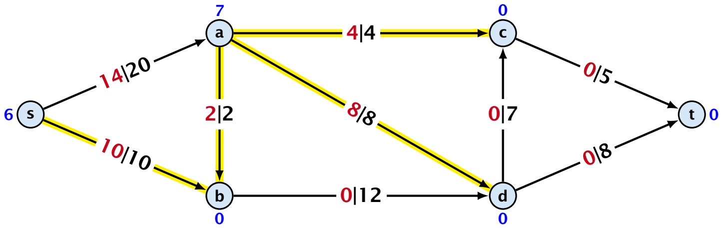

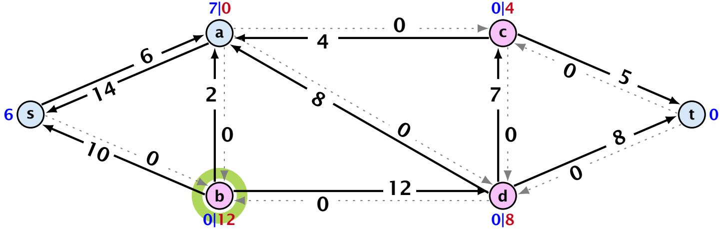

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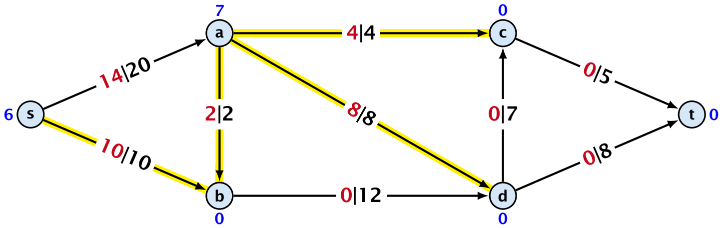

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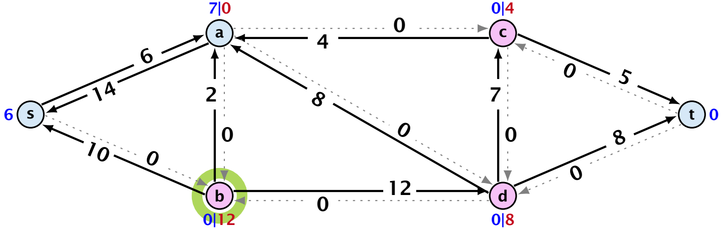

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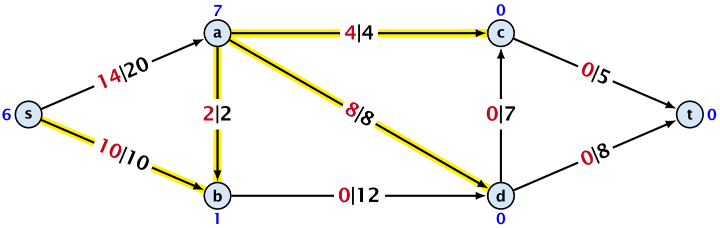

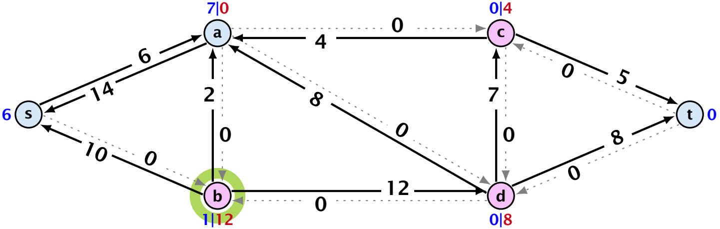

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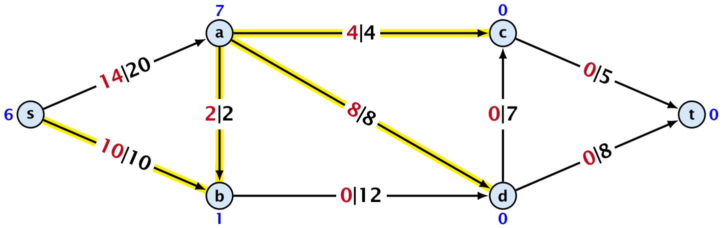
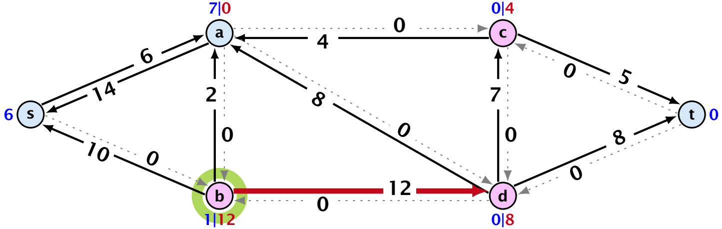


saturating and deactivating push



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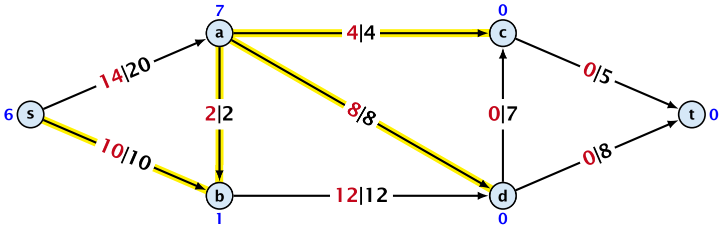

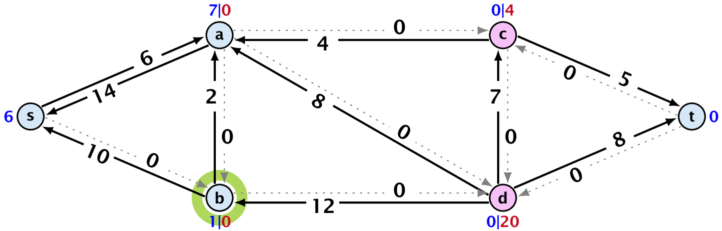

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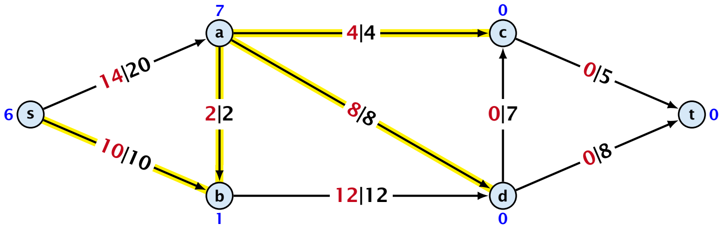

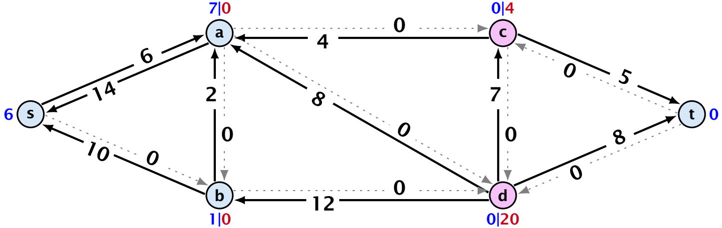

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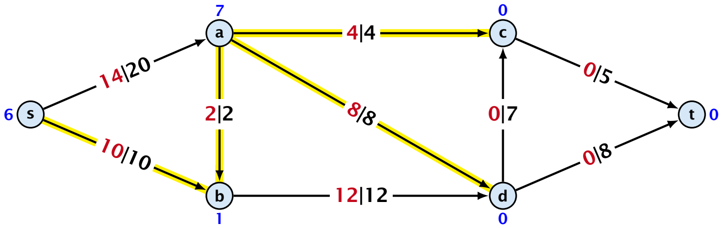

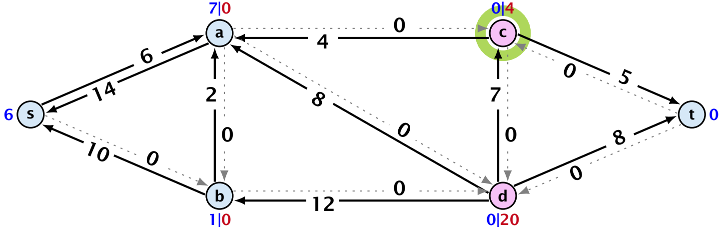

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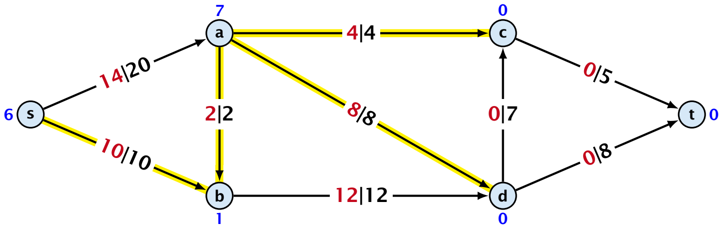

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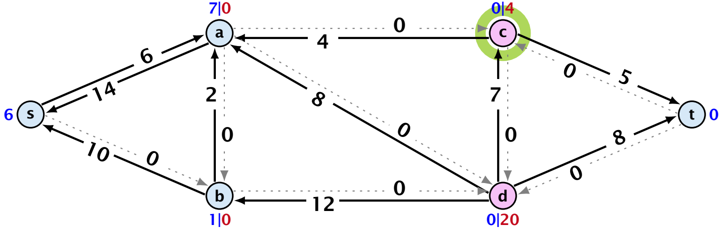

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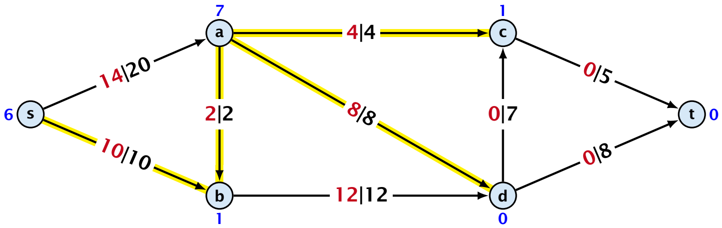

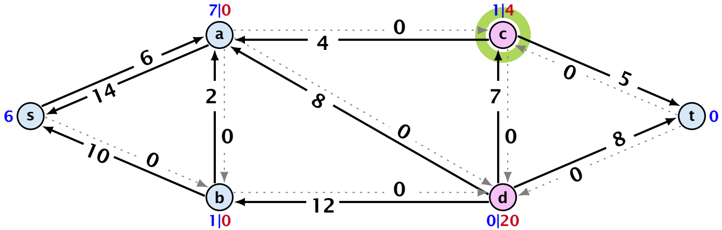

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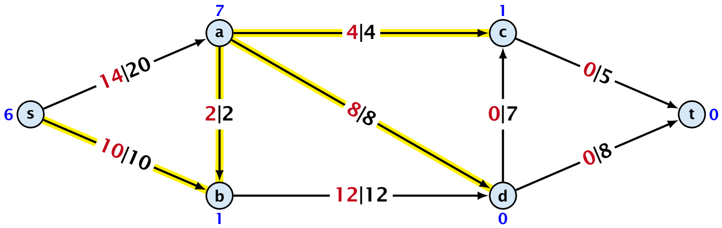
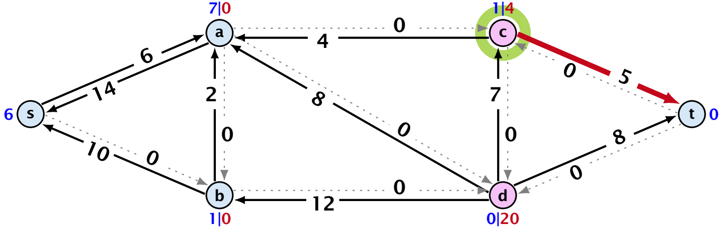


deactivating push



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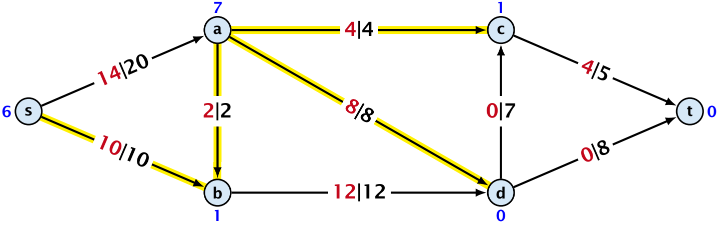

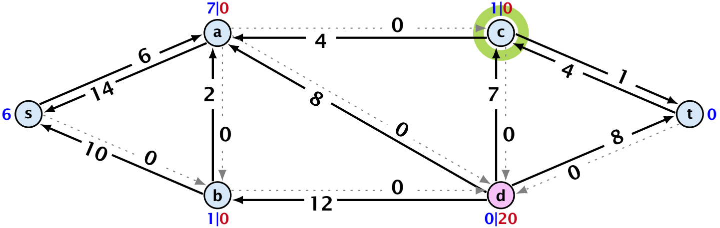

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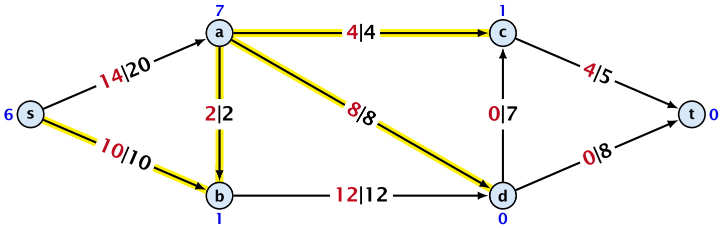

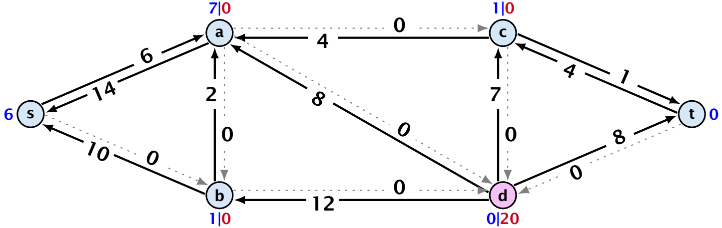

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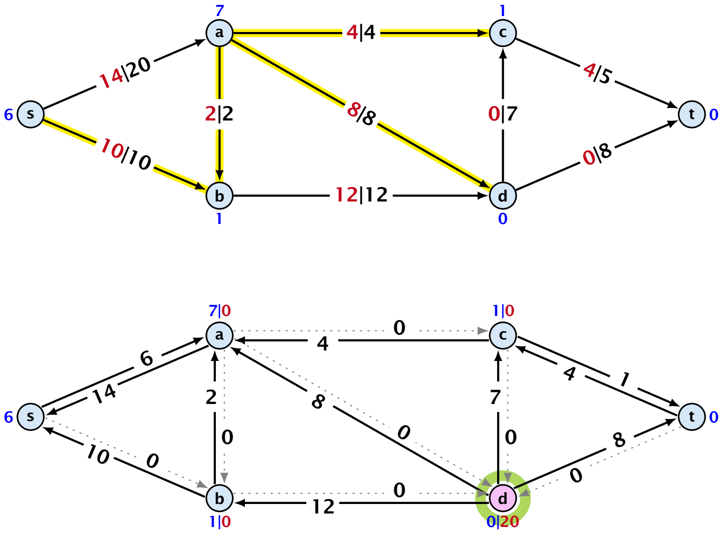

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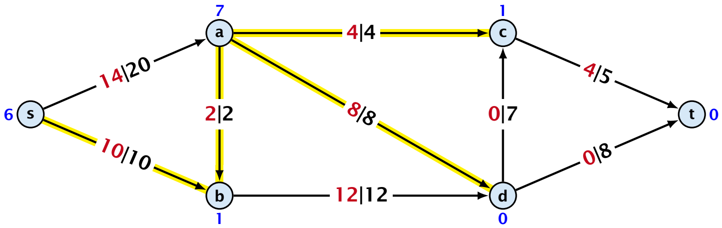

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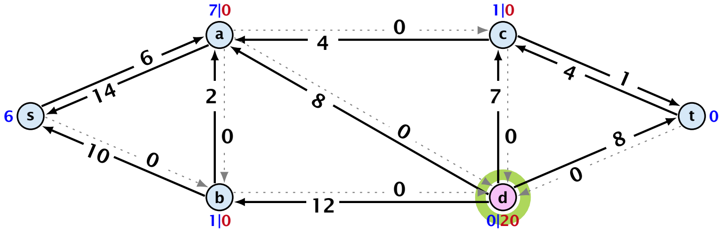

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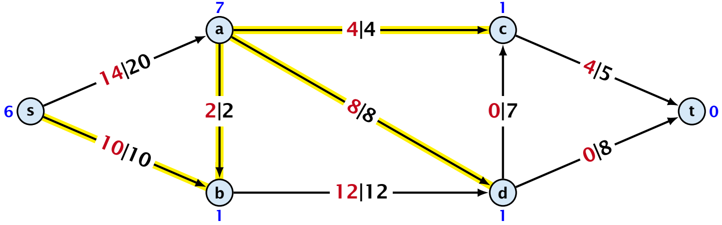

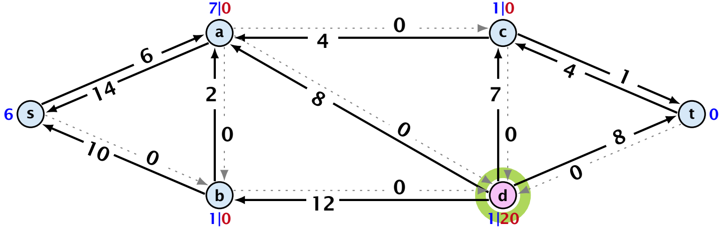

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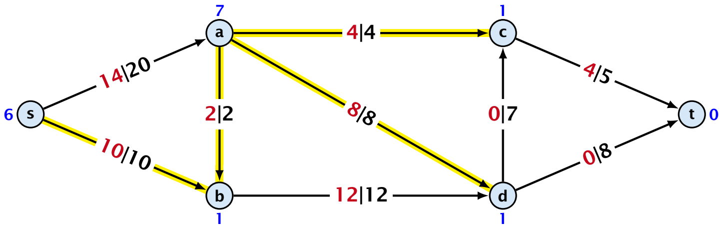

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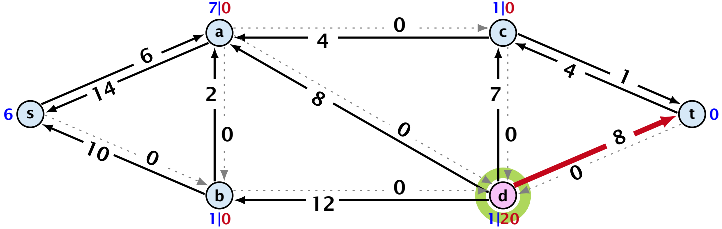

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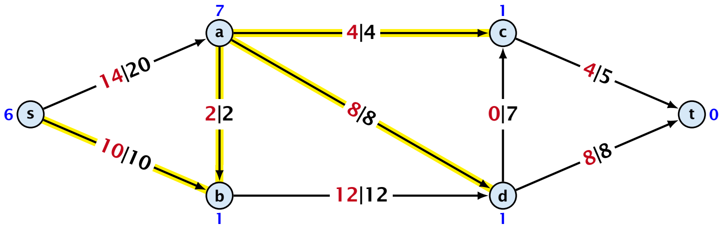

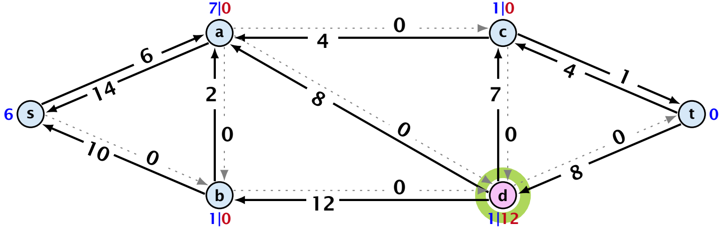

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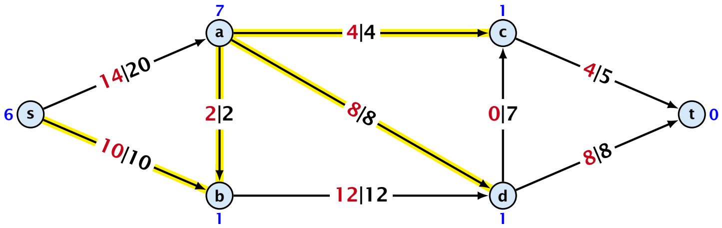

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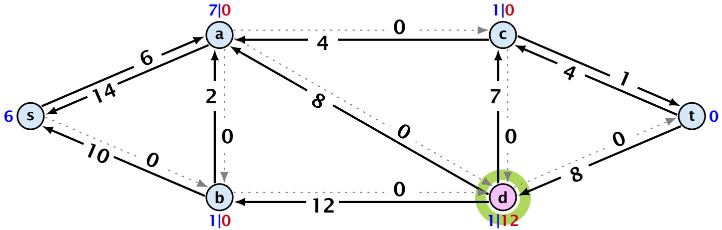

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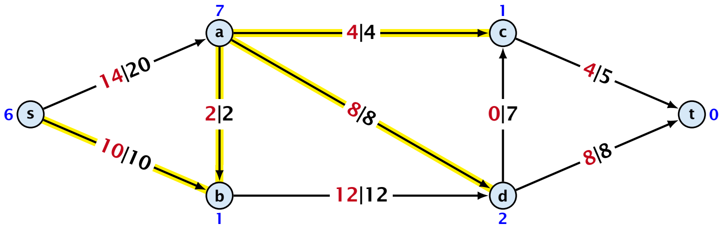

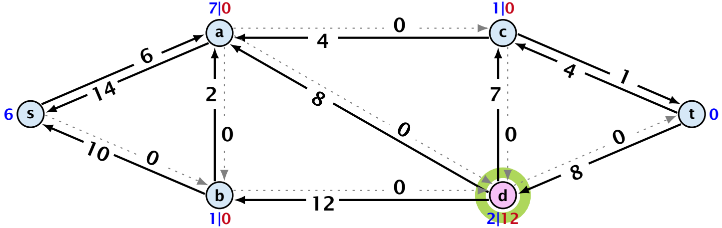

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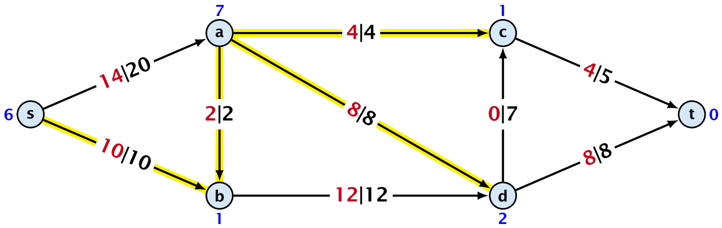
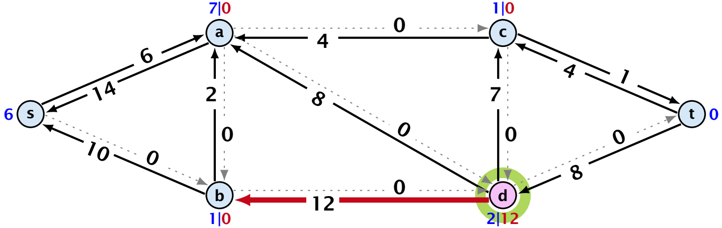


saturating and deactivating push



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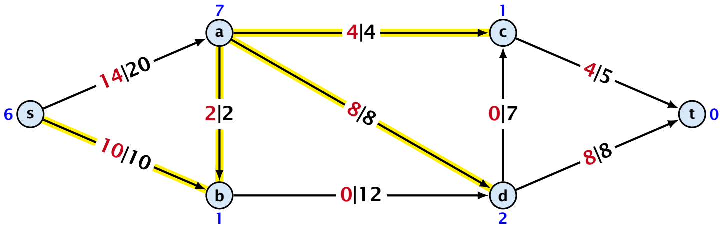

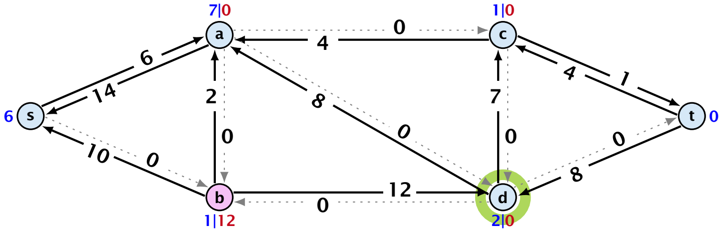

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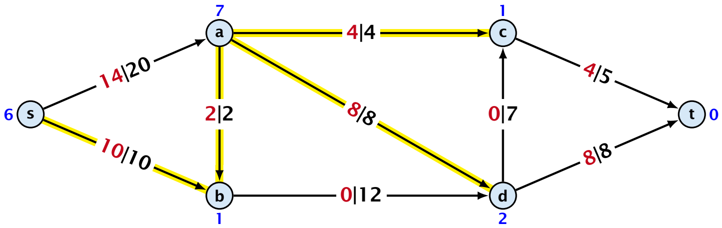

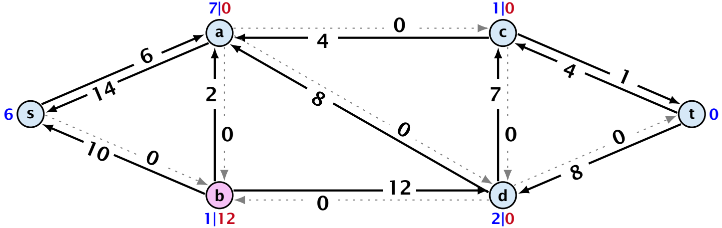

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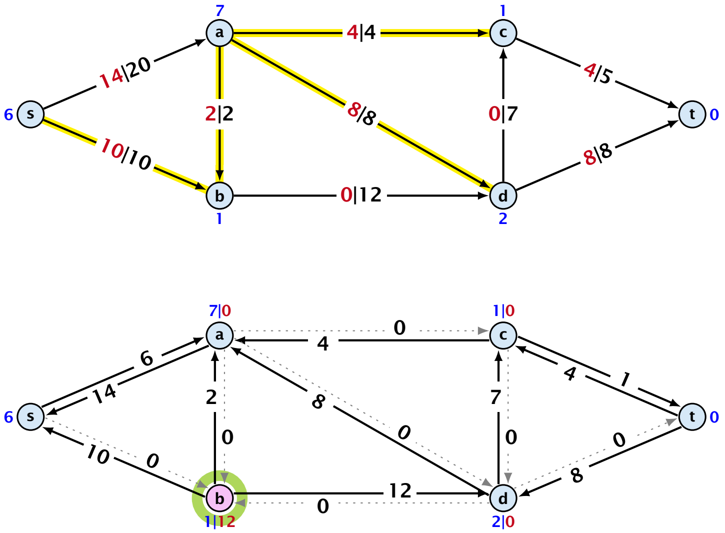

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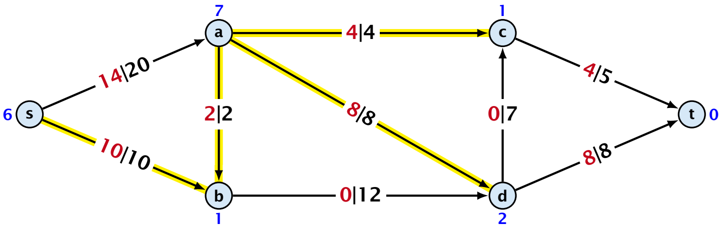
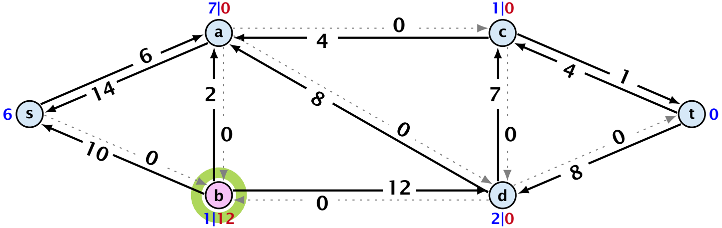


relabel to 3



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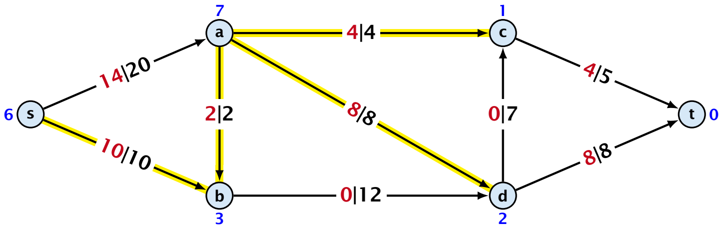

