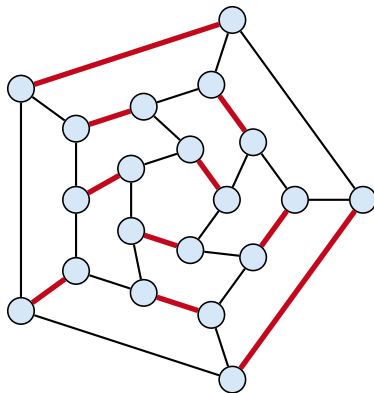


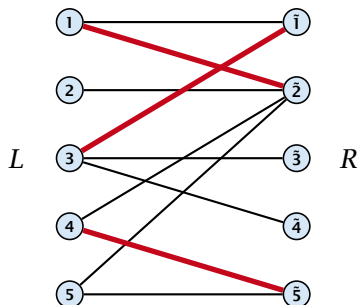
Matching

- ▶ Input: undirected graph $G = (V, E)$.
- ▶ $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- ▶ Maximum Matching: find a matching of maximum cardinality



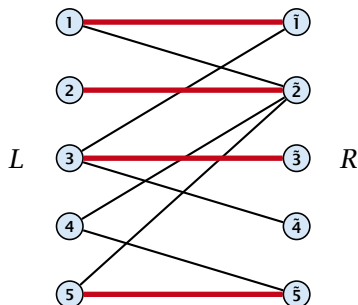
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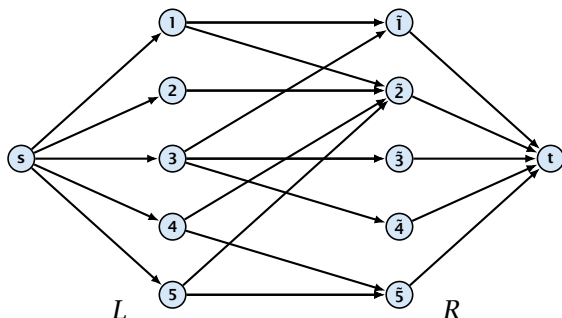
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Maxflow Formulation

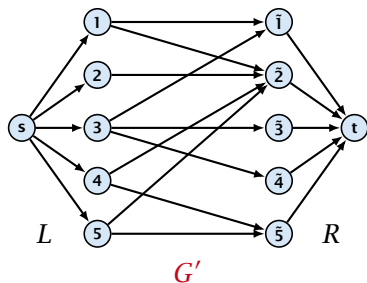
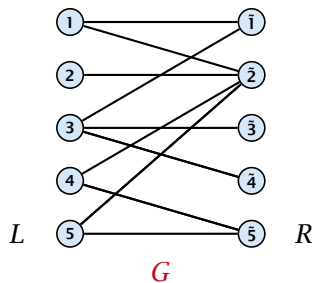
- ▶ Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s, t\}, E')$.
- ▶ Direct all edges from L to R .
- ▶ Add source s and connect it to all nodes on the left.
- ▶ Add t and connect all nodes on the right to t .
- ▶ All edges have unit capacity.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

