

## 5.9 Hashing

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- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
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**Hashing** tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

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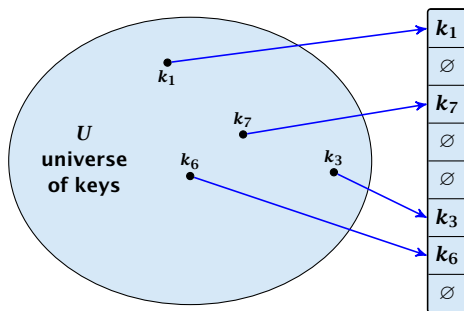
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### The hash-function $h$ should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

# Direct Addressing

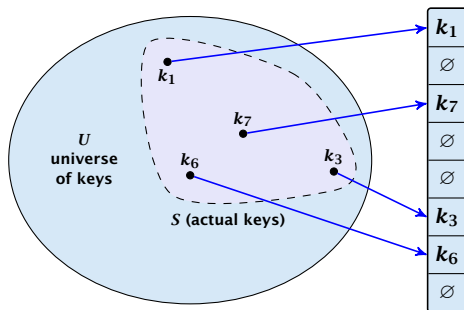
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

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## Problem: Collisions

Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

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Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

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## Lemma 16

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

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## Uniform hashing:

Choose a hash function uniformly at random from all functions  $f : U \rightarrow [0, \dots, n-1]$ .

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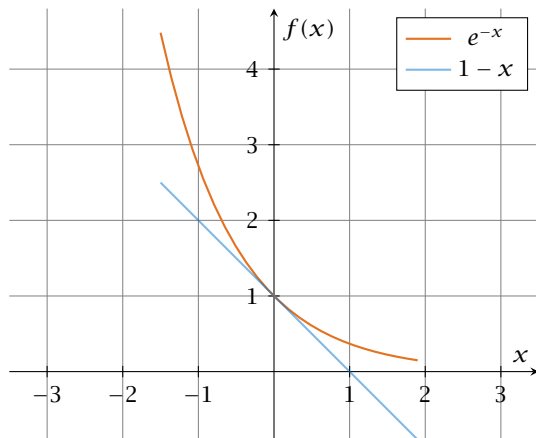
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Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

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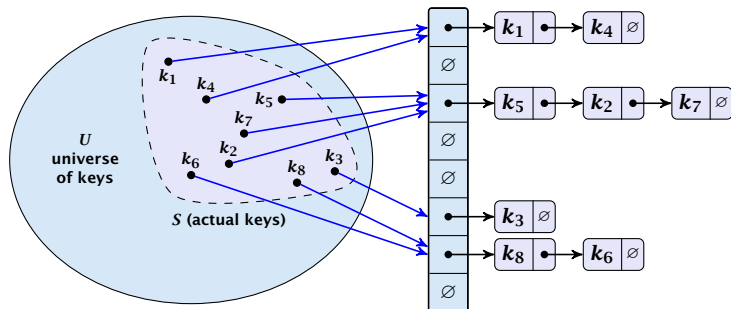
- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.





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We assume **uniform hashing** for the following analysis.

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$$A^- = 1 + \alpha .$$



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Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.



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**Search( $k$ ):** Try position  $h(k, 0)$ ; if it is empty your search fails; otw. continue with  $h(k, 1), h(k, 2), \dots$

**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \bmod n$$

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▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$



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(sometimes:  $h(k, i) = h(k) + ci \bmod n$ ).

- ▶ **Quadratic probing:**

$$h(k, i) = h(k) + c_1i + c_2i^2 \bmod n.$$

- ▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \bmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (**teilerfremd**); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

# Linear Probing

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## Lemma 17

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

# Quadratic Probing

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## Lemma 18

Let  $Q$  be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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## Lemma 19

Let  $D$  be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right)$$

$$D^- \approx \frac{1}{1 - \alpha}$$

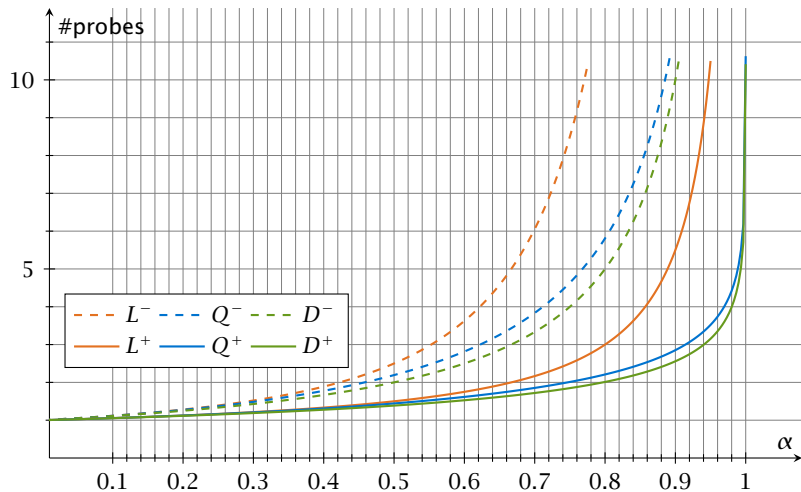


# Open Addressing

Some values:

| $\alpha$ | <i>Linear Probing</i> |       | <i>Quadratic Probing</i> |       | <i>Double Hashing</i> |       |
|----------|-----------------------|-------|--------------------------|-------|-----------------------|-------|
|          | $L^+$                 | $L^-$ | $Q^+$                    | $Q^-$ | $D^+$                 | $D^-$ |
| 0.5      | 1.5                   | 2.5   | 1.44                     | 2.19  | 1.39                  | 2     |
| 0.9      | 5.5                   | 50.5  | 2.85                     | 11.40 | 2.55                  | 10    |
| 0.95     | 10.5                  | 200.5 | 3.52                     | 22.05 | 3.15                  | 20    |

