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Proof.

- ▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- ▶ $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.



Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f (along non-zero edges).

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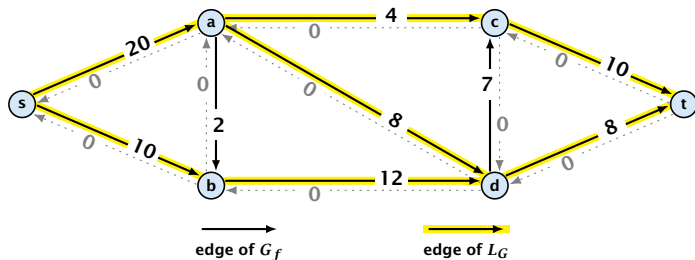
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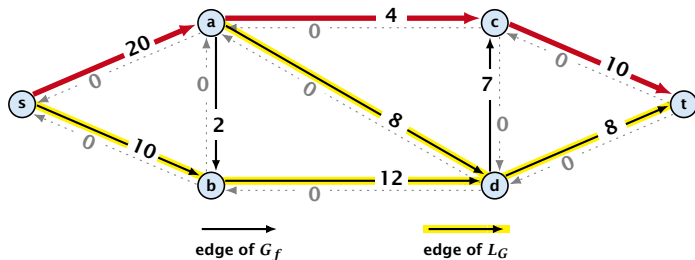


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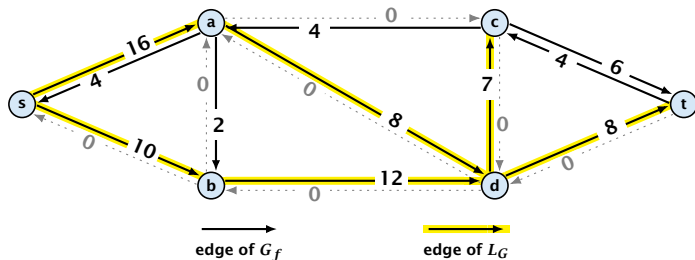


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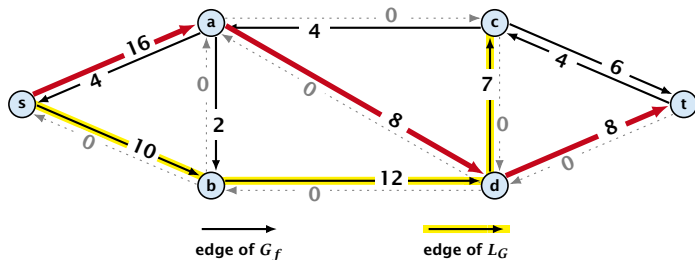


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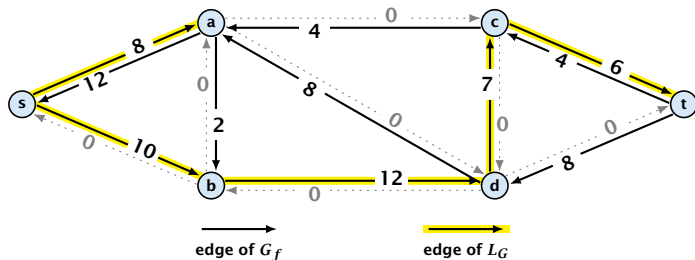


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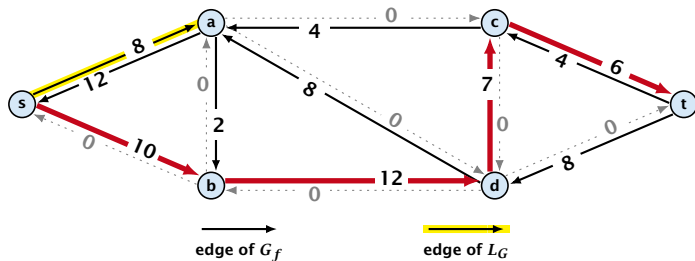