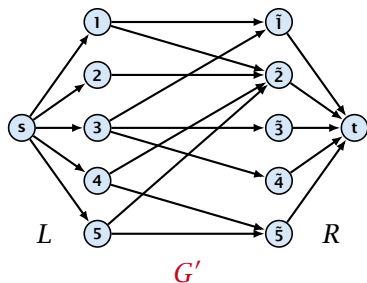
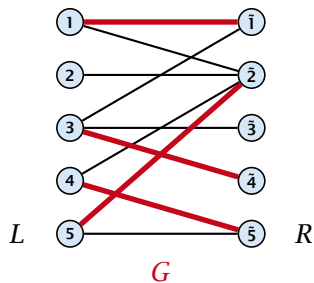
- ▶ Given a maximum matching M of cardinality k .
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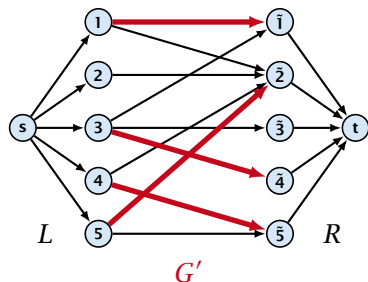
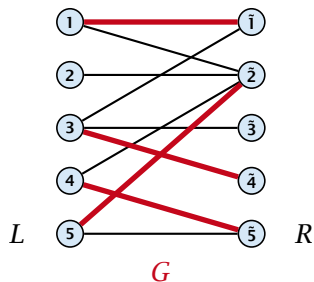
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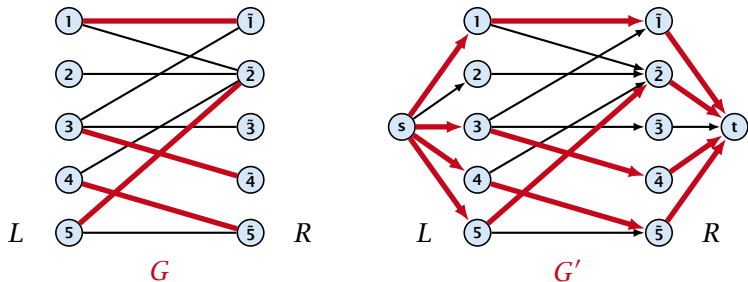
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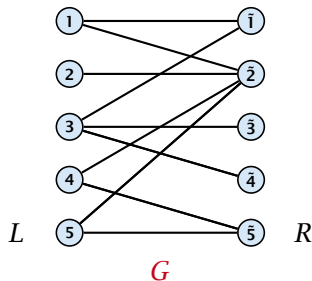
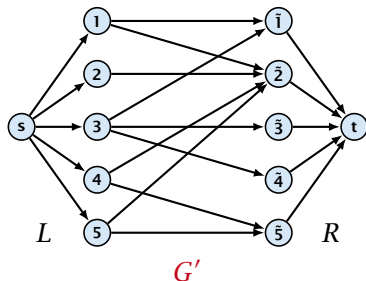
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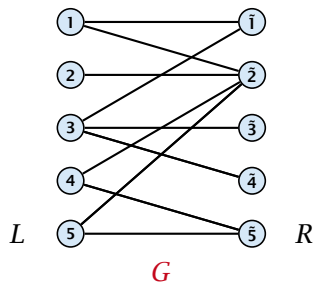
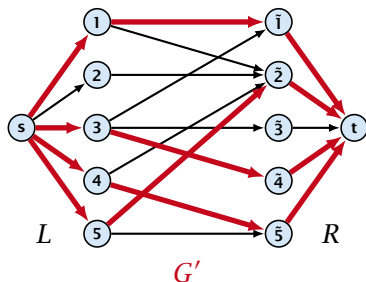
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- ▶ Consider $M =$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M .
- ▶ $|M| = k$, as the flow must use at least k middle edges.



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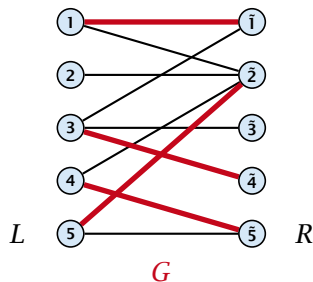
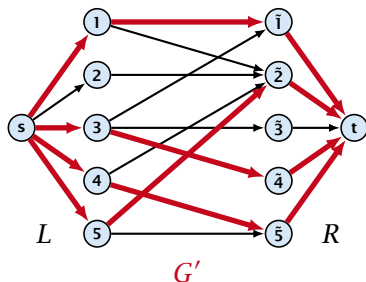
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8.1 Matching

Which flow algorithm to use?

- ▶ Generic augmenting path: $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$.
- ▶ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.
- ▶ Shortest augmenting path: $\mathcal{O}(mn^2)$.

For **unit capacity simple graphs** shortest augmenting path can be implemented in time $\mathcal{O}(m\sqrt{n})$.

A graph is a **unit capacity simple graph** if

- ▶ every edge has capacity 1
- ▶ a node has either at most one leaving edge **or** at most one entering edge

Baseball Elimination

<i>team</i> <i>i</i>	<i>wins</i> w_i	<i>losses</i> ℓ_i	<i>remaining games</i>			
			<i>Atl</i>	<i>Phi</i>	<i>NY</i>	<i>Mon</i>
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