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

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$$\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}]$$

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$$\begin{aligned} & \Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ & \quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}] \end{aligned}$$

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# Analysis of Idealized Open Address Hashing

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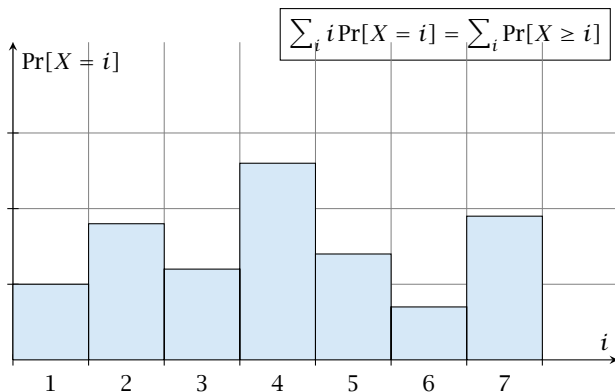


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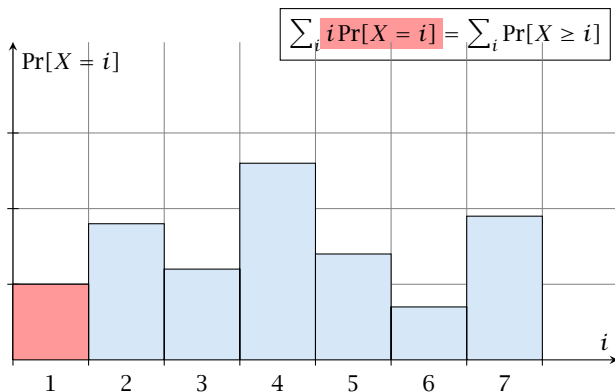
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

# Analysis of Idealized Open Address Hashing



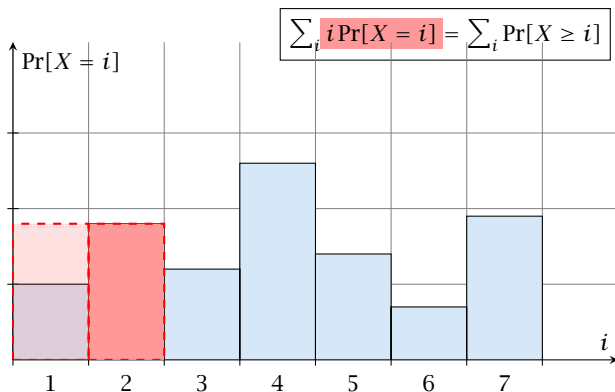
# Analysis of Idealized Open Address Hashing

$i = 1$



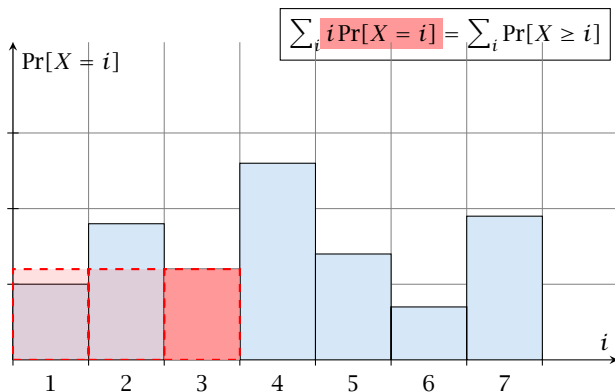
# Analysis of Idealized Open Address Hashing

$i = 2$



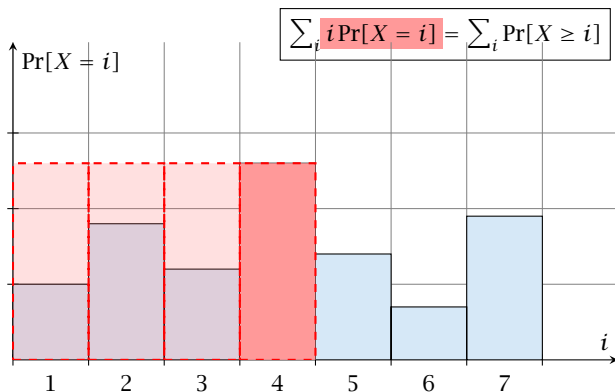
# Analysis of Idealized Open Address Hashing

$i = 3$



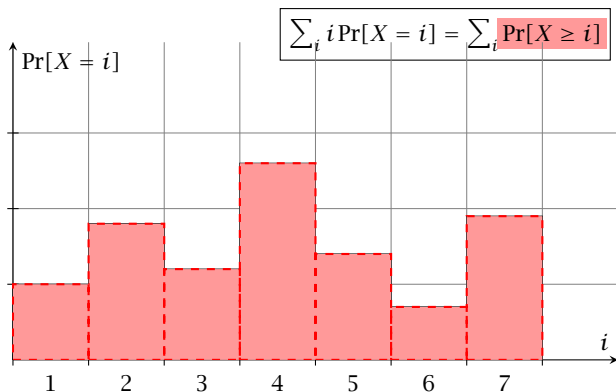
# Analysis of Idealized Open Address Hashing