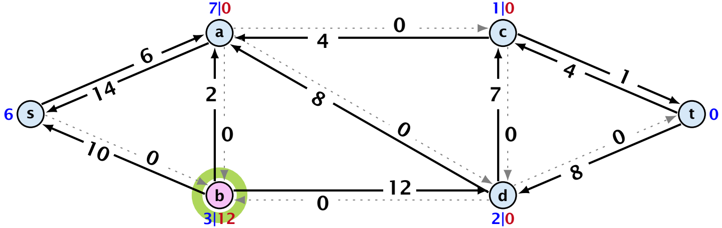

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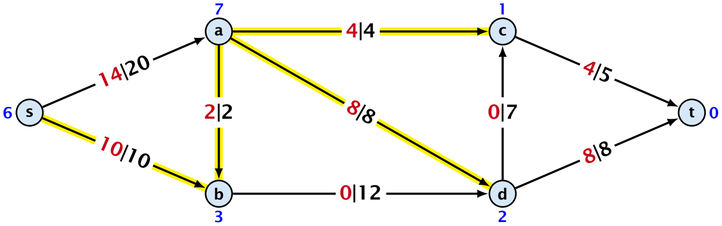
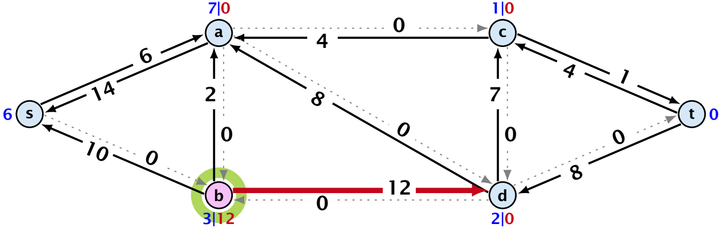


saturating and deactivating push



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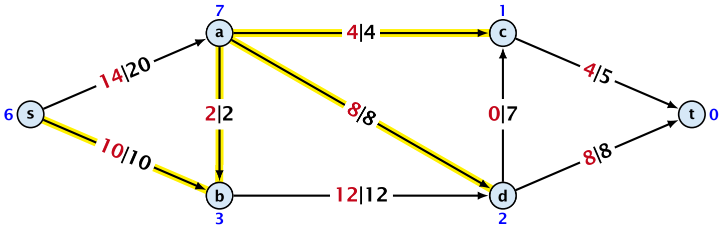

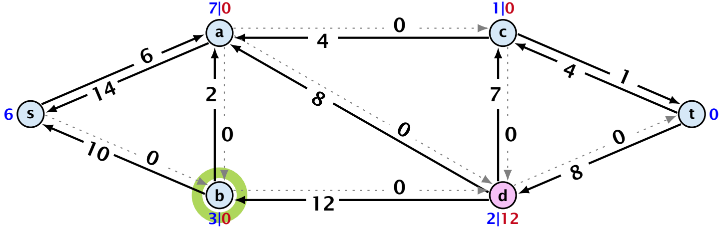

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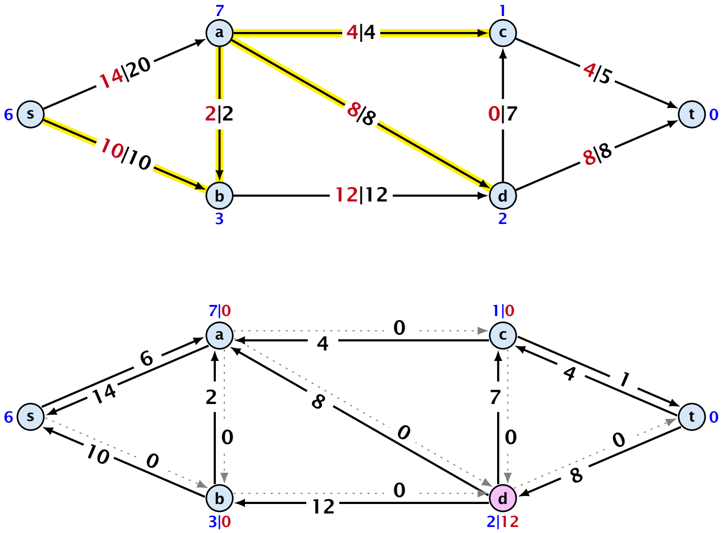

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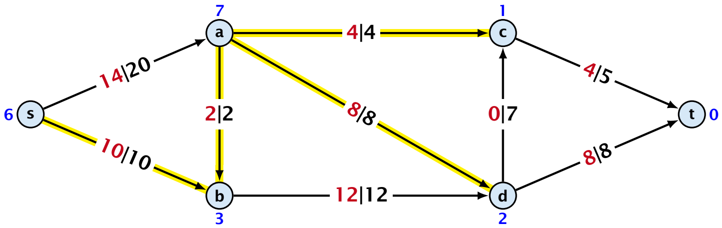

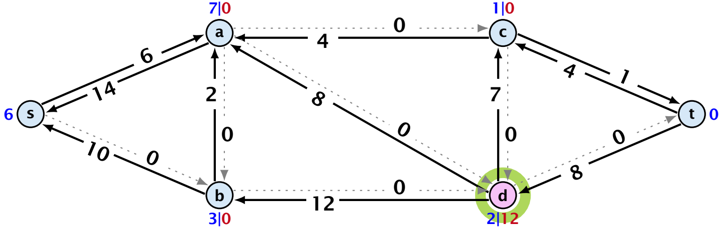

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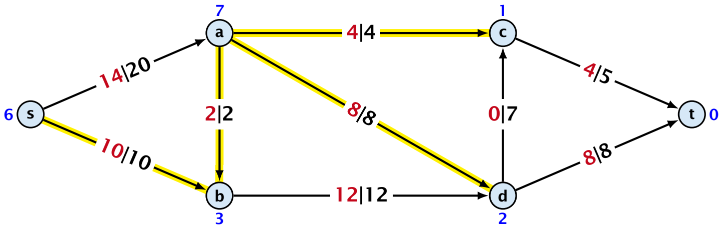

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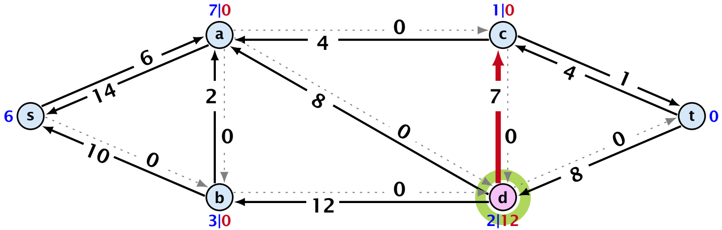

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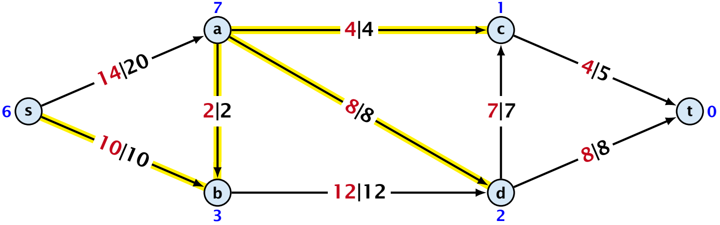

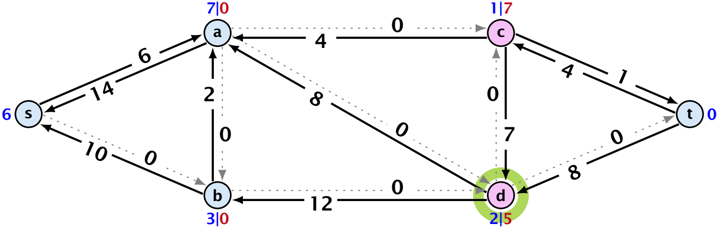

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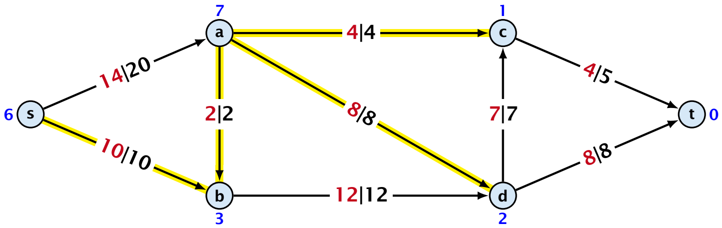

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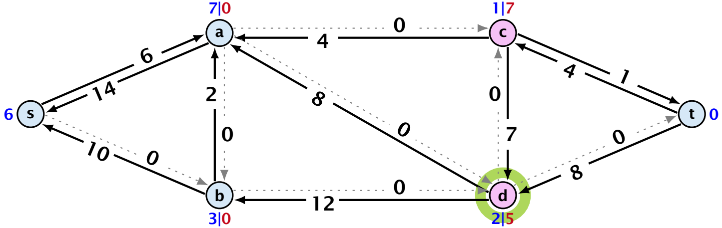

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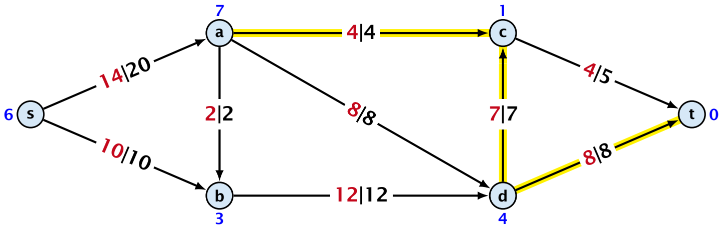

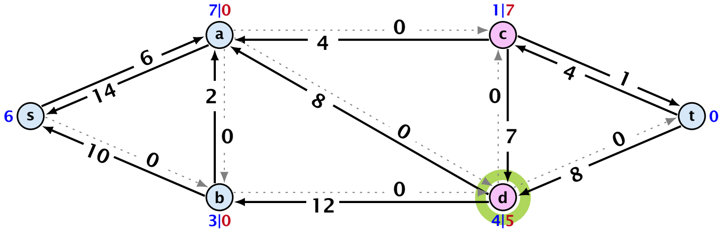

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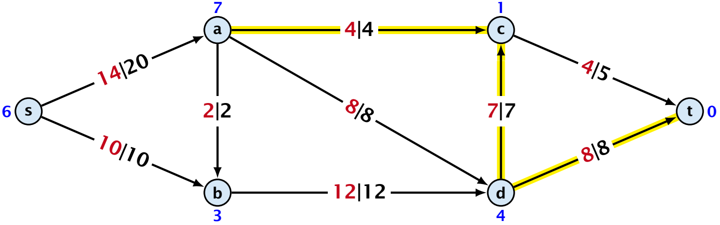

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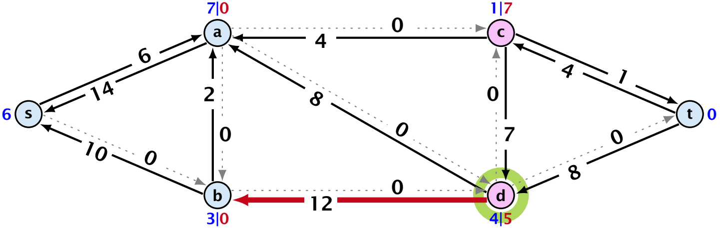

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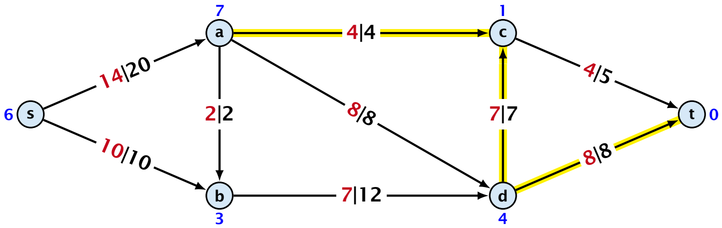

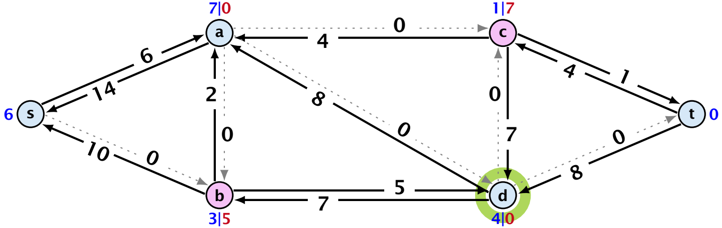

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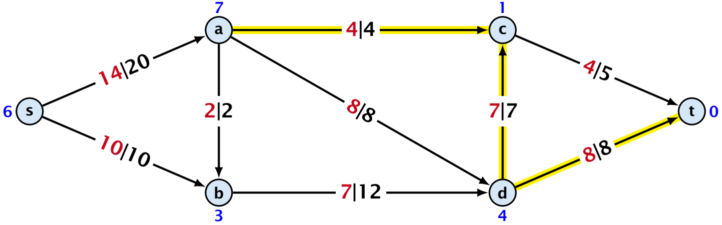

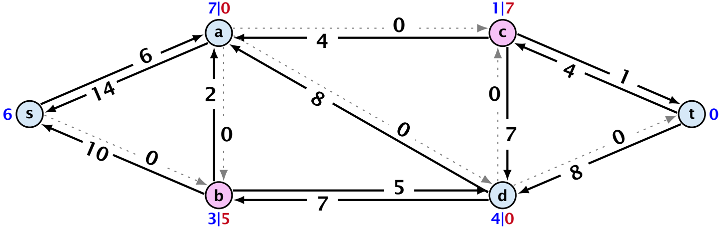

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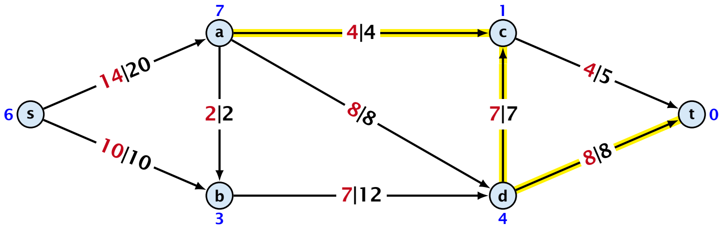

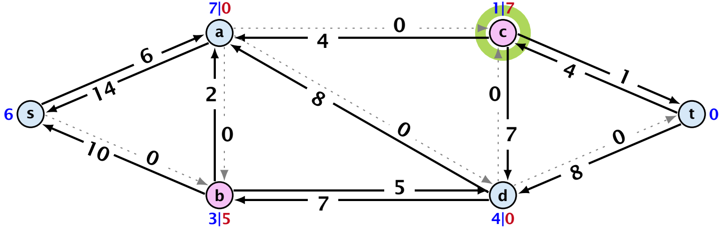

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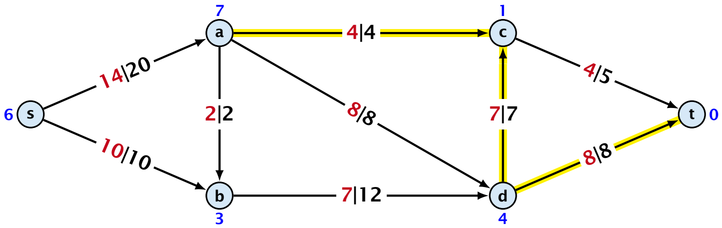
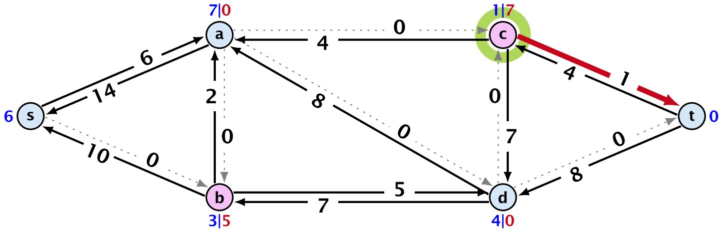


saturating push



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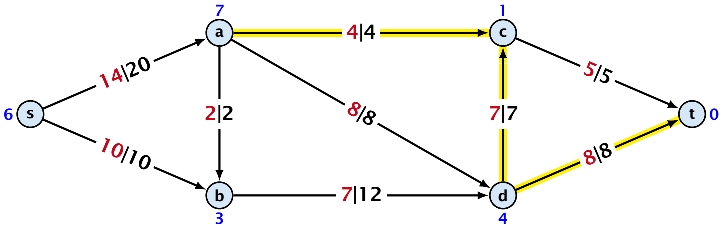

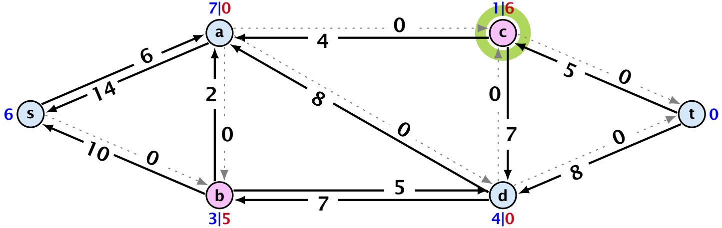

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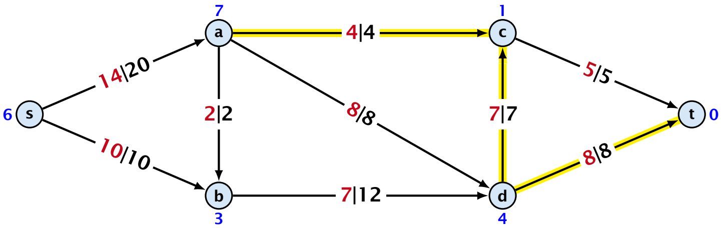
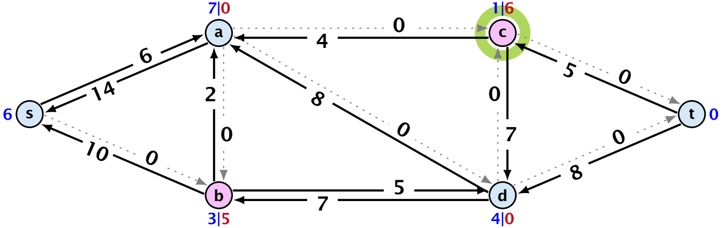


relabel to 5



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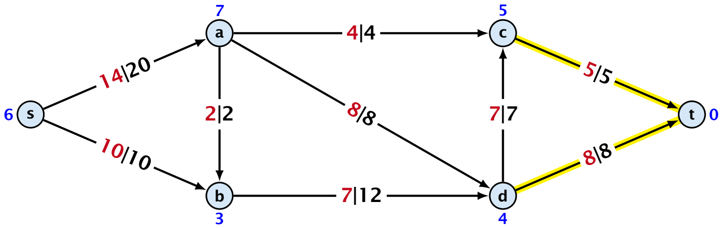

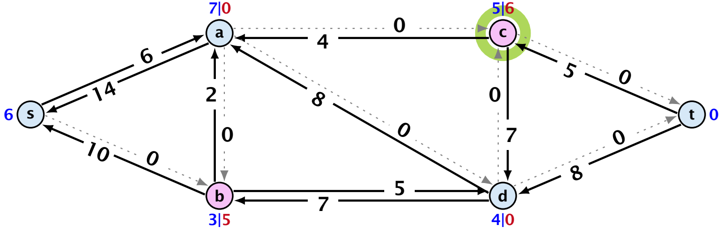

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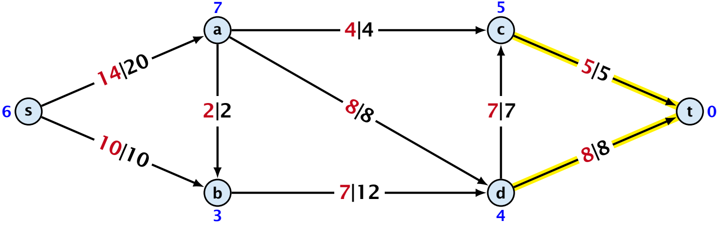
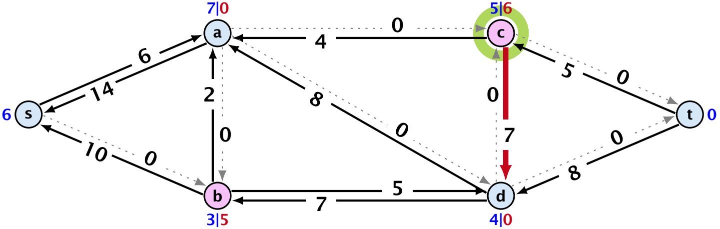


deactivating push



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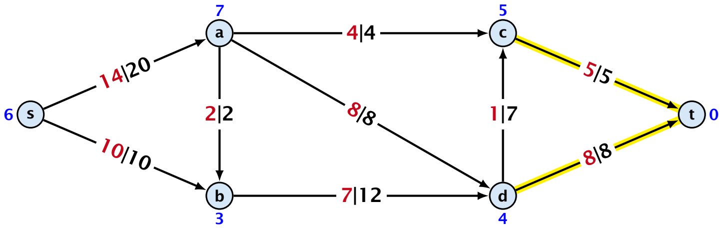

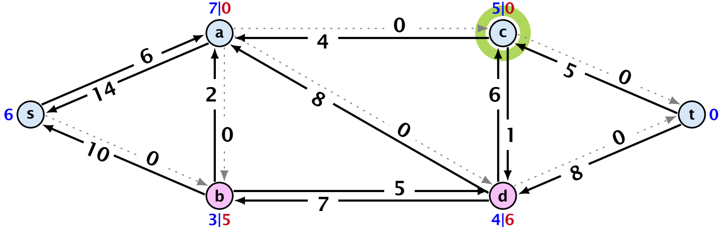

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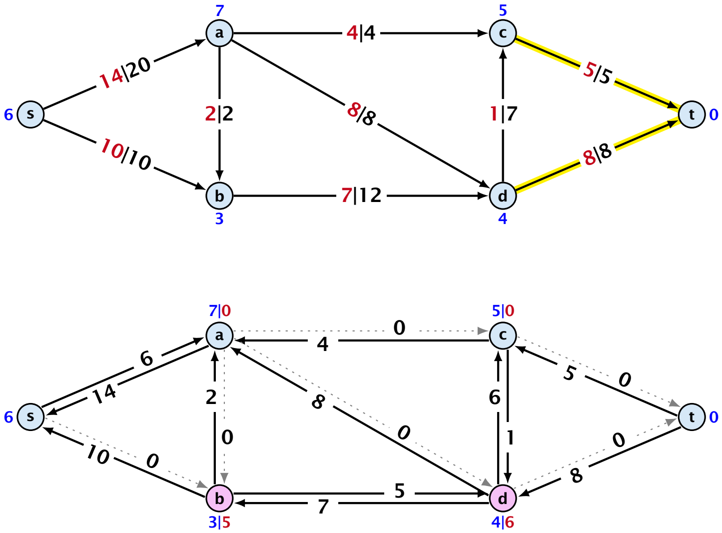

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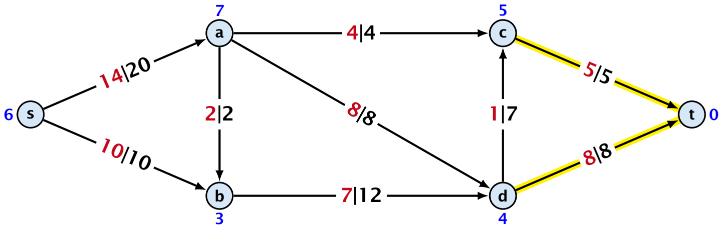

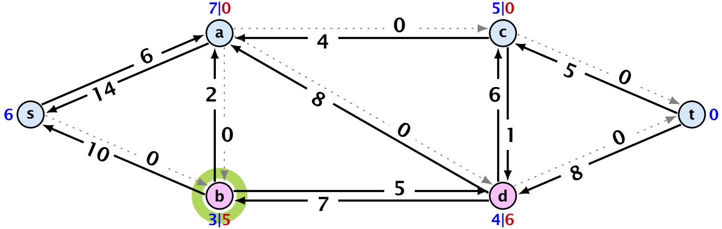

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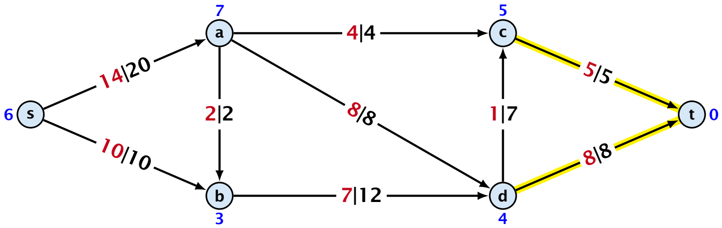

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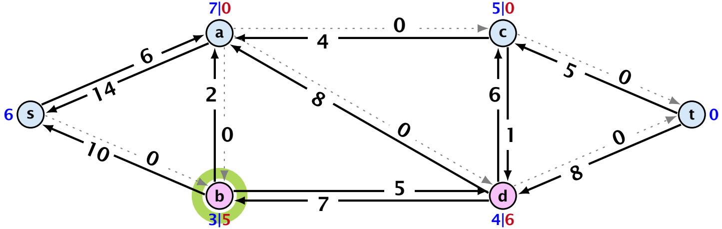

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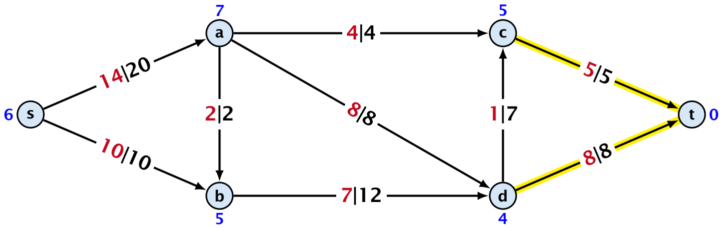

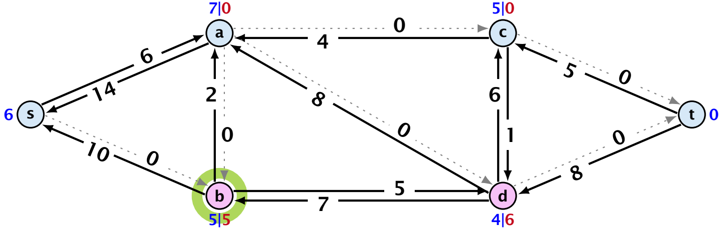

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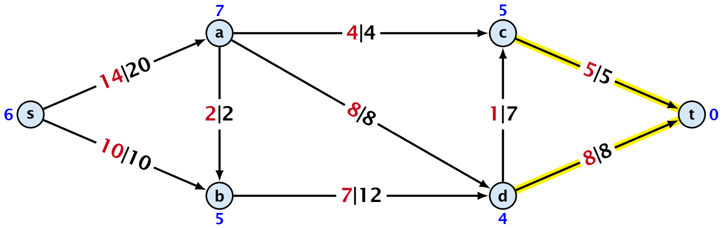
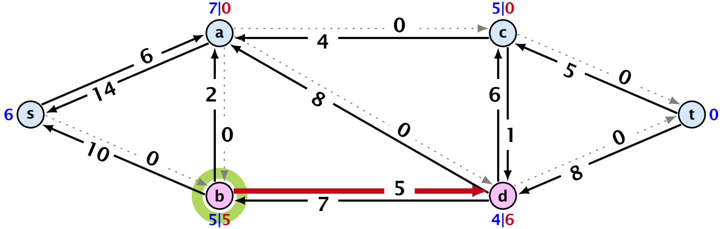


saturating and deactivating push



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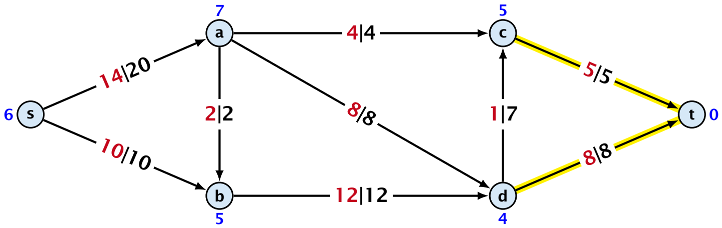