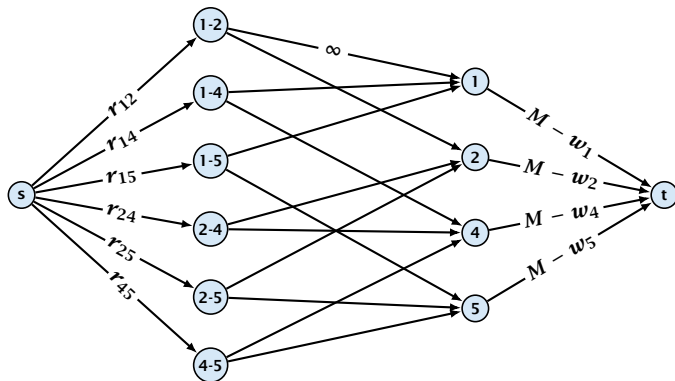
Baseball Elimination

Formal definition of the problem:

- ▶ Given a set S of teams, and one specific team $z \in S$.
- ▶ Team x has already won w_x games.
- ▶ Team x still has to play team y , r_{xy} times.
- ▶ Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.




Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 51

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i \in S \setminus \{z\}, i < j} r_{ij}$.

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- ▶ This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

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- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- ▶ Hence, team z is not eliminated.

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Project selection problem:

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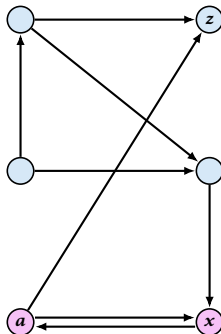
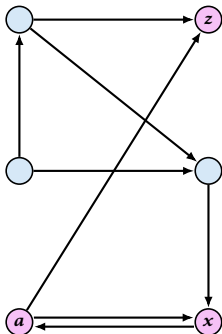
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Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

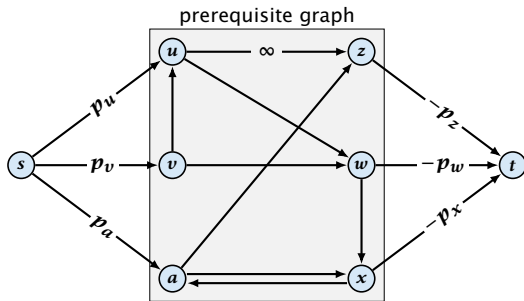
- ▶ $\{x, a, z\}$ is a feasible subset.
- ▶ $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge (s, v) with capacity p_v for nodes v with positive profit.
- ▶ Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



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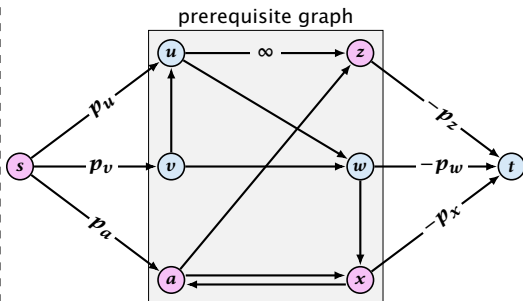
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For the formula we define $p_s := 0$.

The step follows by adding $\sum_{v \in A: p_v > 0} p_v - \sum_{v \in A: p_v > 0} p_v = 0$.

Note that minimizing the capacity of the cut $(A, V \setminus A)$ corresponds to maximizing profits of projects in A .



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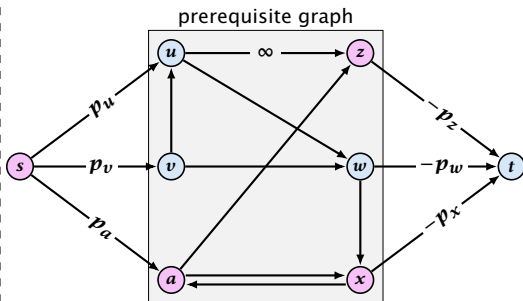
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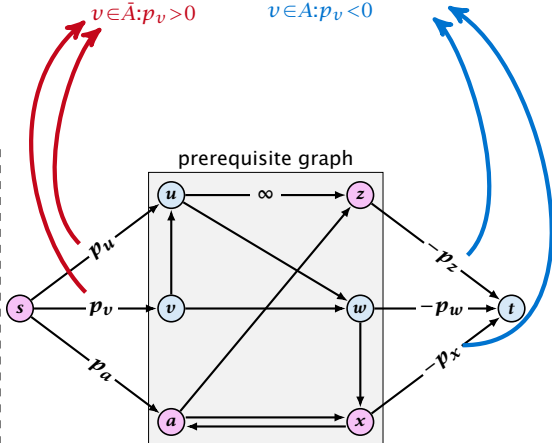
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