$i = 4$



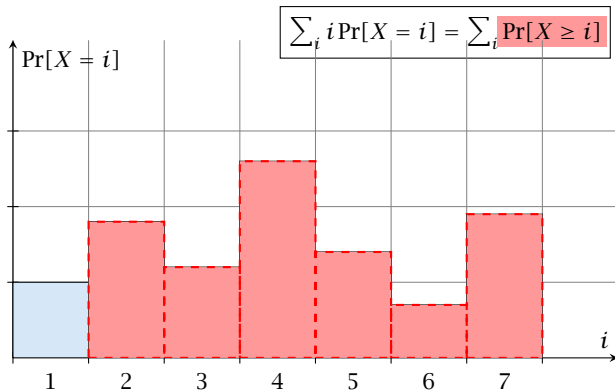
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$i = 1$



# Analysis of Idealized Open Address Hashing

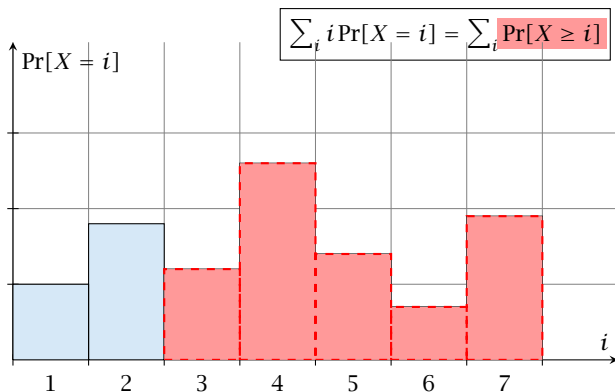
$i = 2$





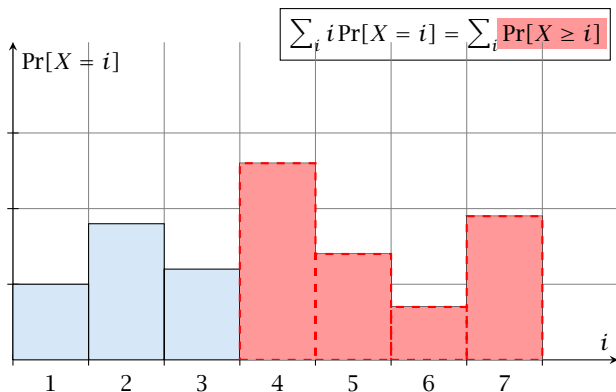
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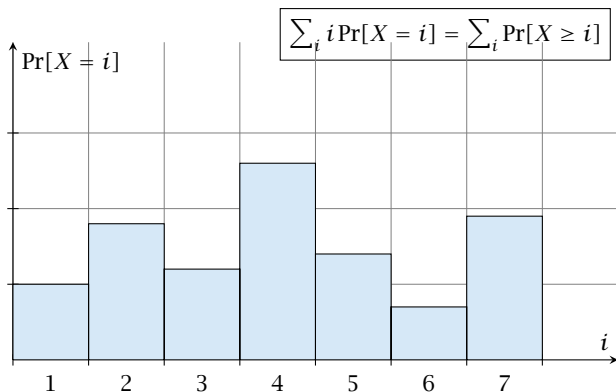


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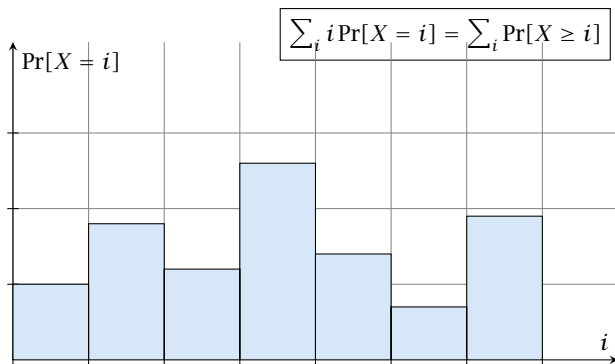
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# Analysis of Idealized Open Address Hashing



# Analysis of Idealized Open Address Hashing



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

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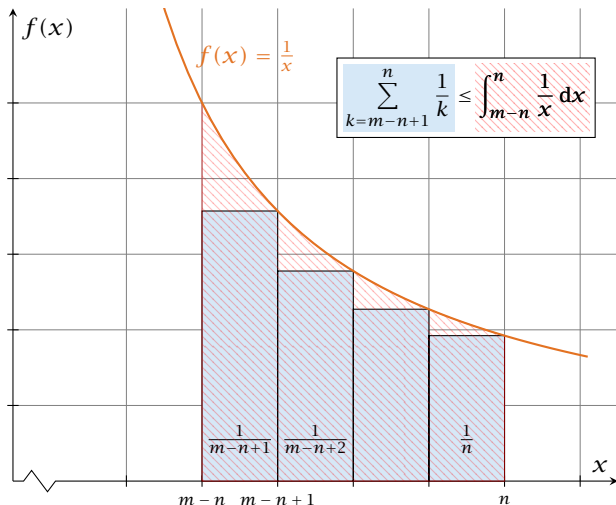
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# Analysis of Idealized Open Address Hashing



## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- ▶ For open addressing this is difficult.



## Deletions in Hashtables

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  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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- ▶ For Linear Probing one can delete elements without using **deletion**-markers.
- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.