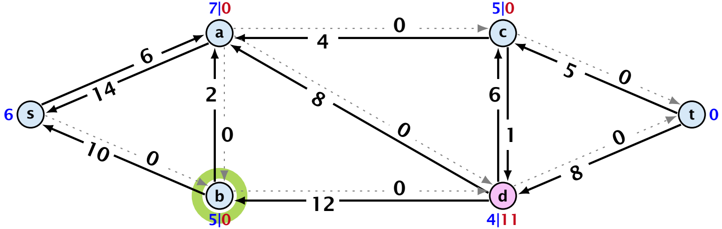

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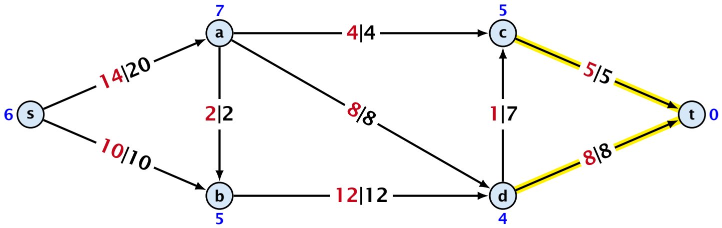

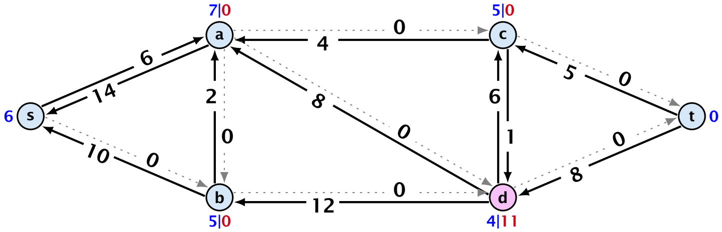

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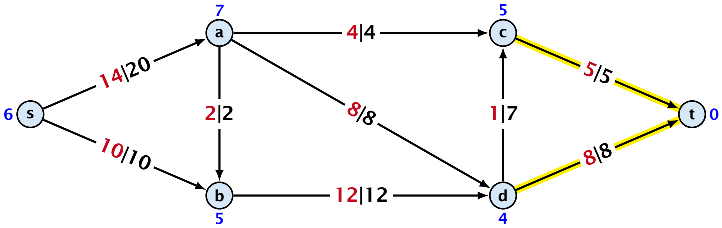

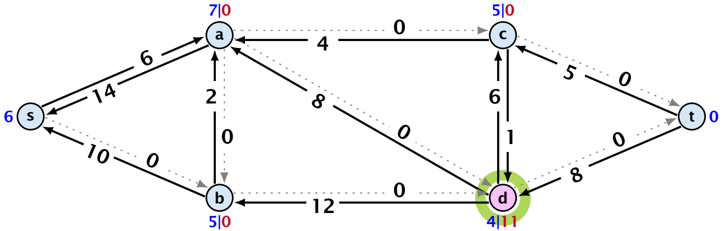

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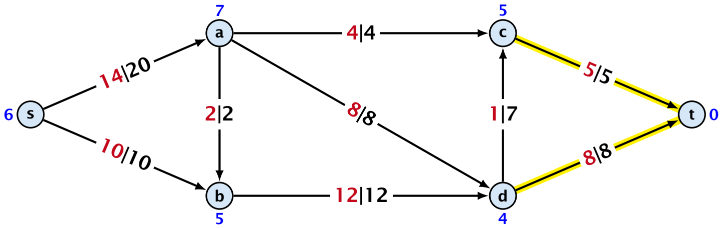
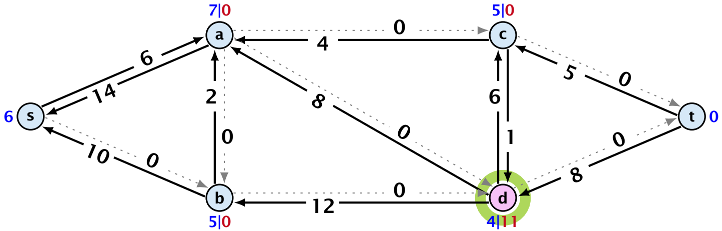


relabel to 6



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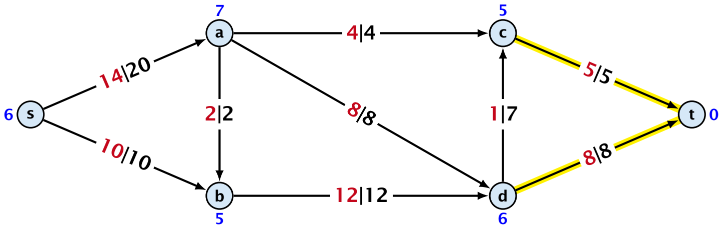

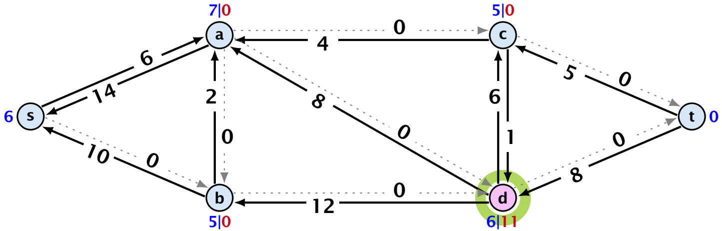

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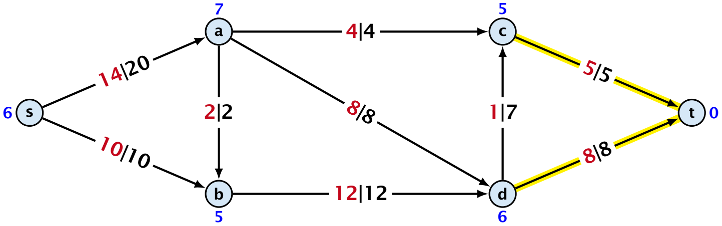
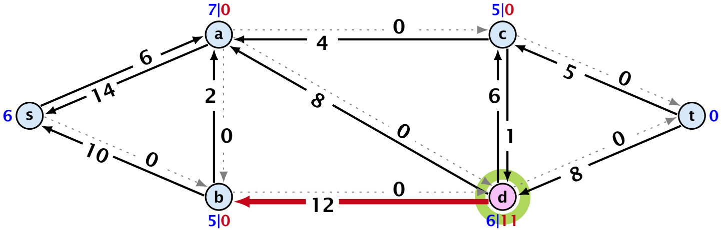


deactivating push



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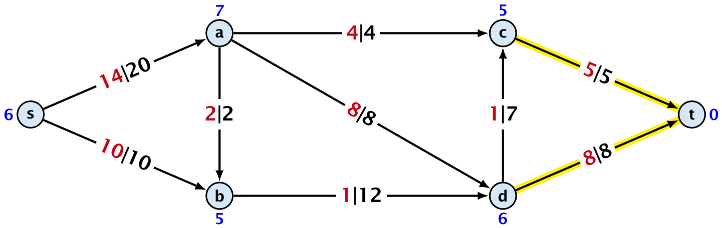

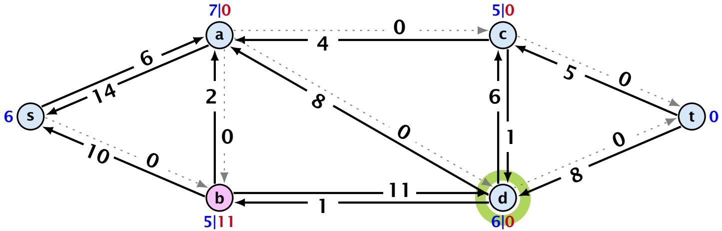

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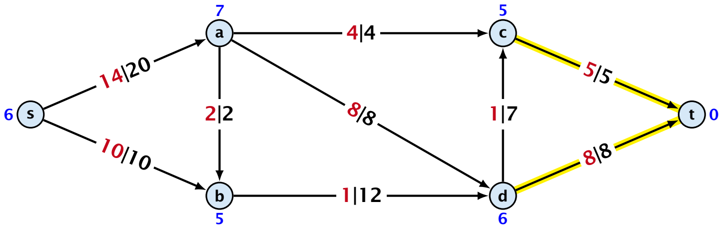

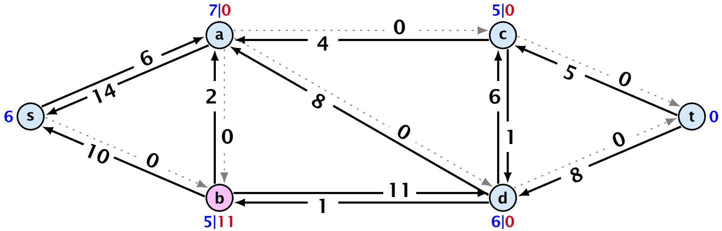

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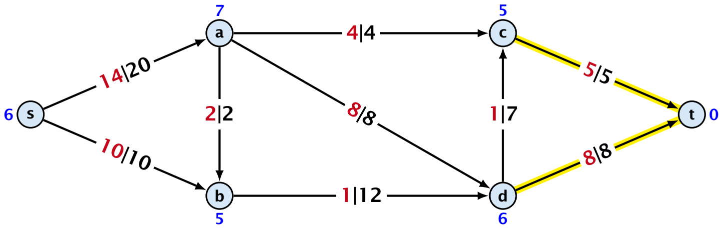

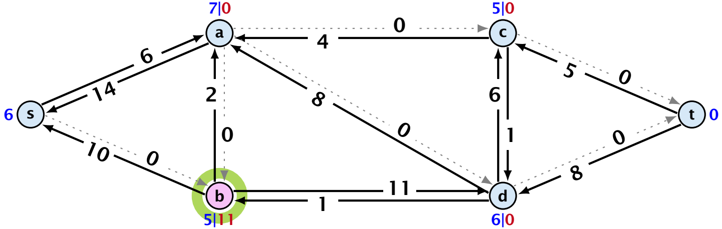

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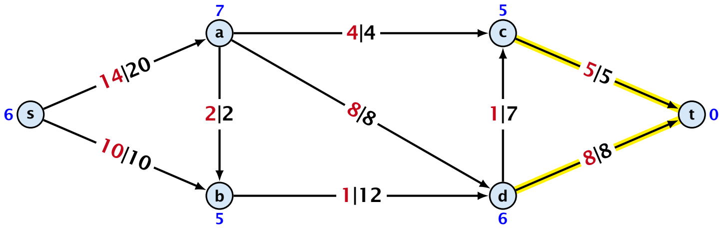
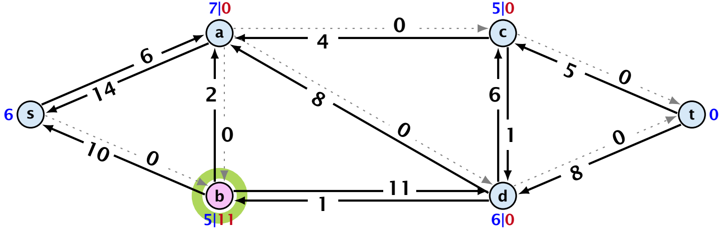


relabel to 7



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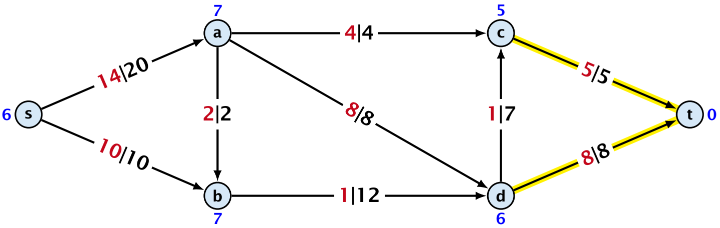

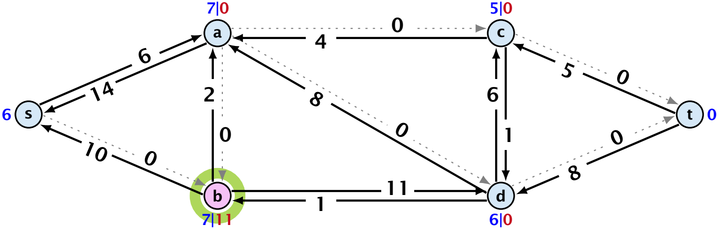

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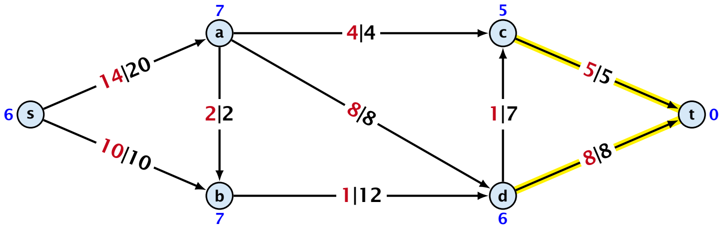
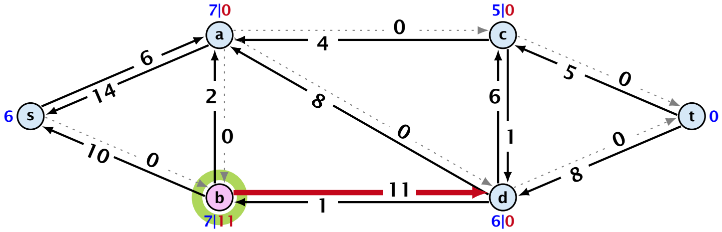


saturating and deactivating push



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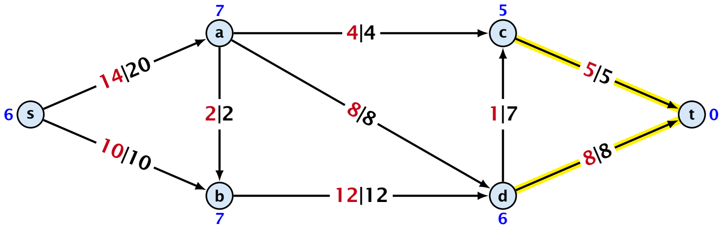

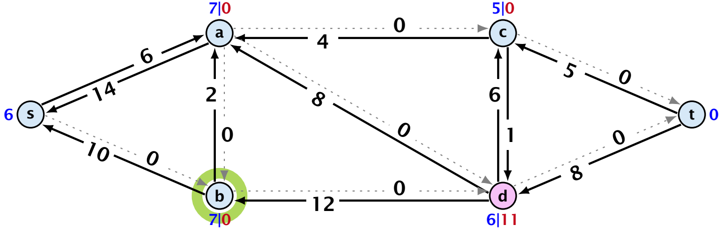

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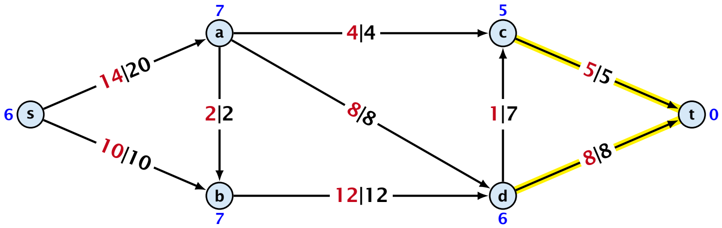

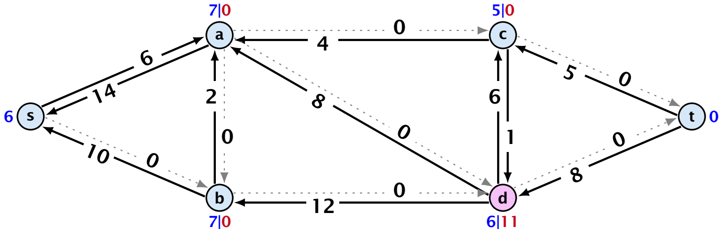

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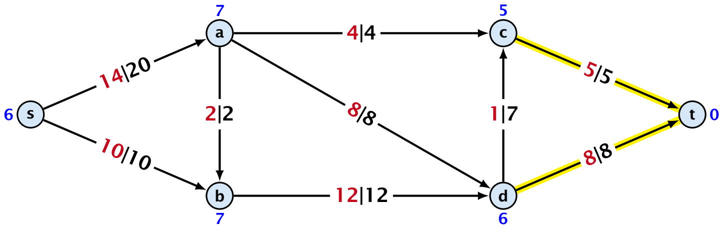

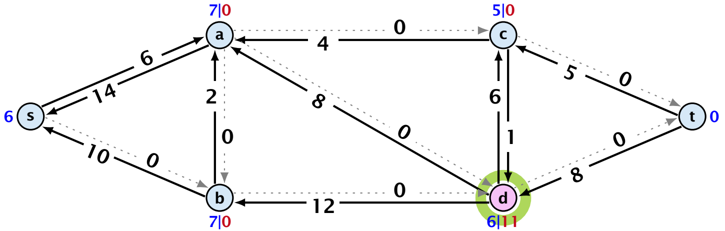

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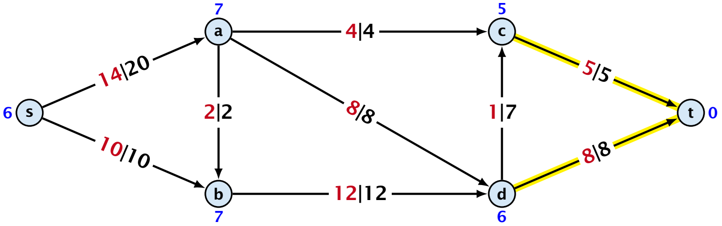

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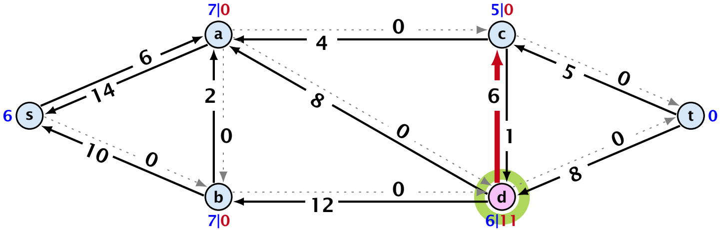

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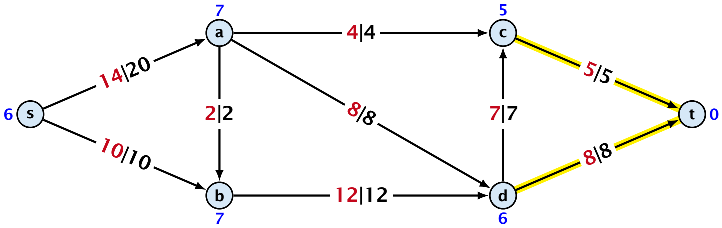

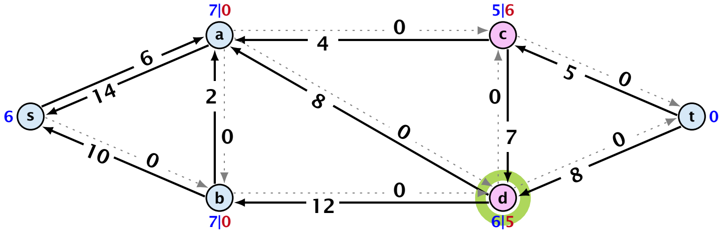

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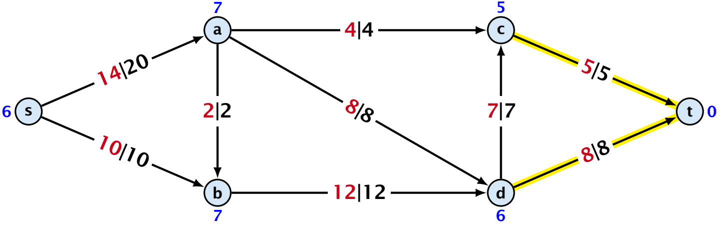
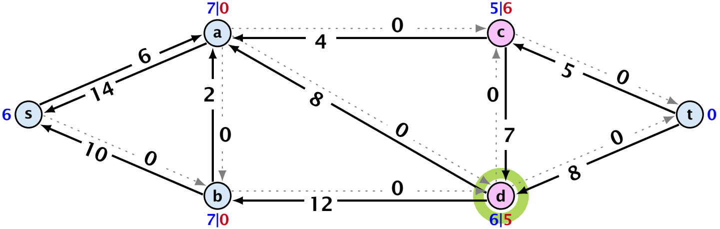


relabel to 8



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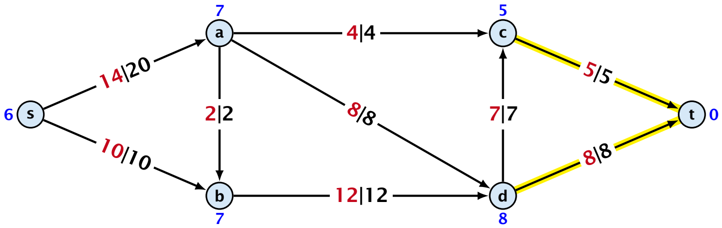
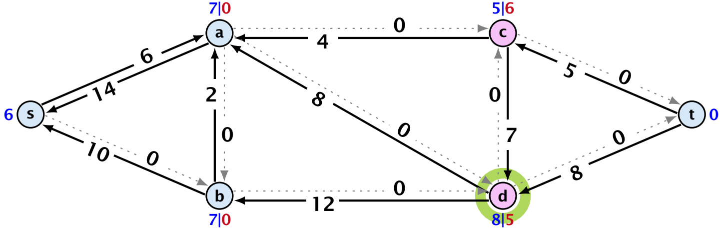
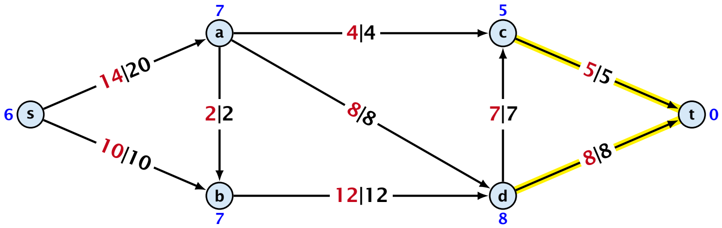




Preflow Push



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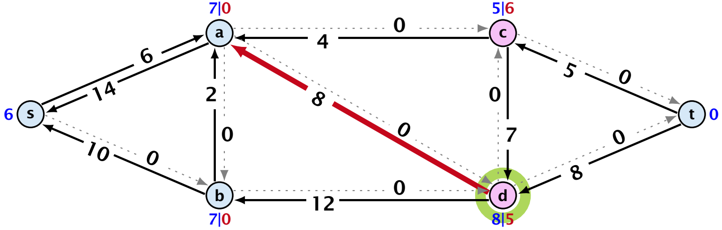

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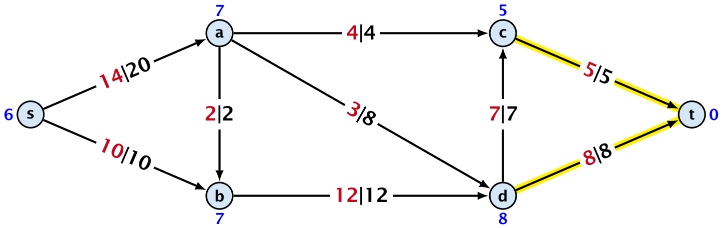

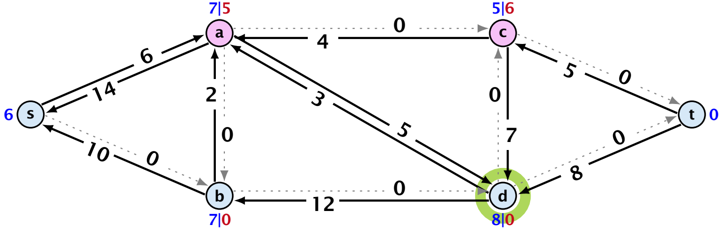

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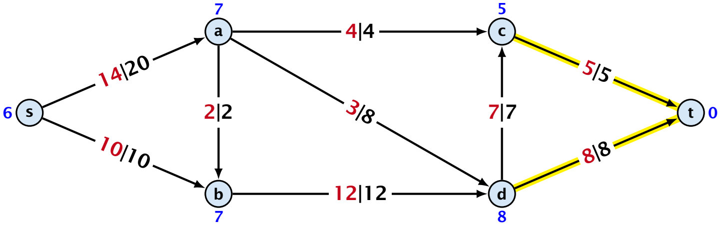

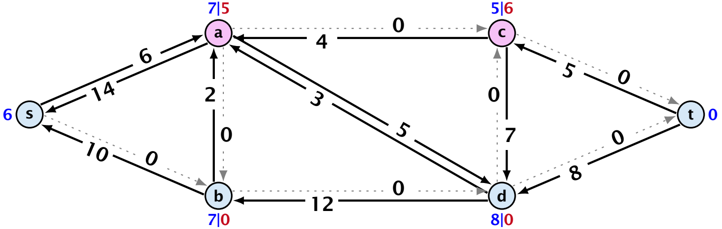

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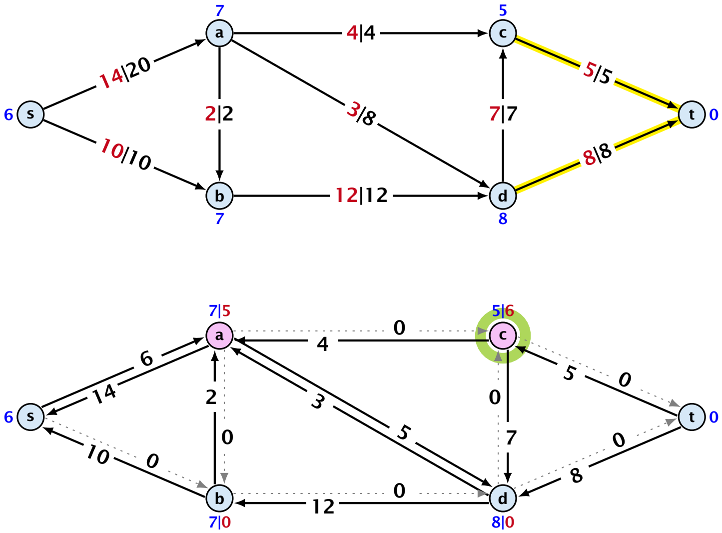

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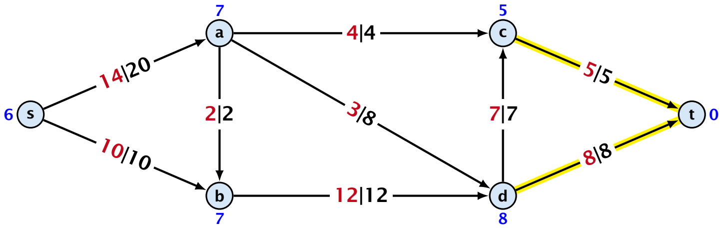
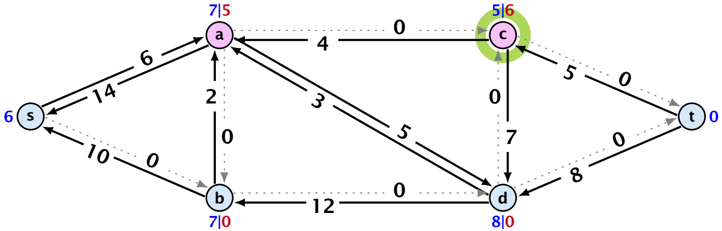


relabel to 8



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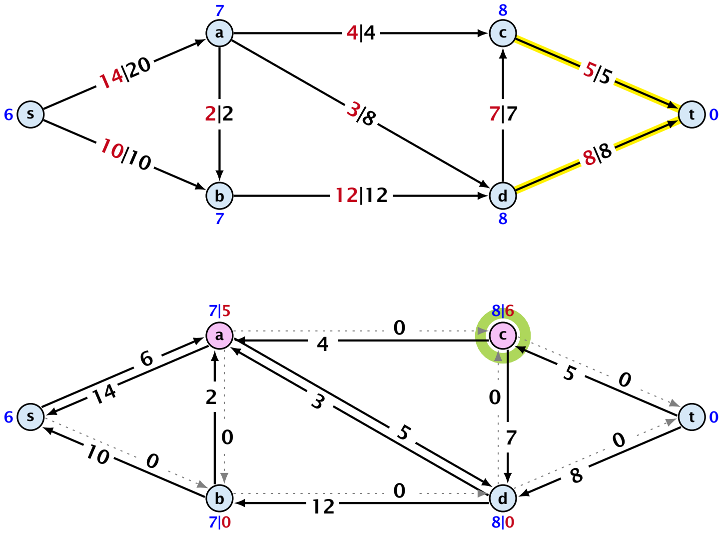