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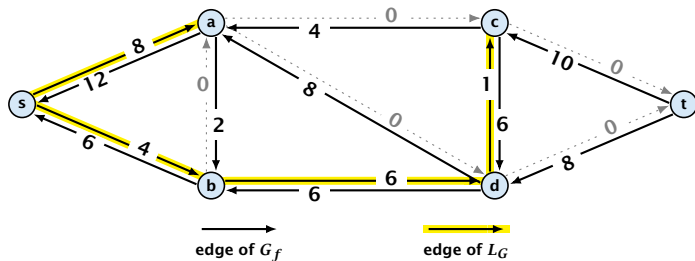


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In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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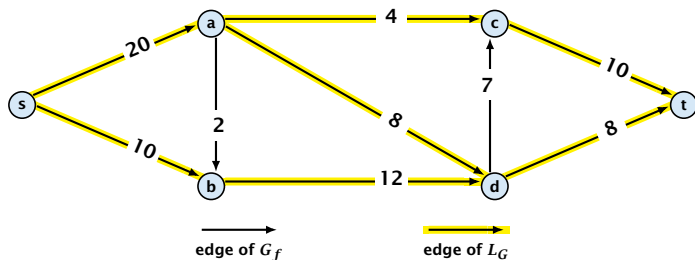
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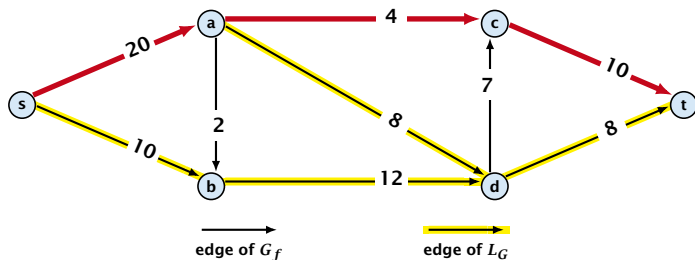
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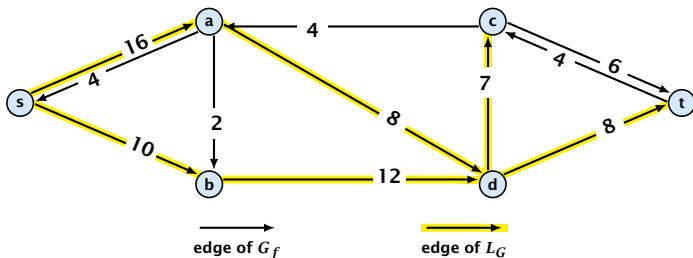
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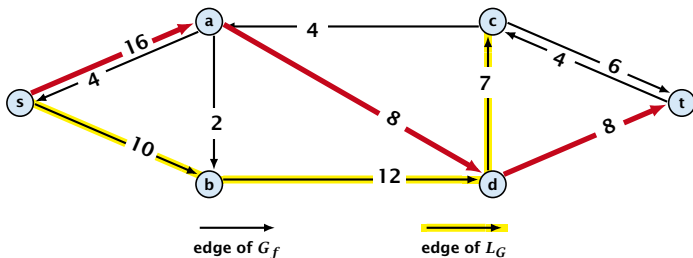
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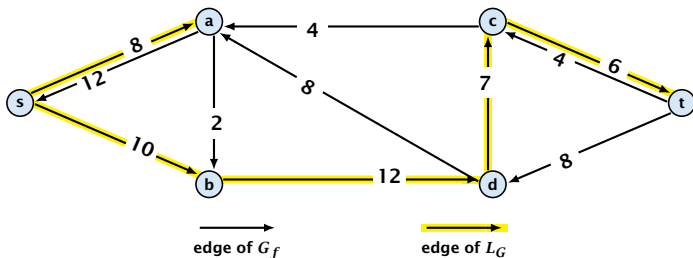
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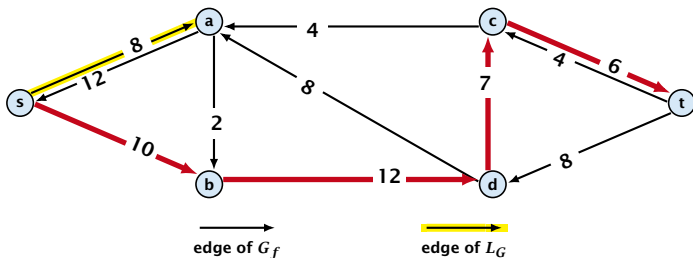
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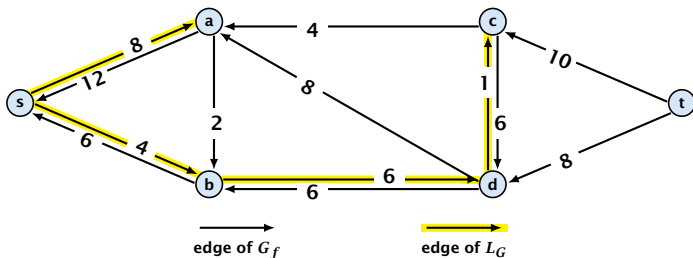
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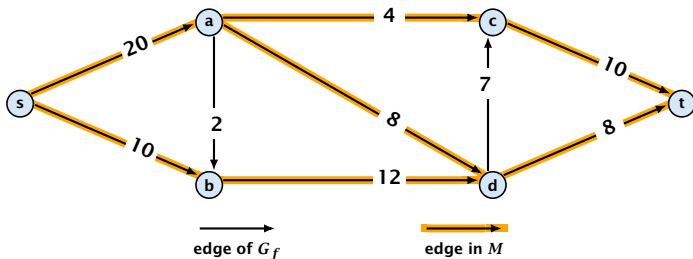