## Deletions for Linear Probing

### Algorithm 37 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:      $y \leftarrow T[p]$ 
5:      $T[p] \leftarrow \text{null}$ 
6:      $p \leftarrow \text{succ}(p)$ 
7:     insert( $y$ )
```

$p$  is the index into the table-cell that contains the object to be deleted.

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Pointers into the hash-table become invalid.

# Universal Hashing

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Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

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However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

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Universal hashing tries to define a set  $\mathcal{H}$  of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from  $\mathcal{H}$ .

# Universal Hashing

## Definition 20

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .



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where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

# Universal Hashing

## Definition 21

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ , i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

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# Universal Hashing

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This requirement clearly implies a universal hash-function.

## Definition 22

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

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Let  $U := \{0, \dots, p - 1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p - 1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p - 1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

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## Lemma 24

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

*is a universal class of hash-functions from  $U$  to  $\{0, \dots, n-1\}$ .*

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where we use that  $\mathbb{Z}_p$  is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

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- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .



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$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

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From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

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It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)}$$

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Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 25

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

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For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

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The polynomial is defined by  $d+1$  distinct points.



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Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

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Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

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Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.



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possibilities to choose  $\bar{a}$  such that  $h_{\bar{a}} \in A_\ell$ .

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Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

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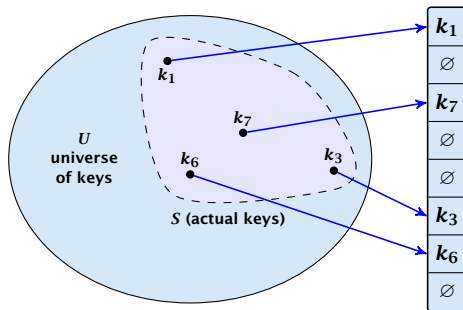
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This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.





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The probability of having **1** or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

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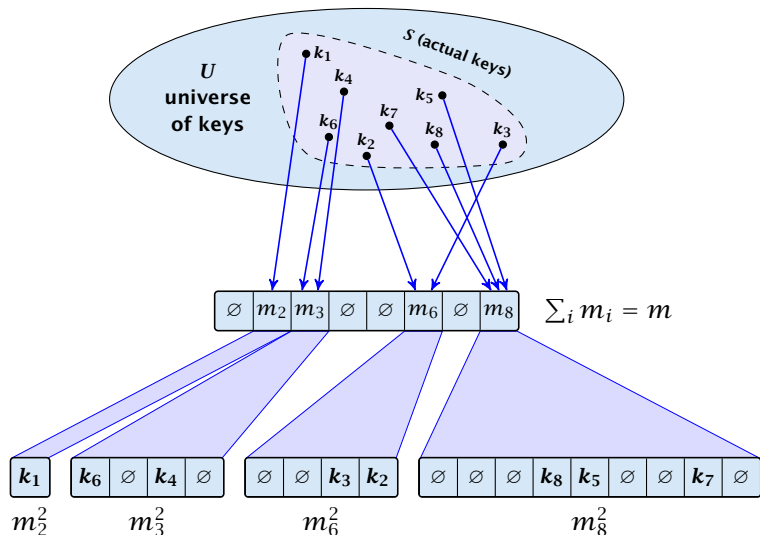
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Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

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The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ .  
Note that  $m_j$  is a random variable.

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The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ . Note that  $m_j$  is a random variable.