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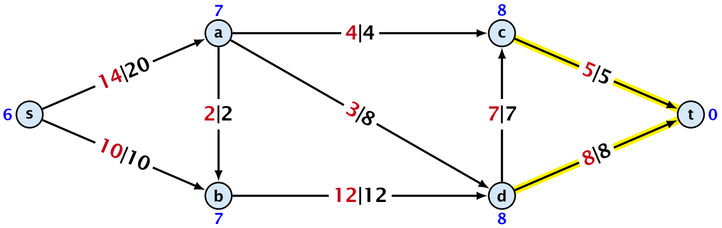

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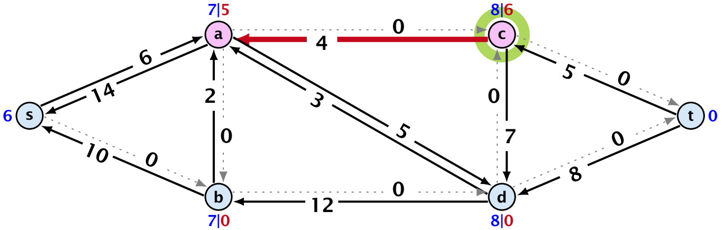

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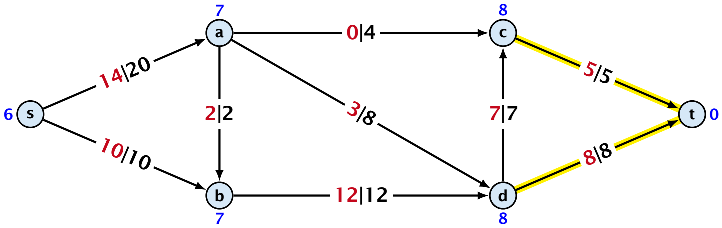

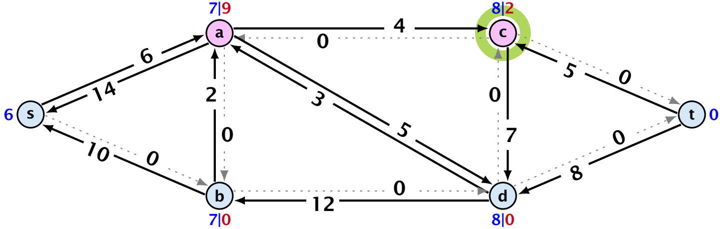

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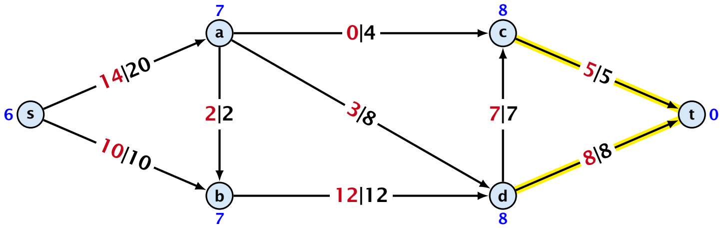
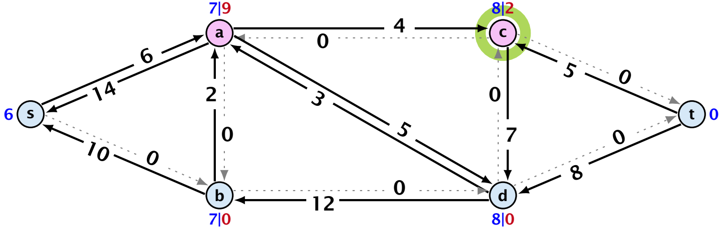


relabel to 9



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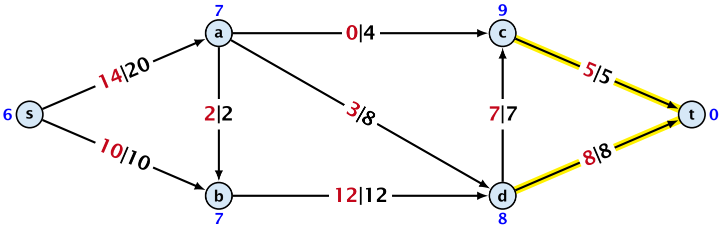

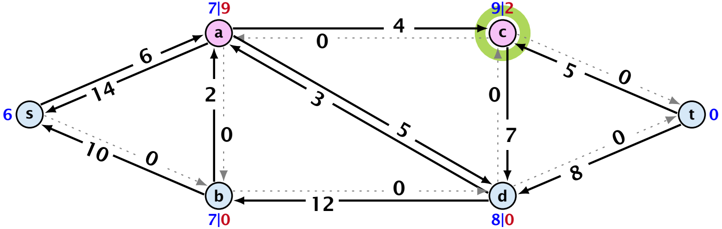

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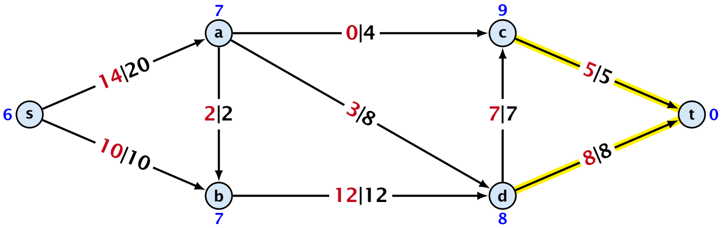
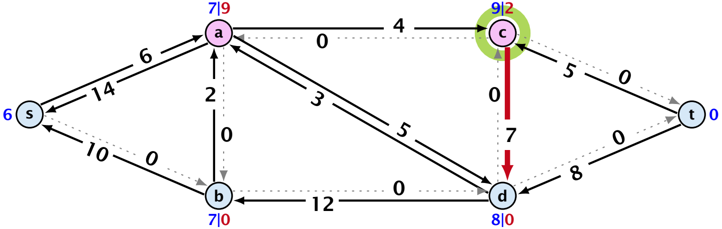


deactivating push



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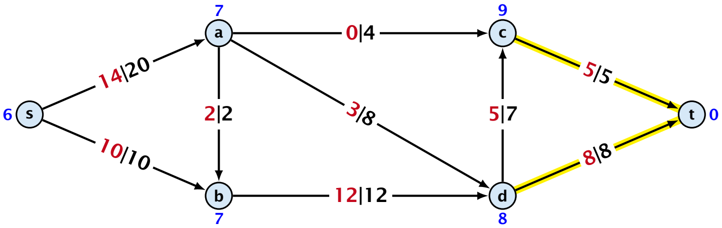

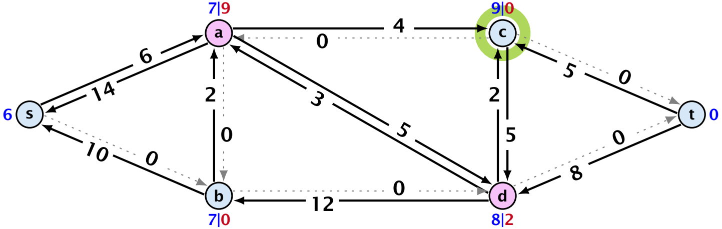

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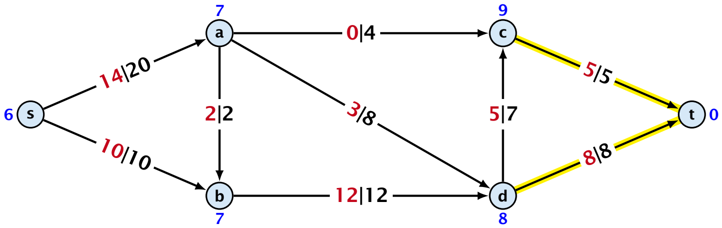

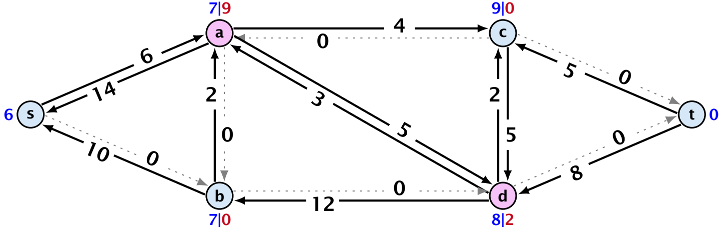

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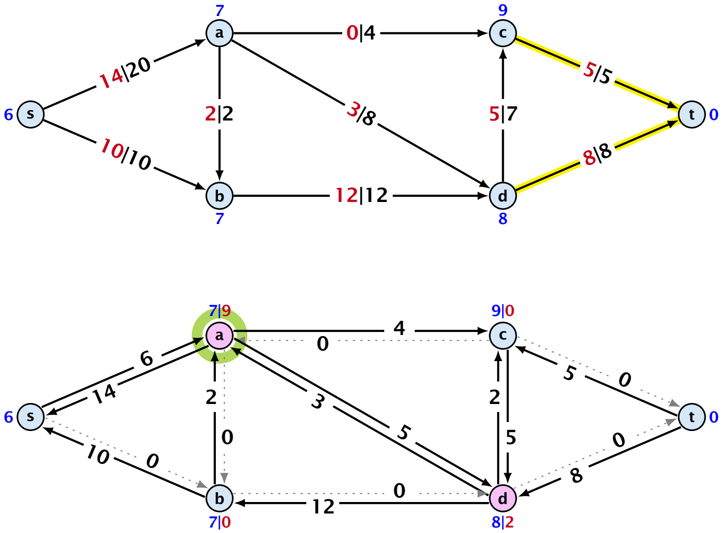

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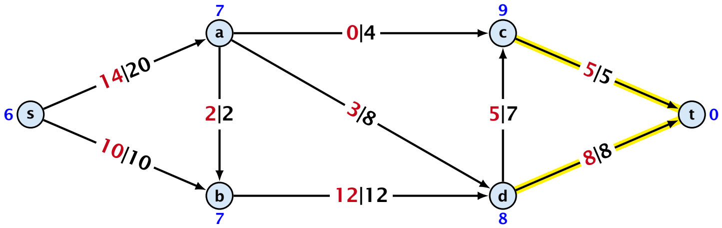

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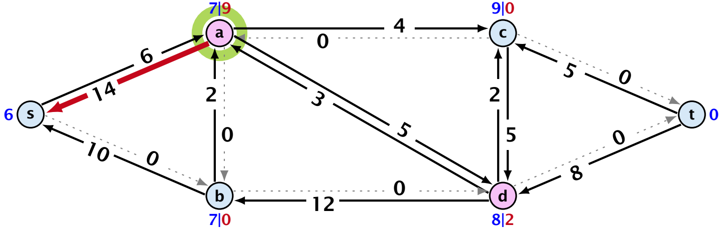

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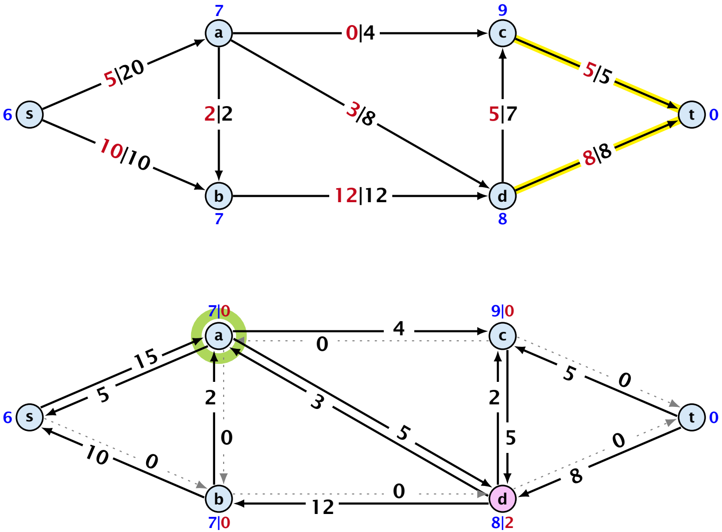

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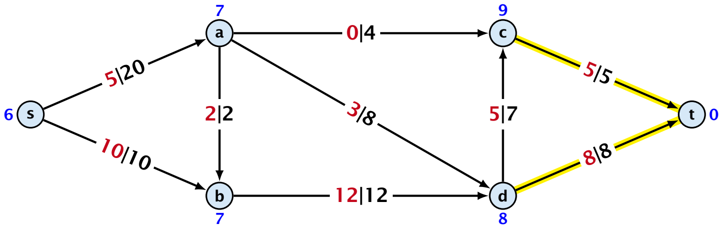

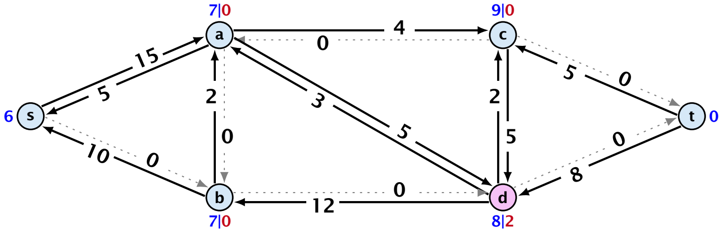

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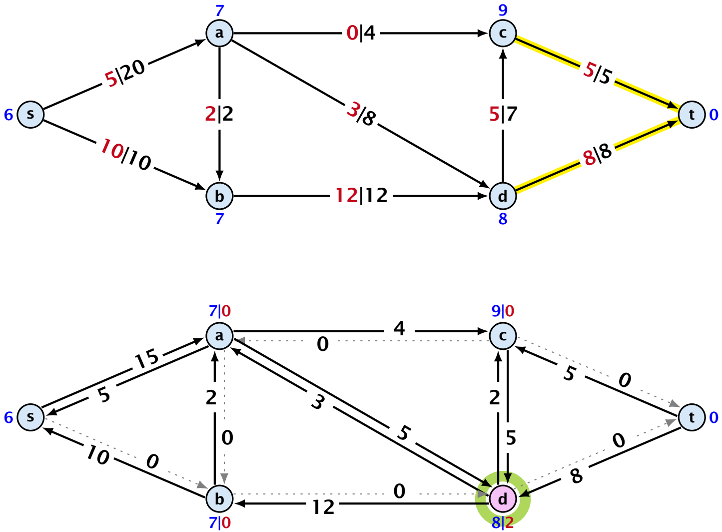

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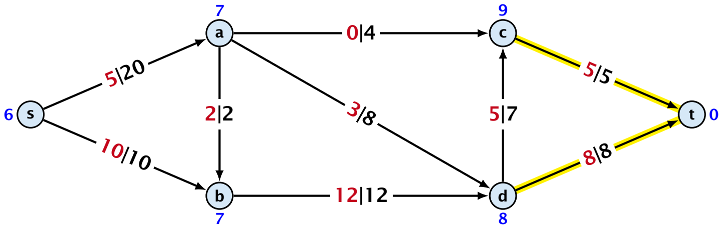

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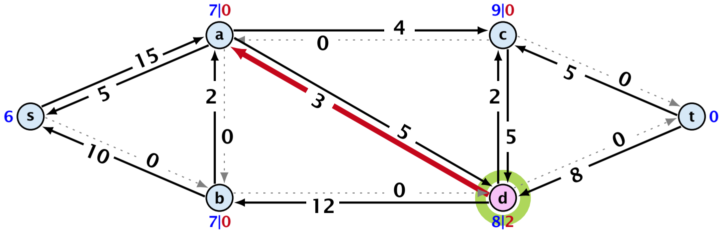

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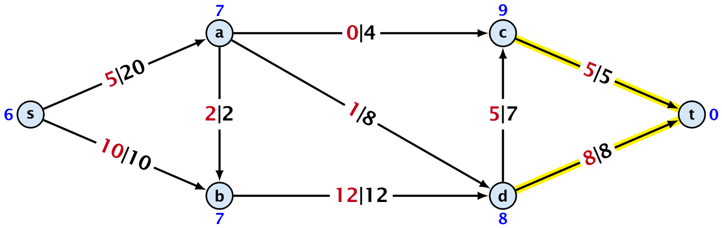

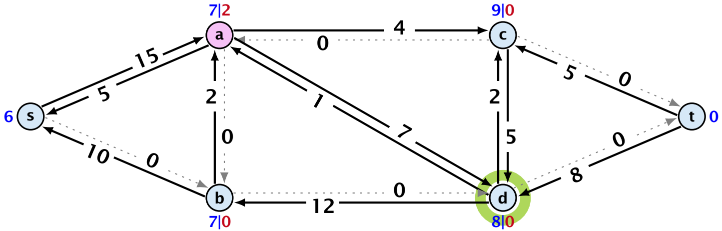

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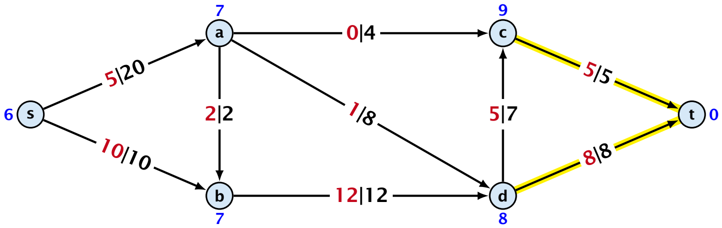

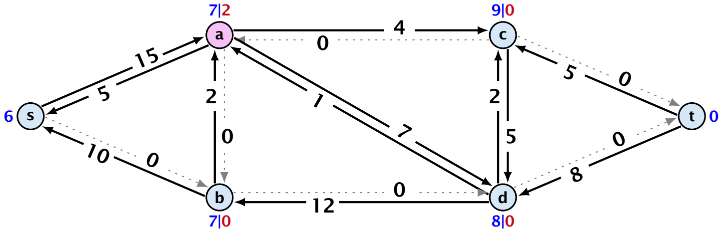

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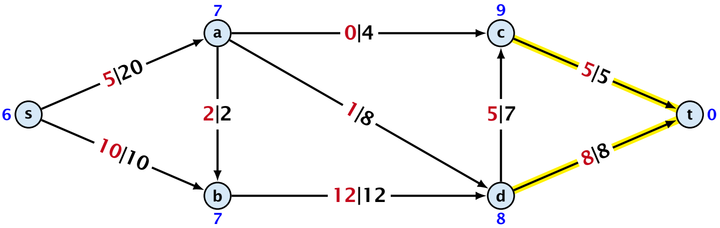

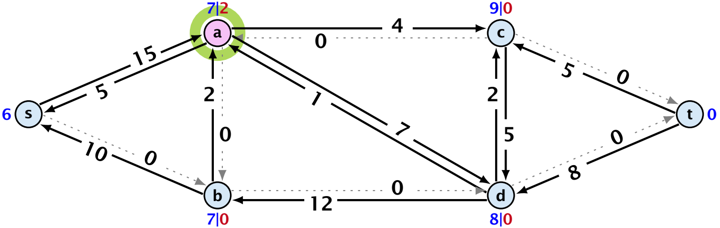

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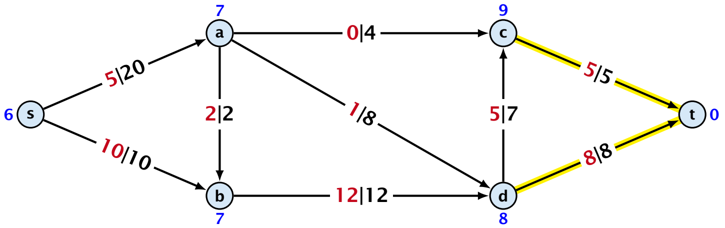

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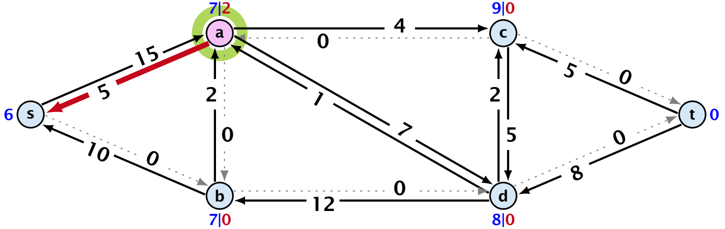

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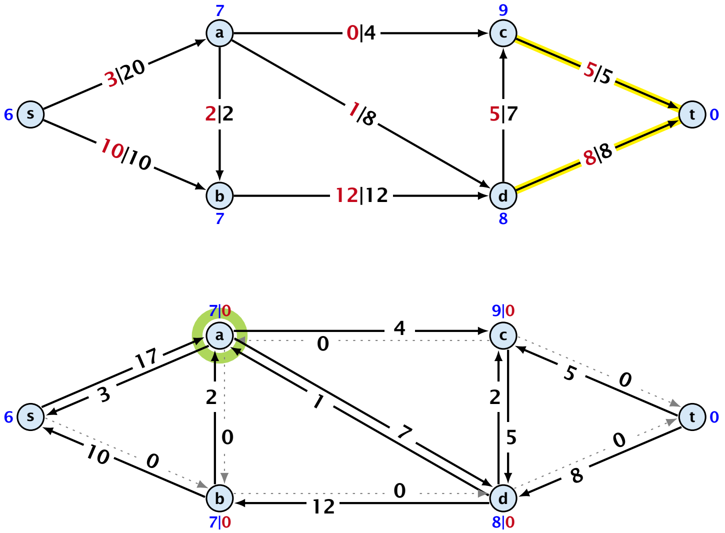

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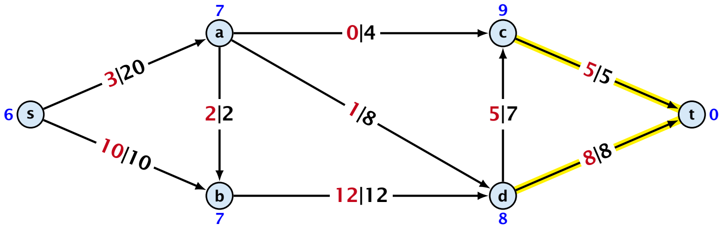

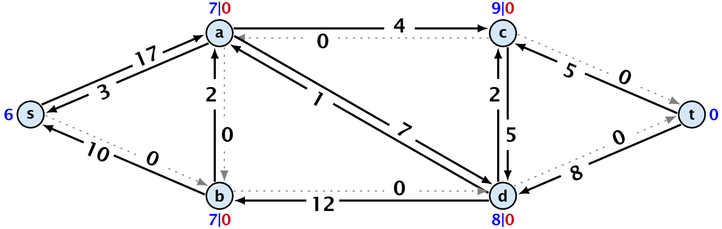

Analysis

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- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

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&= \sum_{b\in B}\sum_{v\in A} f(v,b) - \sum_{b\in B}\sum_{v\in A} f(b,v) + \sum_{b\in B}\sum_{v\in B} f(v,b) - \sum_{b\in B}\sum_{v\in B} f(b,v)
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$$f(x,y) = \begin{cases} 0 & (x,y) \notin E \\ f((x,y)) & (x,y) \in E \end{cases}$$

We have

$$\begin{aligned} f(B) &= \sum_{b \in B} f(b) \\ &= \sum_{b \in B} \left(\sum_{v \in V} f(v,b) - \sum_{v \in V} f(b,v) \right) \\ &= \sum_{b \in B} \left(\sum_{v \in A} f(v,b) + \sum_{v \in B} f(v,b) - \sum_{v \in A} f(b,v) - \sum_{v \in B} f(b,v) \right) \\ &= \qquad\qquad - \sum_{b \in B} \sum_{v \in A} f(b,v) \\ &\leq 0 \end{aligned}$$

Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

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The label of a node cannot become larger than $2n-1$.

Analysis

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Proof.

- When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

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Lemma 59
There are only $\mathcal{O}(n^2)$ *relabel operations.*

Analysis

Lemma 60

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

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Proof.

- Suppose that we just made a saturating push along (u,v).
- Hence, the edge (u,v) is deleted from the residual graph.
- For the edge to appear again, a push from v to u is required.

Analysis

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Proof.

- Suppose that we just made a saturating push along (u, v).
- Hence, the edge (u, v) is deleted from the residual graph.
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- Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.

Analysis

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- Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.
- For a push from v to u the edge (v, u) must become admissible. The label of v must increase by at least 2.
- Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

Lemma 61

The number of deactivating pushes performed is at most $\mathcal{O}(n^2m)$.

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Proof.

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- A relabel increases Φ by at most 1.
- A deactivating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- Hence,

$$
\begin{aligned}
\#\text{deactivating\_pushes} &\leq \#\text{relabels} + 2n \cdot \#\text{saturating\_pushes} \\
&\leq \mathcal{O}(n^2 m) \ .
\end{aligned}
$$

Analysis

Theorem 62
There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2m)$.

Analysis

Proof:

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For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

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A push along an edge (u, v) can be performed in constant time

- check whether edge (v, u) needs to be added to G_f

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- check whether edge (v, u) needs to be added to G_f
- check whether (u, v) needs to be deleted (saturating push)

Analysis

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- check whether edge (v,u) needs to be added to G_f
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- check whether u becomes inactive and has to be deleted from the set of active nodes

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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissible

Analysis

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For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissible
- check for all incoming edges if they become non-admissible

Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 2 discharge(u)

```
1: while u is active do
2:     v ← u.current-neighbour
3:     if v = null then
4:         relabel(u)
5:         u.current-neighbour ← u.neighbour-list-head
6:     else
7:         if (u, v) admissible then push(u, v)
8:         else u.current-neighbour ← v.next-in-list
```

Note that $u.\textit{current-neighbour}$ is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 63

If $v = \mathrm{null}$ in Line 3, then there is no outgoing admissible edge from u.

Proof.

- While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissible.
- The only thing that could make the edge admissible again would be a relabel at u.
- If we reach the end of the list ($v = \mathrm{null}$) all edges are not admissible. □

This shows that discharge(u) is correct, and that we can perform a relabel in Line 4.

9.2 Relabel to Front

Algorithm 1 relabel-to-front(G, s, t)

```
initialize preflow
initialize node list L containing V \ {s, t} in any order
foreach u ∈ V \ {s, t} do
    u.current-neighbour ← u.neighbour-list-head
u ← L.head
while u ≠ null do
    old-height ← ℓ(u)
    discharge(u)
    if ℓ(u) > old-height then // relabel happened
        move u to the front of L
    u ← u.next
```

9.2 Relabel to Front

Lemma 64 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1.
     - Pushes do no create any new admissible edges. Therefore, if discharge$()$ does not relabel u, L is still topologically sorted.
     - After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).

       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

9.2 Relabel to Front

Proof:

- Maintenance:
  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

     For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

Lemma 65
There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#\mathit{relabels} + 1) = \mathcal{O}(n^3)$.

9.2 Relabel to Front

Lemma 66
The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.\mathit{current\text{-}neighbour}$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

9.2 Relabel to Front

Recall that a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can also be a deactivating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 67

*The cost for all saturating push-operations that are **not** deactivating is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.\mathit{current\text{-}neighbour}$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $\mathit{degree}(u) + 1$ many entries ($+1$ for null-entry).

9.2 Relabel to Front

Lemma 68
The cost for all deactivating push-operations is only $\mathcal{O}(n^3)$.

A deactivating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 69
The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

9.3 Highest Label

Algorithm 1 highest-label(G, s, t)

1: initialize preflow
2: **foreach** $u \in V \setminus \{s, t\}$ **do**
3: $\quad u.\mathit{current\text{-}neighbour} \leftarrow u.\mathit{neighbour\text{-}list\text{-}head}$
4: **while** $\exists$ active node u **do**
5: $\quad$ select active node u with highest label
6: $\quad$ discharge(u)

9.3 Highest Label

Lemma 70

When using highest label the number of deactivating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only "activate" nodes on levels strictly less than ℓ.