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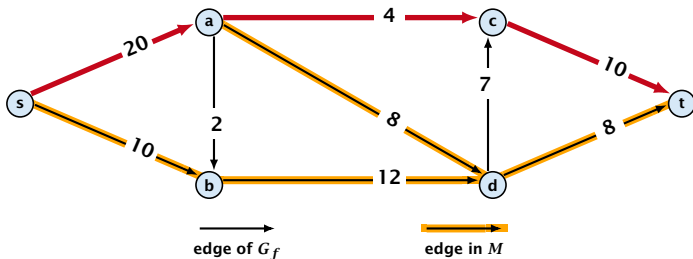
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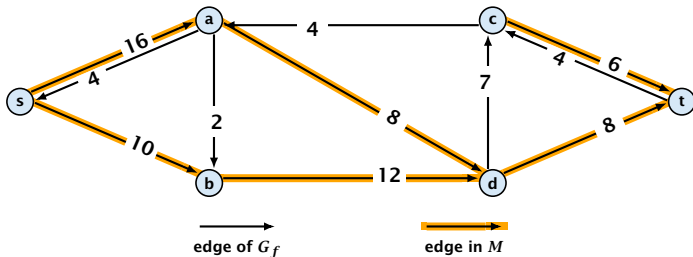
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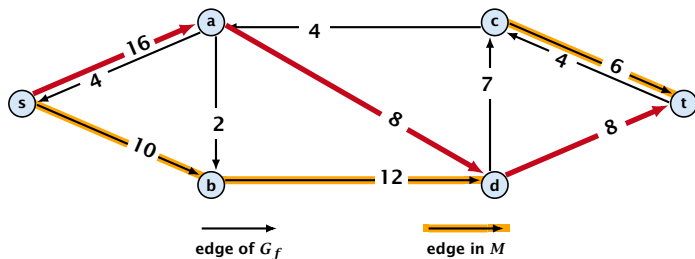
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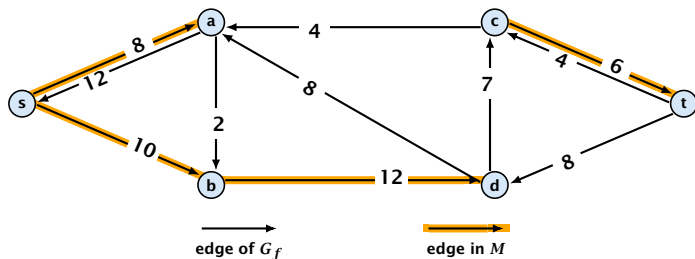
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Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

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However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

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Note that M is not the set of edges of the level graph but a subset of level-graph edges.

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You can delete incoming edges of v from M .

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There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.