$$\begin{aligned} \mathbb{E} \left[ \sum_j m_j^2 \right] &= \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[ \sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[ \sum_j m_j \right] \end{aligned}$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

# Cuckoo Hashing

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## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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- ▶ A search clearly takes constant time if the above constraint is met.



# Cuckoo Hashing

Insert:



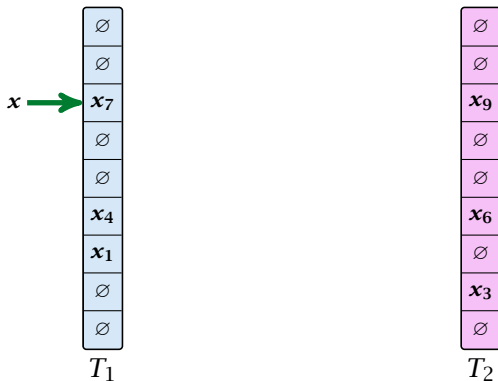
$T_1$



$T_2$

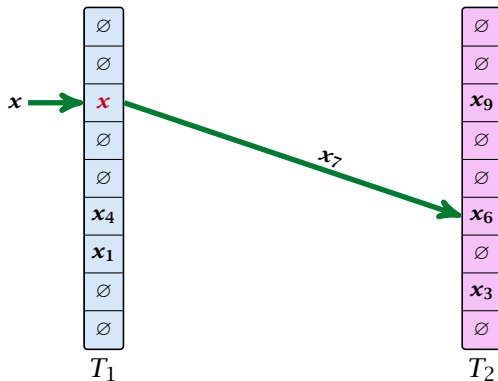
# Cuckoo Hashing

Insert:



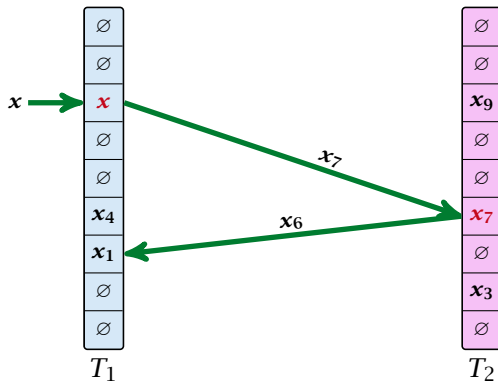
# Cuckoo Hashing

Insert:



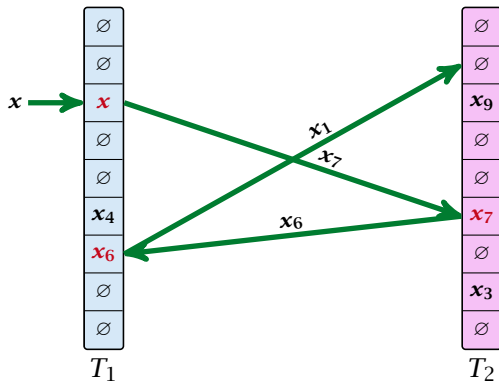
# Cuckoo Hashing

Insert:



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## Algorithm 38 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow 1$   
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.

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- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because  $x = \text{null}$ .

# Cuckoo Hashing

**What is the expected time for an insert-operation?**

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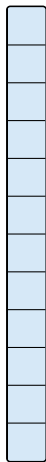
We first analyze the probability that we end-up in an infinite loop (that is then terminated after **maxsteps** steps).

## What is the expected time for an insert-operation?

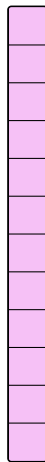
We first analyze the probability that we end-up in an infinite loop (that is then terminated after  $\text{maxsteps}$  steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

# Cuckoo Hashing: Insert

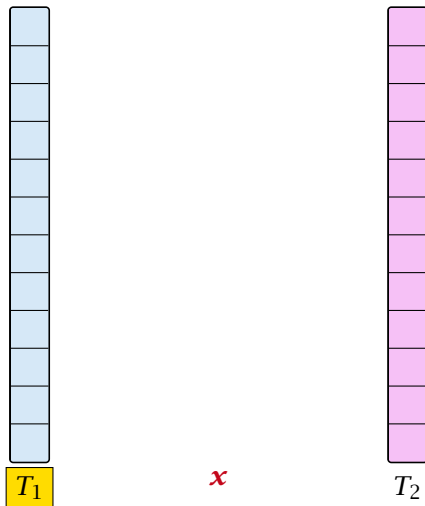


$T_1$

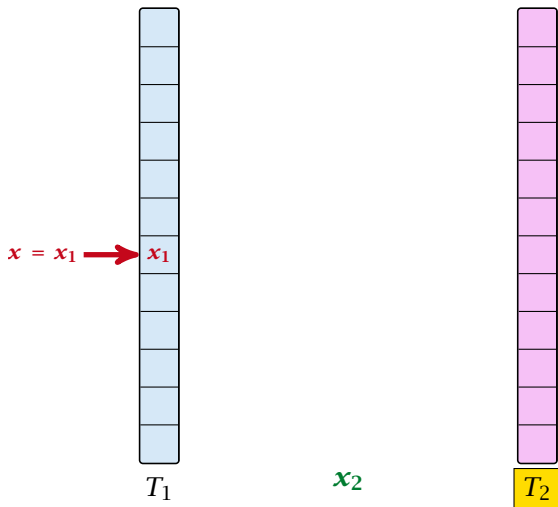


$T_2$

# Cuckoo Hashing: Insert

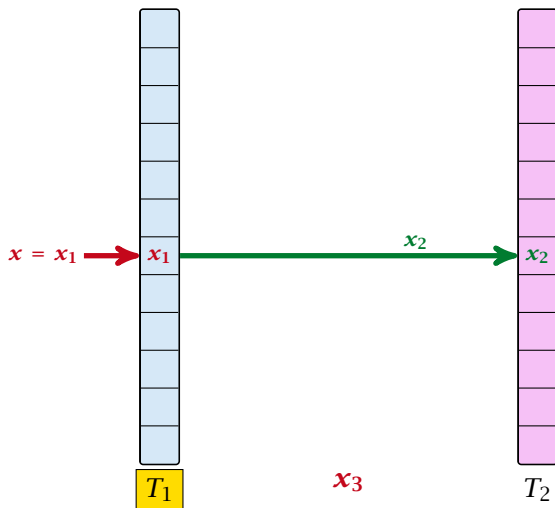


# Cuckoo Hashing: Insert

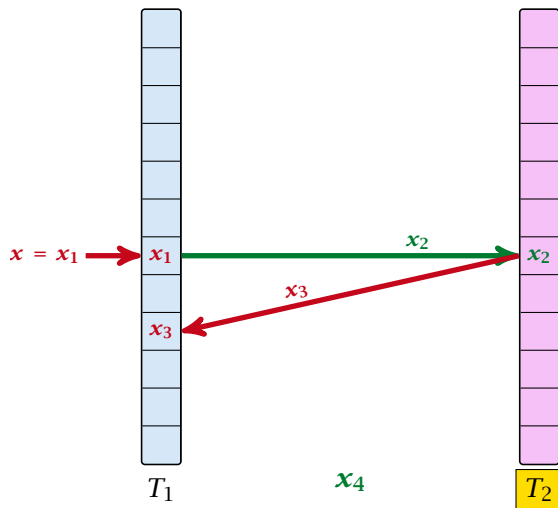




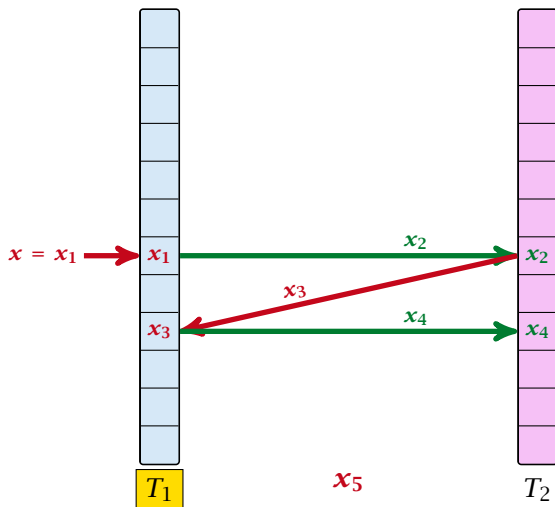
# Cuckoo Hashing: Insert



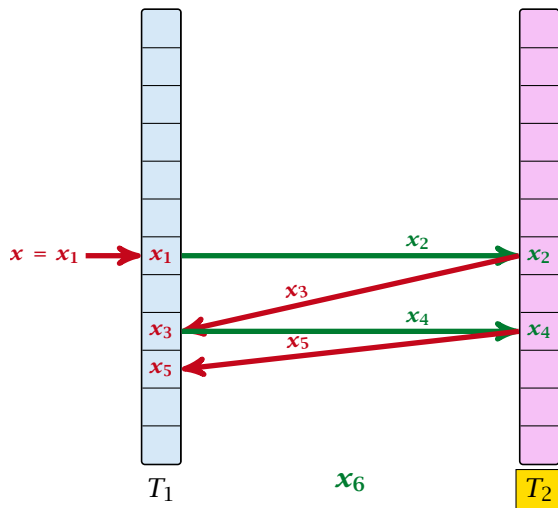
# Cuckoo Hashing: Insert



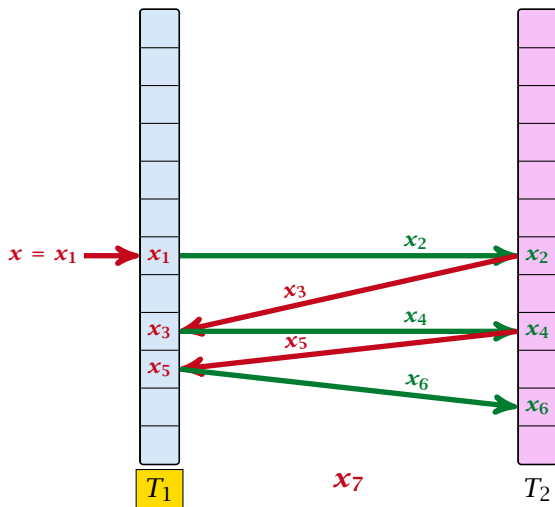
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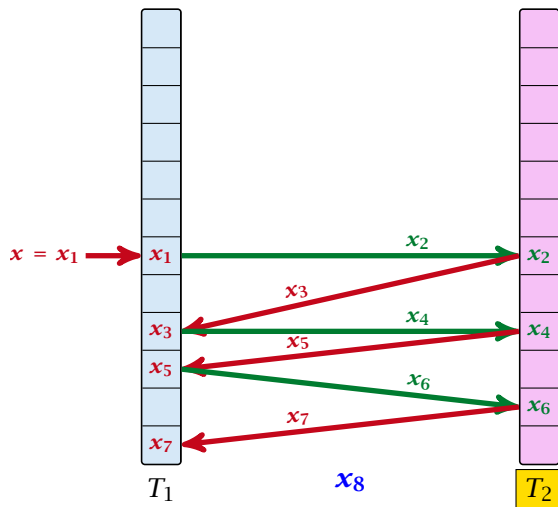
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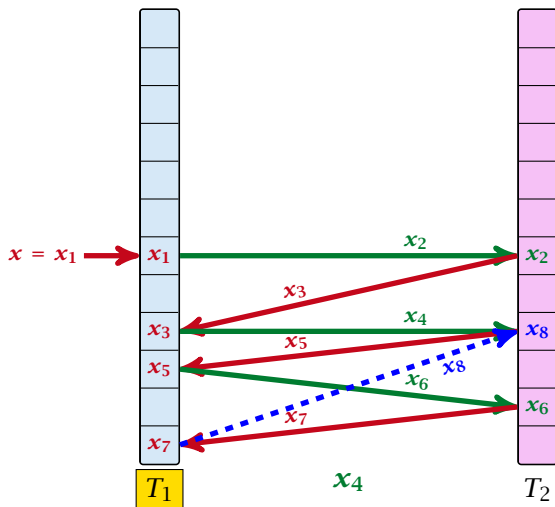