This means, after a deactivating push from u a relabel is required to make u active again.

Hence, after n deactivating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of deactivating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

9.3 Highest Label

Since a discharge-operation is terminated by a deactivating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:
How do we find the next node for a discharge operation?

9.3 Highest Label

Maintain lists L_i, $i \in \{0, \ldots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \ldots, L_0$, (in that order) until you find a non-empty list.

Unless the last (deactivating) push was to s or t the list $k-1$ must be non-empty (i.e., the search takes constant time).

9.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#deactivating\text{-}pushes\text{-}to\text{-}s\text{-}or\text{-}t)$$

Lemma 71

The number of deactivating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 72

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

9.3 Highest Label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n + 1$) made a deactivating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

Mincost Flow

Problem Definition:

$$\begin{array}{lll}\text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & 0 \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v)\end{array}$$

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- $G = (V, E)$ is a directed graph.

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- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.

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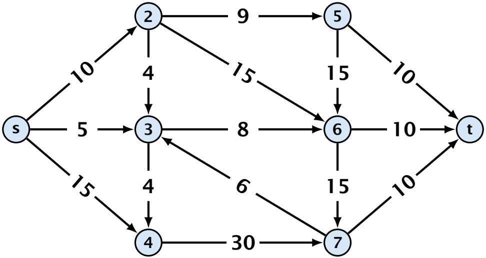
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow



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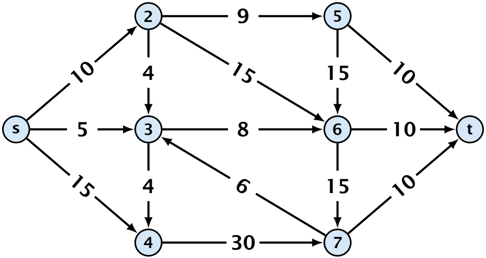
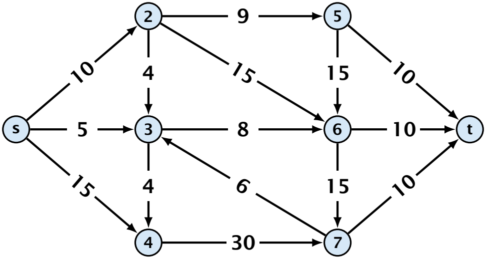


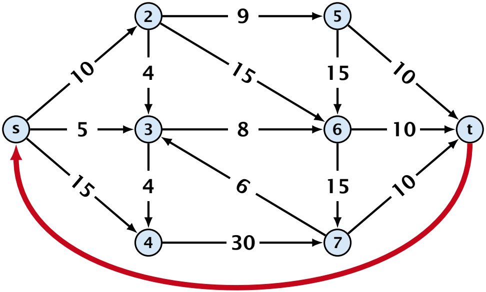
- Given a flow network for a standard maxflow problem.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.

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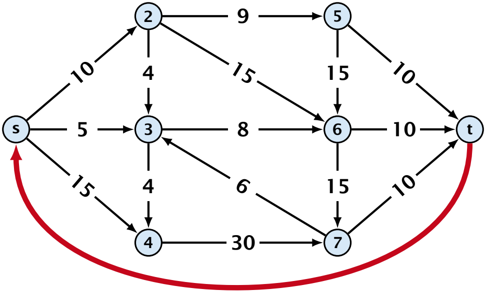

- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.

Solve Maxflow Using Mincost Flow

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- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.

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- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value at least k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E:\;\; 0 \le f(e) \le u(e) \\ & \forall v \in V:\;\; f(v) = b(v) \end{array}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

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A more general model?

$$\begin{array}{lll} \min & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$ $c : E \to \mathbb{R}$;

Generalization

Differences

- Flow along an edge e may have non-zero lower bound $\ell(e)$.
- Flow along e may have negative upper bound $u(e)$.
- The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

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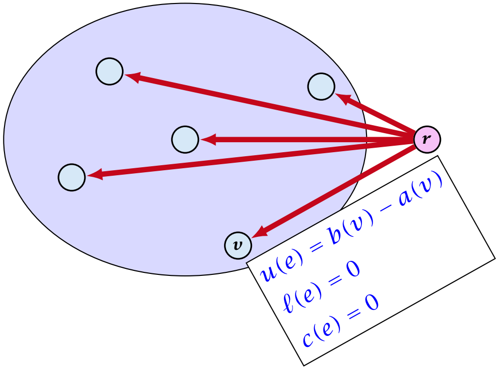

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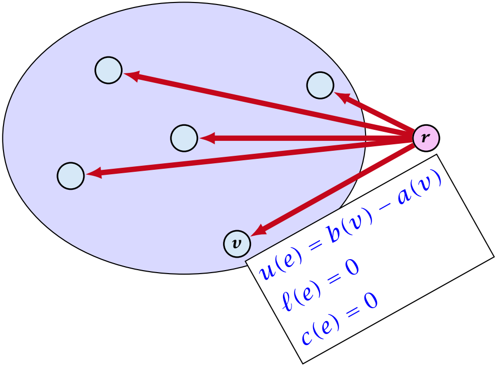
Add new node r.



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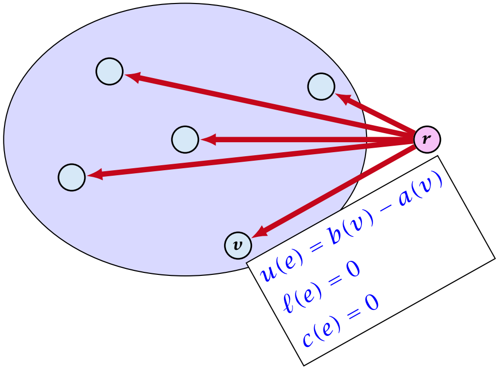
We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.



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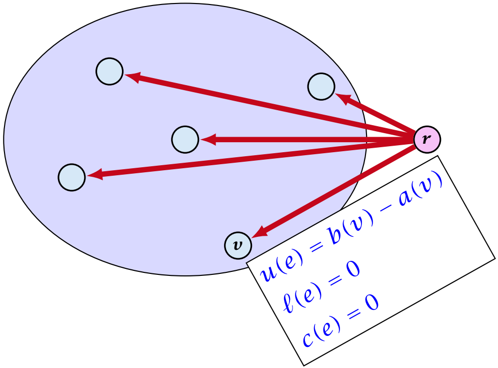
Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.



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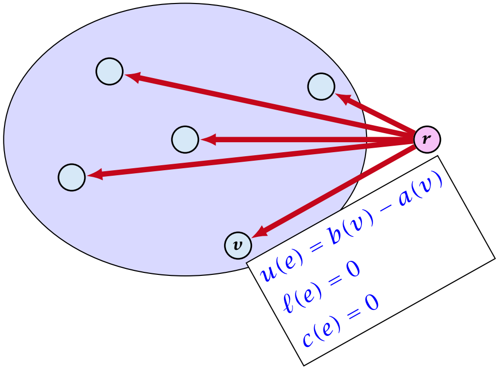

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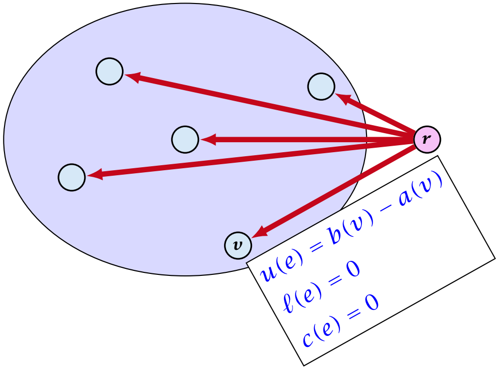

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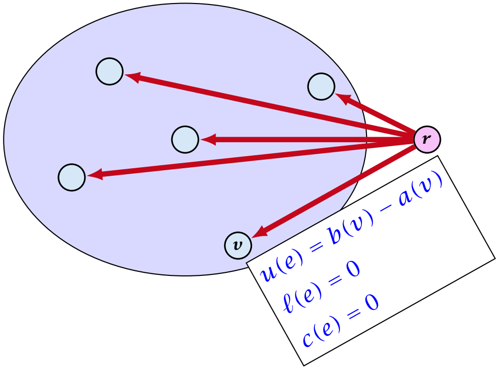

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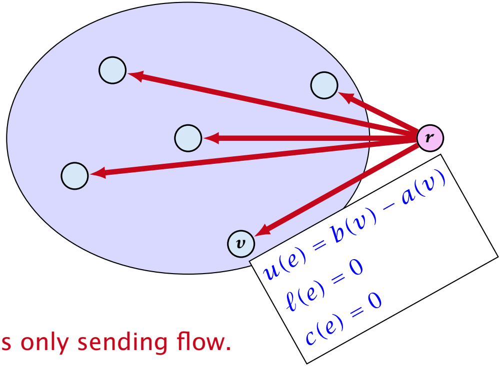
$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction II

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

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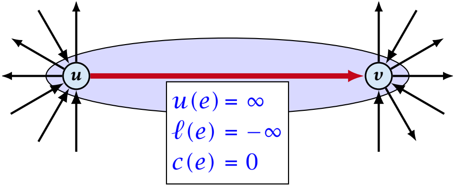

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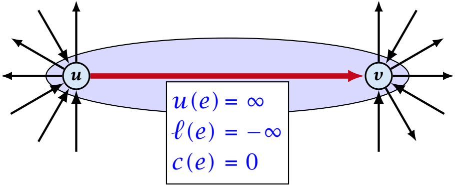

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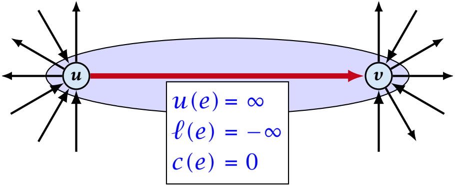
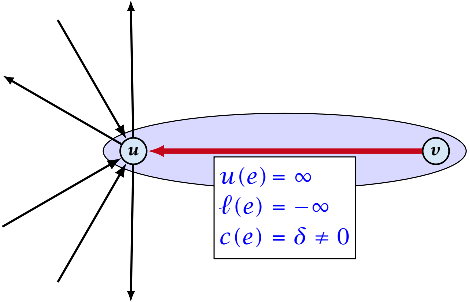


If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:



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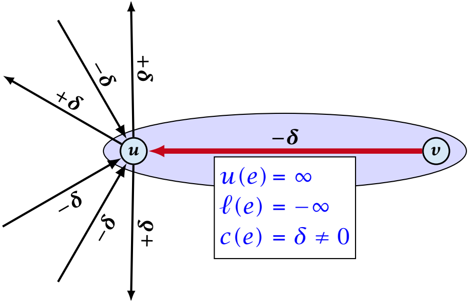

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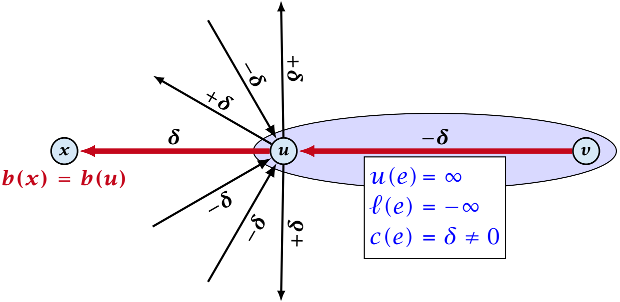

Additionally we set $b(u) = 0$.

Reduction III

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

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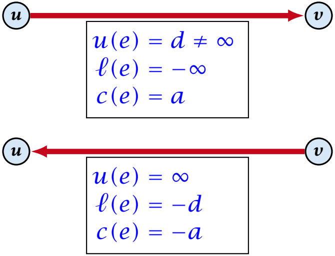

Replace the edge by an edge in opposite direction.

Reduction IV

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

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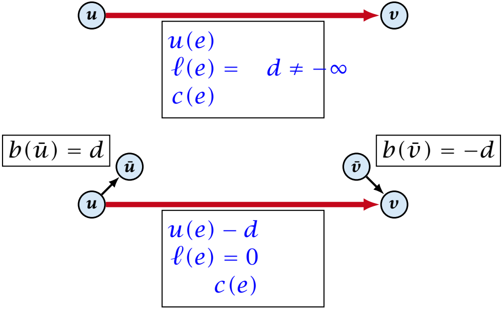

The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.

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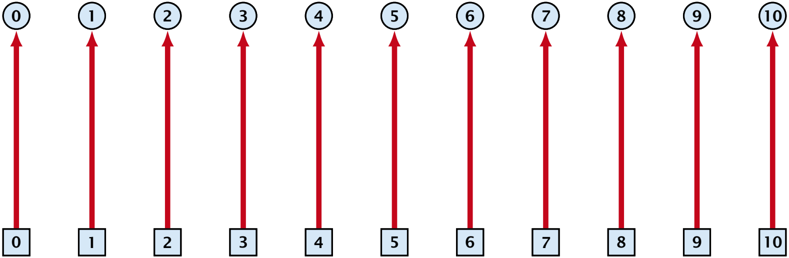
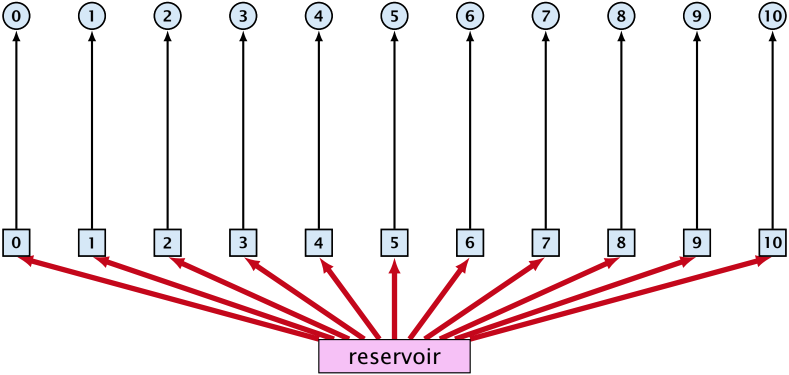
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.



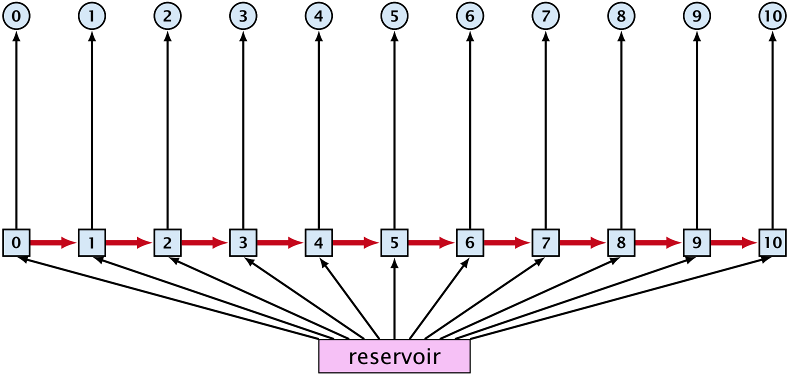
day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



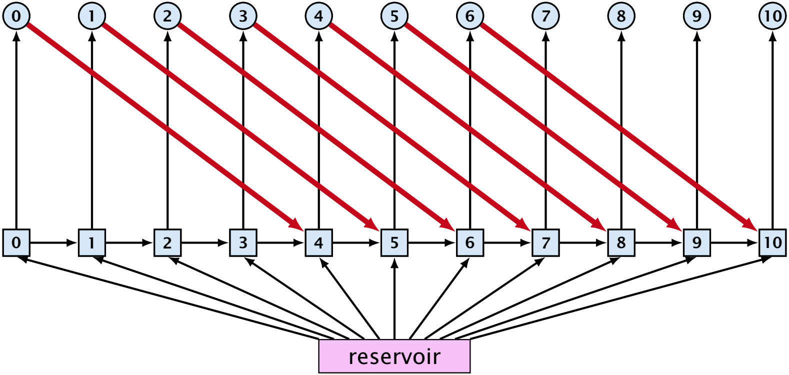
buy edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = p$



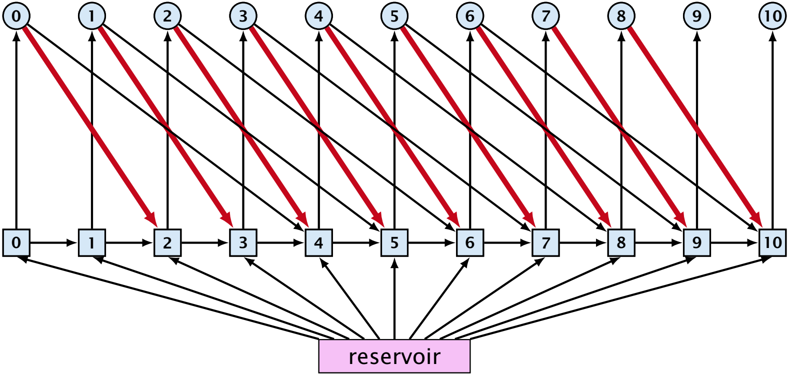
forward edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$



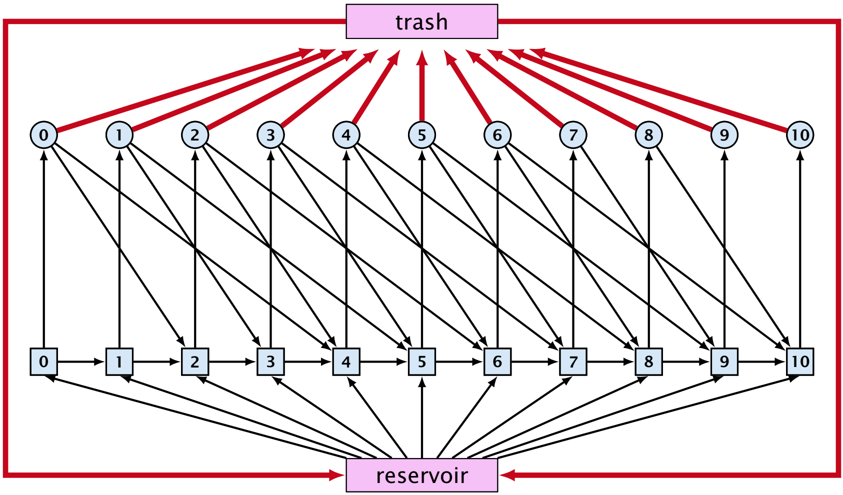
slow edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = s$



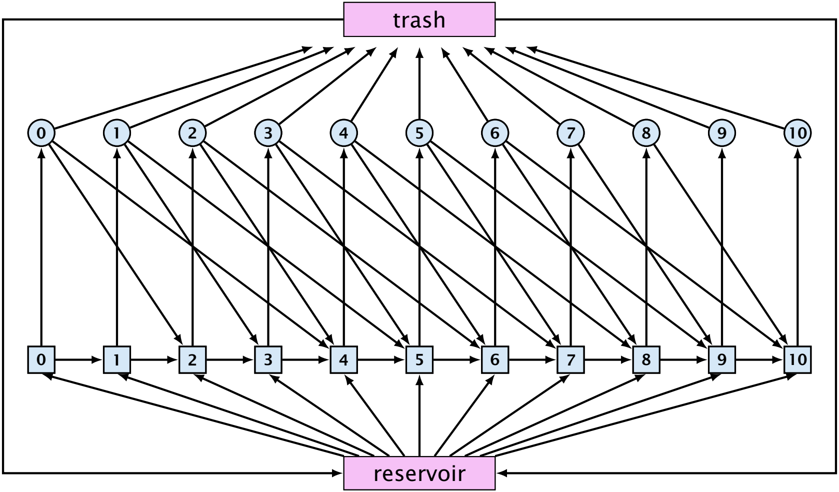
fast edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = f$



trash edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$



trash
0
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5
6
7
8
9
10
reservoir

Residual Graph

Version A:
The residual graph G' for a mincost flow is just a copy of the graph G.

If we send $f(e)$ along an edge, the corresponding edge e' in the residual graph has its lower and upper bound changed to $\ell(e') = \ell(e) - f(e)$ and $u(e') = u(e) - f(e)$.

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Version B:
The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

10 Mincost Flow

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A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \leq u(e)$ for every edge of G.

Lemma 73

A given flow is a mincost-flow if and only if the corresponding residual graph G_f *does not have a feasible circulation of negative cost.*

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  Then $f + g$ is a feasible flow with cost $\mathrm{cost}(f) + \mathrm{cost}(g) < \mathrm{cost}(f)$. Hence, f is not minimum cost.

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$\Leftarrow$ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

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A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

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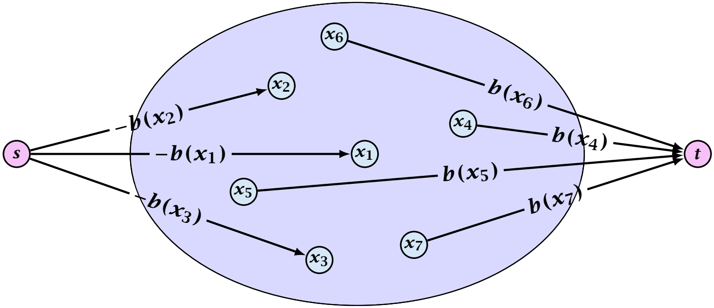
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

10 Mincost Flow

Algorithm 45 CycleCanceling$(G = (V, E), c, u, b)$

1: establish a feasible flow f in G
2: **while** G_f contains negative cycle **do**
3: use Bellman-Ford to find a negative circuit Z
4: $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
5: augment δ units along Z and update G_f

How do we find the initial feasible flow?



- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \ .$$

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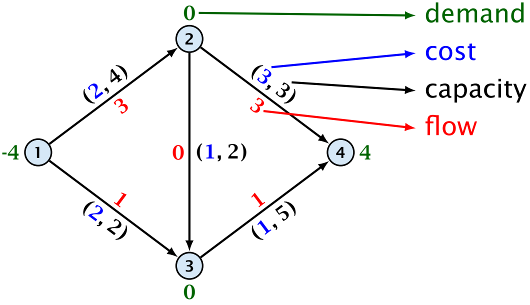

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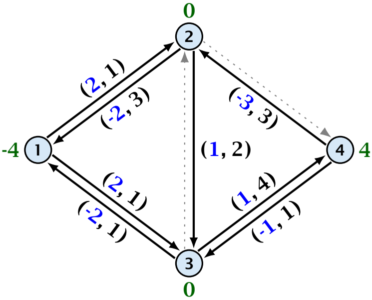

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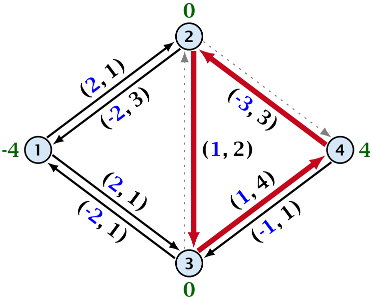

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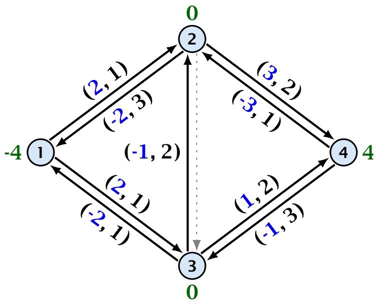

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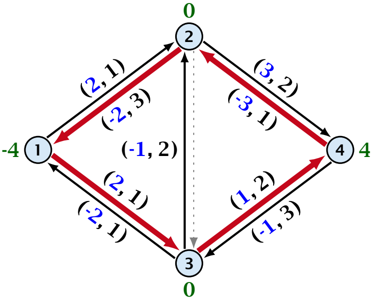

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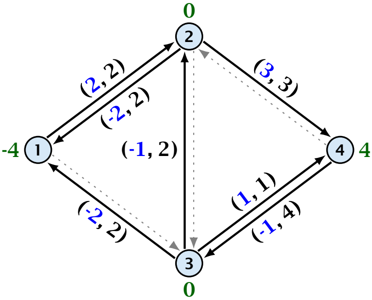

Lemma 75
The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, *for integer capacities and costs, when for all edges* e, $|c(e)| \le C$ *and* $|u(e)| \le U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \ldots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

10 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \ \ \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \ \ a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$ $c : E \to \mathbb{R}$;

Lemma 76 (without proof)

A general mincost flow problem can be solved in polynomial time.

11 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set F and capacities w that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f(s, t)$ in G is equal to $f_T(s, t)$.
2. **Cut Property:** A minimum s-t cut in T is also a minimum cut in G.

Here, $f(s, t)$ is the value of a maximum s-t flow in G, and $f_T(s, t)$ is the corresponding value in T.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \dots, S_t$), and a spanning tree T on the vertex set $\{S_1, \dots, S_t\}$.

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- In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.

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In the end this gives a tree on the vertex set V.

