What do you measure?

- Memory requirement
- Running time
- Number of comparisons
- Number of multiplications
- Number of hard-disc accesses
- Program size
- Power consumption



How do you measure?

- Implementing and testing on representative inputs
 - How do you choose your inputs?
 - May be very time-consuming.
 - Very reliable results if done correctly.
 - Results only hold for a specific machine and for a specific set of inputs.
- Theoretical analysis in a specific model of computation.
 - Gives asymptotic bounds like "this algorithm always runs in time $\mathcal{O}(n^2)$ ".
 - Typically focuses on the worst case.
 - Can give lower bounds like "any comparison-based sorting algorithm needs at least Ω(n log n) comparisons in the worst case".



Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

- the size of the input (number of bits)
- the number of arguments

Example 1

Suppose *n* numbers from the interval $\{1, ..., N\}$ have to be sorted. In this case we usually say that the input length is *n* instead of e.g. $n \log N$, which would be the number of bits required to encode the input.



Model of Computation

How to measure performance

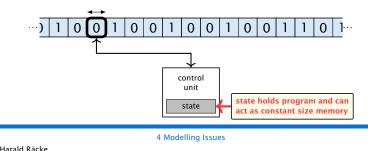
- Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
- 2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.



Turing Machine

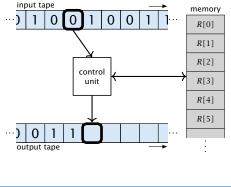
- Very simple model of computation.
- Only the "current" memory location can be altered.
- Very good model for discussing computabiliy, or polynomial vs. exponential time.
- Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.
- \Rightarrow Not a good model for developing efficient algorithms.





Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers R[0], R[1], R[2],
- Registers hold integers.
- Indirect addressing.



Note that in the picture on the right the tapes are one-directional, and that a READ- or WRITE-operation always advances its tape.



4 Modelling Issues

9. Jan. 2023

Random Access Machine (RAM)

Operations

• input operations (input tape $\rightarrow R[i]$)

▶ READ *i*

• output operations ($R[i] \rightarrow$ output tape)

▶ WRITE *i*

- register-register transfers
 - $\blacktriangleright R[j] := R[i]$
 - $\blacktriangleright R[j] := 4$
- indirect addressing
 - $\blacktriangleright R[j] := R[R[i]]$

loads the content of the R[i]-th register into the *j*-th register

 $\blacktriangleright R[R[i]] := R[j]$

loads the content of the j-th into the R[i]-th register



Random Access Machine (RAM)

Operations

branching (including loops) based on comparisons

jump x jumps to position x in the program; sets instruction counter to x; reads the next operation to perform from register R[x]
jumpz x R[i] jump to x if R[i] = 0 if not the instruction counter is increased by 1;
jumpi i jump to R[i] (indirect jump);
arithmetic instructions: +, -, ×, /

```
R[i] := R[j] + R[k];
R[i] := -R[k];
```

The jump-directives are very close to the jump-instructions contained in the assembler language of real machines.



Model of Computation

- uniform cost model
 Every operation takes time 1.
- logarithmic cost model

The cost depends on the content of memory cells:

- The time for a step is equal to the largest operand involved;
- The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w , where usually $w = \log_2 n$.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.



Example 2

Algorithm 1 RepeatedSquaring(n) 1: $r \leftarrow 2$; 2: for $i = 1 \rightarrow n$ do 3: $r \leftarrow r^2$ 4: return r

- running time (for Line 3):
 - uniform model: n steps
 - Iogarithmic model:

 $2 + 3 + 5 + \dots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$

- space requirement:
 - uniform model: $\mathcal{O}(1)$
 - logarithmic model: $\mathcal{O}(2^n)$



There are different types of complexity bounds:

best-case complexity:

 $C_{\rm bc}(n) := \min\{C(x) \mid |x| = n\}$

Usually easy to analyze, but not very meaningful.

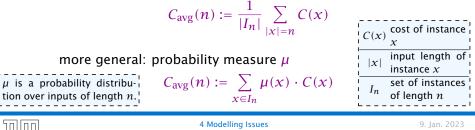
worst-case complexity:

```
C_{\rm wc}(n) := \max\{C(x) \mid |x| = n\}
```

Usually moderately easy to analyze; sometimes too pessimistic.

```
average case complexity:
```

larald Räcke



There are different types of complexity bounds:

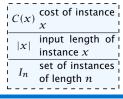
amortized complexity:

The average cost of data structure operations over a worst case sequence of operations.

randomized complexity:

The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x.

Then take the worst-case over all x with |x| = n.





is a probability distribu-

tion over inputs of length n.

Bibliography

- [MS08] Kurt Mehlhorn, Peter Sanders: Algorithms and Data Structures — The Basic Toolbox, Springer, 2008
- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein: Introduction to algorithms (3rd ed.), McGraw-Hill, 2009

Chapter 2.1 and 2.2 of [MS08] and Chapter 2 of [CLRS90] are relevant for this section.