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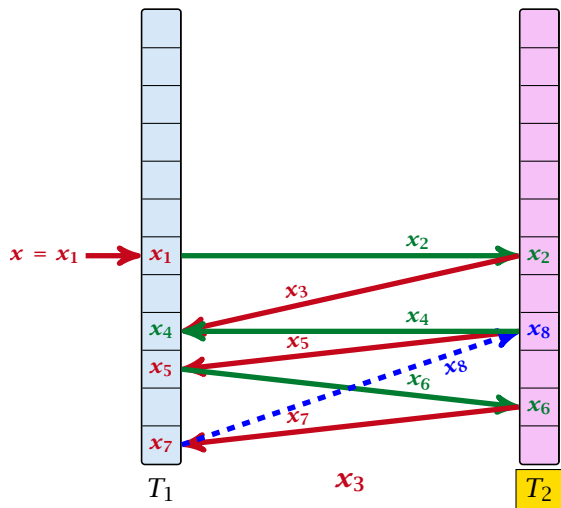
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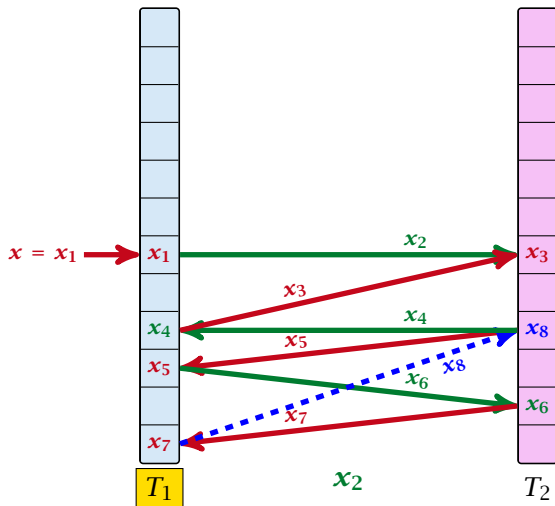


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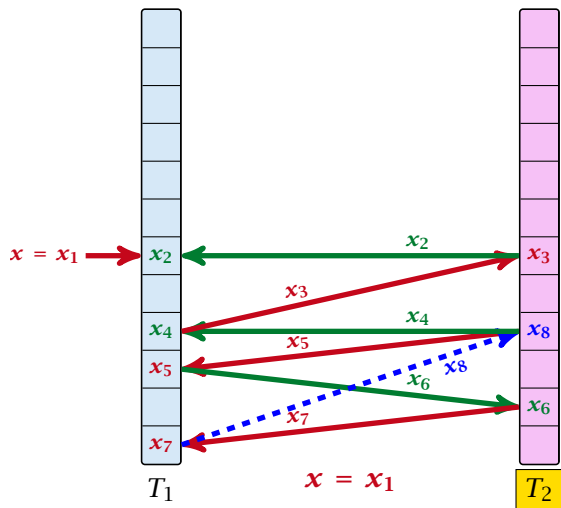




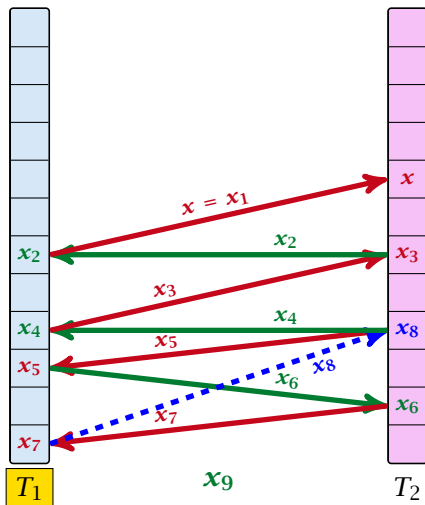
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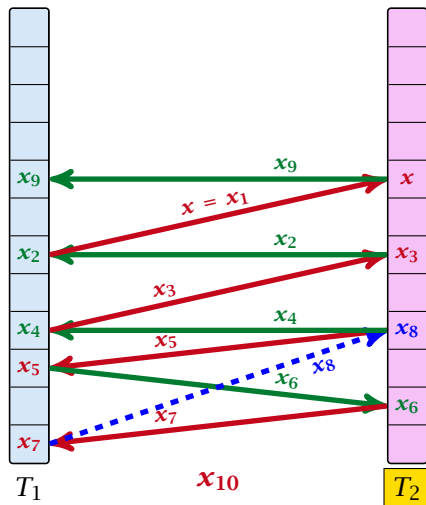
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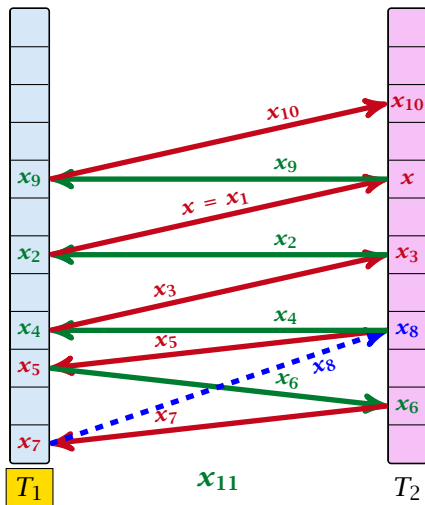
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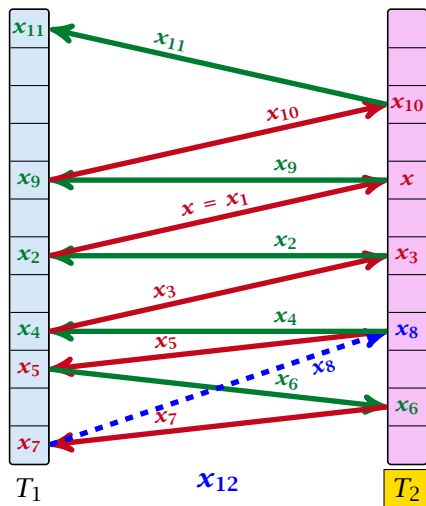
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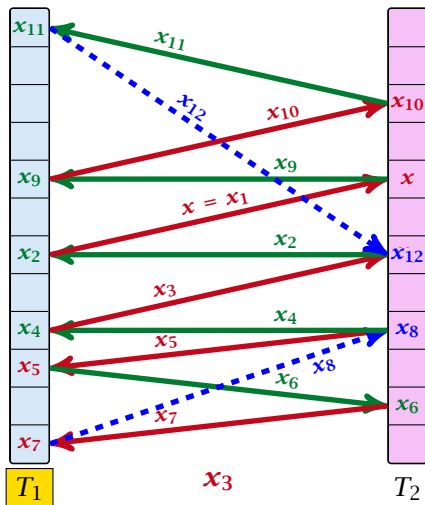
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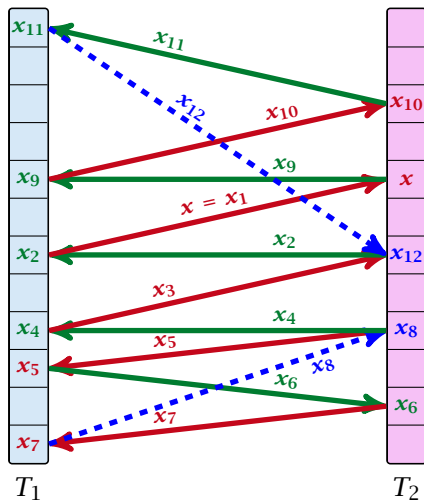
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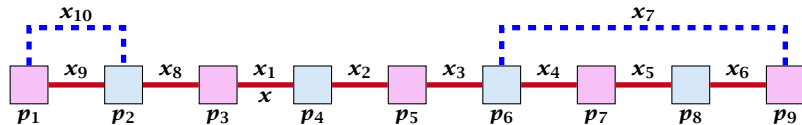


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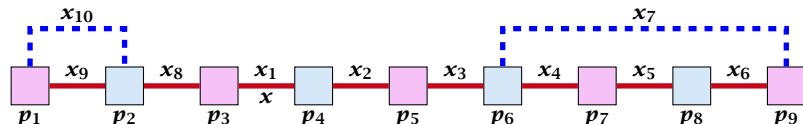


# Cuckoo Hashing



A cycle-structure of size  $s$  is defined by

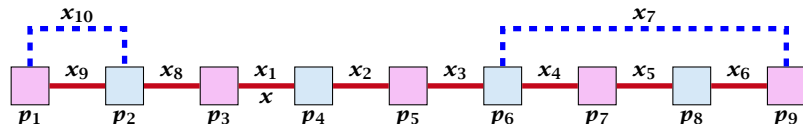
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A cycle-structure of size  $s$  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).

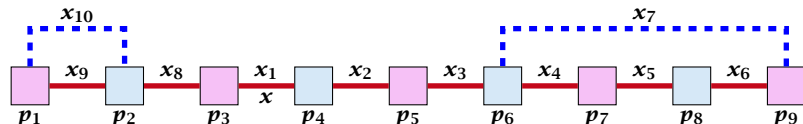
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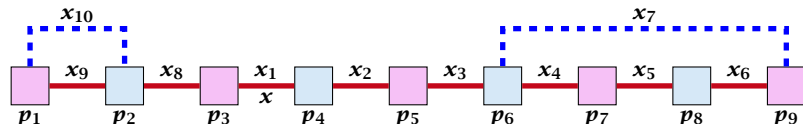
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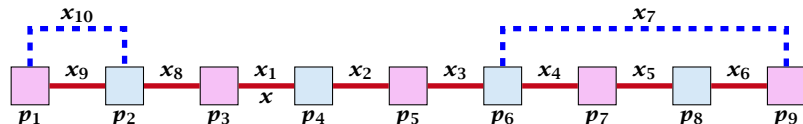
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- ▶ The leftmost cell is “linked forward” to some cell on the right.
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- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

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## Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .



# Cuckoo Hashing

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These events are independent.

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The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

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What is the probability that **there exists** an active cycle structure of size  $s$ ?