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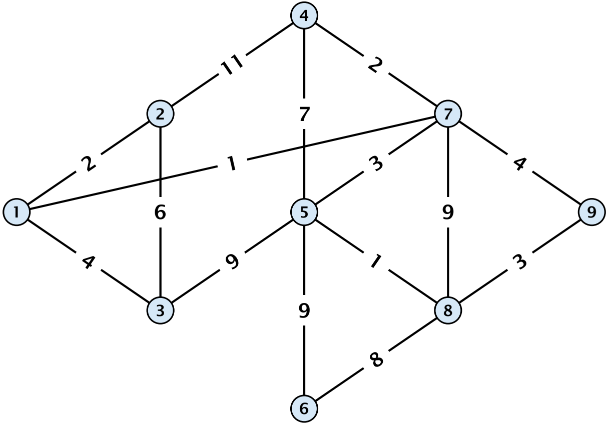
- Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

Example: Gomory-Hu Construction



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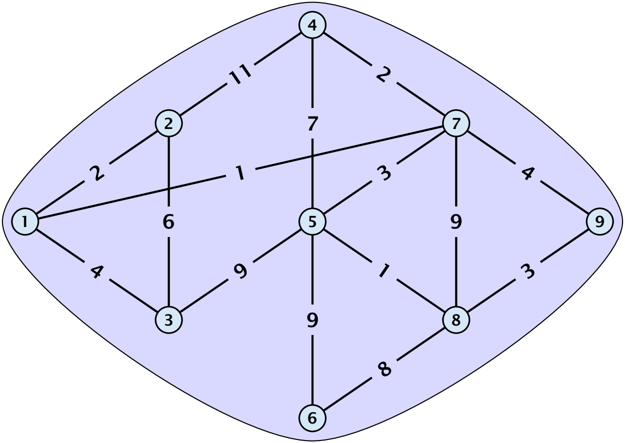

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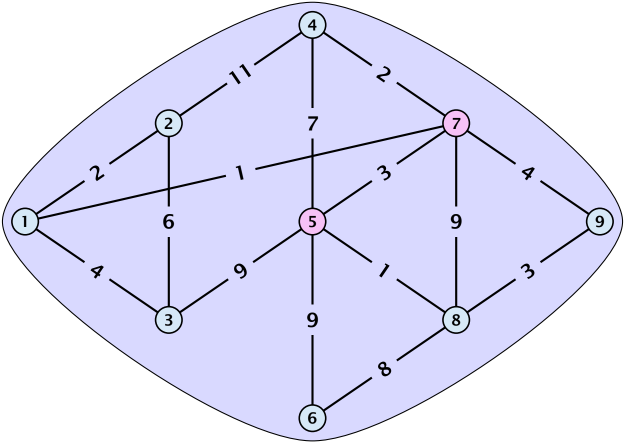

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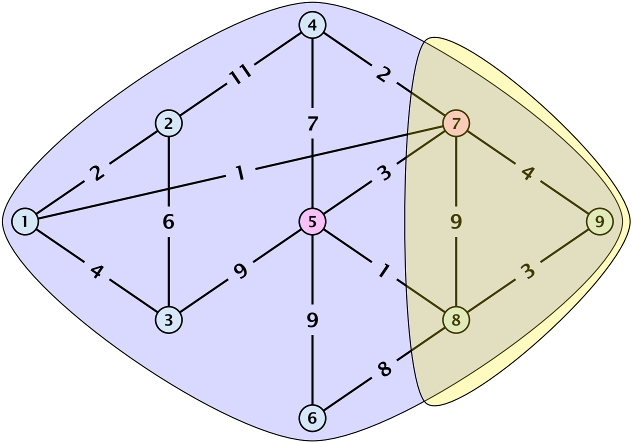

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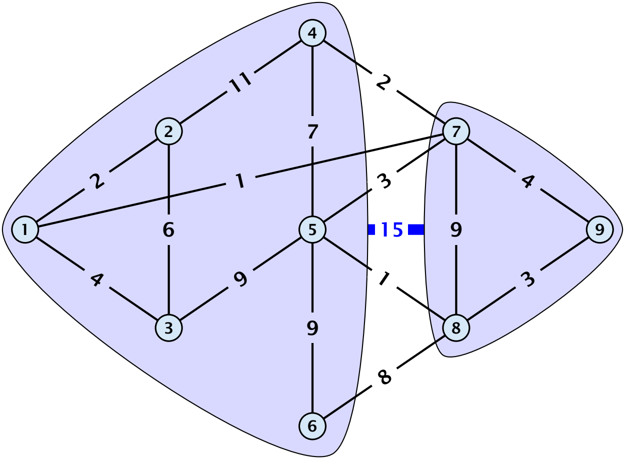

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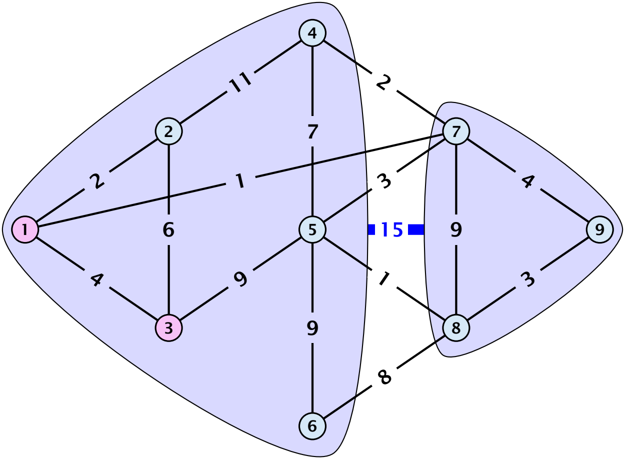

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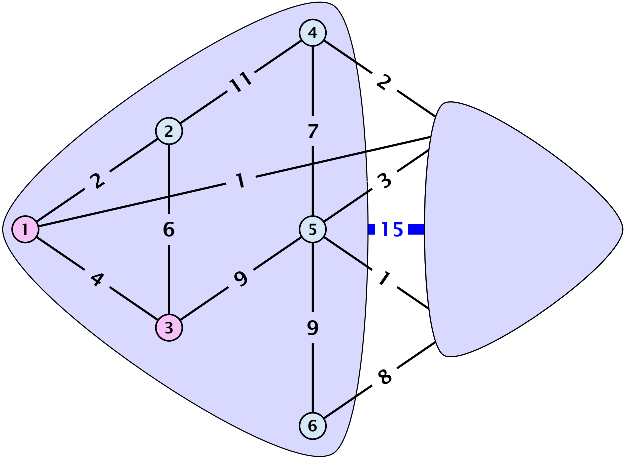

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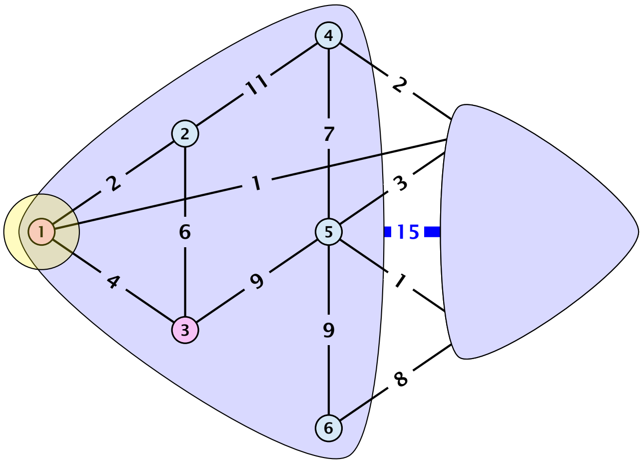

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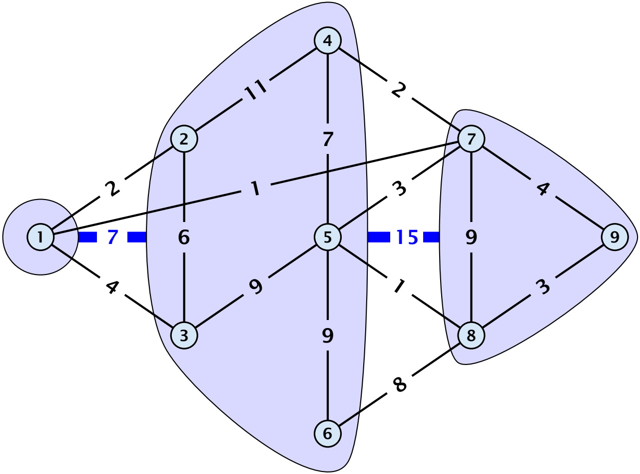

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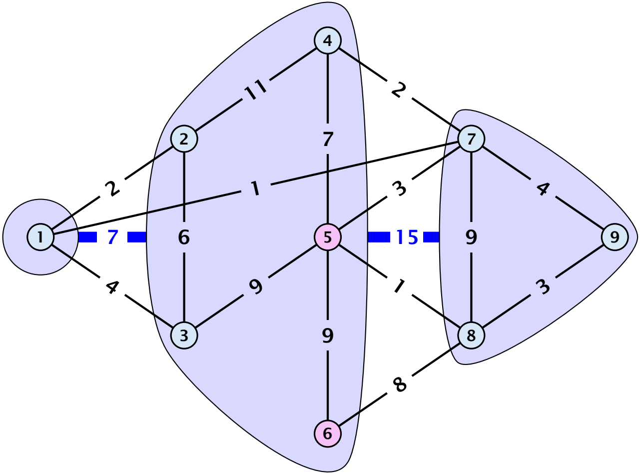

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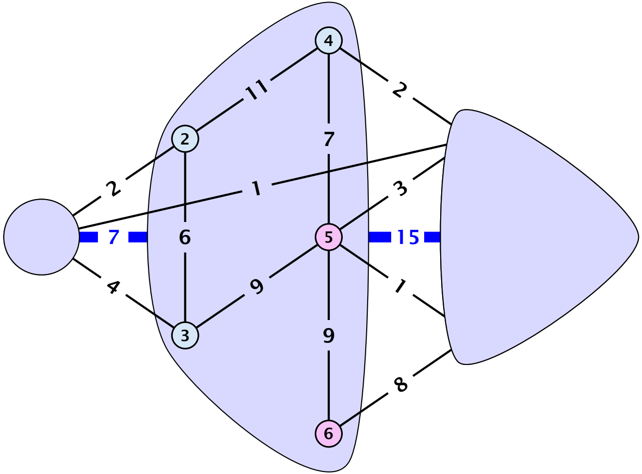

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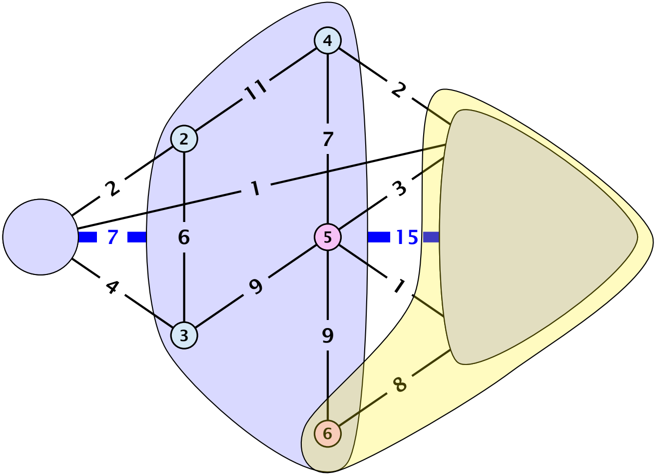

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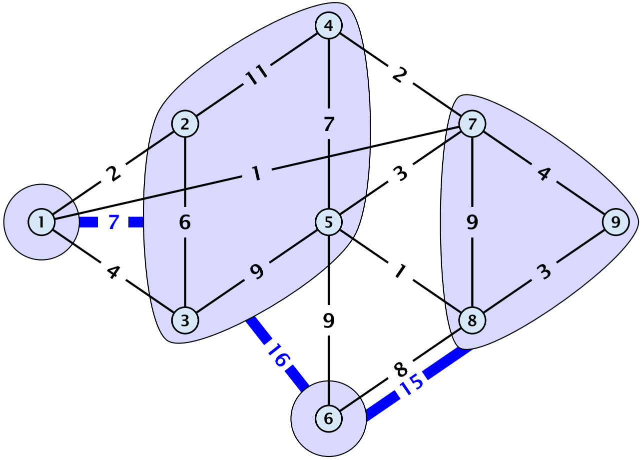

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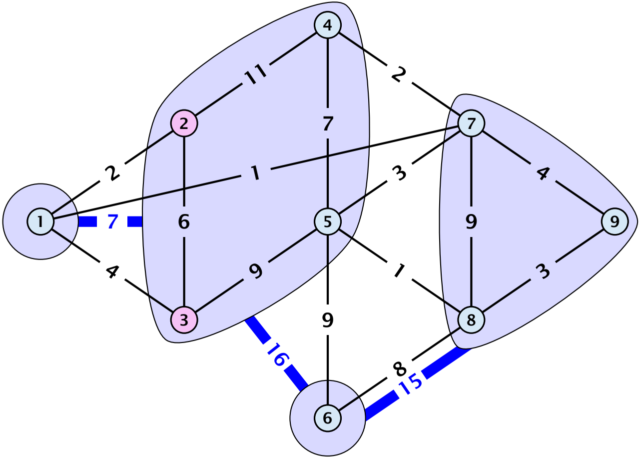

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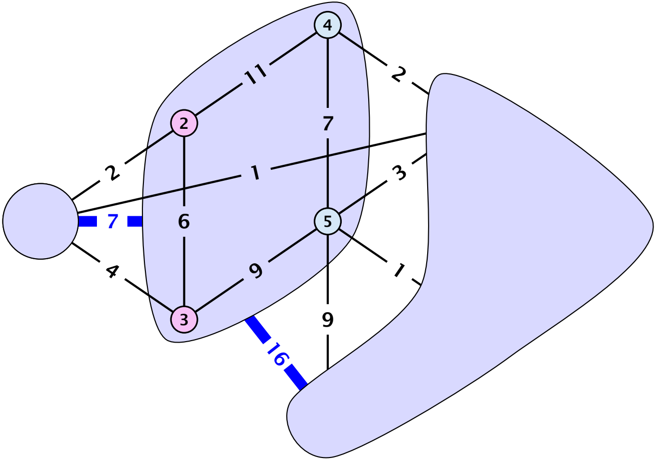

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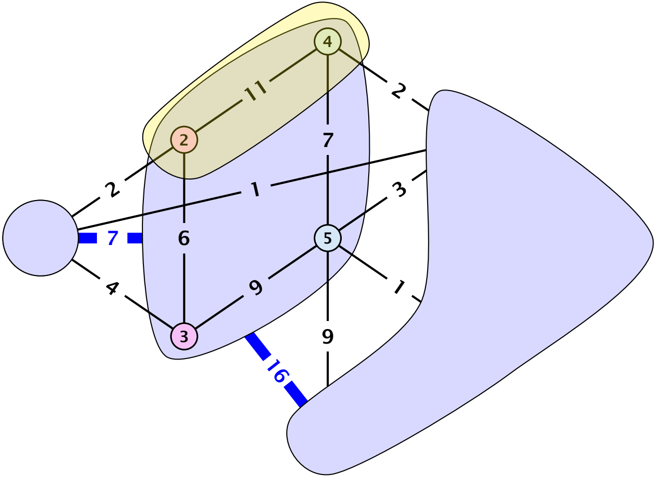

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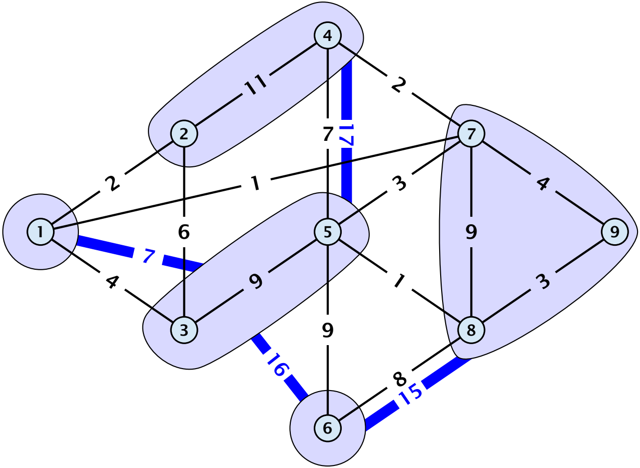

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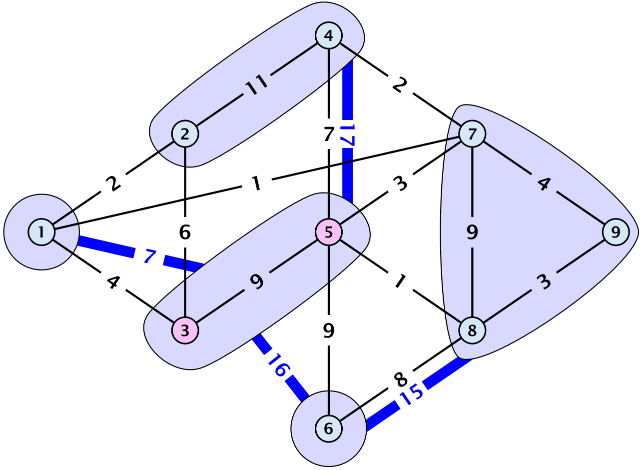

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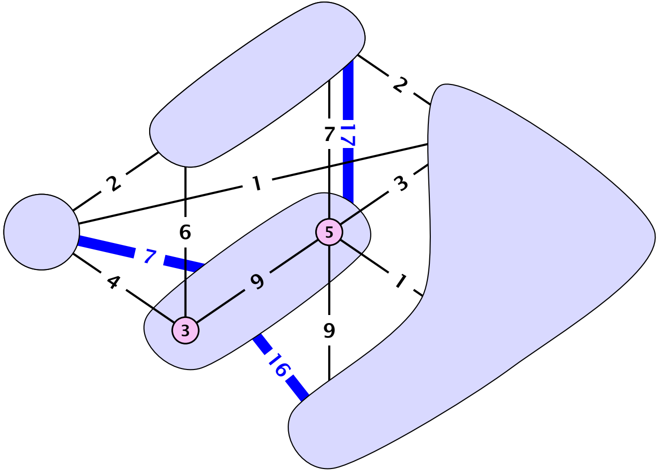

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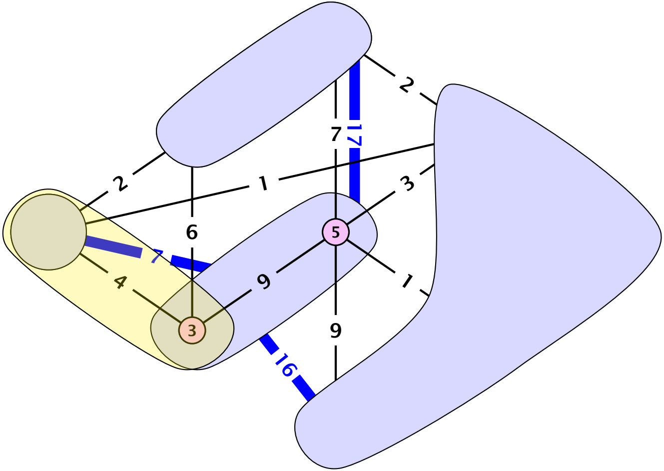

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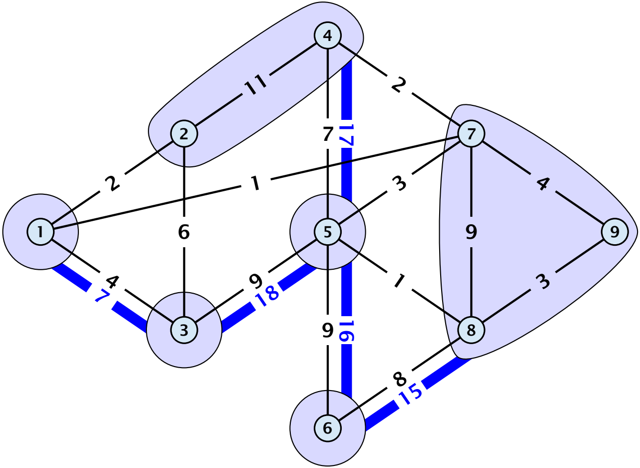

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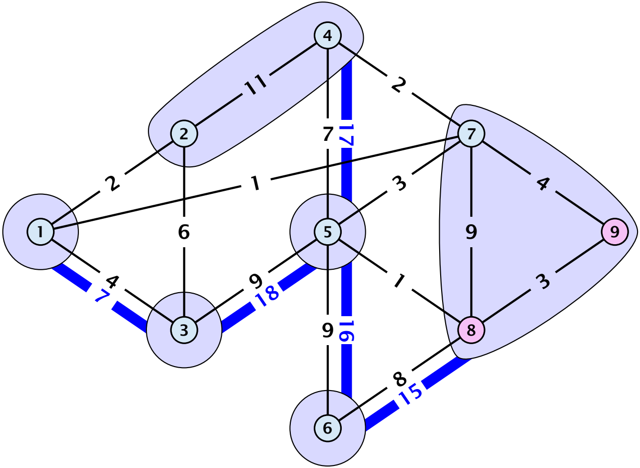

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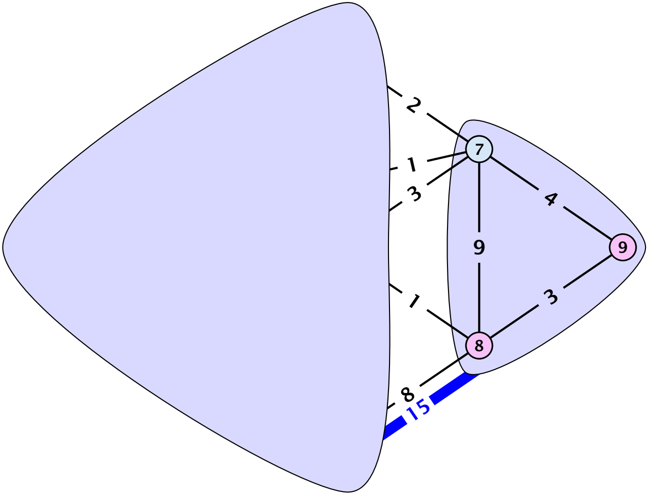

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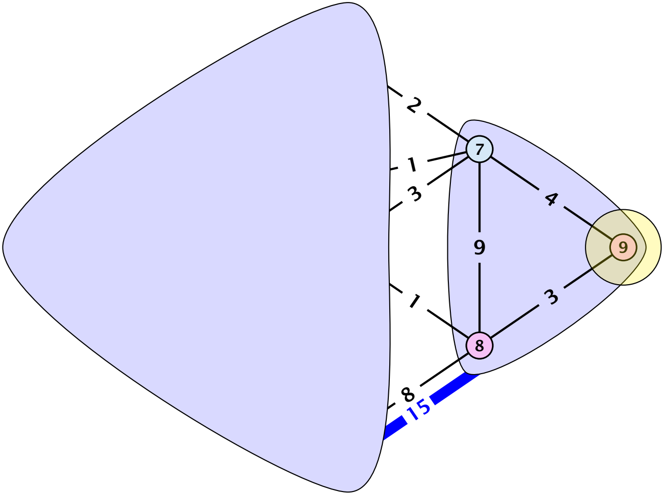

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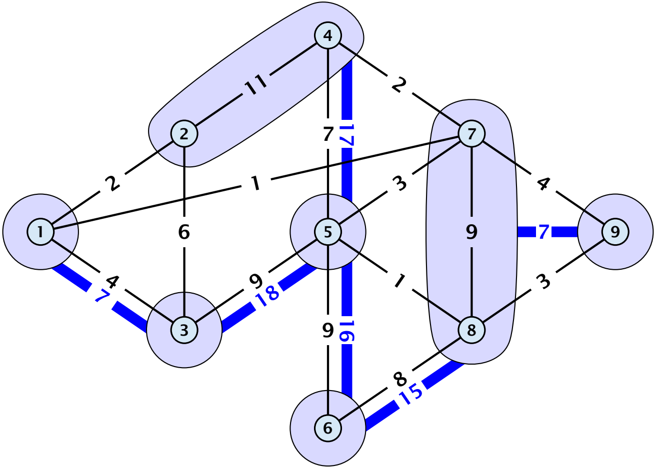

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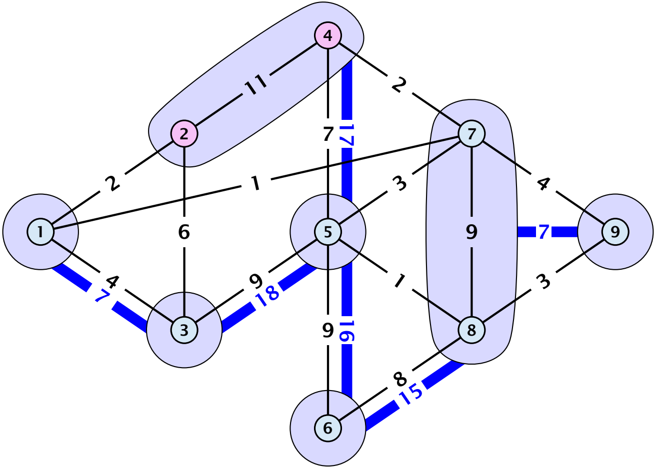

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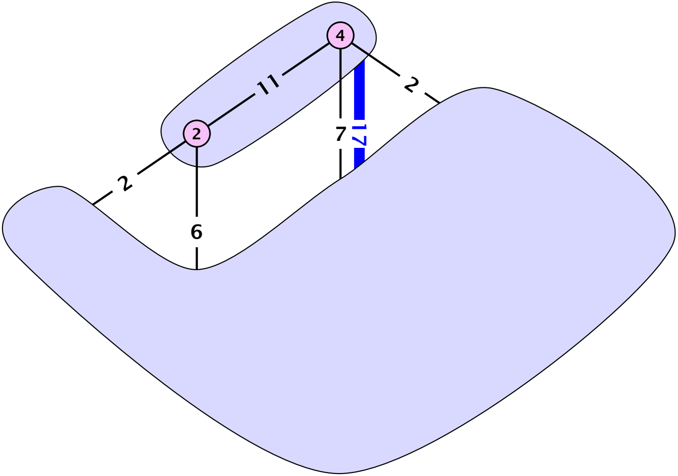

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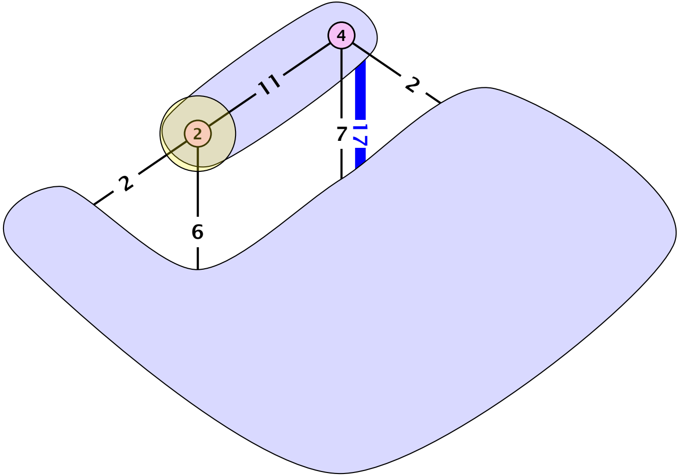

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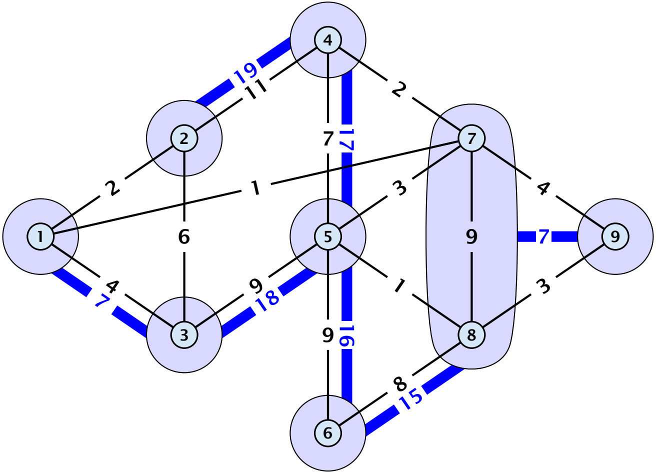

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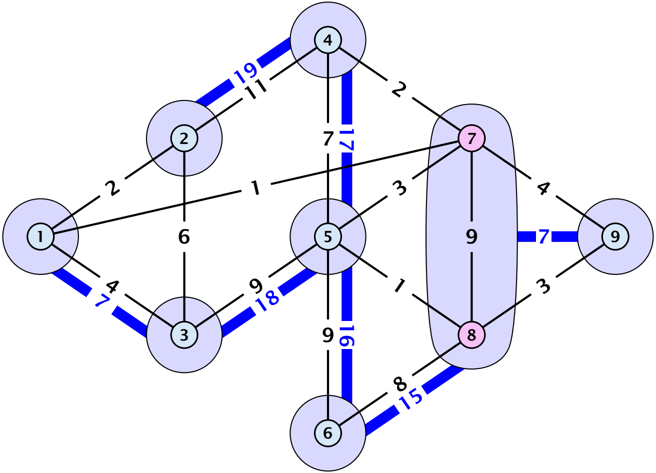

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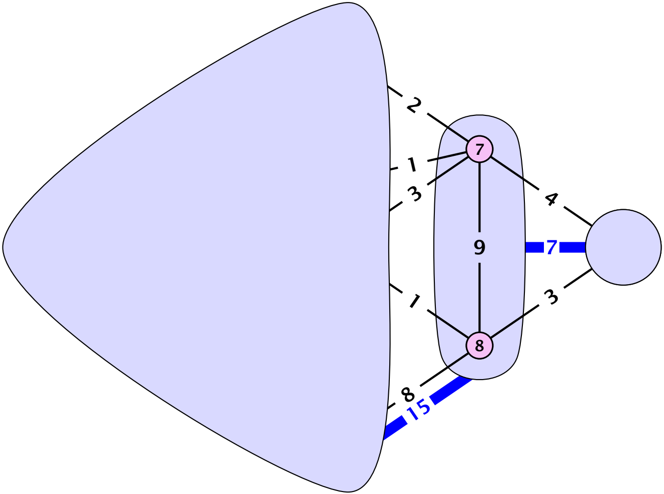

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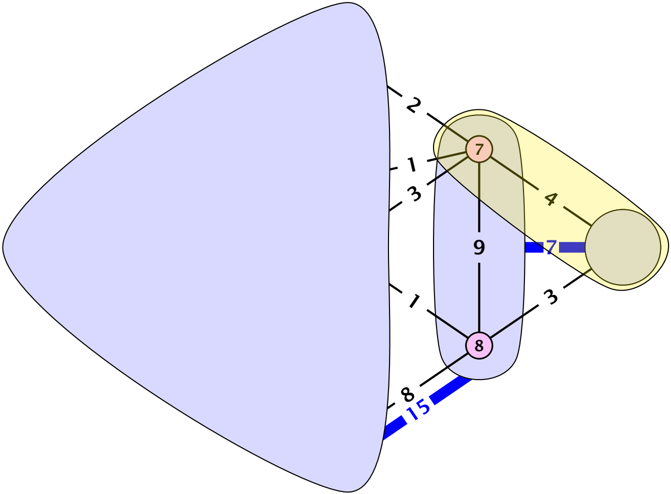

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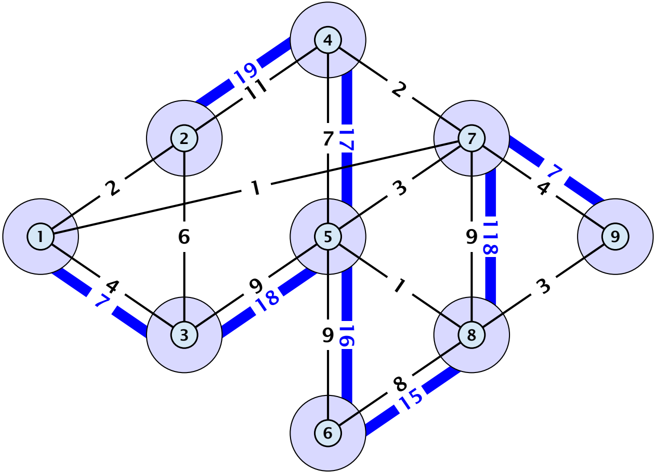

Analysis

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For nodes $s, t, x \in V$ *we have* $f(s,t) \geq \min\{f(s,x), f(x,t)\}$

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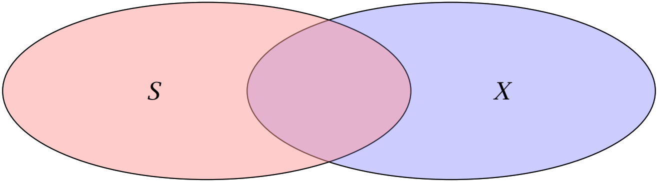

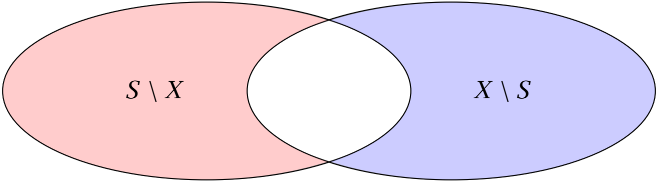

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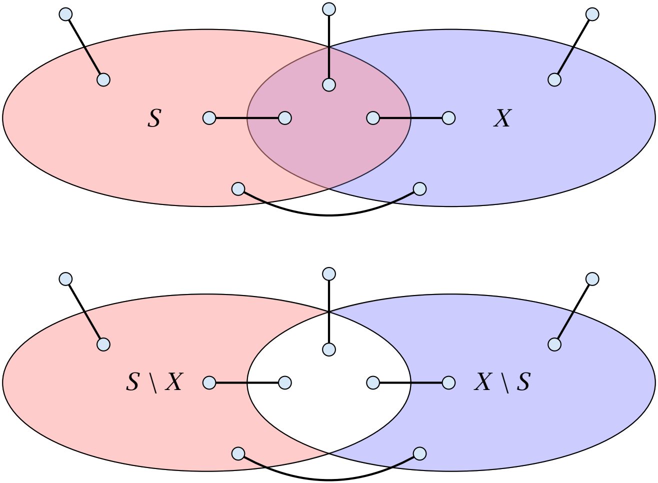

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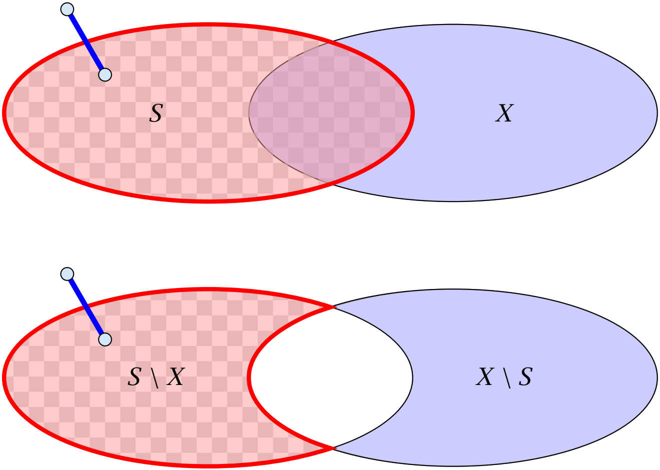

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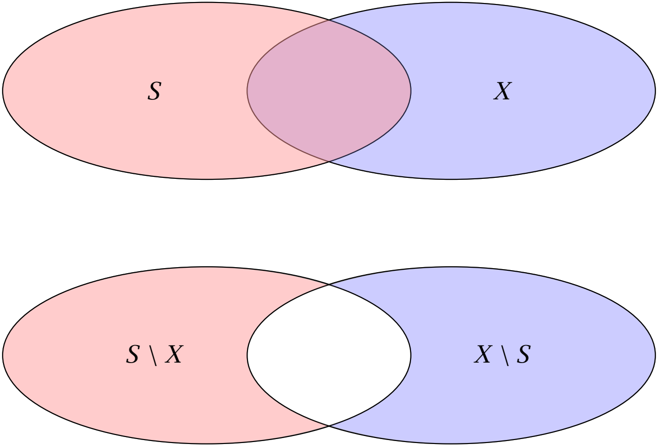

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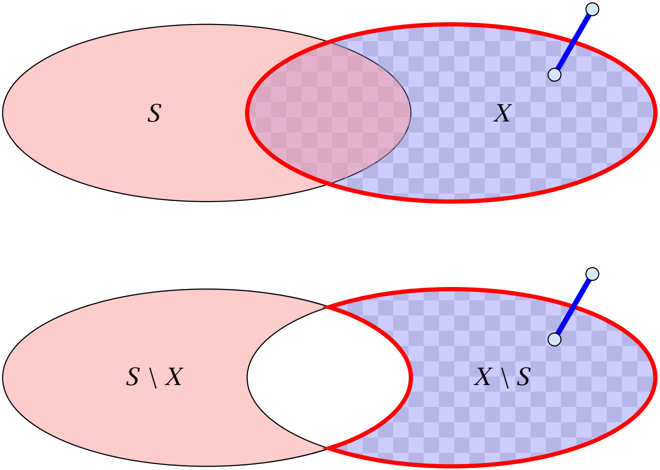

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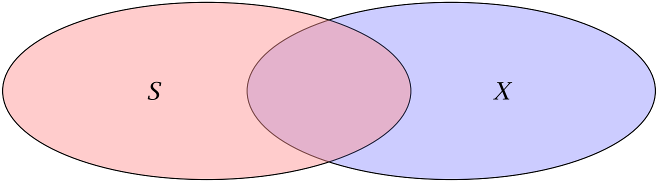

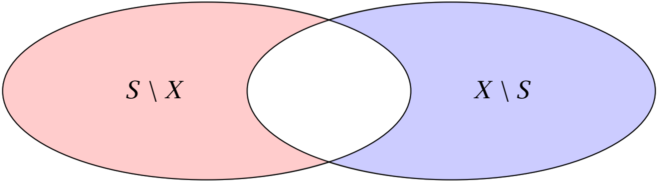

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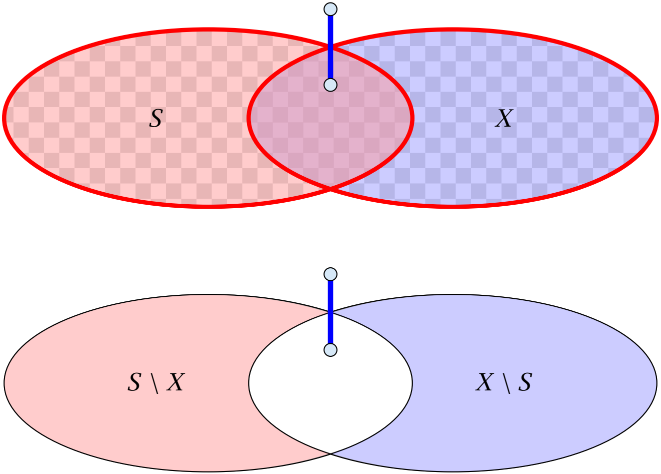

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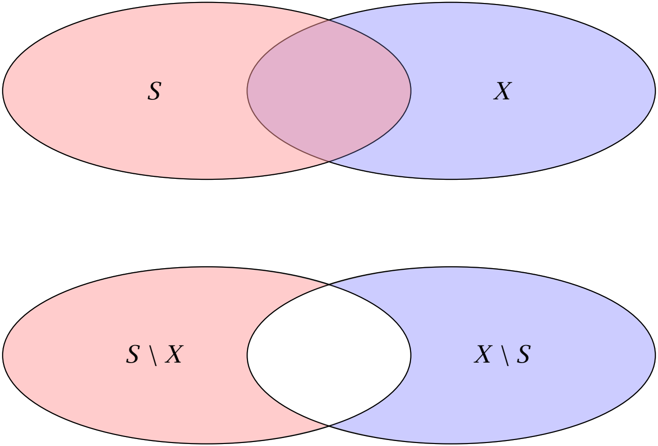

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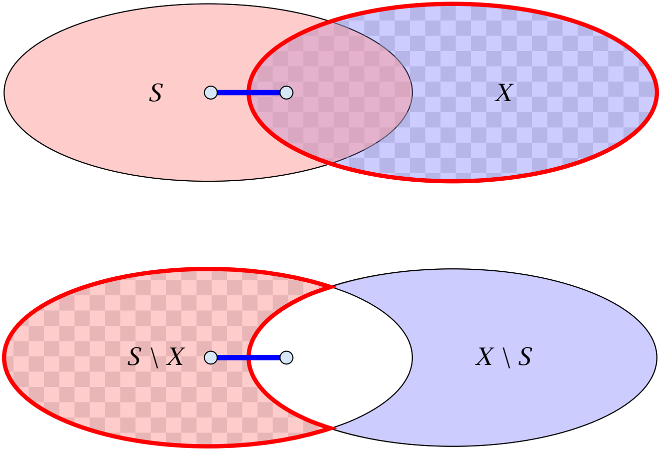

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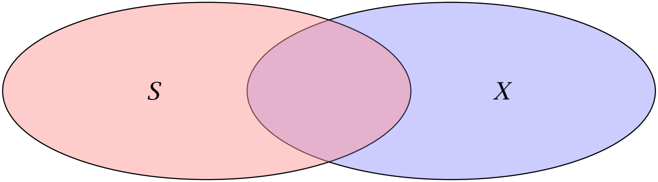

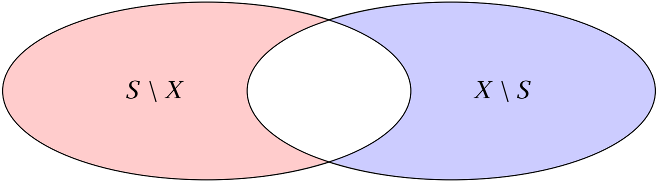

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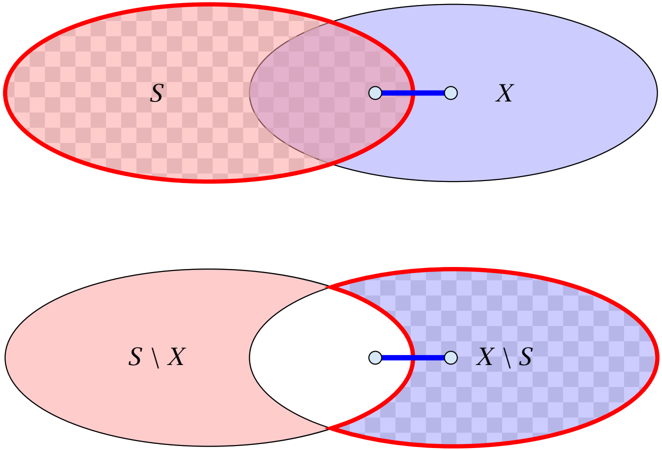

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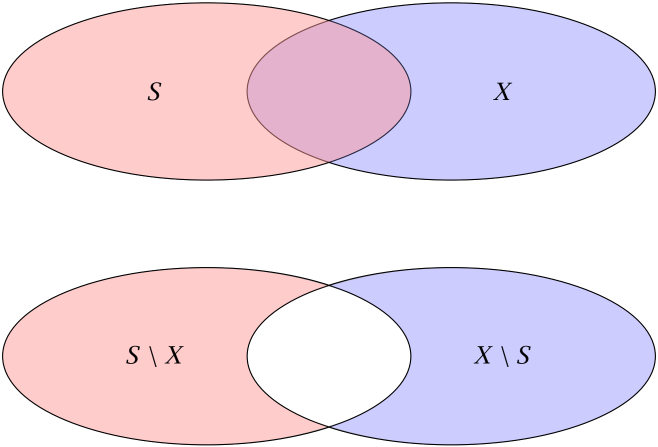

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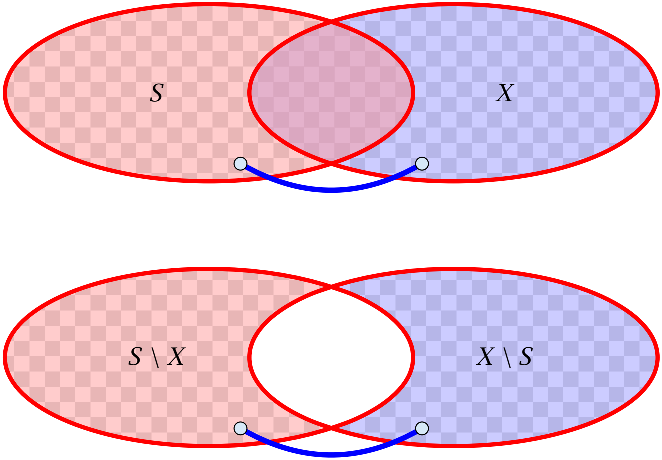

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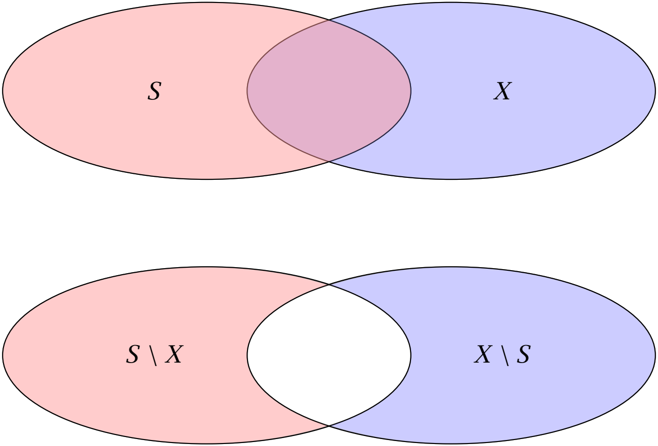

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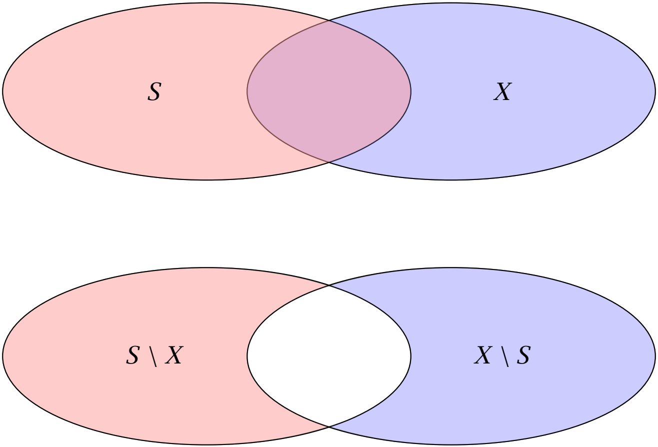

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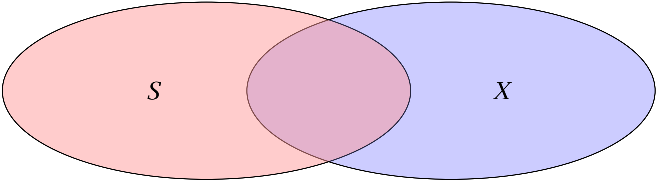

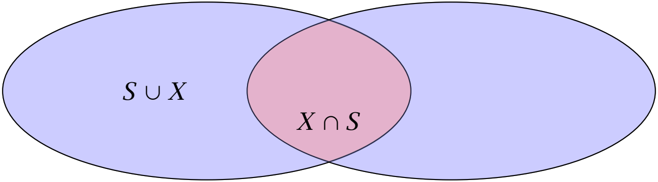

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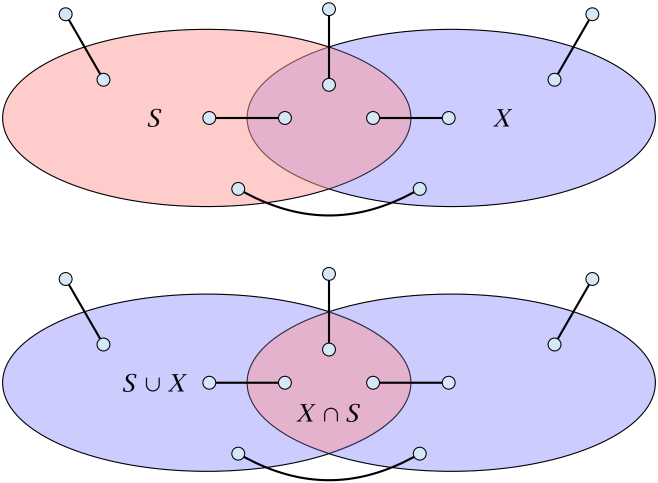

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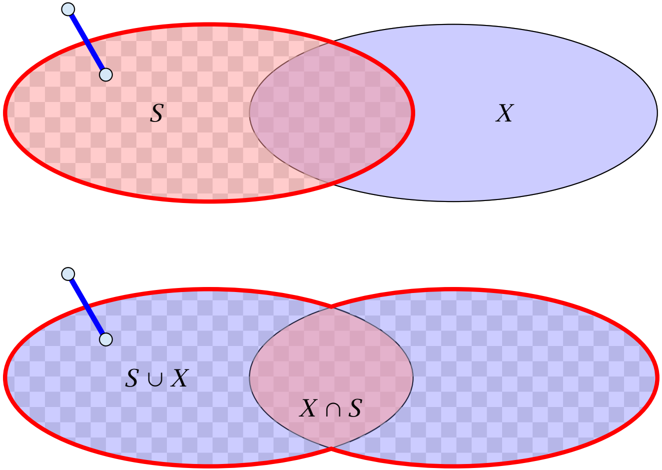

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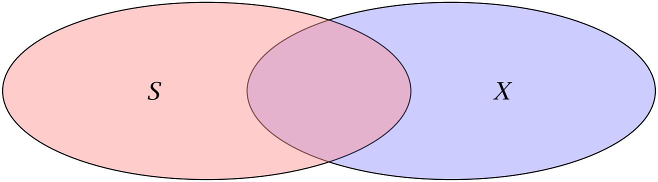

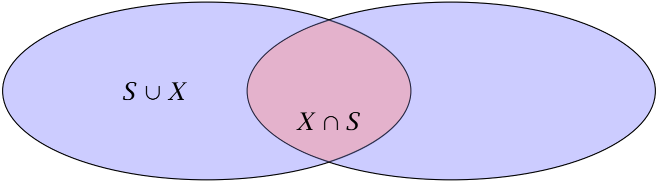

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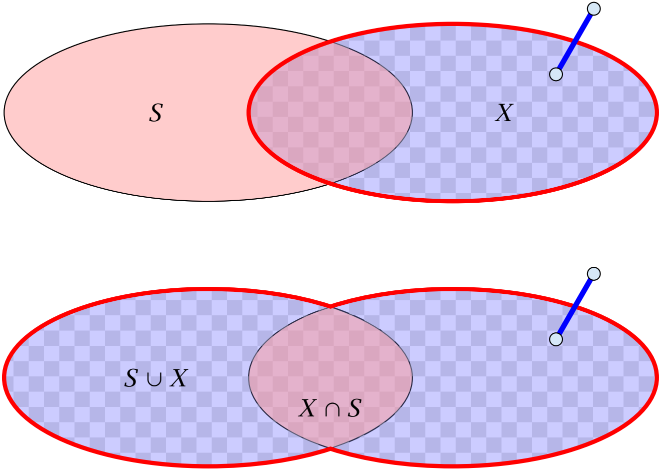

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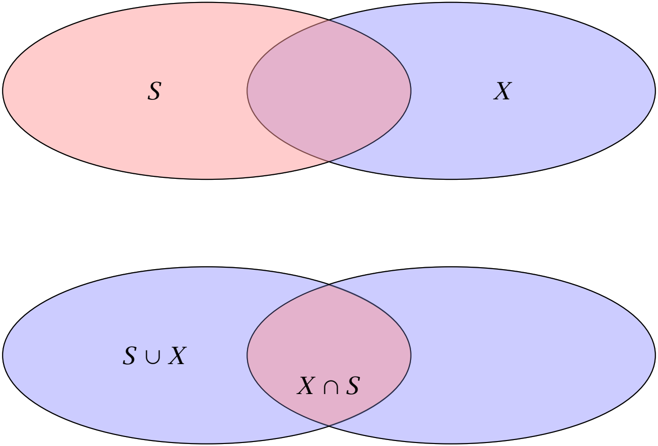

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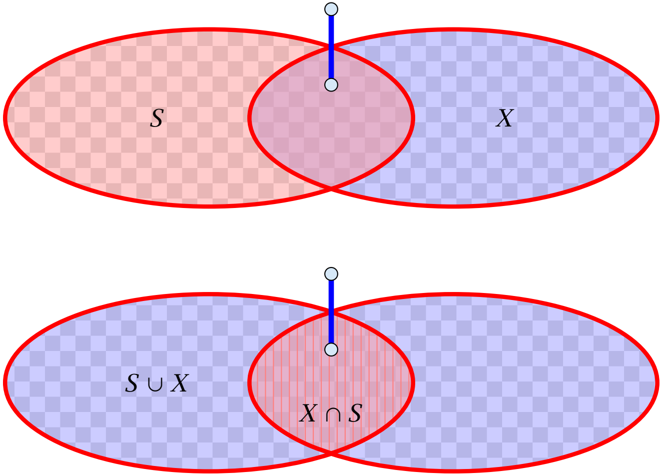

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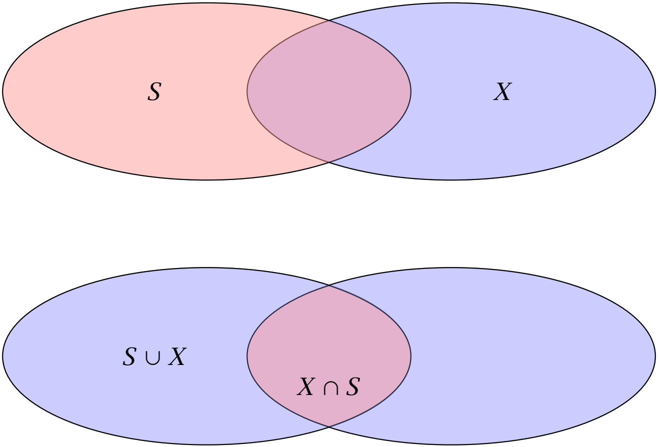

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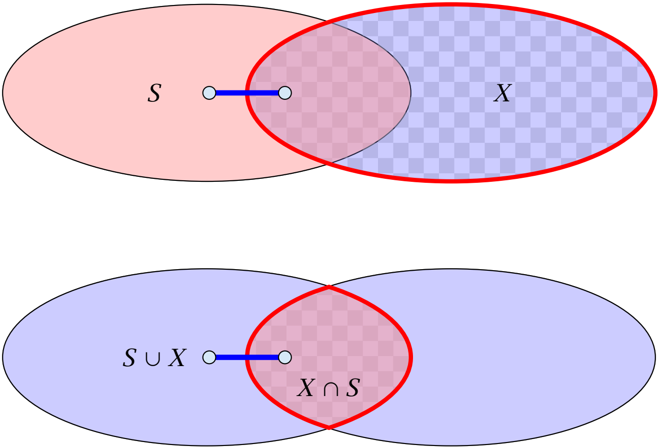

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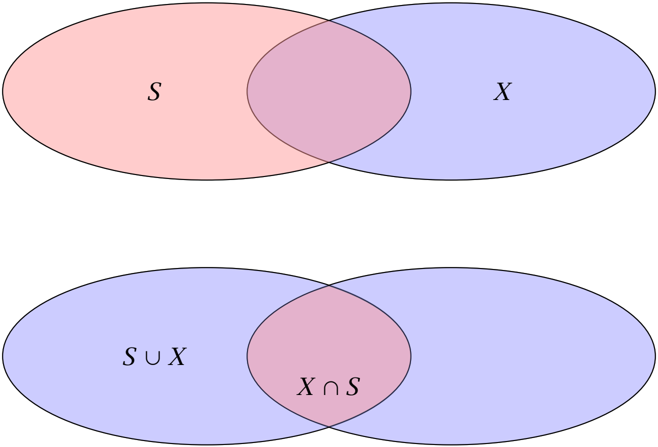

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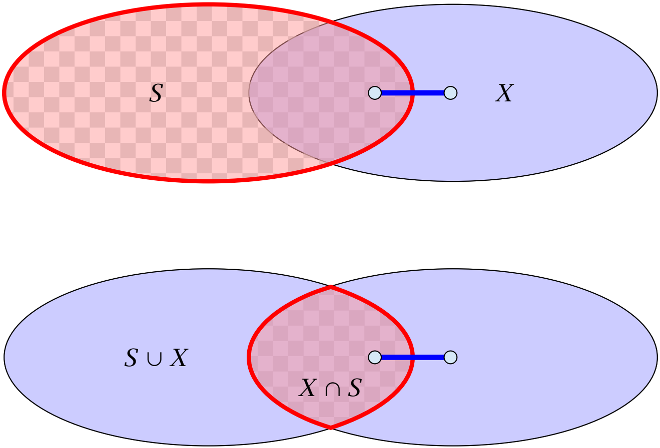

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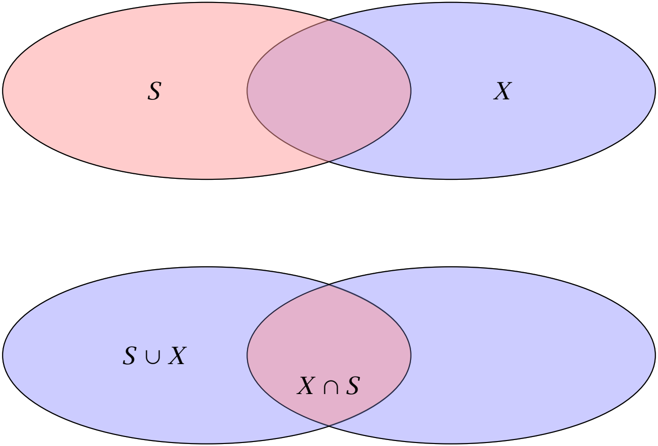

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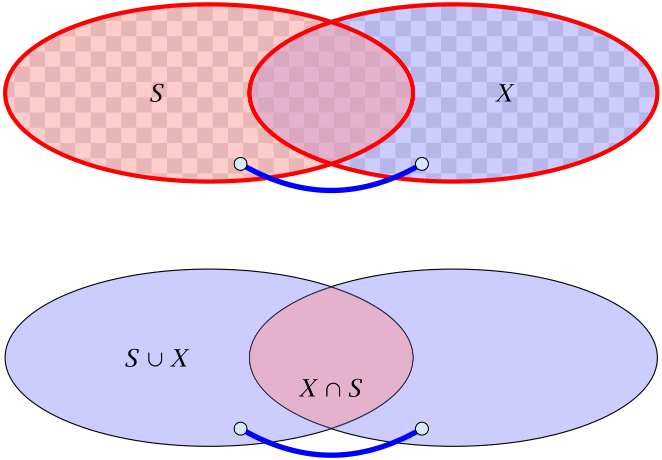

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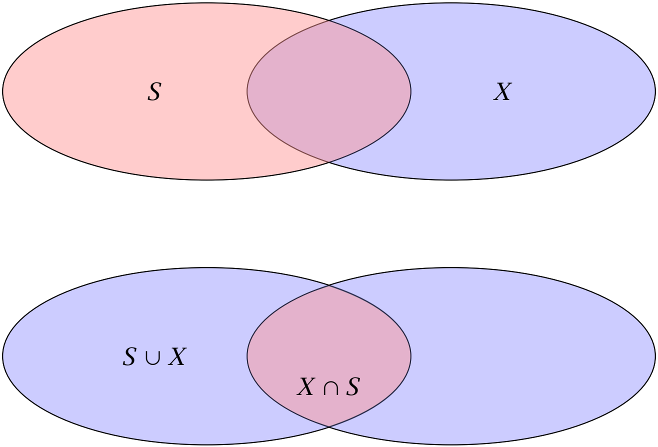

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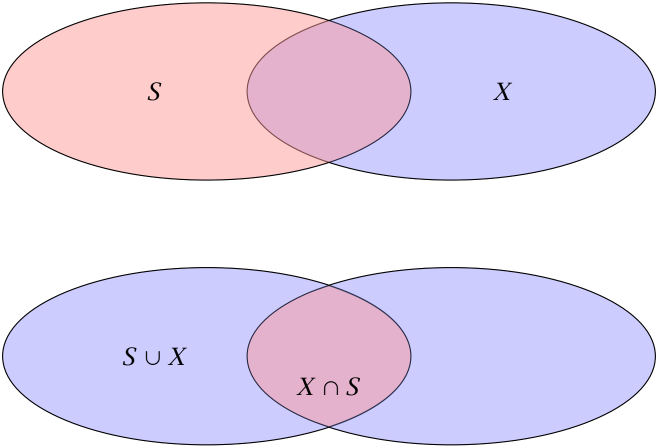

Analysis

Lemma 79 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s,t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s,t) = f(s,t)$, where $f_H(s,t)$ is the value of a minimum s-t mincut in graph H.