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

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- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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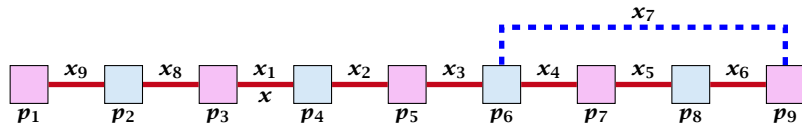
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

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## Lemma 26

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

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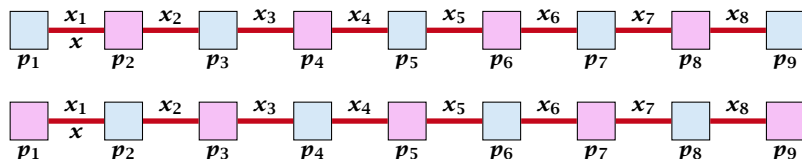
As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

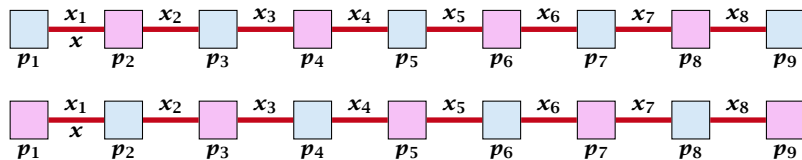


# Cuckoo Hashing



A path-structure of size  $s$  is defined by

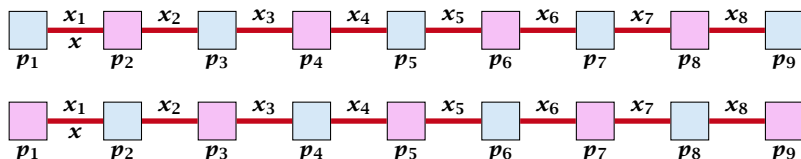
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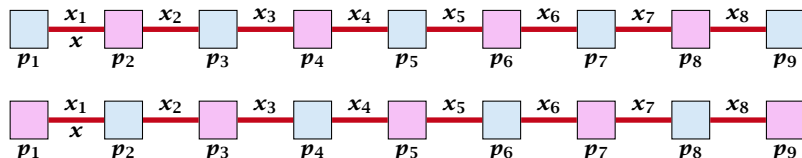
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- ▶ The leftmost cell is either from  $T_1$  or  $T_2$ .

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A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

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This gives  $\text{maxsteps} = \Theta(\log m)$ .

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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for a suitable constant  $c > 0$ .

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).



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Therefore the expected cost for re-hashes is  $\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$ .

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The expected cost for all rehashes is

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Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

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Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

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- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

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Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .

The  $1/(2(1+\epsilon))$  fill-factor comes from the fact that the total hash-table is of size  $2n$  (because we have two tables of size  $n$ ); moreover  $m \leq (1+\epsilon)n$ .