Analysis

Invariant [existence of representatives]:
For any edge $\{S_i, S_j\}$ in T, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum a-b cut in G.

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We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

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- Since by the invariant this edge induces an s-t cut with capacity $f(x_j, x_{j+1})$ we get $f(s,t) \le f(x_j, x_{j+1}) = f_T(s,t)$.

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- The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T.
- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).
- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. s and t, this is an s-t mincut (cut property).

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After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a, b)$ we can simply choose a and b as representatives.

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If $s \in S_i^a$ we can keep x and s as representatives.

Otherwise, we choose x and a as representatives. We need to show that $f(x, a) = f(x, s)$.

Proof of Invariant

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The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 79 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

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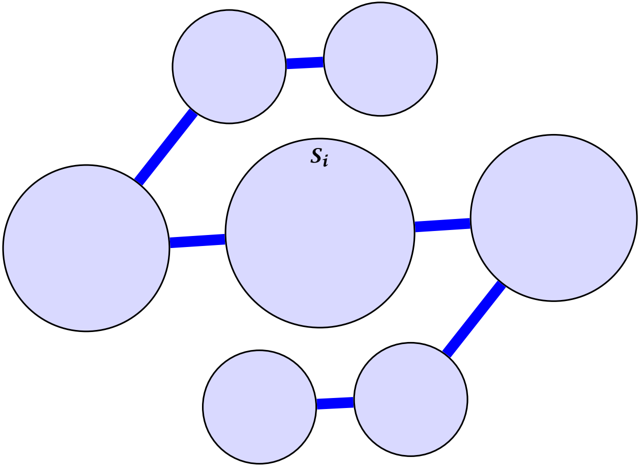
We further have $f'(x,a) \ge \min\{f'(x,v_B), f'(v_B,a)\}$.

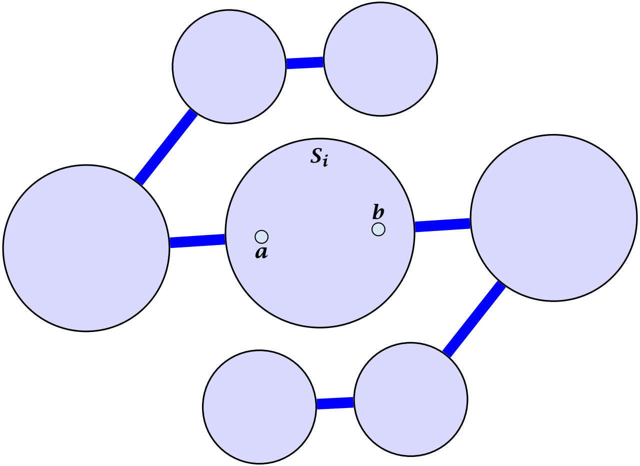
Since $s \in B$ we have $f'(v_B,x) \ge f(s,x)$.

Also, $f'(a,v_B) \ge f(a,b) \ge f(x,s)$ since the a-b cut that splits S_i into S_i^a and S_i^b also separates s and x.

Analysis



Analysis



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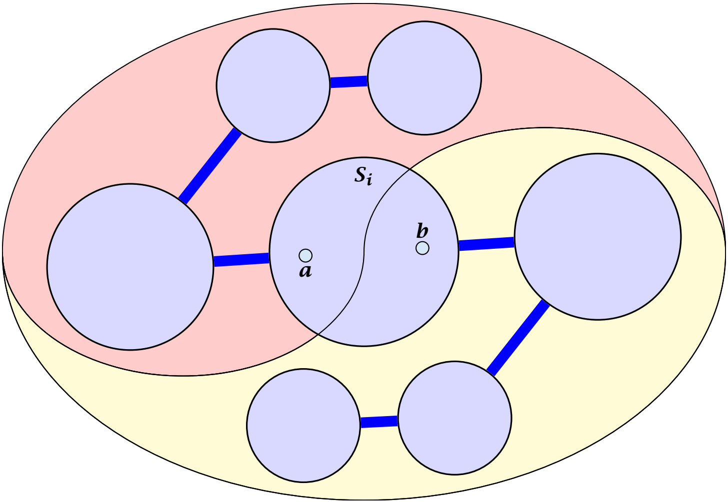

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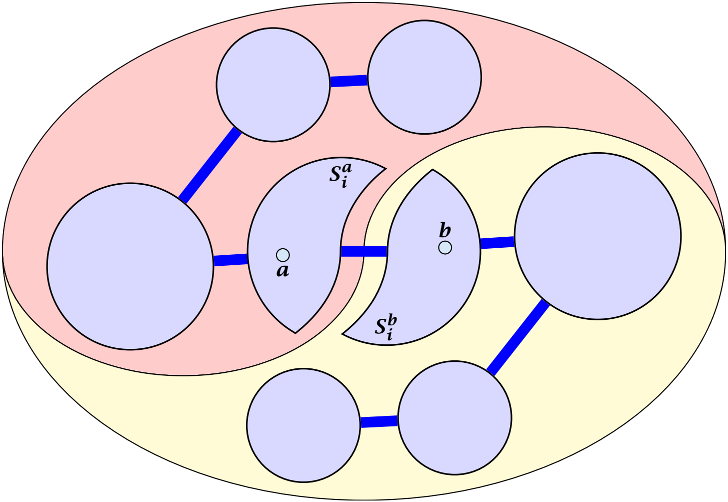

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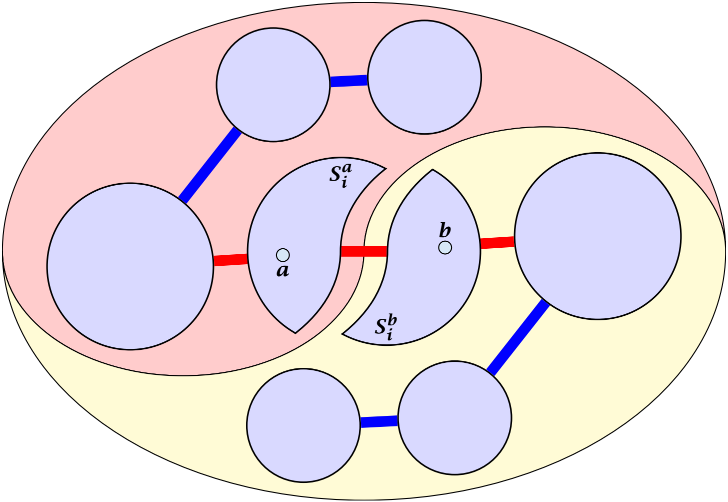

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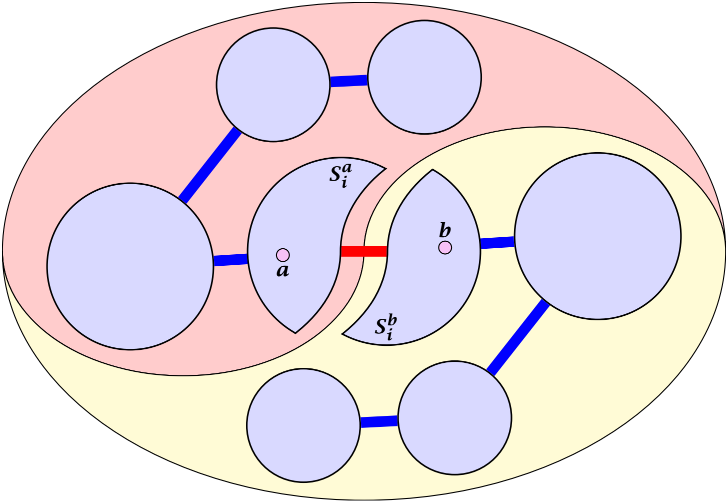

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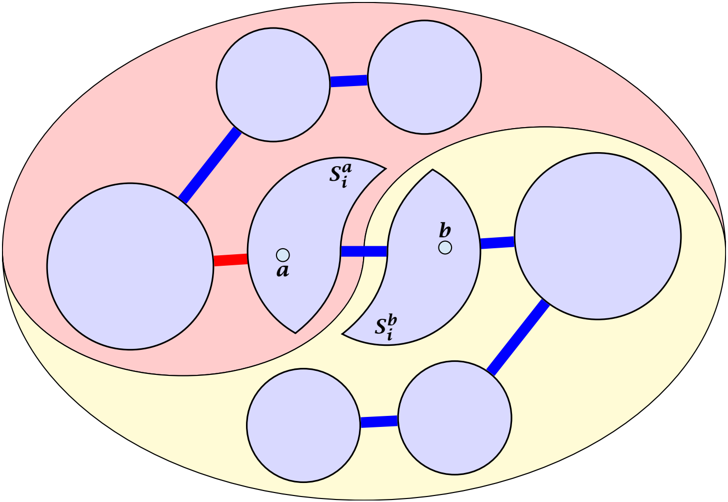

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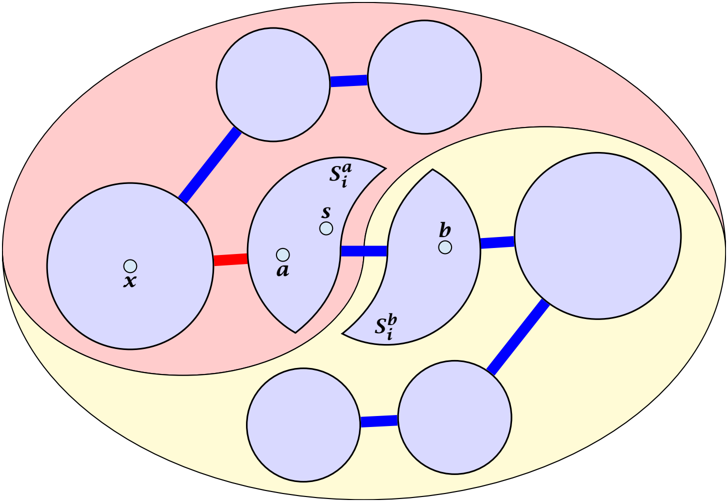

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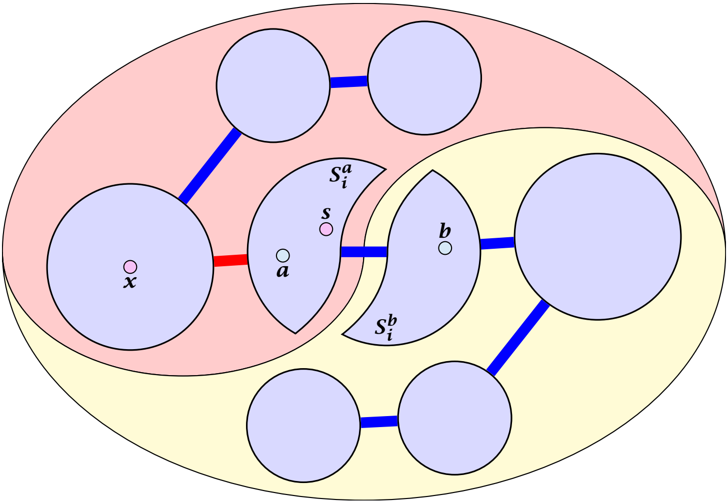

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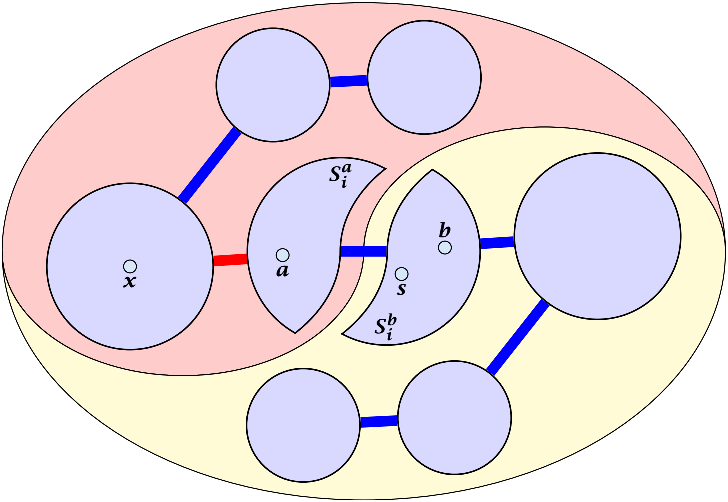

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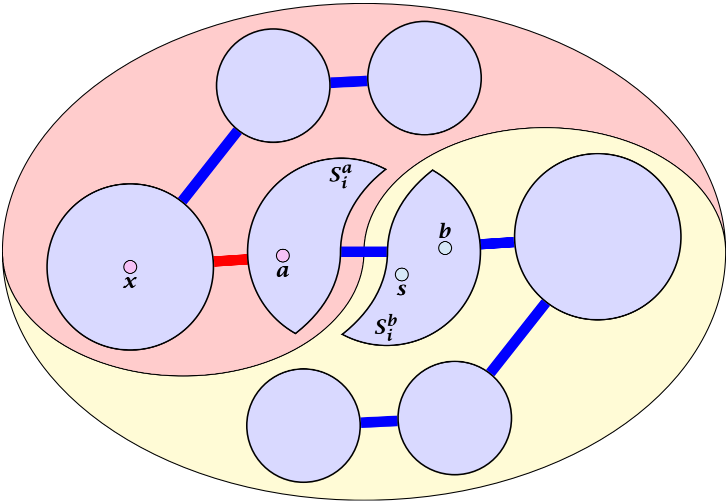

12 Global Mincut

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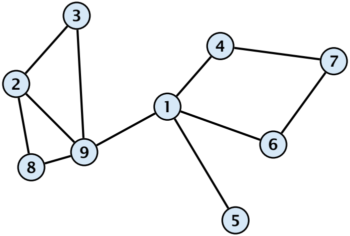

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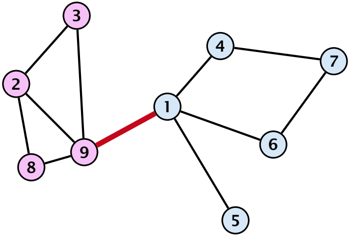

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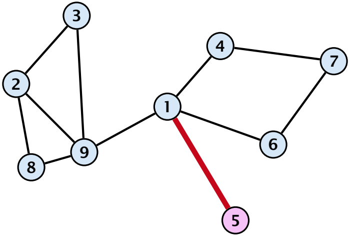

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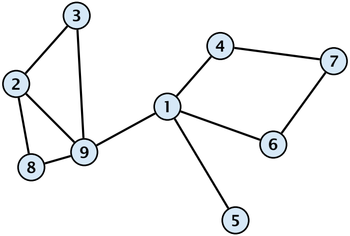

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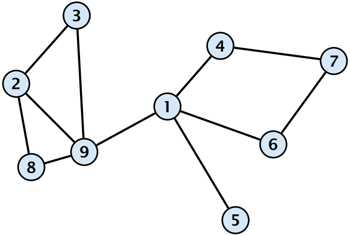

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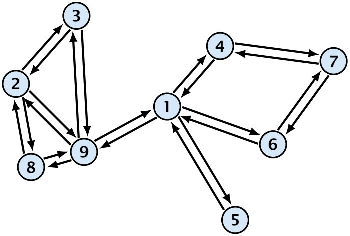

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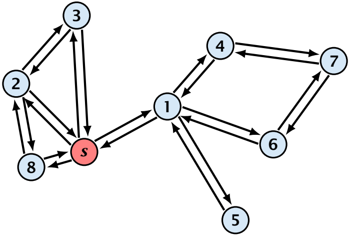

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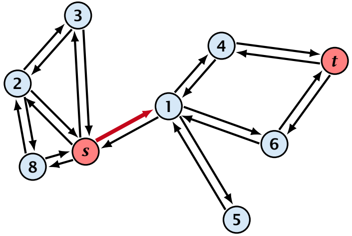

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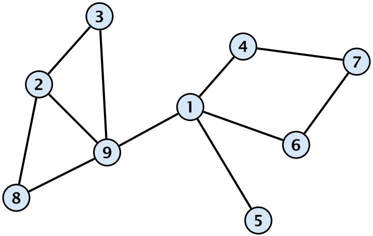

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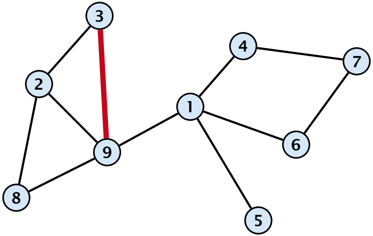

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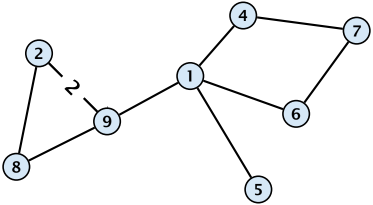

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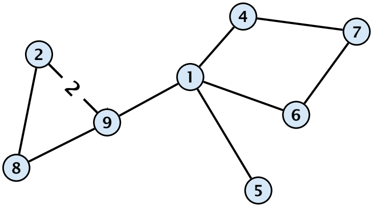

- Edge-contractions do not decrease the size of the mincut.

Edge Contractions

We can perform an edge-contraction in time $\mathcal{O}(n)$.

Randomized Mincut Algorithm

Algorithm 1 KargerMincut($G = (V, E, c)$)

1: **for** $i = 1 \to n - 2$ **do**
2: choose $e \in E$ randomly with probability $c(e)/c(E)$
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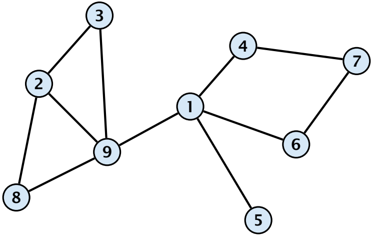
- What is the probability that this algorithm returns a mincut?

Example: Randomized Mincut Algorithm



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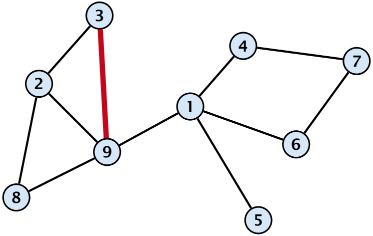

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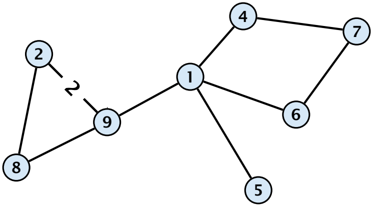

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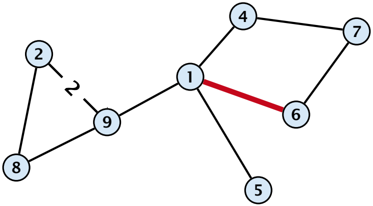

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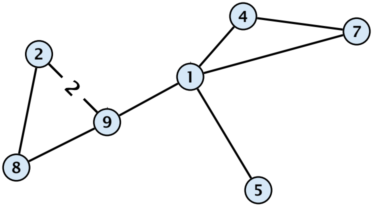

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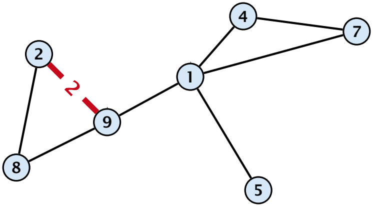

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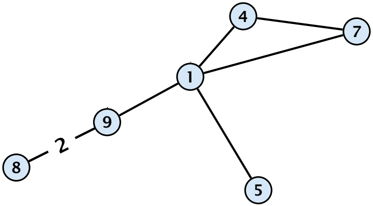

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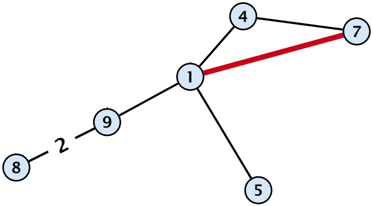

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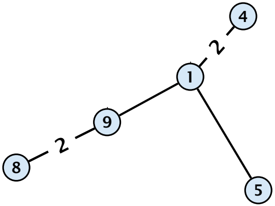

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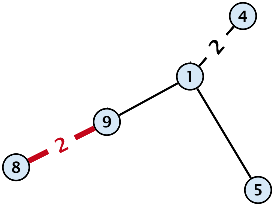

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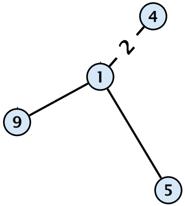

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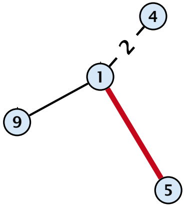

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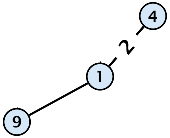

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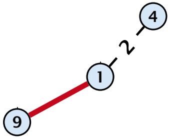

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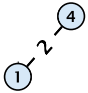

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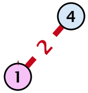

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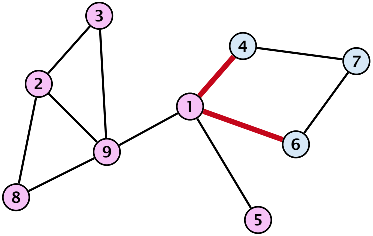

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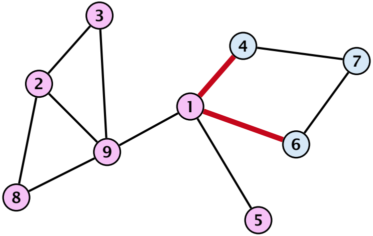

What is the probability that this algorithm returns a mincut?

Analysis

What is the probability that a given mincut A is still possible after round i?

- It is still possible to obtain cut A in the end if so far no edge in $(A, V \setminus A)$ has been contracted.

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- Hence, the probability of choosing an edge from the cut is at most $\min / c(E) \leq 2/(n-i+1)$.

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Choosing $t=2$ gives that with probability $1/\binom{n}{2}$ the algorithm computes a mincut.

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Theorem 81

The randomized mincut algorithm computes an optimal cut with high probability. The total running time is $\mathcal{O}(n^4\log n)$.

Improved Algorithm

Algorithm 2 RecursiveMincut$(G = (V, E, c))$

```
for i = 1 → n − n/√2 do
    choose e ∈ E randomly with probability c(e)/c(E)
    G ← G/e
if |V| = 2 return cut-value;
cuta ← RecursiveMincut(G);
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return min{cuta, cutb}
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3: $G \leftarrow G/e$
4: **if** $|V| = 2$ **return** cut-value;
5: $cuta \leftarrow$ RecursiveMincut(G);
6: $cutb \leftarrow$ RecursiveMincut(G);
7: **return** $\min\{cuta, cutb\}$

Running time:

- $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + \mathcal{O}(n^2)$

Improved Algorithm

Algorithm 2 RecursiveMincut$(G = (V, E, c))$

```
for i = 1 → n − n/√2 do
    choose e ∈ E randomly with probability c(e)/c(E)
    G ← G/e
if |V| = 2 return cut-value;
cuta ← RecursiveMincut(G);
cutb ← RecursiveMincut(G);
return min{cuta, cutb}
```

Running time:

- $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + \mathcal{O}(n^2)$
- This gives $T(n) = \mathcal{O}(n^2 \log n)$.

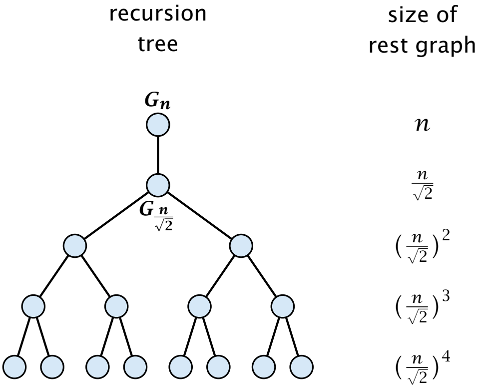
Probability of Success

The probability of not contracting an edge from the mincut during one iteration through the for-loop is at least

$$\frac{t(t-1)}{n(n-1)} \geq \frac{t^2}{n^2} = \frac{1}{2} \ ,$$

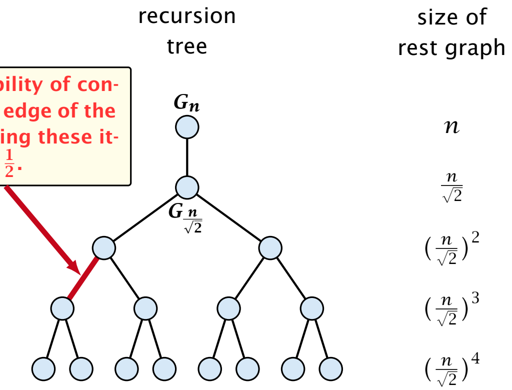
as $t = \frac{n}{\sqrt{2}}$.

Probability of Success

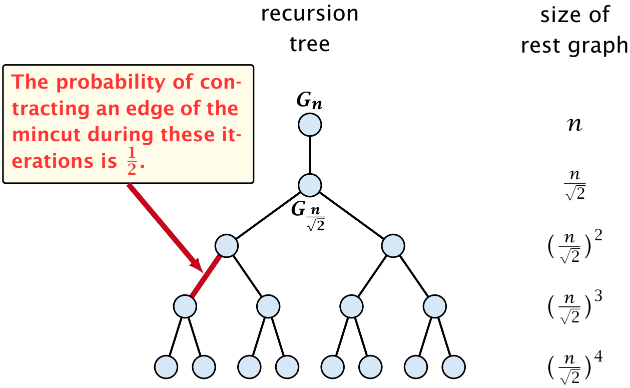


Probability of Success

The probability of contracting an edge of the mincut during these iterations is $\frac{1}{2}$.



Probability of Success



We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability $\frac{1}{2}$. If in the end you have a path from the root to at least one leaf node you are successful.

Probability of Success

Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

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Lemma 82

The probability that an edge e is alive is at least $\frac{1}{h(e)+1}$.

Probability of Success

Proof.

- An edge e with $h(e) = 1$ is alive if and only if it is not deleted. Hence, it is alive with proability at least $\frac{1}{2}$.

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Lemma 83

One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ *and has a success probability of* $\Omega(\frac{1}{\log n})$.

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One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ *and has a success probability of* $\Omega(\frac{1}{\log n})$.

Doing $\Theta(\log^2 n)$ *runs gives that the algorithm succeeds with high probability. The total running time is* $\mathcal{O}(n^2 \log^3 n)$.